Contests with Heterogeneous Agents*

Sérgio O. Parreiras†
Anna Rubinchik–Pessach‡

November 2005

Abstract

We study tournaments with many ex-ante asymmetric contestants, whose valuations for the prize are independently distributed. First, we characterize the equilibria in monotone strategies, second, we provide sufficient conditions for the equilibrium uniqueness and, finally, we reconcile the experimental evidence documenting the 'workaholic' behavior in contests with the related theory by introducing heterogeneity among participants. It is a 'weak' participant that might become a 'workaholic' in an equilibrium, that is, his effort density might *increase* at the highest valuation – weak, either because he is more risk averse or because his rivals consider that it is very unlikely that he has a high value for the prize. In contrast, effort densities are always decreasing in case of symmetry with identically distributed values for the prize and identical attitudes towards risk in case of CARA, as well as in contests with only two participants. Moreover, we show that for low valuations more risk averse agents are less likely to exert low effort than their 'strong' rivals, while those with dominated distribution of the prize valuation are more likely to do so. An explicit solution for the uniform distribution case with contestant-specific support is provided as well.

*We would like to thank participants of the 15th International Conference of Game Theory; the World Game Theory Congress, seminar participants at Duke University, UNC at Chapel Hill, EPGE-FGV-RJ, Queen's University, University of CO at Boulder. Gary Biglaiser, Jim Friedman, Jennifer Lamping, Vijay Krishna, Eric Maskin, Humberto Moreira, Claudio Mezzetti provided useful remarks.

†Department of Economics, UNC at Chapel Hill, sergiop@unc.edu

‡Department of Economics, UC at Boulder, rubinchi@colorado.edu

1 Introduction

It is hard to imagine an area of human activity that does not involve contests. Students striving to be the best in their class, employees awaiting promotion, sportsmen fighting for a gold medal, researchers competing for grants, R&D firms racing to capture monopoly profits in their markets. In most of the cases, losers suffer a loss associated with the invested effort and this is the scenario we consider in this paper. Casual observation suggests that often-times the potential participants can be roughly divided into two main categories: those who invest a minimal effort, being rationally sceptical about the possibility of winning, and the others – workaholics, as Schotter and Muller (2003) call them – who fight with all their might to win a prize. Experimental evidence supports this observation (Schotter and Muller 2003, Noussair and Silver 2005), suggesting that relatively frequent are "low" and "high" effort levels chosen by contestants, while the effort levels in-between the two are picked less often. This finding might seem puzzling in the view of the recent developments in the all-pay auction literature. In a model with two contestants, as in Amann and Leininger (1996), equilibrium bid (effort) density is decreasing, and the same is true in the model with symmetrically distributed valuations for winning the prize, as in Gavious, Moldovanu, and Sela (2002). We show that with more than two contestants whose valuations are distributed differently, but independently, the observed phenomenon – relatively intense effort exerted by contestants with high valuations, – can be explained. Stronger yet, in our set-up *equilibrium* effort density might be increasing at the top.

Interestingly, ex-ante asymmetry of the participants is a sensible description of many of the contests that we observe. In most of the environments mentioned at the outset, for example, contestants have some information about the ability of the others based on commonly observable characteristics, such as general background, previous experience, gender, age, etc. Characteristics differ across individuals, so, naturally, such environments are asymmetric. Often not all the information needed to induce individual valuation is publicly available. Thus, there is some residual uncertainty that contestants might have about each other. In other words, conditional on observable characteristics of an individual, her valuation is still non-deterministic in the eyes of the others. In line with most of the literature on the subject, we assume that contestants' beliefs about the value associated with winning by a particular individual can be reduced to a probability distribution, which is commonly shared by all. In other words, any two contestants share a common prior about distribution of value for the prize held by any third contestant.

We explore two possible interpretations of ex-ante comparative strengths of contestants: their perceived desire to win and the degree of their risk aversion; consequently, these are the two sources of heterogeneity in our model.

In this paper we achieve two goals. First, we characterize equilibria of the asymmetric contests with N contestants. Second, we show that introducing asymmetry might qualitatively change the density of equilibrium effort levels, in particular, a "weak" contestant facing "stronger" contestants might very often either put in a negligible effort level, or – at the other extreme – work extremely hard. This is in contrast with the symmetric model, which generates a monotonically decreasing effort density.

Experiments with more than two contestants (Schotter and Muller 2003, Noussair and Silver 2005) have shown that the empirical distribution of effort has its mode located near the lowest observed effort level and a local mode near the highest observed effort level compatible with rational play. Following, Schotter and Muller (2003), we call the former the drop-out effect and the latter, the race-to-the-bottom effect.

A natural explanation for the drop-out and race-to-the-bottom effects is risk-aversion. In a symmetric model, Fibich, Gavious, and Sela (2004) show that a small increase in the risk-aversion parameter makes low valuations contestants exert less effort and high valuation contestants exert more effort in comparison with the risk-neutral benchmark. A small perturbation of the risk-neutral model, however, does not change the qualitative features of the equilibrium effort (bid) distribution: in particular, the distribution of effort remains always decreasing for small values of the risk-aversion parameter. We show that no matter how risk averse are the contestants, their equilibrium effort density is decreasing if all of them have identical attitudes towards risk with preferences represented by the constant absolute risk aversion function (CARA).

In sum, risk-aversion per se, at least in the CARA case, is incapable of explaining the local mode of high effort. Key ingredients to account for race-to-the-bottom effect are the heterogeneity of the contestants (differences in risk attitudes or in distributions of abilities/value for the prize) and there being more than two contestants.

We show that the equilibrium distribution of effort in an asymmetric, but only two-contestant, contest is also decreasing. In accord with this finding is the available experimental evidence by Kirchamp (2004) showing a negligible race-to-the-bottom effect in two-player contests.

Related literature. Apart from the contributions mentioned above, we will briefly mention some of the related others, clearly, not even attempting to provide an adequate survey of the literature on tournaments and all-pay auctions. Contests under complete information about individual valuations with several participants were analyzed by Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996): apart from knife-edge cases, the two individuals with the highest values for the prize enter the competition, while the rest drop out. Although the full information models might account for a massive drop-out effect, it is hard to interpret the results as generating the race-to-the-bottom effect. Besides, as follows from the examples mentioned at the outset, it might be desirable – from a positive perspective – to leave some uncertainty about rivals' intentions in the model.

In all the literature known to the authors dealing with contests under incomplete information, the contestants are either ex-ante identical, or there are only two participants. The latter case with independent valuations was studied by Amann and Leininger (1996), who, in particular, demonstrated that distribution of effort of one of the contestants might have a mass point at zero (provided the lowest possible valuation is zero), thus, generating a drop-out effect. Lizzeri and Persico (1998) analyzed the two-person contest with affiliated signals. The contests with ex-ante identical participants whose valuations are affiliated were examined (in addition to the contributions mentioned earlier) by Krishna and Morgan (1997). The incomplete information case with many symmetric participants independent signals was extended by Gavious, Moldovanu, and Sela (2002) to allow for non-linear cost of effort, and by Fibich, Gavious, and Sela (2004) to allow for a small degree of risk-aversion. Notably, in the symmetric case it is possible as well to account for the drop-out effect, as the equilibrium effort densities are infinitely high at zero.

With the exception of the models that allow for affiliated signals (Krishna and Morgan 1997, Lizzeri and Persico 1998) all the previous incomplete information all-pay auctions models are nested within our model. Moreover, the mixed strategy equilibrium of the complete information models (Hillman and Riley 1989, Baye, Kovenock, and de Vries 1996) can also be characterized by our Proposition 2.

We present the model next, characterize the equilibrium in Section 2.1, providing sufficient conditions for its uniqueness in Section A.6, and formulate our main results in Section 3. Next, we solve for equilibrium in case the values for winning are distributed uniformly with contestant-specific support and provide some comparative statics for that case, see Section 4. The proofs omitted in the text are in the Appendix.

2 The model

There are $N \geq 2$ individuals competing for a prize. The prize is allocated to the contestant who demonstrates the top performance or achieves the best result. We assume that one's performance fully reflects individual effort. Simply put, effort is observable. The contestants have different values that they associate with receiving the prize, or their desire to win. The payoff to the winner, say, contestant k, who exerts costly effort $b \geq 0$, is

$$u_k\left(v_k - b + w\right),$$

while the losers get $u_j(w-b)$, $j \neq k$, where $w > 0$ is the initial wealth of a contestant, substantial enough, $w > \overline{v}_i$, so that a contestant is never resource-constrained in choosing an effort (bid), the value of which is always bounded by $\overline{v}_i$, her highest possible valuation of the prize. We assume the contestants are weakly risk averse with $u_i : \mathbb{R} \to \mathbb{R}_+$, twice differentiable, strictly increasing and concave.

Before deciding on one's effort, each contestant becomes aware of her own desire to win, v, and, based on the observed characteristics a rival, i, she forms a probabilistic prior with respect to the value, v_i that the rival attaches to winning, the value which is viewed as a random variable, V_i, by all the contestants but i. Naturally, then, in the eyes of all contestants, values of the rivals are distributed independently, but not necessarily identically, $V_i \sim F_i$ on $[\underline{v}, \overline{v}_i]$.

We assume that each F_i is differentiable and that its derivative, the probability density function, f_i, is continuous and is bounded away from zero for all $v \in [\underline{v}, \overline{v}_i]$.[1]

To choose the optimal level of effort, or, simply, a bid, b, any contestant i has to maximize the payoff that will result from placing that bid,

$$\Pi_i(b|v_i) = W_i\left(b; b_{-i}\right) u_i\left(v_i - b + w\right) + \left(1 - W_i\left(b; b_{-i}\right)\right) u_i\left(w - b\right), \qquad (1)$$

[1] We will consider examples in which the latter assumption is relaxed, but will maintain it for the rest of the analysis.

where $W_i(\cdot)$ is the probability of winning, which, in particular, is driven by the effort levels chosen by the others, that is, their strategies.

A strategy for individual i is a Lebesgue–measurable function that maps valuations into effort levels, $b_i : [\underline{v}, \overline{v}_i] \to \mathbb{R}_+$. Athey (2001)'s results imply existence of a Bayes-Nash equilibrium in non-decreasing strategies. Thus, we restrict attention to equilibria in which contestants with higher valuations for the prize expend (weakly) higher effort, or, simply, bid higher. Moreover, if a contestant bids above zero, her strategy is strictly increasing in a Bayes-Nash equilibrium in non-decreasing strategies, as follows from Lemma (16) in the Appendix. This observation enables us to formulate inverse bid functions that associate bid b with the valuation of the contestant who places that bid. Clearly, in an asymmetric environment these functions might vary by contestant.

Let the generalized inverse bid functions be defined as follows:

$$\phi_i(b) = \max\left(\underline{v}, \sup\left\{v : b_i(v) \leq b\right\}\right),\ i = 1, \ldots, N.$$

The generalized inverse bid satisfy the following properties: it agrees with the inverse bid b_i^{-1} whenever the latter is well defined; it is continuous[2]; and it is differentiable almost everywhere since it is a bounded non-decreasing function. Finally given these functions we can determine,

$$F_i(\phi_i(b)) = G_i(b) \equiv \text{Prob}\left[b_i(V_i) \leq b\right],$$

the probability that contestant i bids at or below b. Then the probability of winning by contestant i who bids b can be expressed as the product of cumulative distributions of equilibrium bids,

$$W_i(b) \equiv \prod_{j \neq i} G_j(b).$$

2.1 Equilibrium

Fix the bidding behavior of all the contestants but i. To maximize her payoff (1), contestant i with valuation v_i should choose $b \leq v_i$ to equate the marginal benefit from bidding b and the marginal cost (if such value b exists),

$$\begin{aligned} MB_i(b) &= MC_i(b), \quad (2) \\ MB_i(b) &\equiv \left[u_i(v_i - b + w) - u_i(w - b)\right] W_i'(b) \\ MC_i(b) &\equiv u_i'(w - b)\left(1 - W_i(b)\right) + u_i'(v_i - b + w) W_i(b) \end{aligned}$$

[2]Lemma 17 in the Appendix, section A.3.

where the marginal probability of winning is

$$W_i'(b) = \sum_{j \neq i} \prod_{k \neq i,j} G_k(b) g_j(b).$$

Remark 1 *Since contestants are weakly risk-averse, $MB_i(b) - MC_i(b)$ is strictly increasing in v_i, for $b > 0$. In other words, $\Pi_i(b|v_i)$ satisfies the strict single-crossing property.*

If the marginal benefit is below marginal cost for any choice of $b \in (0, v_i]$, then contestant i with valuation v_i should drop out, or choose $b_i(v_i) = 0$.

To derive the characterization of equilibrium, therefore, we need to identify the "active participants." For this, given effort level $b > 0$ we define an equilibrium set of contestants who choose this effort for some realizations of their valuations for the prize,

$$J(b) = \{ j \in \{1, .., N\} \, | \exists v_j \in [\underline{v}, \overline{v}_j] : b_j(v_j) = b \} .$$

It is important to keep in mind that, in contrast with the symmetric model, this set might not include all the contestants for some $b > 0$; in other words, it might happen in an equilibrium that the contestants choose different bidding intervals. It not hard to see, however, that for any bid b in the support of equilibrium bids there should be at least two contestants who, for some realization of their valuations, choose to bid b, in other words the set $J(b)$ always contains at least two elements. Naturally, if contestant j never exerts as high an effort as some $b > 0$, so that her highest equilibrium bid, $\overline{b}_j$, is strictly below b, then $G_j(b) = 1$.

If satisfied with equality, the system of the first order conditions (2) can be re-arranged in the following form,

$$\sum_{j \neq i} \frac{g_j(b)}{G_j(b)} = S_i(b), \; i \in J(b); \tag{3}$$

where $g_j(b) \equiv f_j(\phi_j(b))\phi_j'(b)$ is the probability density function of the bids placed by contestant i and

$$S_i(b) \equiv \frac{MC_i(b)}{W_i(b)\left(u_i(v_i - b + w) - u_i(w - b)\right)} > 0. \tag{4}$$

Also, in an equilibrium it has to be the case that if contestant i with valuation v_i exerts effort b, then $v_i = \phi_i(b)$.

By inspecting the first order conditions (3), one can easily notice that the right hand side, $S_i(b)$, does not involve effort density functions for any contestant, so the system of equations (3) is linear in $g_i(b)$ for any $b > 0$. The following proposition provides a solution to the system in terms of bid densities. These conditions are necessarily satisfied in an equilibrium.

Proposition 2 *For almost all*[3] $b > 0$*, the system of first order conditions* (3) *can be represented as*

$$g_i(b) = \begin{cases} \frac{G_i(b)}{(K(b)-1)} \left(\sum_{j \in J(b) \setminus \{i\}} S_j(b) - (K(b)-2) S_i(b) \right), & i \in J(b) \\ 0, & \textit{otherwise} \end{cases} \tag{5}$$

where $K(b) = \#J(b)$ *is the number of contestants with a type who bids* b.

This representation of the necessary conditions simplifies the problem of finding an equilibrium in an asymmetric environment. Indeed, the derivative of the inverse bid density, $\phi_i'(b) = \frac{g_i(b)}{f_i(\phi_i(b))}$ of contestant i (when it exists) is expressed in terms of own and rivals' inverse bid densities, $\phi_j(b)$, that determine the corresponding terms $S_j(b)$. Notice the term

$$\Delta S_i(b) \equiv \sum_{j \in J(b) \setminus \{i\}} S_j(b) - (K(b)-2) S_i(b)$$

is negative only if $MC_i(b) > MB_i(b)$, so that it is not optimal for contestant of type v_i to offer that bid. Moreover, as noticed before, $K(b) \geq 2$. Thus, the density is well defined.

Observe that necessary conditions (5) do not assure that bid functions $b_i(v_i)$ are continuous in an equilibrium. In the Appendix, section A.4, we provide a set of sufficient conditions for the continuity of equilibrium strategies in case contestants have linear utility functions, maintaining the assumption that their values can be drawn from different distributions. Along with the continuity, conditions (5) coupled with the condition $b_i(\overline{v}_i) = \overline{b}_i$ fully determine a unique solution, see argument in the Appendix, section A.6. Clearly, in those cases, necessary conditions (5) fully characterize the equilibrium.

Characterization of the equilibrium enables us to formulate the following important properties of the contestants' behavior at the lower bound of their valuations.

[3] More exactly, (5) holds for all $b > 0$ where the inverse bid functions are differentiable and the set of active contestants, $J(b)$, is constant in some neighborhood of b.

Proposition 3 *All contestants choose zero as their lowest bid,* $b_i(\underline{v}) = 0$ *for all* i*. Moreover, if* $\underline{v} > 0$ *and* $b_i(v) > 0$ *for some* v *then either*

1. $G_i(0) = 0$ *and* $\lim_{b \searrow 0} g_i(b) = +\infty$ *for all* i *or*

2. $G_k(0) = 0$ *for some* k *and* $G_j(0) > 0$ *for all* $j \neq k$*.*

Proposition 3 says that only two scenarios are possible. Either a contestant is infinitely more likely to drop-out than to exert any given positive effort level, though drop-out does not occur with positive probability, or exactly $N - 1$ contestants drop-out (for some range of low valuations) in an equilibrium. The proposition requires a contestant to participate actively, i.e., bid a positive amount, for at least some valuations for the prize.

This finding suggests robustness of the result by Amann and Leininger (1996) demonstrating possibility of drop-out behavior in the model with two contestants. Also, drop-out behavior happens with positive probability under complete information (Baye, Kovenock, and de Vries 1996). In those equilibria, contestant j may never exert effort in $(0, e_j)$ while exerting zero effort and an effort above e_j with positive probability. Note that in that case an equilibrium strategy is a map from the values to a distribution over the set of bids and for the bids belonging to the support of a contestant's mixed strategy, condition (2) should be satisfied (with the probability of winning re-defined correspondingly). However, as the first-order conditions of contestant i may not hold with equality in a neighborhood of zero, which is the lowest equilibrium bid, the proof of Proposition 3 can not be employed.[4]

The system (5) can be solved for equilibrium inverse bids. In general, only numerical solutions can be obtained. Nevertheless, we derive closed-form solutions for the case in which valuations are distributed uniformly with the contestant-specific support, see Section 4. Moreover, we can use the sign of the expression $\Delta S_i(b)$ to construct an indicator of individual participation. This enables us to derive the results in Section 3.1, describing sufficient conditions for a complete drop-out of some contestants. Finally, and – most importantly – characterization (5) provides a way to demonstrate our main results resting on the shape of the effort density, in particular, the sign of its derivative at the top bid. We identify cases in which that sign is positive, that is, some of the contestants intensify their efforts provided their valuation is close enough to the top, thus providing a rationalization for

[4]Under incomplete information, $\frac{\partial}{\partial b}\Pi(b|\phi_i(b)) = 0$ for all $b \geq 0$.

the phenomena demonstrated experimentally by Schotter and Muller (2003) and Noussair and Silver (2005), – the phenomena that can not be explained either within a model with symmetric distributions of individual valuations, or with two contestants only.

3 Empirical Predictions

3.1 Participation in Asymmetric Contests

It is known that either in symmetric or two-contestant asymmetric environments, the highest ability type of any contestant exerts the highest equilibrium level of effort. In contrast, asymmetric environments with more than two contestants may give rise to a complete drop-out behavior, that is, in equilibrium a contestant exerts zero effort regardless of his valuation. If contestants are risk-neutral, personal valuation attached to the prize can be alternatively viewed as a reciprocal of one's cost of effort, or just as individual ability.

The following proposition states that if all the contestants are risk-neutral and are likely to have high abilities; if among them there is a contestant i whose highest possible ability is substantially below the average; and if any highest-ability rival exerts the highest equilibrium effort level, then contestant i should not participate in the contest at all, no matter how able he is (or how much he desires the prize).

Proposition 4 *Assume that contestants are risk-neutral. If*

1. *For all j, F_j first-order stochastically dominates $U[0, \overline{v}_j]$;*
2. *There is an i such that the inequality $\overline{v}_i^{-1} > \frac{\sum_{j \neq i} \overline{v}_j^{-1}}{N-2}$ holds;*
3. *For all $j \neq i$, $\phi_j(\overline{b}) = \overline{v}_j$;*

Then $b_i(v) = 0$ for all $v \in [\underline{v}, \overline{v}_i]$.

This proposition can be also used for cases in which some of the $N-1$ never bid. In this case N has to be replaced with the number of active contestants. In other words, the proposition can be applied to rule out participation of several contestants in a consecutive manner.

Proposition 4 has the following partial converse, which provides conditions for participation. It establishes that common support for individual abilities implies common bidding interval in an equilibrium, provided that (1) one of the contestants, who share the same support, places the highest equilibrium bid when she has the highest possible ability; and (2) their equilibrium strategies are continuous.

Proposition 5 *Assume that the contestants are risk-neutral and assume valuations of contestants i and j have common support, that is, $\overline{v}_i = \overline{v}_j$. If $b_i(\overline{v}_i) = \overline{b}$, then $b_j(\overline{v}_j) = \overline{b}$.*

3.2 Main Results: Distribution of Bids under Asymmetry

In the introduction we mentioned that the two-contestant model and the symmetric model with many contestants, that were analyzed in the previous literature, can be tackled within the current framework. Let us start with these cases to assert that neither of them can generate workaholic behavior. Indeed, the effort density is monotonically decreasing for each contestant under symmetry or if there are two players only, simply meaning that higher efforts are chosen less frequently by all participants.

Proposition 6 *Assume that contestants are risk-neutral and at least one of the following conditions is satisfied: (1) the distribution of abilities is the same for all contestants (symmetric model); (2) there are only two contestants. Then, the equilibrium bid probability density function of any contestant is non-increasing.*

We can even strengthen the previous finding by allowing the agents to be risk averse. Provided the distribution of valuations is the same for all and their risk attitudes are the same as well (and satisfy CARA), higher efforts are still more rare in equilibrium, thus, eliminating the second "mode" of the distribution of bids discussed in the experimental literature. The same result is true in the presence of only two contestants with potentially different coefficients of risk aversion.

Proposition 7 *Assume contestants' utility functions exhibit constant absolute risk aversion and the ex-ante distribution of valuations is the same* *for all. Also, assume that at least one of the following conditions is satisfied: (1) the risk-aversion parameter is the same for all contestants; (2) there are only two contestants. Then, the equilibrium bid probability density function of any contestant is non-increasing.*

In contrast, in the presence of asymmetry competition for the prize might become fierce. We start with the risk-neutral case in which distributions of abilities differ across contestants, e.g., some might be perceived as "strong" opponents, while the others are viewed as "weak". Interestingly enough, it is the "weak" contestant – the rivals of whom dismiss almost completely the possibility of her having a high value for the prize, or of her being of the top ability, – it is she who might race to the bottom. The ex-ante weak contestant will do so, provided her ability is, in fact, very high, or close to be the highest – exactly the case almost entirely "overlooked" by her opponents. Provided they are competing mainly among themselves, almost ignoring their weak rival (when placing high bids), it is in the interest of that weak contestant to exert high effort as the chance of winning from doing so for her is sufficiently high. Of course, the rivals are fully aware of the equilibrium strategy of the weak contestant, but in their eyes the likelihood of their opponent being very able and aggressive is sufficiently small, thus, for each of the strong contestants, standing against other strong rivals is relatively more important, and they, indeed, almost ignore the presence of the weak. That is the core intuition behind the following proposition.

Proposition 8 *Assume contestants are risk-neutral, there are more than two contestants and, at least two of them have distinct distributions of abilities. Then, a contestant's effort density function may be increasing at high effort levels.*

In the section that immediately follows we illustrate this proposition with an example and plot the equilibrium effort density of the weak contestant, which in that case is, indeed, bimodal.

Next result demonstrates that if the contestants differ by their attitudes towards risk, similar conclusion can be obtained. Here a weak contestant, – in this case the contestant who is more risk averse – faces stronger rivals, who are, say, risk-neutral. In this case, we show that the weaker one intensifies her effort at higher valuations.

Proposition 9 *Assume contestants' utility functions exhibit constant absolute risk aversion and the ex-ante distribution of valuations is the same for all. Also assume there are more than two contestants and, at least two of them have distinct risk-aversion parameters. Then, a contestant's effort density function may be increasing at high effort levels.*

Interestingly, if we look at a general case with risk-averse (not necessarily CARA) contestants, the 'weak' agent is always more likely to choose high effort levels.

Proposition 10 *Assume contestant i is more risk averse than contestant j and the highest equilibrium effort level of all contestants is $\overline{b}$. Then there exists $\delta > 0$ such that for all $0 < \varepsilon < \delta$, $G_i\left(\overline{b} - \varepsilon\right) < G_j\left(\overline{b} - \varepsilon\right)$. Moreover, $G_i(\varepsilon) < G_j(\varepsilon)$ provided that*[5] $G_i(0) = G_j(0) = 0$.

Proposition 10 also says that the more risk-averse agent is less likely to exert low effort levels, that is, for the tails of the effort distribution, the more risk-averse agent is more aggressive than the less risk-averse agent.

So far, the behavior of the 'weak' agent at the high tail of the effort distribution does not depend on wether the source of asymmetry is the difference in risk-attitudes or if it is the difference in distributions of valuations. For the lower tail of the effort distribution, however, the nature of the heterogeneity matters and this is the main contrast between Proposition 10 and Proposition 11, which follows below.

Before comparing Propositions 10 and 11, let's define formally what means for a contestant to be 'weak' when agents are asymmetric with respect the distribution of valuations: For any two distributions of abilities/valuations, F_j and F_i distributed in the same support $[\underline{v}, \overline{v}]$, we say that F_j strictly first-order stochastic dominates F_i – equivalently, $F_j \succ F_i$, when for all $v \in (\underline{v}, \overline{v})$, we have that $F_i(v) < F_i(v)$. As before, for risk-neutral contestants, when $F_j \succ F_i$, we refer to i and j as respectively the weak and the strong contestants.

[5]The condition $G_i(0) = G_j(0) = 0$ holds true, for example, whenever agents' strategies are continuous and, for any agent, there is another agent who has identical risk preferences, that is, for any k there is l such that $u_k(\cdot) = u_l(\cdot)$. Notice that this example allows for i and j to have different preferences.

Proposition 11 *Assume agents are risk-neutral. If the distributions F_i and F_j have the same support $[\underline{v}, \overline{v}]$, $F_j \succ F_i$, and $f_i(\overline{v}) < f_j(\overline{v})$ holds* [6] *then $G_j \succ G_i$.*

Here, the probability that the weak contestant exerts effort above than b is less than the probability that a strong contestant exerts effort above b. In contrast to Proposition 10, the weak contestant's probability of drop-out (effort levels close to zero) is greater than the strong contestant's drop-out probability.

It is important to remember that although the weak contestant is more likely to exert high effort at the top, that is, his effort density is increasing, it is still below the (possibly decreasing) effort density of a strong contestant. The next section illustrates this implication of Proposition 11.

3.2.1 Increasing Bid Densities: An Example

We want to demonstrate an example of an environment in which the effort density of some contestants is increasing for high abilities, which can not happen in a symmetric environment, or in a two-contestant contest.

Consider the following specification. Abilities are distributed on the unit interval, $[0, 1]$. contestant 1's ability is distributed according to

$$F_1(\phi_1) = 2\phi_1 - \phi_1^2 \tag{6}$$

and the ability of contestant $j \in \{1, 2\}$ follows an uniform distribution, that is

$$F_j(\phi_j) = \phi_j. \tag{7}$$

We define the auxiliary function $Q : [0, 1] \longrightarrow [0, 1]$ by

$$Q(v) = \frac{1}{v^2} \frac{e^2}{e^{2/v}}. \tag{8}$$

The function Q is a bijection because, $Q(0) = 0$, $Q(1) = 1$ and $Q' > 0$. Hence, its inverse Q^{-1} exists and, it is differentiable in $(0, 1]$. In the Appendix, we prove:

[6] Notice that $F_j \succ F_i$ implies the weak inequality $f_i(\overline{v}) \leq f_j(\overline{v})$, the strict inequality is needed only to simplify the proof.

Proposition 12 *The strategies* $b_1(v) = \dfrac{32 - 8x - 4x^2 - x^3}{32} \dfrac{\exp\left(\frac{4(x-1)}{x}\right)}{x^3}$ *and* $b_j(v) = b_1(Q^{-1}(v))$*,* $j \in \{2, 3\}$ *are a Bayes–Nash equilibrium for the contest game.*

The effort density of contestant 1 is,

$$g_1(b) = f_1(\phi_1(b))\,\phi_1'(b) = \frac{f_1(\phi_1(b))}{b_1'(\phi_1(b))} = \frac{\phi_1(b)^5}{2} \exp\left(\frac{4}{\phi_1(b)} - 4\right),$$

which is increasing for high effort levels: its derivative,

$$g_1'(b) = \phi_1'(b)\frac{\phi_1(b)^4}{2} \exp\left(\frac{4}{\phi_1(b)} - 4\right)\left[5 - \frac{4}{\phi_1(b)}\right],$$

evaluated in a neighborhood of $\bar{b}$ is positive, since $\phi_1' > 0$ and $\phi_1(\bar{b}) = 1$.

Since contestants 2 and 3 have identical valuations distributions and their equilibrium strategies are also identical, without any loss of generality we shall, hereafter, refer to contestant 2 only instead of contestants 2 and 3.

Observe that the function Q maps that type of contestant 1 to the type of contestant 2 who bids the same amount, that is, $\phi_{j2}(b) = Q(\phi_1(b))$. is Thus, the effort density of contestant 2 is,

$$g_2(b) = f_2(\phi_2(b))\,\phi_2'(b) = Q'(\phi_1(b))\phi_1'(b)$$

We use a parametric plot, $[x(v), y(v)] = [b_1(v), g_1(b_1(v))]$, to display the graph of the effort density of contestant 1 without solving explicitly for $\phi_1(b)$. In the same manner, we plot the density effort for contestants 2. Whereas the density of contestant 1 increases for high effort levels corresponding to high abilities realizations, the density of contestants 2 is decreasing. Also for high effort levels, the density of contestant 1 is below the density of contestant 2, which is depicted by the dotted line in Figure 1.

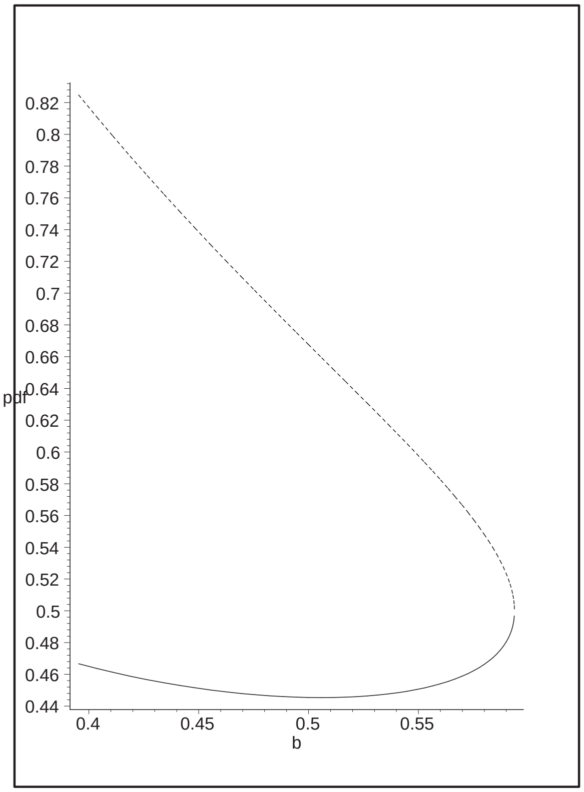

Figure 1: Effort densities, $g_1(b)$ and $g_2(b)$, evaluated at high bids.

To illustrate Proposition 11 we display the cumulative effort densities and show that $G_1 \prec G_2$. The CDF of contestant 2 is depicted by the dotted line in Figure 2.

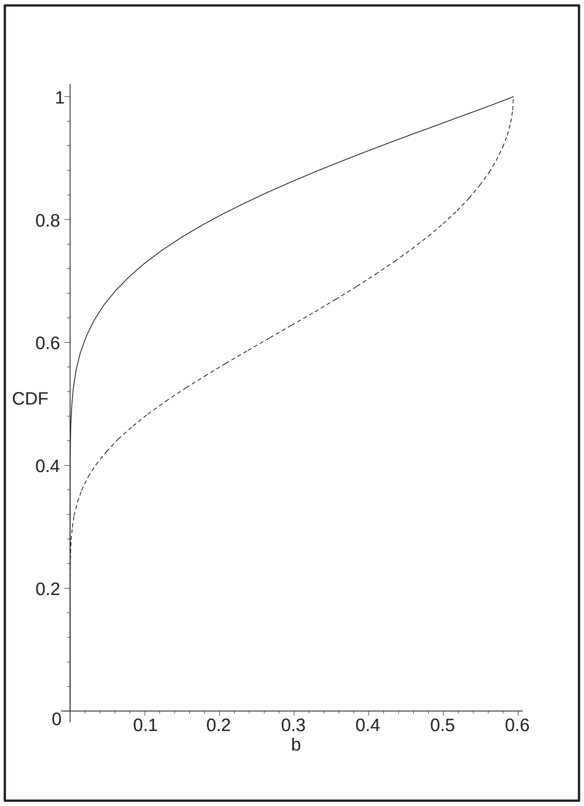

Figure 2: Effort CDFs, $G_1(b)$ and $G_2(b)$.

4 The Uniform Model

In this section we use the characterization of equilibrium derived in the first section to explicitly solve for equilibrium in a model where individual abilities are drawn from uniform distributions, $F_i \sim U[0, \alpha_i]$ for $i = 1, \ldots, N$.

Albeit the distribution of abilities in the uniform model fails to satisfy the assumptions of Proposition 20, we prove in the Appendix the following result.

Proposition 13 *Equilibrium bids are continuous in the uniform model.*

Without any loss of generality, we assume that $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N$ and introduce the auxiliary notation, $\kappa_i \equiv \frac{\sum_{j \neq i} \alpha_j^{-1} - (N-2)\alpha_i^{-1}}{\sum_{j \neq 1} \alpha_j^{-1} - (N-2)\alpha_1^{-1}}$.

When all the other contestants always participate, a necessary and sufficient condition for contestant i to always participate is $\kappa_i > 0$, which always holds when contestants are not too asymmetric, because it is equivalent to:

$$\frac{\sum_{j \neq i} \alpha_j^{-1}}{N-2} > \alpha_i^{-1}. \tag{9}$$

Also, in the case where all the others always participate, Proposition 4 implies that if (9) is violated, no type of contestant i participates. These results also carry to the more general case when just a subset of contestants participate. In the uniform model, either a contestant always participates or, he never participates. The set of active contestants is characterized by the following proposition:

Proposition 14 *Contestants $i = 1, 2, \ldots, K^*$ always participate, while contestants $i > K^*$ do not participate. The number of active contestants is $K^* = arg\min_K \{\kappa(K) : \kappa(K) > 0\}$, where $\kappa(K) \equiv \frac{\sum_{j=1}^{K-1} \alpha_j^{-1} - (K-2)\alpha_K^{-1}}{\sum_{j=2}^{K} \alpha_j^{-1} - (K-2)\alpha_1^{-1}}$.*

4.1 The Equilibrium

For simplicity, we compute explicitly the equilibrium profile of strategies when all contestants participate, that is (9) holds for all $i = 1, \ldots, N$. The characterization of the equilibrium in the general case is almost identical as in the case when all contestants participate, however additional notation is required.

Proposition 15 *The unique equilibrium strategy profile is given by*

$$b_i(v) = \frac{N-1}{\sum_{j=1}^{N} \alpha_j^{-1}} \left[\frac{v}{\alpha_i}\right]^{\frac{1+\kappa}{\kappa_i}} \quad \textit{for } i = 1, \ldots, N.$$

The lowest and highest equilibrium effort levels are $\underline{b} = 0$ and $\overline{b} = \frac{N-1}{\sum_{j=1}^{N} \alpha_i^{-1}}$. The CDF of contestant i's effort is,

$$G_i(b) = \left[\frac{\sum_{j=1}^{N} \alpha_j^{-1}}{N-1} b\right]^{\frac{\kappa_i}{1+\kappa}}, \tag{10}$$

which is decreasing in α_i since κ_i is increasing in α_i and also $\frac{\sum_{j=1}^{N}\alpha_j^{-1}}{N-1}b < 1$ for any $b \in [0, \overline{b}]$. Although in the uniform model the support of valuations is not identical as required by Proposition 11, also here – a strong agent is more likely to choose high effort levels than a weak one.

The interim and expected aggregate effort levels are:

$$R(v_1, ..., v_N) = \frac{N-1}{\sum \alpha_j^{-1}} \sum_{i=1}^{N} \left[\frac{v_i}{\alpha_i}\right]^{\frac{1+\kappa}{\kappa_i}} \text{ and } R(\alpha_1, ..., \alpha_N) = \frac{N-1}{\sum \alpha_j^{-1}} \sum_{i=1}^{N} \frac{\sum \alpha_j^{-1} - (N-1)\alpha_i^{-1}}{2\sum \alpha_j^{-1} - (N-1)\alpha_i^{-1}}.$$

Contestant i's interim and expected payoffs are,

$$\Pi_i(v_i) = \left[\alpha_i - \frac{N-1}{\sum \alpha_j^{-1}}\right] \left[\frac{v_i}{\alpha_i}\right]^{\frac{1+\kappa}{\kappa_i}}, \quad \text{and} \quad \Pi_i = \left[\alpha_i - \frac{N-1}{\sum \alpha_j^{-1}}\right] \frac{\sum \alpha_j^{-1} - (N-1)\alpha_i^{-1}}{2\sum \alpha_j^{-1} - (N-1)\alpha_i^{-1}}. \tag{11}$$

We also obtain the following comparative statics results. When i becomes more likely to have a high ability (put simply, α_i increases) and all the other contestants remain active, i's payoff increases. It may happen that, as α_i increases, 'weak' contestants drop out – in this instance, i's payoff increases continuously; however, it is not differentiable. We also show that i's payoff is decreasing in α_j. All the proofs for these results are in the Appendix.

Figure 3 below illustrates the comparative statics results. It depicts the contestants' payoffs for $N = 3$, $\alpha_1 = 6$ and $\alpha_2 = 3$ as α_3 increases. As in Proposition 4, for low values of α_3 contestant 3 drops-out and so her payoff is zero. As α_3 increases, contestant 3 starts participating and her payoff increases. When α_3 becomes sufficiently high, contestant 2, who is more likely to have low valuations, drops-out.

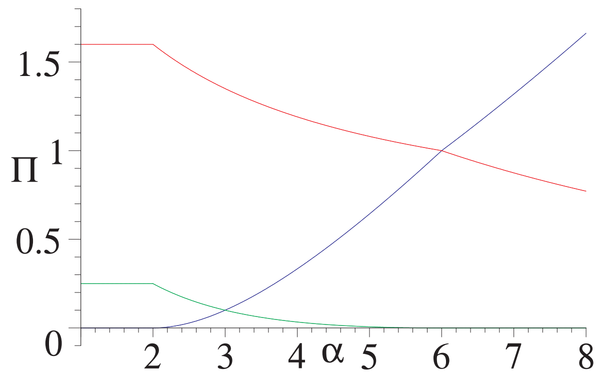

Figure 3: The contestants' payoffs as functions of α_3

5 Conclusions

Having characterized equilibrium in contests with many contestants, we have shown that "weak contestants" might bid very aggressively provided their valuation for winning is close to the top.

Indeed, if all contestants are risk neutral and if all of them, except one, believe that their weak rival ('underdog') is very unlikely to have a high valuation for the prize or be of a high ability, their equilibrium behavior will be almost unaffected by the presence of the underdog, at least at the top valuations, that is, they will compete mainly against each other. However, in case the underdog does have a high ability, he has a decent chance of winning by exerting high effort, and so he does, in equilibrium. Ironically, it is the pessimistic belief about the abilities of this contestant, the belief shared by his rivals, that endow the high ability underdog with informational rent that makes the race to the bottom worthwhile.

We also show that a sufficiently risk averse (CARA) contestant facing enough risk neutral rivals will bid aggressively at the top and, as in the previous case, her bid density will be increasing at the top valuations.

Having suggested a possible explanation of the past experimental evidence, our results offer directions for future experimental investigations.

References

AMANN, E., AND W. LEININGER (1996): "Asymmetric All-Pay Auctions with Incomplete Information: The Two Player Case," *Games and Economic Behavior*, 14, 1–18.

ATHEY, S. (2001): "Single Crossing Properties and The Existence of Pure Strategy Equilibria In Games of Incomple Information," *Econometrica*, 60, 861–889.

BAYE, M. R., D. KOVENOCK, AND C. G. DE VRIES (1996): "The All-Pay Auction With Complete Information," *Economic Theory*, 8, 291–305.

FIBICH, G., A. GAVIOUS, AND A. SELA (2004): "All-Pay Auctions With Weakly Risk-Averse Bidders," mimeo Tel-Aviv University.

GAVIOUS, A., B. MOLDOVANU, AND A. SELA (2002): "Bid Costs and Endogenous Bid Caps," *Rand Journal of Economics*, 33(4), 709–722.

HILLMAN, A. L., AND J. G. RILEY (1989): "Politically Contestable Rents and Transfers," *Economics and Politics*, 1, 17–39.

KIRCHAMP, O. (2004): "Volunteering Stabilisation, Experiments With Wars of Attrition and All-Pay Auctions," mimeo Manheim Univesity.

KRISHNA, V., AND J. MORGAN (1997): "An Analysis of the War of Attrition and the All-Pay Auction," *Journal of Economic Theory*, 72, 343–362.

LEBRUN, B. (1999): "First Price Auctions in The Asymmetric N Bidder Case," *International Economic Review*, 40(1), 125–142.

LIZZERI, A., AND N. PERSICO (1998): "Uniqueness and Existence of Equilibrium In Auctions With A Reserve Price," *Games and Economic Behavior*, 30, 83–114.

NOUSSAIR, C., AND J. SILVER (2005): "Behavior in All-Pay Auctions with Incomplete Information," *Games and Economic Behavior*, online release.

SCHOTTER, A., AND W. MULLER (2003): "Workaholics and Drop Outs in Optimal Organizations," NY University.

TSIRELSON, B. (2000): "Probabilistic Aspects of Economic Games," mimeo Tel Aviv University.

Contents

A Appendix

A.1 The System of First Order Conditions

Again, the bidding strategies of the contestants are denoted by $b_i : [\underline{v}, \overline{v}_i] \to \mathbb{R}_+$. For every bid $b \geq 0$ denote the set of active contestants at b, that is, contestants that choose this bid for some of their type realizations:

$$J(b) = \{j \in \{1,..,N\} \,|\, \exists v_j \in [\underline{v}, \overline{v}_i] : b_j(v_j) = b\},$$

and write $K(b) = \#J(b)$ for the cardinality of J, the number of active contestants at b. Also consider the set of bids where the number of active contestants does not change,

$$\mathcal{B} = \left\{ b \in \mathbb{R}_{++} : \frac{\partial}{\partial b} K(b) = 0 \right\}.$$

Proof of Proposition 2. Consider contestant $i \in \{1,..,N\}$. Fix the strategies of other contestants, thereby determining set $J(b) \setminus \{i\}$ for any b. If at some given $b > 0$ condition

$$\begin{aligned} &[u_i(v_i - b + w) - u_i(w - b)] \sum_{j \in J(b) \setminus \{i\}} \prod_{k \neq i,j} G_k(b) g_j(b) + \\ &- [u_i'(v_i - b + w) - u_i'(w - b)] \prod_{j \in J(b) \setminus \{i\}} G_j(b) - u_i'(w - b) \leq 0, \; i \in J(b) \quad (12) \end{aligned}$$

is satisfied with equality, then let $b_i(v_i) = b$. Otherwise, $b_i(v_i) = 0$.

As we restrict attention to non-decreasing strategies, the highest bid of contestant i is the optimal bid for the highest type of that contestant, $b_i(\overline{v}_i) = \overline{b}_i$. In addition, there are at least two contestants, k, l whose highest types place the highest equilibrium bid, $\overline{b}_k = \overline{b}_l = \max_{i \in \{1,..,N\}} \{\overline{b}_i\}$. Then for any $b \in (0, \overline{b}_k], K(b) \geq 2$.

Secondly, for $b \in b_i([\underline{v}, \overline{v}_i]) \setminus \{0\}$, the system (12) is satisfied as equality. Rewrite it as

$$\sum_{j \neq i} \frac{g_j(b)}{G_j(b)} = S_i(b),$$

for $i \in \left\{ j \in \{1, .., N\} \,|\bar{b}_i > 0 \right\}$. Recall,

$$\begin{aligned} S_i(b) &\equiv \frac{W_i(b)\, u_i'(v_i - b + w) + (1 - W_i(b))\, u_i'(w - b)}{W_i(b)\,(u_i(v_i - b + w) - u_i(w - b))} > 0, \\ W_i(b) &= \prod_{j \neq i} G_j(b). \end{aligned}$$

The system of equations is linear in $\frac{g_j(b)}{G_j(b)}$ that allows us to solve it as follows:

$$\left(\frac{g_j(b)}{G_j(b)}\right)_{i \in J(b)} = M^{-1}\left(S_i(b)\right)_{i \in J(b)}, \tag{13}$$

$$\text{where } M = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \ldots & 1 & 0 \end{pmatrix}, \tag{14}$$

Note that the inverse of the K by K matrix M, with $K \geq 2$ is

$$\frac{1}{K-1}\begin{pmatrix} -(K-2) & 1 & \cdots & 1 \\ 1 & -(K-2) & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \ldots & 1 & -(K-2) \end{pmatrix}.$$

Therefore, $g_i(b) = \frac{G_i(b)}{(K(b)-1)}\left(\sum_{j \in J(b)\setminus\{i\}} S_j(b) - (K(b)-2)S_i(b)\right)$ for $b \in B$.
■

A.2 Monotonicity Of Equilibrium Bid Functions

Let b a Nash equilibrium profile in non-decreasing strategies. For any Borel set $A \subset \mathbb{R}$, define $\mu_{G_i}(A) = \Pr[b_i(V_i) \in A]$ and $\mu_{W_i}(A) = \Pr[\max_{j \neq i} b_j(V_j) \in A] = \prod_{j \neq i} G_j(A)$.

That is, μ_{G_i} (respectively μ_{W_i}) is the measure associated to the cumulative probability distribution function, G_i (respectively W_i).

Lemma 16 *The measure μ_{G_i} has no atoms at $b > \underline{b}$.*

Proof. If positive mass of types of contestant i bids b then $\lim_{e \nearrow b} W_j(b) < W_j(b) < W_j(b+\delta)$ for any $\delta > 0$ because of the tie braking rule – when a tie happens, the object is randomly allocated, with equal probabilities, among all contestants who exert the highest effort level. As a result, the left and right derivatives 'explode', that is, $W'_{j-}(b) = W'_{j+}(b) = +\infty$. Therefore, the type of, say contestant j, who bids b in equilibrium will be strictly better of by raising its bid marginally above b. The marginal cost of raising the bid, $MC(b) = u'_j(w-b)(1-W_j(b)) + u'_j(v_j - b + w)W_j(b)$ increases discontinuously, yet it remains bounded while the marginal benefit of increasing the bid is unbounded, $MB(b) = [u_j(v_j - b + w) - u_j(w-b)]\, W'_j(b)$. ∎

As a result of the above lemma, when $v > \phi_i(0)$, bids must be strictly increasing and so the inverse bid functions are strictly increasing.

A.3 Continuity of The Generalized Inverse Bids

Lemma 17 *For any contestant i, ϕ_i is continuous.*

Proof. If ϕ_i were discontinuous at $b > \underline{b}$ then G_i would have an atom at b contradicting Lemma 16. Hence, we must establish that ϕ is right continuous at $\underline{b}$. Suppose that ϕ_i fails to be right continuous at any b – that is, there is a $\delta > 0$ such that $\phi_i(b) < \phi_i(b) + \delta < \phi_i(b+\varepsilon)$ for any $\varepsilon > 0$. In other words, type $\phi_i(b) + \delta$ bids strictly above b and strictly below $b + \varepsilon$ for any $\varepsilon > 0$, which is a contradiction. ∎

A.4 Continuity of Equilibrium Strategies

In this section we offer sufficient conditions for the continuity of equilibrium strategies in case contestants have linear utility functions, maintaining the assumption that their values can be drawn from different distributions. For risk-neutral environments, we can interpret v_i both as i's ability, or the inverse of the cost of effort, and as her value of winning.

We will start with a lemma implying that the probability of winning for each contestant can not be constant over any open interval included in his equilibrium bidding range.

Lemma 18 *The support of μ_{W_i} has no gaps.*

Proof. If there exist: $\alpha < \beta$, $0 < \varepsilon < \frac{\alpha-\beta}{4}$ such that $\alpha - \varepsilon \in \text{supp}(\mu_{W_i})$, $\beta + \varepsilon \in \text{supp}(\mu_{W_i})$ and $(\alpha, \beta) \bigcap \text{supp}(\mu_{W_i}) = \emptyset$ then contestant i will not place any bid in (α, β). As a result, $(\alpha, \beta) \bigcap \text{supp}(\mu_{W_j}) = \emptyset$ and, also contestant j, where $j \neq i$, will not bid in (α, β). In sum, no contestants will bid in the interval (α, β). Consider the type v_j who bids above and closest to β (if this type is not well defined, consider any type who bids arbitrarily close to β). This type has strict incentives to bid below β. ■

A gap in the support of μ_{G_i} corresponds to a jump in the equilibrium bidding strategy of contestant i. For the two-contestant case, $\mu_{G_i} \equiv \mu_{W_j}$ for $j \neq i$. Therefore, for $N = 2$, the above lemma implies that the equilibrium bids are continuous. For $N \geq 3$ however, the above reasoning does not rule out gaps in the support of G_i.

Proposition 19 *Let all N contestants be ex-ante identical, with their abilities being independently and identically distributed, then equilibrium strategies are continuous.*

Proof. Assume that a symmetric equilibrium equilibrium strategy $b(\cdot)$ is discontinuous at some valuation v^*. Define $\underline{e} = lim_{v \nearrow v^*}$ and $\overline{e} = lim_{v \searrow v^*}$. In this equilibrium, no contestant exerts effort in the non-empty, open interval $(\underline{e}, \overline{e})$. Hence, the winning probability of any contestant, $W(\cdot)$, is constant within this interval and that contradicts Lemma 18 ■

Next, we adapt an argument of Lebrun (1999) to derive a sufficient condition for the continuity of equilibrium strategies.

Proposition 20 *Assume the contestants are risk neutral. Equilibrium strategies are continuous at $\underline{v}$ and if either:*

1. *for all contestants, $\dfrac{F_i(v)}{v}$ is strictly decreasing in v; or*

2. *for all contestants, $\dfrac{F_i(v)}{v}$ is strictly increasing;*

then equilibrium strategies are continuous.

Proof. We shall write $W_i(b) = \mu_{W_i}((-\infty, b])$ and $G_i(b) = \mu_{G_i}((-\infty, b])$. In words, $W_i(b)$ is the probability that contestant i wins when she bids b and $G_i(b)$ is the probability that she bids at or bellow b, that is $W_i(b) = \prod_{j \neq i} G_j(b)$ and $G_j(b) = F_j(\phi_j(b))$.

Lemma 21 *Lebrun (1999) If bids have discontinuities then the there is a minimal interval* $(\underline{\beta}, \overline{\beta})$ *with the property that for contestant, say* i, $\lim_{v \nearrow v_i} b_i(v) = \underline{\beta} < \overline{\beta} = \lim_{v \searrow v_i} b_i(v)$, *and contestants* $j \neq i$ *either bid continuously in the interval or they do not bid in it.*

Proof. Assume that contestant i's bid function jumps at v_i: $b_i^+ = \lim_{v \searrow v_i} b_i(v)$, $b_i^- = \lim_{\phi_i \nearrow v_i} b_i(v)$ and $b_i^- < b_i^+$. Notice that the type v_i must be indifferent between bidding b_i^- and b_i^+:

$$v_i \, W_i(b_i^-) - b_i^- = v_i \, W_i(b_i^+) - b_i^+ \Longrightarrow \triangle W_i = \frac{\triangle \, b}{v_i}, \tag{15}$$

Since both b_i^- and b_i^+ are best responses, for any $b \in \left(b_i^-, b_i^+\right)$:

$$v_i \, W_i(b_i^-) - b_i^- \geq v_i \, W_i(b) - b \Longrightarrow W_i(b) - W_i(b_i^-) \leq \frac{b - b_i^-}{v_i} \tag{16}$$

$$v_i \, W_i(b_i^+) - b_i^+ \geq v_i \, W_i(b) - b \Longrightarrow W_i(b_i^+) - W_i(b) \geq \frac{b_i^+ - b}{v_i} \tag{17}$$

The probability $W_i(b) - W_i(b_i^-)$ corresponds to the event that all contestants other than i bid in the interval $(b_i^-, b]$. Analogously, $W_j(b) - W_j(b_i^-)$ corresponds to the event that all contestants other than j bid in the interval $(b_i^-, b]$, however since i does bid in the range (b_i^-, b_i^+), $W_j(b) - W_j(b_i^-)$ corresponds to the event that all contestants other than j and i bid in the interval $(b_i^-, b]$. Clearly then, the inequality $W_i(b) - W_i(b_i^-) \geq W_j(b) - W_j(b_i^-)$ holds true. And as a result,

$$W_j(b) - W_j(b_i^-) \leq \frac{b - b_i^-}{v_i} \tag{18}$$

If there is at least one bid $b \in \left(b_i^-, b_i^+\right)$ such that b is the best response for some type of contestant j, say v_j:

$$v_j \, W_j(b_i^-) - b_i^- \leq v_j \, W_j(b) - b \Longrightarrow W_j(b) - W_j(b_i^-) \geq \frac{b - b_i^-}{v_j} \tag{19}$$

Combining the last two equations:

$$\frac{b - b_i^-}{v_i} \geq W_j(b) - W_j(b_i^-) \geq \frac{b - b_i^-}{v_j}, \tag{20}$$

and therefore $v_i \leq v_j$. Put simply, we proved an useful auxiliary result:

Lemma 22 *(Lebrun 1999) Whenever contestant j bids in the gap of i's bids then j's valuation must be equal or higher than the valuation of i at which i's bid jumps.*

If the bid function of j also jumps at v_j, say from b_j^- to b_j^+ and there is an overlap of gaps, $(b_i^-, b_i^+) \bigcap (b_j^-, b_j^+) \neq \emptyset$, a variation of the above argument shows that $v_j \leq v_i$. Therefore, if such overlap existed, it must be $v_i = v_j$.

Let's assume the existence of such overlap, which implies $v_i = v_j$. Without loss of generality we take $b_j^- \in \left(b_i^-, b_i^+\right)$. In this case, since

$$\frac{b_j^- - b_i^-}{v_i} \geq W_i(b_j^-) - W_i(b_i^-) \geq W_j(b_j^-) - W_j(b_i^-) \geq \frac{b_j^- - b_i^-}{v_j} = \frac{b_j^- - b_i^-}{v_i}, \quad (21)$$

we must also have that $W_i(b_j^-) - W_i(b_i^-) = W_j(b_j^-) - W_j(b_i^-)$, which implies that contestant j also does not bid in (b_i^-, b_j^-). It follows then that b_j^- is an isolated point in the support of the bid distribution of j, G_j. And this implies that a positive mass of types of j bid b_j^- but that contradicts the fact G_j is non atomic.

In sum, since gaps of bids don't overlap. Two gaps are either disjoint or ordered by inclusion. Therefore, there exists a minimal gap that does not contains any other gap. ■

Denote the minimal gap by $\left(\underline{\beta}, \overline{\beta}\right)$, and let J be the set of active contestants who bid continuously in the range $\left(\underline{\beta}, \overline{\beta}\right)$. While, contestants $\{1, \ldots, N\} \setminus J$ do not bid in the range $\left(\underline{\beta}, \overline{\beta}\right)$.

Lemma 23 *For any i, the function W_i is differentiable almost everywhere and for all $b \in \left[\underline{\beta}, \overline{\beta}\right]$, the right and left derivatives, $W'_{i+}(b)$ and $W'_{i-}(b)$, exist.*

Proof. We have that: $W_i = \prod_{j \neq i} F_j(\phi_j(b))$ and for all j, ϕ_j is differentiable almost everywhere and, F_j is differentiable. Hence, W_i is differentiable almost everywhere. A lemma of Tsirelson (2000) establishes that $\frac{1}{W_i}$ is convex in $\left(\underline{\beta}, \overline{\beta}\right)$ and therefore its right and left derivatives always exist in $\left(\underline{\beta}, \overline{\beta}\right)$. By the chain rule, the right and left derivative of W_i always exist since, the reciprocal is a smooth function in $\mathbb{R} \setminus \{0\}$. ■

For $b \in \left(\underline{\beta}, \overline{\beta}\right)$ where W_i is differentiable, we have that:

$$\begin{aligned}
W_i'(b) &= \sum_{j \neq i} g_j(b) \prod_{k \neq i,j} G_k(b) = \sum_{j \in \mathcal{J} \setminus \{i\}} g_j(b) \prod_{k \neq i,j} G_k(b) = \\
&= \sum_{j \in \mathcal{J} \setminus \{i\}} f_j(\phi_j(b)) \phi_j'(b) \prod_{k \neq i,j} F_k(\phi_k(b)) = \\
&= \sum_{j \in \mathcal{J} \setminus \{i\}} \frac{\sum_{k \in J(b) \setminus \{j\}} \frac{F_k(\phi_k(b))}{\phi_k(b)} - (K(b)-2) \frac{F_j(\phi_j(b))}{\phi_j(b)}}{(K(b)-1) F_i(\phi_i(b))} = \\
&= \frac{\sum_{j \in J(b)} \frac{F_j(\phi_j(b))}{\phi_j(b)}}{(K(b)-1) F_i(\phi_i(b))} \text{ if } i \notin \mathcal{J} \text{ and} \qquad (22) \\
&= \frac{1}{\phi_i(b)} \text{ if } i \in \mathcal{J}.
\end{aligned}$$

The passage from the second to the third line in (22) follows from the characterization in (2).

Assume that the minimal gap corresponds to the jump of contestant i's bid at v_i. Optimality requires that:

$$W_{i+}'(\underline{\beta}) \leq \frac{1}{v_i} \leq W_{i-}'(\overline{\beta}) \qquad (23)$$

and so,

$$\sum_{j \in J} \frac{F_j(\phi_j(\underline{\beta}))}{\phi_j\left(\underline{\beta}\right)} \leq (K(b)-1) \frac{F_i}{v_i} \leq \sum_{j \in J} \frac{F_j(\phi_j(\overline{\beta}))}{\phi_j\left(\overline{\beta}\right)} \qquad (24)$$

Since $i \notin J$, $F_i(\phi(b))$ is constant for bids in $\left[\underline{\beta}, \overline{\beta}\right]$.

Notice that equation (24) can not hold true if $\frac{F_j(y)}{y}$ is strictly decreasing in v for all j. In contrast, when $\frac{F_j(y)}{y}$ is strictly decreasing, equation (24) hold true and moreover, the winning probability W_i is convex in $\left(\underline{\beta}, \overline{\beta}\right)$ and in this instance, the existence of gaps can not be ruled out by this result.

We re-write equation (24) as,

$$\sum_{j \in \mathcal{J}} \frac{F_j(\phi_j(\underline{\beta}))}{\phi_j\left(\underline{\beta}\right)} - (K(b)-1) \frac{F_i}{v_i} \leq 0 \leq \sum_{j \in \mathcal{J}} \frac{F_j(\phi_j(\overline{\beta}))}{\phi_j\left(\overline{\beta}\right)} - (K(b)-1) \frac{F_i}{v_i}, \quad (25)$$

When $\underline{\beta} > 0$ then the left-had size inequality in (25) must be binding. The left-hand side is always non-positive. If it were negative, contestant i would not bid below $\underline{\beta}$ because by continuity, $\sum_{j\in\mathcal{J}} \frac{F_j(\phi_j(\underline{\beta}))}{\phi_j(\underline{\beta})} - (K(b)-1)\frac{F_i}{v_i} < 0$ implies $\sum_{j\in\mathcal{J}} \frac{F_j(\phi_j(\underline{\beta}-\varepsilon))}{\phi_j(\underline{\beta}-\varepsilon)} - (K(b)-1)\frac{F_i(\phi_i(\underline{\beta}-\varepsilon))}{\phi_i(\underline{\beta}-\varepsilon)}$ and so, $\phi_i'(\underline{\beta}-\varepsilon) < 0$ for some $\varepsilon > 0$.

Moreover, it can not be the case that, $\underline{\beta} = 0$. As the next section shows, in equilibrium, any contestant with the lowest valuation does not bid above zero and so, we have that $v_i > \underline{v}$. If it were the case that $\underline{\beta} = 0$, by Lemma 22, for any $0 < b < \overline{\beta}$ and contestant $j \in J(b)$, $\phi_j(b) > v_i$ which can not hold. Because, it would imply $lim_{b\searrow 0}\phi_j(b) = \underline{v} \geq v_i$, which contradicts $v_i > \underline{v}$.

Also, the left-hand size inequality in (25) is always binds. Since the bidding function of any contestant j who bids in the gap ($j \in J(\overline{\beta})$ is continuous at β, j's marginal probability of winning must be also continuous at β since, $W_j(b)' = \frac{1}{\phi_j(b)}$.

The marginal winning probability is given by

$$W_j'(b) = \sum_{\substack{l\neq j\\ l\in J(b)}} g_l(b) \prod_{\substack{k\neq l\\ k\neq j}} G_k(b),$$

and since $g_i(b) = 0$ for $b \in \left(\underline{\beta}, \overline{\beta}\right)$ and W_j is continuous at $\overline{\beta}$ it follows that the density of bids of contestant i must also be zero at the upper boundary of the gap, $g_i(\overline{\beta}) = 0$. This is equivalent to the right-hand inequality in (25) being satisfied as equality.

Since both inequalities in (25) are binding, when all $\dfrac{F_j(v)}{v}$ are non-increasing (or if all ratios are non-decreasing), then $\frac{F_j(\phi_j(b))}{\phi_j(b)}$ must be constant for all $b \in (\underline{\beta}, \overline{\beta})$ and $j \in J(b)$. ■

A.5 The Lowest Equilibrium Bid

Lemma 24 *For all $i = 1,\ldots,N$, $b_i(\underline{v}) = 0$.*

Proof. Assume that for $b_i(\underline{v}) = \beta > 0$. Since the equilibrium is monotone, the probability contestant $j \neq i$ wins by bidding at or below β is zero. As a result, either $b_j(v) = 0$ or $b_j(v) \geq \beta$. In sum, $G_j(\beta) = G_j(0)$ for $j \neq i$ ($G_i(\beta) = \Pr[b_i(V) < \beta]$ since G_i is non-atomic). Now, if $G_j(\beta) = G_j(0) > 0$, for all $j \neq i$, then $W_i(\beta - \varepsilon) = W_i(\beta)$ for some $\varepsilon > 0$. Therefore, bidding $\beta - \varepsilon$ yields a higher payoff than bidding β. On the other hand, when $G_j(\beta) = G_j(0) = 0$, for some j, then $W_i(\beta) = 0$ and so, bidding zero yields a higher payoff than bidding β. ■

A.6 Uniqueness

Let $\phi(b) = (\phi_1(b), \ldots, \phi_2(b))$ and write,

$$S_i(b, \phi) \equiv \frac{u_i'(w-b)\left(1 - \prod_{j\neq i} F_j(\phi_j)\right) + u_i'(\phi_i - b + w)\left(\prod_{j\neq i} F_j(\phi_j)\right)}{\prod_{j\neq i} F_j(\phi_j)\left(u_i(\phi_i - b + w) - u_i(w-b)\right)}.$$

Proposition 25 *Assume that: for all i, b_i is continuous, f_i is continuous and uniformly bounded above zero in its support, $[\underline{v}, \overline{v}]$; then, the system of differential equations,*

$$\frac{\partial}{\partial b}\phi_i = \frac{\prod_{j\neq i} F_j(\phi_j)}{f_i(\phi_i)}\left[\sum_{j\neq i} S_j(b, \phi) - (N-2)S_i(b, \phi)\right], i = 1, \ldots, N\ ,$$

has a unique solution that satisfies the terminal condition, $\phi(\overline{b}) = (\overline{v}, \ldots, \overline{v})$.

Proof. There is a neighborhood of $\left(\overline{b}, \phi(\overline{b})\right)$ such that the system satisfies the Lipschitz condition because, for all i: f_i is continuous and bounded away from zero and; S_i is continuous in (b, ϕ) and bounded in a small neighborhood of $\left(\overline{b}, \phi(\overline{b})\right)$. Consequently, the solution $\phi(b)$ is locally (restricted to this neighborhood) unique.

Furthermore, as long as $\phi_i(b) > \underline{v}$ for all i, there is a neighborhood of $(b, \phi(b))$ where the Lipschitz condition is satisfied. Therefore, $\phi(b)$ can be further extended by continuity, in a unique way, from $\overline{b}$ to $\underline{b}$ where, $\underline{b}$ is defined as the largest $b < \overline{b}$ such that there is at least one contestant, say k, such that $\phi_k(\underline{b}) = \underline{v}$. Lemma 24 above establishes that $\underline{b} = 0$ and that allow us to pin-down the value of $\overline{b}$ using the condition, $\phi_k(0) = \underline{v}$. It is important to notice that there is no guarantee that the above unique solution corresponds to an equilibrium. For example, it is conceivable that $\phi'(b) < 0$ for some b and i. One needs to prove that all contestants are active in order to show that the above solution corresponds to an equilibrium, that is, $J(b) = \{1, \ldots, N\}$ for any $b \in (0, \overline{b})$. Indeed, since for any contestant i, $v_i = \overline{v}$, Proposition 5 implies that $b_i(\overline{v}) = \overline{b}$. In addition, since strategies are continuous (by assumption) and strictly increasing, for any $b \in\in (0, \overline{b})$ and any i, there is v such that $b_i(v) = b$. ■

A.7 No Atoms at Zero

Proof of Proposition 3. First, the lowest effort should be zero by the Lemma 24. Since the winning probabilities might be zero at $b = 0$, the effort densities may 'explode' at the lowest effort level. Clearly, there should be at least one contestant that does not choose $b = 0$ with a strictly positive probability. Call this contestant k. Consider a pair of contestants i, j who are different from k. For these contestants the winning probability approaches zero as $b \to 0$ in the presence of contestant k. By definition of winning probability, we have the following identity $W_i(b)G_i(b) = W_j(b)G_j(b)$, so

$$\lim_{b\searrow 0} \frac{G_j(b)}{G_i(b)} = \lim_{b\searrow 0} \frac{W_i(b)}{W_j(b)} = \lim_{b\searrow 0} \frac{W_i'(b)}{W_j'(b)}$$

where the last equality follows from the L'Hôpital's Rule. Using the first order conditions (2)

$$\lim_{b\searrow 0} \frac{W_i'(b)}{W_j'(b)} = \frac{u_i'(w)}{u_i(\phi_i(0)+w)-u_i(w)} \frac{u_j(\phi_j\,(0)+w)-u_j(w)}{u_j'(w)} < \infty,$$

since ϕ_i and ϕ_j are right continuous, $\phi_i \geq \underline{v}$, $\phi_j \geq \underline{v}$ and, $\underline{v} > 0$ by assumption.

As a result,

$$\lim_{b\searrow 0} \frac{G_j(b)}{G_i(b)} = \frac{u_i'(w)}{u_i(\phi_i(0)+w)-u_i(w)} \frac{u_j(\phi_j\,(0)+w)-u_j(w)}{u_j'(w)} < \infty. \tag{26}$$

It follows that only two scenarios are possible. First, i and j start bidding at zero, so that $G_i\,(0) = G_j\,(0) = 0$. Second, for both $G_i\,(0) > 0$, $G_j\,(0) > 0$, so that both bid distributions have an atom at zero. As our choice of i and j not k was arbitrary, either $N-1$ contestants choose zero bid with positive probability or none of them does. ■

A.8 Participation Results

Proof of Proposition 4. In the case where the contestants are risk neutral the system of first order conditions admits the following solution for almost all $b \in \bigcap_{i=1}^{N} b_i([\underline{v}, \overline{v}_i])$

$$g_i\,(b) = \frac{\sum_{k\neq i} \frac{G_k(b)}{\phi_k(b)} - (N-2)\frac{G_i(b)}{\phi_i(b)}}{(N-1)\prod_{j\neq i} G_j\,(b)}, \tag{27}$$

indeed, in this case $S_i\,(b) = \frac{1}{v_i \prod_{j\neq i} G_j(b)}$.

As F_j first-order stochastically dominates $U[0,\overline{v}_j]$ for all j, it follows that $\frac{\sum_{j\neq i}\frac{F_j(\phi_j)}{\phi_j}}{N-2} < \frac{\sum_{j\neq i}\overline{v}_j^{-1}}{N-2}$, which implies that $\sum_{j\neq i}\frac{G_j(b)}{\phi_j(b)} < (N-2)\frac{1}{\overline{v}_i}$, and, therefore, the highest ability type of contestant i will not bid $b>0$ since his first-order condition is negative, so $b_i(\overline{v}_i)=0$. Moreover, since the equilibrium is monotone, $b_i(v)=0$ for all v. ■

Proof of Proposition 5. If $b_j(\overline{v}_j)=\beta<\overline{b}$ and $b_i(\overline{v}_i)=\overline{b}$ then, by revealed preferences, it must be that $\overline{v}_jW_j(\beta)-b_j(\overline{v}_j)\geq\overline{v}_j-\overline{b}=\overline{v}_i-\overline{b}>\overline{v}_iW_i(\beta)-b_j(\overline{v}_j)>\overline{v}_iW_j(\beta)-b_j(\overline{v}_j)=\overline{v}_jW_j(\beta)-b_j(\overline{v}_j)$ since $W_j(b_j(\overline{v}_j))\,1=W_i(\beta)G_i(\beta)$. A contradiction. ■

A.9 Proofs of the Main Results

A.9.1 Ex-ante Asymmetry

Proof of Propositions 6 and 8.

Assume all contestants are bidding in the same interval.

When all the contestants have identical distributions, equation (27) reads:

$$g(b) \;=\; \frac{\frac{G(b)}{\phi(b)}}{(N-1)G(b)^{N-1}} = \frac{1}{(N-1)\phi(b)\,G(b)^{N-2}}.$$

Note that both ϕ and G are increasing in b, so g must be decreasing in b.

Allowing for different distributions and setting $N=2$, equation (27) give us,

$$g_i(b) = \frac{\frac{G_j(b)}{\phi_j(b)}}{G_j(b)} = \frac{1}{\phi_j(b)},$$

so the density for $i=1,2$ is decreasing as well. This completes the proof of Proposition (6) . Next,

$$\begin{aligned} g_i'(b) \;=\;& \frac{\sum_{k\neq i} g_k(b)\frac{\phi_k(b)-G_k(b)/f_k(\phi_k(b))}{(\phi_k(b))^2}-(N-2)g_i(b)\frac{\phi_i(b)-G_i(b)/f_i(\phi_i(b))}{(\phi_i(b))^2}}{(N-1)\prod_{j\neq i}G_j(b)} \\ & -\frac{\sum_{k\neq i}\frac{G_k(b)}{\phi_k(b)}-(N-2)\frac{G_i(b)}{\phi_i(b)}}{(N-1)\left(\prod_{j\neq i}G_j(b)\right)^2}\sum_{j\neq i}\prod_{k\neq i,j}G_k(b)g_j(b). \end{aligned}$$

Then

$$
\begin{aligned}
g_i'\left(\bar{b}\right) &= \frac{1}{N-1}\left(\sum_{k\neq i} g_k(\bar{b})\frac{\overline{v}-1/f_k\left(\overline{v}\right)}{\overline{v}^2} - (N-2)g_i(\bar{b})\frac{\overline{v}-1/f_i\left(\overline{v}\right)}{\overline{v}^2}\right) \\
&\quad -\frac{1}{N-1}\frac{1}{\overline{v}}\sum_{j\neq i} g_j(\bar{b})
\end{aligned}
$$

Also,

$$
g_j(\bar{b}) = \frac{1}{(N-1)\overline{v}}, \text{ for all } j,
$$

so

$$
\begin{aligned}
g_i'\left(\bar{b}\right) &= \frac{1}{(N-1)^2\overline{v}^2}\left(\sum_{k\neq i}\left(\overline{v}-1/f_k\left(\overline{v}\right)\right) - (N-2)\left(\overline{v}-1/f_i\left(\overline{v}\right)\right) - (N-1)\,\overline{v}\right) \\
&= \frac{1}{(N-1)^2\overline{v}^2}\left(\frac{N-2}{f_i\left(\overline{v}\right)} - \sum_{k\neq i}\frac{1}{f_k\left(\overline{v}\right)} - (N-2)\overline{v}\right) \qquad (28)
\end{aligned}
$$

which is positive if $f_i(\overline{\phi})$ is sufficiently small (relative to $f_k(\overline{\phi})$). ■

Proof of Proposition 11.

1. Assume that G_i and G_j cross or are tangent at some point in the interior of support of equilibrium effort levels, $b^* \in (0,\bar{b})$. In this case, it follows that $G_i(b^*) = G_j(b^*)$ together with $F_j \succ F_i$ imply that $\phi_i(b^*) < \phi_j(b^*)$. Moreover, from $G_i(b^*) = G_j(b^*)$, $\phi_i(b^*) < \phi_j(b^*)$, and the characterization of the effort densities (27), it follows that $g_i(b^*) < g_j(b^*)$. In sum, G_i and G_j can not be tangent at any $b \in (0,\bar{b})$ and moreover, if G_i and G_j cross then G_j must intersect G_i from below.

2. At the boundaries of the support of the equilibrium effort levels, 0 and $\bar{b}$, the distributions of effort may be tangent. In particular, a direct inspection of (27) reveals that the they are tangent at $\bar{b}$, that is, $G_i(\bar{b}) = G_j(\bar{b}) = 1$ and $g_i(\bar{b}) = g_j(\bar{b})$, where $\bar{b} = b_i(\overline{v}) = b_j(\overline{v})$ as established by Proposition 5. These equalities and the expression for the derivative of the effort density, (5), yield $g_i'(\bar{b}) \geq g_j'(\bar{b})$, if and only if, $f_i(\overline{v}) \leq f_j(\overline{v})$. But, $F_j \succ F_i$ implies $f_i(\overline{v}) \leq f_j(\overline{v})$. Moreover, by assumption $f_i(\overline{v}) < f_j(\overline{v})$ and therefore $g_i'(\bar{b}) > g_j'(\bar{b})$. As a result, there is an $\delta > 0$ such that for any $\varepsilon < \delta$, $g_i(b-\varepsilon) < g_j(b-\varepsilon)$. This last result implies that $G_i(b-\varepsilon) > G_j(b-\varepsilon)$. Put simply, also at the top $\underline{b}$, G_j must intersect G_i from below.

The conclusions of 1 and 2 above imply that G_i and G_j can never intersect in the interior of the support and, G_i is always above G_j. ■

A.9.2 Different Attitudes Towards Risk

Proof of Propositions 7 and 9. In the model with CARA agents, $u_i(x) = -\frac{\exp(-\rho_i x)}{\rho_i}$. To avoid cumbersome notation, we present the proof for the case in which all the agents bid in the same interval. For this environment, (4) becomes,

$$\begin{aligned} S_i(b) &= \frac{u_i'(w-b)(1-W_i(b)) + u_i'(v_i-b+w)W_i(b)}{W_i(b)(u_i(v_i-b+w)-u_i(w-b))} \\ &= \frac{\rho_i}{W_i(b)}\left(\frac{1}{(1-e^{-\rho_i\phi_i(b)})} - W_i(b)\right), \end{aligned} \tag{29}$$

so, by proposition 2,

$$\begin{aligned} g_i(b) &= \frac{G_i(b)}{N-1}\left(\sum_{j\neq i} S_j(b) - (N-2)S_i(b)\right) = \\ &= \frac{G_i(b)}{(N-1)}\left\{\sum_{j\neq i}\frac{\rho_j}{W_j(b)}\left(\frac{1}{1-\exp(-\phi_j\rho_j)} - W_j(b)\right)\right. \\ &\quad \left. -(N-2)\frac{\rho_i}{W_i(b)}\left(\frac{1}{1-\exp(-\phi_i\rho_i)} - W_i(b)\right)\right\} \end{aligned} \tag{30}$$

When $N = 2$, this reduces to:

$$g_j(b) = \frac{\rho_i}{1-\exp(-\rho_i\phi_i(b))} - \rho_i G_j(b), \; i \neq j \tag{31}$$

Since both $\phi(\cdot)$ and $G_j(\cdot)$ are increasing in b, it follows that g_j is decreasing in b. This proves half of Proposition 7.

When agents are symmetric with respect to risk aversion parameter, that is, when $\rho_i = \rho$ for all $i \in \{1, .., N\}$, then in a symmetric equilibrium, inverse bids are the same, $\phi_i = \phi$, and so is the effort density, $g_i = g$ for all $i \in \{1, .., N\}$. Then (30) reduces to:

$$g(b) = \frac{\rho}{[1-\exp(-\rho\phi(b))](N-1)G^{N-2}(b)} - \frac{\rho G(b)}{N-1}. \tag{32}$$

Once more, since both $\phi(\cdot)$ and $G(\cdot)$ are increasing in b, it follows that g is decreasing in b. This concludes the proof of Proposition 7.

To prove Proposition 9 we present conditions under which the density of effort is increasing in a neighborhood of the highest equilibrium effort. For than note that by (29)

$$S_i'(b) = -\frac{\rho_i}{W_i(b)^2}\frac{\frac{\partial W_i(b)}{\partial b}}{1-\frac{1}{\exp(\rho_i\phi_i(b))}} - \frac{\rho_i^2}{W_i(b)}\frac{\frac{\partial \phi_i(b)}{\partial b}}{\exp(\rho_i\phi_i(b))\left(1-\frac{1}{\exp(\rho_i\phi_i(b))}\right)^2}.$$

Given the definition of the winning probability of contestant i and (33),

$$\begin{aligned} W_i'(\bar{b}) &= \sum_{j\neq i}\prod_{k\neq i,j} G_k(\bar{b})\, g_j(\bar{b}) = \\ &= \sum_{j\neq i} g_j(\bar{b}) = S_i(\bar{b}). \end{aligned}$$

Also, by definition of g_i, we have $\frac{\partial\phi_i(b)}{\partial b} = \frac{g_i(\bar{b})}{f(\bar{\phi})}$, therefore,

$$S_i'(\bar{b}) = -\left(\exp\left(\rho_i\overline{\phi}\right)\right) S_i^2\left(1+\frac{g_i(\bar{b})}{f(\overline{\phi})}\right)$$

using the identity,

$$\frac{1}{1-\frac{\rho_i}{S_i(\bar{b})+\rho_i}} = \exp\left(\rho_i\overline{\phi}\right),$$

we have

$$S_i'(\bar{b}) = -\left(1+\frac{g_i(\bar{b})}{f(\overline{\phi})}\right)\left(S_i(\bar{b})+\rho_i\right) S_i(\bar{b}).$$

By first order conditions (3),

$$\begin{aligned} \sum_{j\neq i} g_j'(\bar{b}) &= S_i'(\bar{b}) + \left(\sum_{j\neq i} g_j(\bar{b})\right)^2 \\ &= S_i(\bar{b})^2 - \left(1+\frac{g_i(\bar{b})}{f(\bar{b})}\right)\left(S_i(\bar{b})+\rho_i\right) S_i(\bar{b}) \\ &= -S_i(\bar{b})\left(\frac{g_i(\bar{b})}{f(\bar{b})}\left(S_i(\bar{b})+\rho_i\right)+\rho_i\right) \end{aligned}$$

It follows that

$$\begin{aligned}(N-1)\,g_2'\left(\overline{b}\right) &= -S_1\left(\overline{b}\right)\left(\frac{g_1\left(\overline{b}\right)}{f\left(\overline{b}\right)}\left(S_1\left(\overline{b}\right)+\rho_1\right)+\rho_1\right)<0 \\ (N-2)\,g_2'\left(\overline{b}\right)+g_1'\left(\overline{b}\right) &= -S_2\left(\overline{b}\right)\left(\frac{g_2\left(\overline{b}\right)}{f\left(\overline{b}\right)}\left(S_2\left(\overline{b}\right)+\rho_2\right)+\rho_2\right)\end{aligned}$$

Therefore,

$$\begin{aligned}g_1'\left(\overline{b}\right) &= -S_2\left(\overline{b}\right)\left(\frac{g_2\left(\overline{b}\right)}{f\left(\overline{\phi}\right)}\left(S_2\left(\overline{b}\right)+\rho_2\right)+\rho_2\right) \\ &\quad +\frac{(N-2)}{(N-1)}S_1\left(\overline{b}\right)\left(\frac{g_1\left(\overline{b}\right)}{f\left(\overline{\phi}\right)}\left(S_1\left(\overline{b}\right)+\rho_1\right)+\rho_1\right)\end{aligned}$$

Note that if $\rho_2=0$, then $S_2\left(\overline{b}\right)=\lim_{\rho_i\to 0}\dfrac{\rho_i\exp(-\rho_i\overline{\phi})}{1-\exp(-\rho_i\overline{\phi})}=\frac{1}{\overline{\phi}}$, also $g_2\left(\overline{b}\right)=\frac{S_1\left(\overline{b}\right)}{(N-1)}$, so

$$g_1'\left(\overline{b}\right)\frac{(N-1)}{(N-2)}=S_1\left(\overline{b}\right)\left(S_1\left(\overline{b}\right)\frac{g_1\left(\overline{b}\right)}{f\left(\overline{\phi}\right)}+\rho_1\left(\frac{g_1\left(\overline{b}\right)}{f\left(\overline{\phi}\right)}+1\right)-\frac{1}{\overline{\phi}^2\,(N-2)}\right),$$

which is positive, provided N is large enough or ρ_1 is sufficiently big. Note also, that the latter does not prevent the first contestant from participating, as his effort density at the top is still positive (recall $S_i\left(\overline{b}\right)$ is decreasing in ρ_i)

$$g_1\left(\overline{b}\right)=S_2\left(\overline{b}\right)-\frac{(N-2)}{(N-1)}S_1\left(\overline{b}\right)\geq\frac{S_2\left(\overline{b}\right)}{(N-1)}=\frac{1}{\overline{\phi}\,(N-1)}\geq 0,$$

which is consistent with him bidding at the top. This concludes the proof. ■

Proof of Proposition 10.

1. To prove the first assertion, which echoes Fibich, Gavious, and Sela (2004), let us evaluate the necessary conditions (5) $b=\overline{b}$, noting that

$W_i\left(\bar{b}\right) = 1$ for all i,

$$g_i(\bar{b}) = \frac{1}{N-1}\left[\sum_{j\neq i} S_j\left(\bar{b}\right) - (N-2)S_i\left(\bar{b}\right)\right] \tag{33}$$

$$S_i\left(\bar{b}\right) = \frac{u_i'\left(\overline{v}-\overline{b}+w\right)}{u_i\left(\overline{v}-\overline{b}+w\right)-u_i\left(w-\overline{b}\right)} \tag{34}$$

Without loss of generality we can normalize the utility functions of i and j such that $u_i'\left(\overline{v}-\overline{b}+w\right) = u_j'\left(\overline{v}-\overline{b}+w\right)$ and $u_i\left(\overline{v}-\overline{b}+w\right) = u_j\left(\overline{v}-\overline{b}+w\right)$. As i is more risk averse than j, u_i is a concave transformation of u_j, so after the transformation $u_i\left(x\right) \leq u_j\left(x\right)$ for any x, in particular, $u_i\left(w-\overline{b}\right) < u_j\left(w-\overline{b}\right)$. It follows that $S_i\left(\overline{b}\right) < S_j\left(\overline{b}\right)$, so $g_i(\bar{b}) > g_j(\bar{b})$, as required.

2. By (26),

$$\lim_{b\searrow 0}\frac{G_j(b)}{G_i(b)} = \frac{u_i'(w)}{u_i(\underline{v}+w)-u_i(w)}\frac{u_j(\underline{v}+w)-u_j(w)}{u_j'(w)}$$

Normalizing the Bernoulli functions, again, such that $u_i'(w) = u_j'(w)$ and $u_i(w) = u_j(w)$, and given that i is more risk averse, $u_i(\underline{v}+w) \leq u_j(\underline{v}+w)$, so $\lim_{b\searrow 0}\dfrac{G_j(b)}{G_i(b)} \geq 1$, as required.

■

A.10 Derivation of Examples

Sketch of the Proof of Proposition 12. It suffices to establish that the inverse bid functions satisfy the following system of differential equations:

$$\phi_1'(b) = \frac{\phi_1(b)}{(2-2\phi_1(b))\phi_2(b)^2}$$

$$\phi_2'(b) = \frac{2-\phi_1(b)}{\phi_1(b)\phi_2(b)}$$

It is easy to show that using the identities $Q(\phi_1(b)) = \phi_2(b)$ and $b_j(\phi_j(b)) = b$, $j = 1, 2$. The complete proof is available on request. ■

A.10.1 Continuity of bidding function for the Uniform Model

Proof of Proposition 13. As in the proof of Proposition 20, consider a minimal gap where contestant i bids discontinuously. From (24), we have that $\sum_{j\in J}\alpha_j^{-1}=(K-1)\alpha_i^{-1}$, where K is the number of active contestants in the minimal gap. The first-order condition of contestant i implies that $\phi_i'(b)=0$ for $b\in(\underline{\beta}-\varepsilon,\underline{\beta})\cup(\overline{\beta},\overline{\beta}+\varepsilon)$, for some $\varepsilon>0$. This is so because, in this range of bids on a neighborhood of the exterior boundary of the minimal gap, the number of active contestants is $K+1$, and so $K(b)-2=K+1-2=K-1$. But $\phi_i'=0$ on a neighborhood of the exterior boundary of the minimal gap contradicts that ϕ_i is strictly increasing in this neighborhood. ■

A.10.2 Participation Results for the Uniform Model

Proof of Proposition 14. We want to establish, for the uniform model, that:

1. if $b_i(\alpha_i)>0$ then $b_i(\alpha_i)=\overline{b}$;
2. if $b_i(\alpha_i)=\overline{b}$ and $a_j\geq a_i$ then $b_j(\alpha_j)>0$;
3. either $b_i(v)>0$ for all $v>0$ or $b_i\equiv 0$.

Assume that 1 does not hold. Let $i\in I\equiv arg\,max\left\{b_i(\alpha_i):b_i(\alpha_i)<\overline{b}\right\}$. Let J denote the subset of contestants who bid in $\left(b_i(\alpha_i),\overline{b}\right]$ and $K=\#J$ the number of such contestants. By (22), for any bid b in this interval, i's marginal probability of winning is: $\frac{\sum_{j\in J}\alpha_j^{-1}}{K-1}$. Furthermore, for it to be optimal that i does not bid in the interval, it must be that,

$$\sum_{j\in J}\alpha_j^{-1}<(K-1)\alpha_i^{-1}\text{for all } i\in I. \tag{35}$$

Some additional notation is required: let $i^*\in arg\,min\left\{\alpha_i:i\in I\right\}$ and $L=\#I$. The previous equation implies

$$\sum_{\substack{j\neq i^*\\ j\in J\cup I}}\alpha_j^{-1}<(K+L-2)\alpha_{i^*}^{-1}\Leftrightarrow\phi_{i^*}(b_i(\alpha_i)-\varepsilon)<0,\ \text{for some } \varepsilon>0, \tag{36}$$

which contradicts that bids are non-decreasing.

Part 2 follows immediately from a revealed preference argument: If $b_i(\alpha_i) > b_j(\alpha_j) = 0$ then j's payoff is zero while i's payoff is positive. Nevertheless since $\alpha_j \geq \alpha_i$, contestant i can guarantee a positive payoff by bidding $b_i(\alpha_i)$.

Assume that part 3 does not hold. Therefore, it must be that there is a $v^* > 0$ such that $b_i(v) = 0$ for $v < v^*$ and $b_i(v) > 0$ for $v > v^*$. In other words, $1 > G_i(0) > 0$ or equivalently,

$$\lim_{b \searrow 0} \phi_i(b) = v^* > 0.$$

From the system of first-order conditions, the following relation can be obtained:

$$\forall i, j \in J(b),\ \frac{\phi_i'(b)}{\phi_i(b)} = C_{ij} \frac{\phi_j'(b)}{\phi_j(b)},$$

where $C_{ij} > 0$ is a constant. The solution to the system of ordinary differential equations with initial condition $\left(\phi_j(\bar{b}\right) = (\alpha_j)_{j \in J(\bar{b})}$ is unique. Integrating the above equation yields,

$$\log(\phi_i(b)) = C_{ij} \log(\phi_j(b)) + C,$$

where C is a constant. Hence, $\lim_{b \searrow 0} \phi_j(b) = 0$ if and only if $\lim_{b \searrow 0} \phi_i(b) = 0$. In sum, whenever one of the active contestants bids zero with positive probability, all the other contestants bid zero with positive probability. Therefore, $W_i(0) > 0$ and contestant i with valuation $v_i^* > 0$ would obtain a positive payoff by biding $\varepsilon > 0$. ■

A.10.3 Aggregate Effort and Payoffs in the Uniform Model

The aggregated effort (revenue) is simply,

$$R(\phi_1, ..., \phi_N) = \sum_{i=1}^{N} b_i(\phi_i) = \frac{N-1}{\sum \alpha_j^{-1}} \sum_{i=1}^{N} \left[\frac{\phi_i}{\alpha_i}\right]^{\frac{1+\kappa}{\kappa_i}}$$

and so, the expected aggregated effort level is:

$$R(\alpha_1, ..., \alpha_N) = \frac{N-1}{\sum \alpha_j^{-1}} \sum_{i=1}^{N} \frac{\kappa_i}{\kappa_i + \kappa + 1} = \frac{N-1}{\sum \alpha_j^{-1}} \sum_{i=1}^{N} \frac{\sum_{j \neq i} \alpha_j^{-1} - (N-2)\alpha_i^{-1}}{2\sum_{j \neq i} \alpha_j^{-1} - (N-3)\alpha_i^{-1}} \tag{37}$$

Contestant i's interim payoff is,

$$\Pi_i(\phi_i) = \phi_i \prod_{j\neq i} G_j(b_i(\phi_i)) - b(\phi_i) = \phi_i \prod_{j\neq i} \left[\frac{\phi_i}{\alpha_i}\right]^{\frac{\kappa_j}{\kappa_i}} - \frac{N-1}{\sum \alpha_j^{-1}} \left[\frac{\phi_i}{\alpha_i}\right]^{\frac{1+\kappa}{\kappa_i}} =$$
$$= \left[\alpha_i - \frac{N-1}{\sum \alpha_j^{-1}}\right] \left[\frac{\phi_i}{\alpha_i}\right]^{\frac{1+\kappa}{\kappa_i}},$$

and consequently, i's expected payoff is

$$\Pi_i = \left[\alpha_i - \frac{N-1}{\sum \alpha_j^{-1}}\right] \frac{\kappa_i}{\kappa_i + \kappa + 1} = \left[\alpha_i - \frac{N-1}{\sum \alpha_j^{-1}}\right] \frac{\sum \alpha_j^{-1} - (N-1)\alpha_i^{-1}}{2\sum \alpha_j^{-1} - (N-1)\alpha_i^{-1}}.$$

A.10.4 Comparative Statics for the Uniform model

Proposition 26 *If $\kappa_i > 0$ then $\frac{\partial}{\partial \alpha_i}\Pi_i > 0$ and $\frac{\partial}{\partial \alpha_j}\Pi_i < 0$.*

Proof.

$$\frac{\partial}{\partial \alpha_i} \log\left(\Pi_i(\alpha)\right) =$$
$$= \left\{\alpha_i \sum_{j\neq i} \alpha_j^{-1} \left[4 + 5\alpha_i \sum_{j\neq i} \alpha_j^{-1} - N + 2\alpha_i^2 (\sum_{j\neq i} \alpha_j^{-1})^2\right]\right.$$
$$\left. + \left(\alpha_i \sum_{j\neq i} \alpha_j^{-1} - N + 2\right)^2 \left(2\alpha_i \sum_{j\neq i} \alpha_j^{-1} - N + 3\right)\right\}$$
$$\Big/ \left\{\alpha_i \left(\alpha_i \sum_{j\neq i} \alpha_j^{-1} - N + 2\right)\left(2\alpha_i \sum_{j\neq i} \alpha_j^{-1} - N + 3\right)\left(1 + \alpha_i \sum_{j\neq i} \alpha_j^{-1}\right)\right\} \tag{38}$$

As long as $\kappa_i > 0$, both, the denominator and the numerator of (38) are positive.

$$\frac{\partial}{\partial \alpha_j} \log\left(\Pi_i(\alpha)\right) = \frac{\partial}{\partial \alpha_j} \sum_{j\neq i} \alpha_j^{-1} \times$$
$$\times \frac{\alpha_i \left[N - (N-2)^2 + 3(N-1)\alpha_i \sum_{j\neq i} \alpha_j^{-1}\right]}{\left(\alpha_i \sum_{j\neq i} \alpha_j^{-1} - N + 2\right)\left(2\alpha_i \sum_{j\neq i} \alpha_j^{-1} - N + 3\right)\left(1 + \alpha_i \sum_{j\neq i} \alpha_j^{-1}\right)}$$

■

Proposition 27 *At the values of α_i where 'weak' contestants drop out, the marginal return of increasing α_i decreases discontinuously.*

Proof. Let $\alpha = (\alpha_j)_{j=1}^N$ be such that all contestants are participating and consider the critical $\alpha_i^* = \dfrac{1}{(N-2)\alpha_N^{-1} - \sum_{j\neq i,N} \alpha_j^{-1}}$. If α_i is increased but kept below α_i^*, all contestants still participate. But if α_i increases above α_i^*, contestant N drops out. contestant N bids zero in equilibrium. The payoff of contestant i is not differentiable at α_i^*. More exactly,

$$0 < \frac{\partial_+}{\partial \alpha_j} \log\left(\Pi_i(\alpha)\right) - \frac{\partial_-}{\partial \alpha_j} \log\left(\Pi_i(\alpha)\right) =$$

$$= \frac{(N-2)\left[3\alpha_i \sum_{j\neq i} \alpha_j^{-1} - (N-5)\right]}{\alpha_i (N-1)\left[\alpha_i \sum_{j\neq i} \alpha_j^{-1} - (N-3)\right]\left[2\alpha_i \sum_{j\neq i} \alpha_j^{-1} - (N-4)\right]\left(\alpha_i \sum_{j\neq i} \alpha_j^{-1} + 1\right)}.$$

■