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BALANCED ALLOCATION METHODS FOR CLAIMS PROBLEMS WITH INDIVISIBILITIES

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Abstract

We study claims problems in which a certain amount of indivisible units (of an homogeneous good) has to be distributed among a group of agents, when this amount is not enough to fully satisfy agents' demands. We are interested in finding solutions satisfying robustness and fairness properties. To do that, we define the *M-down methods*, which are the unique robust (*composition down* and *consistency*) and fair (*balancedness* or *conditional full compensation*) rules. Besides, we also establish the relationships between these *M-down methods* and the *constrained equal awards rule*.

Keywords: claims problems, indivisibilities, monotonic standard, balancedness, down method.

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1 Introduction

A claims problem represents a situation in which a given quantity of a certain commodity has to be distributed among some agents and the available resources fall short of the total demand. The canonical example is the allotment of the liquidation values when a firm goes bankrupt. In this example, as well as in the literature on claims problems, the good to be distributed is perfectly divisible, and so are the awards allotted to agents. The reader is referred to Moulin (2002) and Thomson (2003) for two surveys of the literature. Nonetheless, there are many claims situations involving the distribution of a commodity that comes in indivisible units.

Consider the following examples: In order to carry out the administrative tasks at university departments, a University hires some secretaries. On one hand, the number of hired secretaries depends on the financial capabilities of the University. On the other hand, each department has the right to receive, depending on its size, a certain number of secretaries. It may happen that the total number of secretaries the departments demand is larger than the available amount. How many secretaries should be assigned to each department? Another illustrative situation of such a class of problems is the distribution of radio frequencies among the different broadcasting corporations, whenever there is no auction mechanism. If the amount of frequencies requested by the firms is too large, the Government should decide how many frequencies are allotted to each corporation. The allotment of airport slots among airline companies constitutes another example.

Previous situations illustrate the so-called *claims problems with indivisibilities*. There, departments, media-corporations, or airline companies are referred to as *agents*. The amount of indivisible units to be allotted (secretaries, frequencies, or slots) is called *estate*, and the demands or rights are called *claims*. A *rule* is a way of distributing the available estate among the agents according to their claims. Rules can be either deterministic or probabilistic. We are interested here in deterministic single-valued rules.

In axiomatic theory, rules are defended on the basis of the properties they fulfil. Among those properties, the general equity requirement is *equal treatment of equals*. It defends that agents with equal claims should receive equal awards. In general, this property cannot be met in claims problem with indivisibilities, and thus, the equal treatment of equals principle takes on a different form: whenever two claimants have identical claims, their awards differ by, at most, one unit. This property was introduced by Balinski and Young (1977) under the name of *balancedness* (in the context of apportionment) and was also used by Young (1994) in claims problems with indivisibilities.

Claims problems with indivisibilities have usually been solved by using *priority methods*. For instance, in Moulin (2000) claimants arrive one at a time and they are fulfilled until the available amount is exhausted. The family of rules resulting from this procedure comprises unique solutions fulfilling three interesting procedural properties: *composition up*, *composition down*, and *consistency*. Imagine that, when estimating the estate we were wrong and it is larger than expected. Then two alternatives are open. Either we solve the new problem. Or we consider the problem with the underestimated estate. And then allocate the remaining estate, after reducing the claims by the amounts of the first step. *Composition up* requires the final allocation to be independent of the chosen alternative. Alternatively, imagine that when estimating the estate we were wrong and it is less than expected. Two alternatives are open. Either we solve the new problem. Or we consider a the problem in which the estate is the reduced one, and the claims are the allocation obtained with the overestimated estate. *Composition down* requires for the final allocation to be independent of the chosen alternative. *Consistency* states that when one agent leaves with her allocation, the solution of the reduced problem is such that all remaining agents receive identical awards as in the original problem. It is worth noting that the pure priority methods characterized in this way obviously fail to satisfy *balancedness*.

A different type of priority methods was introduced by Young (1994), and used also by Moulin and Stong (2002). They all use the idea of *standard of comparison*. A standard of comparison is simply a priority order defined over pairs agent-claim. If i, j are two agents, and x, y are their respective claims, if the standard of comparison σ is such that $\sigma(i, x) < \sigma(j, y)$, then agent i demanding x units has priority over agent j demanding y units. Given a particular standard of comparison, two natural allocation procedures arise: *up methods* and *down methods*. The first one proposes, by using *composition up*, allocating the estate unit by unit. Dually, the second procedure proposes, by using *composition down*, allocating the deficit unit by unit.

Moulin and Stong (2002) characterize the family of *down methods* associated to a standard of comparison by means of three properties: *linked claim-resource monotonicity* (there is no harm for the losses of an agent whose claim increases, but all others' claims and the deficit remain the same), *composition down*, and *consistency*.[1] Dual results are obtained for up methods.

The family of *down methods* associated to standards of comparisons contain allocation procedures that violate *balancedness*. In order to ensure this property, we should consider a subfamily of the standard of comparisons. We call them *monotonic standards*, giving priority to larger claims, i. e., $\sigma(i, x + 1) < \sigma(j, x)$ for all i, j, and x.

In this paper, we concentrate on *down methods* associated to *monotonic standards*. We call them *M-down methods*. It happens that these methods, apart from *balancedness*, also satisfy some other fairness properties that previously appeared in the literature on continuous claims problems (see Herrero and Villar (2001), Herrero and Villar (2002), and Yeh (2004)). *Conditional full compensation* is an example of such principles. It proposes that only claimants responsible of the problem should be rationed. We show that M-down methods not only satisfy this property, but they are the unique rules fulfilling *conditional full compensation*, *composition down* and *consistency* together. We also show here that any M-down method can be interpreted as a discrete version of the constrained equal awards rule. This statement is based on two facts. First, for any problem, the allocations prescribed by the constrained equal awards rule is the ex-ante expectation of the agents under the application of M-down methods, if all plausible monotonic standards are equally likely. Second, the allocations prescribed by any M-down method converge to the allocation recommended by the constrained equal awards rule when the size of the indivisibilities goes to zero.

The rest of the paper is structured as follows: In Section 2 we set up the model of claims problems with indivisibilities. In Section 3 we present the properties our rules fulfil. In Section 4 we introduce standards of comparison and the up and down methods as well as we present our main results. In Section 5 we establish the relationships between M-down methods and the constrained equal awards rule. In Section 6 we conclude with some final comments and remarks. Proofs are relegated to an Appendix.

[1] Moulin and Stong (2002) refers to *linked claim-resource monotonicity* and *composition down* as *demand monotonicity** and *lower composition* respectively.

2 Statement of the model

Let $\mathbb{N}$ be the set of all potential agents. Let $\mathcal{N}$ be the family of all non-empty finite subsets of $\mathbb{N}$. In a claims problem, or simply a **problem**, a fixed quantity of indivisible units, $E \in \mathbb{Z}_{++}$ (called **estate**), has to be distributed among a group of **agents**, $N \in \mathcal{N}$, according to their **claims** (represented by $c = (c_i)_{i\in N} \in \mathbb{Z}_+^N$) when E is not enough to fully satisfy all the claims, i.e., $\sum_{i\in N} c_i \geq E$. Therefore, a problem is given by a triple $e = (N, E, c)$ where $\sum_{i\in N} c_i \geq E$. Let $\mathbb{C}$ be the set of all problems.

$$\mathbb{C}^N = \left\{ e = (N, E, c) \in \{N\} \times \mathbb{Z}_{++} \times \mathbb{Z}_+^N : \sum_{i\in N} c_i \geq E \right\}$$

and

$$\mathbb{C} = \bigcup_{N\in\mathcal{N}} \mathbb{C}^N.$$

Let $\boldsymbol{C}, \boldsymbol{L} : \mathbb{C} \longrightarrow \mathbb{Z}_+^N$ be the **aggregate claim** and **aggregate loss** functions respectively:

$$\boldsymbol{C}(e) = \textstyle\sum_{i\in N} c_i, \qquad \boldsymbol{L}(e) = C(e) - E.$$

An **allocation** for $e \in \mathbb{C}$ is a distribution of the estate among the agents, that is, it is a list, $\boldsymbol{x} \in \mathbb{Z}_+^N$, of integer numbers satisfying two conditions: (a) Each agent receives a nonnegative amount non-larger than her claim (for each $i \in N$, $0 \leq x_i \leq c_i$); and (b) the estate is exactly distributed ($\sum_{i\in N} x_i = E$). Let $\boldsymbol{X(e)}$ be the set of all allocations for $e \in \mathbb{C}$. A **rule** is a way of selecting allocation, that is, it is a function, $\boldsymbol{F} : \mathbb{C} \longrightarrow \mathbb{Z}_+^N$, that selects, for each problem $e \in \mathbb{C}$, a unique allocation $F(e) \in X(e)$.

3 Desirable solutions

We wonder now whether it is possible to obtain desirable solutions for claims problems with indivisibilities, that is, rules that satisfy interesting properties both from a *fairness* and *robustness* points of view.

Resource monotonicity requires that nobody should be harmed if more units of the estate are available. In other words, it sets that if the estate increases, remaining the claims fixed, each agent should receive at least as many units as she did initially.

Resource monotonicity: For each $e = (N, E, c)$, $e' = (N, E', c) \in \mathbb{C}$, if $E' > E$, then $F(e') \geq F(e)$.

Linked claim-resource monotonicity refers to changes in claims. It requires that if an agent increases her claim, and the aggregate loss remains unchanged, then this agent's should lose, at least, as many units as she did initially.[2]

Linked claim-resource monotonicity: For each $e \in \mathbb{C}$ and each $i \in N$, if $c_i' > c_i$, then $c_i - F_i(e) \leq c_i' - F_i(N, E', (c_{-i}, c_i'))$, where $E' = E + (c_i' - c_i)$.[3]

It is worth noting that linked claim-resource monotonicity requires three different conditions to be applied: an increase in the claim, an increase in the estate, and both claim and estate has to be increased by exactly the same amount of units. Given though the

[2] This property was formulated by Moulin and Stong (2002) under the name of *demand monotonicity**.

[3] The notation c_{-i} refers to the restriction of c to the set of agents $N \setminus \{i\}$: $c_{N\setminus\{i\}}$.

requirement made by this property seems reasonable from a fairness point of view, it seems very unlikely that these three conditions met at the same time; specially if we take into account that claims and estate usually have different origin.

Next property is extremely useful when some uncertainty over the estate exists. Imagine that when estimating the value of the estate, we were too optimistic, and the actual value is smaller than expected. Now, two alternatives are open. Either we solve the new problem. Or we consider a the problem in which the estate is the reduced one, and the claims are the allocation obtained with the overestimated estate. The property of *composition down* requires for the final allocation to be independent of the chosen alternative.[4]

Composition down: For each $e = (N, E, c) \in \mathbb{C}$ and each $E' \in \mathbb{Z}_+$ such that $C(e) > E' > E$, then $F(e) = F(N, E, F(N, E', c))$.

It is not difficult to check that composition down implies resource monotonicity.

Finally, we consider properties regarding to changes in the set of agents. Suppose that, after solving a problem, $(N, E, c) \in \mathbb{C}$, a proper subset of the set of agents, $S \subset N$, decide to reallocate the total amount they have received. That is, they face the problem $(S, \sum_{i \in S} x_i, c_S)$, where $c_S \equiv (c_i)_{i \in S}$ and x is the allocation selected by the rule for (N, E, c). *Consistency* requires that the reallocation is the restriction, to the subset S, of the initial allocation. The reader is referred to Thomson (2004) for a widely exposition of the notions of *consistency* and its *converse.*

Consistency: For each $e = (N, E, c) \in \mathbb{C}$, each $S \subset N$, and for each $i \in S$, $F_i(e) = F_i(S, \sum_{j \in S} F_j(N, E, c), c_S)$.

Next property refers to situations in which, apart from the allocations in the two-agent case, we can recover the allocation for the general case. Let us consider an allocation for a problem, $(N, E, c) \in \mathbb{C}$, with the following feature: For each two-agent subset, the rule chooses the restriction of that allocation for the associated reduced problem to this agent subset. *Converse consistency* requires that the allocation is selected by the rule for the original problem (N, E, c).[5]

Let $c.con(e; F) \equiv \{x \in \mathbb{Z}_+^N : \sum_{i \in N} x_i = E$ and for all $S \subset N$ such that $|S| = 2, x_S = F(S, \sum_{i \in S} x_i, c_s)\}$

Converse consistency: For each $(N, E, c) \in \mathbb{C}$, $c.con(e; F) \neq \phi$, and if $x \in c.con(e; F)$, then $x = F(N, E, c)$.

Thomson (2004) formulate a very useful result involving consistency and its converse.

Lemma 3.1 (Elevator Lemma). *If a rule F is consistent and coincides with a conversely consistent rule, F', in the two agent case, then it coincides with F' in general.*

It is worth noting that converse consistency implies consistency. Chun (1999) proves the following result for claims problems with perfectly divisible good. It is also valid in the presence of indivisibilities.

Lemma 3.2. *Resource monotonicity and consistency together imply converse consistency.*

Moulin and Stong (2002) study the class of rules fulfilling claims monotonicity, composition up, and consistency together. In order to describe such a class of rules we use the notion of standard of comparison introduced by Young (1994).

[4]This property was formulated by Moulin (2000).

[5]This property was formulated by Chun (1999).

4 Standards of comparison. Two families of rules

A standard of comparison is a linear order (complete, antisymmetric and transitive) defined over the cartesian product of agents and claims, increasing in claims.

Standard of comparison: $\sigma : \mathbb{N} \times \mathbb{Z}_{++} \longrightarrow \mathbb{Z}_{++}$ such that for each $i \in \mathbb{N}$, and each $a \in \mathbb{Z}_{++}$, $\sigma(i, a+1) < \sigma(i, a)$. Let Σ denote the class of all standards of comparison.[6]

Consider a problem involving two agents, i and j, whose claims are c_i and c_j respectively. Imagine, for instance, that $\sigma(j, c_j) < \sigma(i, c_i)$. The standard of comparison may be interpreted in two different ways, in terms of gains and in terms of losses. (a) In terms of gains: If there is only one unit available, the standard of comparison would determine who gets the unit. In this particular case, the allocation would be $(x_i, x_j) = (0, 1)$. (b) IN terms of losses: If the estate were such that there is only one unit of deficit ($E = c_i + c_j - 1$), the standard of comparison would determine who loses the unit. In this particular case, the allocation would be $(x_i, x_j) = (c_i, c_j - 1)$.

Associated to any standard of comparison, two natural methods for solving claims problems can be constructed. The first option is allocating all units of the estate one by one. The second one is subtracting all units of deficit one by one, after giving (temporarily) all agents their claims. We shall call them *up methods* and *down methods*, respectively.

Let $\sigma \in \Sigma$ be a standard of comparison. For each problem $e \in \mathbb{C}$, the **pair with the highest priority** in e, according to σ, is the pair (i, c_i) such that $\sigma(i, c_i) < \sigma(j, c_j)$ for all $j \in N \setminus \{i\}$.

Up method associated to $\boldsymbol{\sigma}$, $\boldsymbol{U^\sigma}$ (Moulin and Stong (2002)): Let $e \in \mathbb{C}$. Give one unit of the estate to the agent corresponding to the pair with the highest priority, according to σ, among all those involved in e. Reduce the claim of this agent by one unit. Identify the agent corresponding to the pair with the highest priority, according to σ, in the resulting problem; and proceed in the same way. Repeat this process until the estate runs out.

Down method associated to $\boldsymbol{\sigma}$, $\boldsymbol{D^\sigma}$ (Moulin and Stong (2002)): Let $e \in \mathbb{C}$. Start by fully compensating all agents. Subtract one unit from the agent corresponding to the pair with the highest priority, according to σ, among all those involved in e. Reduce the claim of this agent by one unit. Identify the agent corresponding to the pair with the highest priority, according to σ, in the resulting problem; and proceed in the same way. Repeat this process until reaching the estate.

Next example illustrates how both methods work.

Example 4.1. Let $N = \{1, 2, 3\}$, and assume that the standard of comparison is such that, restricted to agents in N, $\sigma(2, x) < \sigma(1, y) < \sigma(3, z)$, for all $x, y, z \in \mathbb{Z}_{++}$. Now, consider the problem e where $E = 6$, and $c = (1, 5, 5)$. For the pairs involved in the aforementioned problem, we have

$$\sigma(2,5) < \sigma(2,4) < \sigma(2,3) < \sigma(2,2) < \sigma(2,1) < \sigma(1,1)$$
$$< \sigma(3,5) < \sigma(3,4) < \sigma(3,3) < \sigma(3,2) < \sigma(3,1).$$

We start by pairing each agent with her claim, that is, we consider the pairs $(1, 1)$, $(2, 5)$, and $(3, 5)$. According to σ, the pair with the highest priority is $(2, 5)$. Then we give one unit of the estate to agent 2, we reduce her claim in one unit, and we consider the new problem with claims $(1, 4, 5)$. We now consider the pairs $(1, 1)$, $(2, 4)$, and $(3, 5)$. According

[6]If $\sigma(i, a) < \sigma(j, b)$ we will understand that the pair (i, a) has priority over the pair (j, b).

to σ, the pair with the highest priority is $(2,4)$. Then we give one unit of the estate to agent 2, we reduce her claim in one unit. The table shows the rest of the procedure until the estate is completely allotted. The first column gives the k.th unit of the estate. The second column gives the allocation up to that unit, $x^{(k)}$. The third column gives the updated vector of claims, $c^{(k)}$.

E	$x^{(k)}$	$c^{(k)}$
	(0,0,0)	(1,5,5)
1	(0,1,0)	(1,4,5)
2	(0,2,0)	(1,3,5)
3	(0,3,0)	(1,2,5)
4	(0,4,0)	(1,1,5)
5	(0,5,0)	(1,0,5)
6	(1,5,0)	(0,0,5)

Similarly, in the next table we show the functioning of the down method for this same problem. In this case we start by fully compensating all agents. This implies allocating 9 units, but we only have 6 available. Thus, we need to remove 3 units. To do that we procede as follows. We pairing each agent with her claim, that is, we consider the pairs $(1,1)$, $(2,5)$, and $(3,5)$. According to σ, the pair with the highest priority is $(2,5)$. Then we subtract one unit of the estate from agent 2, we reduce her claim in one unit, and we consider the new problem with claims $(1,4,5)$. We now consider the pairs $(1,1)$, $(2,4)$, and $(3,5)$. According to σ, the pair with the highest priority is $(2,4)$. Then we subtract one unit of the estate from agent 2, we reduce her claim in one unit. The table shows the rest of the procedure until the 3 units have been removed. The first column gives the k.th unit of the estate. We start from 9 and we remove unit by unit up to reach 6 units. The second column gives the allocation up to that unit, $x^{(k)}$. The third column gives the updated vector of claims, $c^{(k)}$.

E	$x^{(k)}$	$c^{(k)}$
11	(1,5,5)	(1,5,5)
10	(1,4,5)	(1,4,5)
9	(1,3,5)	(1,3,5)
8	(1,2,5)	(1,2,5)
7	(1,1,5)	(1,1,5)
6	(1,0,5)	(1,0,5)

Down methods satisfy resource monotonicity, linked claim-resource monotonicity, composition down, consistency and converse consistency. Moreover, Moulin and Stong (2002) show that only those methods satisfy the aforementioned properties.

Theorem 4.1. *A rule F satisfies linked claim-resource monotonicity, composition down, and consistency if and only if there exists a standard of comparison $\sigma \in \Sigma$ such that $F = D^{\sigma}$.*

The previous example illustrates how the down methods work. Additionally, it shows that these methods may result in very unfair allocations. In Example 4.1 both the second and the third agent are claiming 5 units each. Nevertheless the amount they receive are far away. Under the down method, while agent 3 gets her whole claim, agent 2 gets nothing.

Therefore, down methods may violate a very desirable notion of fairness: *equal treatment of equals*. It requires that, if two claimants have equal claims, they should receive equal amounts. Obviously, in our context, no rule can fulfill this property (it is enough to consider a problem with two agents with equal claims, but only one unit to distribute). Balinski and Young (1977) and Young (1994) consider a weaker version of this condition: **balancedness**. If two agents have equal claims, then their allocations differ, at most, by one unit (that unit is representing precisely the size of the indivisibility).

Balancedness: For each $e \in \mathbb{C}$ and each $\{i, j\} \subseteq N$, if $c_i = c_j$ then $|F_i(e) - F_j(e)| \leq 1$.

As we mentioned above, the application of the down methods with some standards of comparisons results in allocations violating balancedness. We introduce now a requirement on the standards of comparison to recover such a property.

Monotonic standard of comparison: For each $\{i, j\} \subseteq \mathbb{N}$, and each $x, y \in \mathbb{Z}_+$, if $x > y$, then $\sigma(i, x) < \sigma(j, y)$. Let Σ^M denote the subfamily of all monotonic standards of comparison.

In other words, monotonic standards of comparison always give priority to larger demands: the higher the claim, the higher the priority. Next example applies the down method to the same problem as Example 4.1 did but, unlike there, now we consider a monotonic standard of comparison.

Example 4.2. Let $N = \{1, 2, 3\}$, and assume that the standard of comparison σ is monotonic. Furthermore, $\sigma(1, x) < \sigma(2, x) < \sigma(3, x)$ if x is odd, and $\sigma(1, x) < \sigma(3, x) < \sigma(2, x)$ if x is even. Now, let again $E = 6$, and $c = (1, 5, 5)$. The next table shows how the down method associated to this standard of comparison works

E	$x^{(k)}$	$c^{(k)}$
11	(1,5,5)	(1,4,5)
10	(1,4,5)	(1,4,5)
9	(1,4,4)	(1,4,4)
8	(1,4,3)	(1,4,3)
7	(1,3,3)	(1,3,3)
6	(1,2,3)	(1,2,3)

We obtained the following result. All proofs are relegated to Appendix B.

Theorem 4.2. *A rule F satisfies balancedness, linked claim-resource monotonicity, composition down, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = D^\sigma$.*

We shall call **M-down methods** the down methods associated to monotonic standards of comparison. This subfamily of down methods, apart from the properties in Theorem 4.2, satisfies some others principles of fairness. Next property exploits the idea that only claimants responsible for the problem should be rationed.

Consider the problem in with $N = \{1, 2, 3\}$, the claims are $c = (2, 30, 50)$, and there are only $E = 10$ units available. The problem comes obviously from the fact that the aggregate claim is larger than the estate. But in this example, it is particularly due to the second and third agents, whose claims are so high. On the other hand, the first agent's claim is reasonable for the estate in the following sense. If the rest of the agents were demanding, at most, the same as agent 1 is, the aggregate demand would be 6 units, and then the estate would be enough to fully satisfy all the agents. In this sense, agent 1 is not responsible of the problem and she should be excluded in the rationing. More generally, *conditional full compensation* refers to how small a claim should be for its owner to be fully honored. One way to decide that threshold of smallness in a problem is the following. Substitute it for the claim of any other agents whose claim is higher, and check whether there would then be enough to compensate everyone.[7]

Conditional full compensation: For each $e \in \mathbb{C}$, if $\sum_{j=1}^{n} \min\{c_i, c_j\} \leq E$, then $F_i(e) = c_i$.

It is worth noting that two of the fairness principles presented here are closely related. Conditional full compensation implies balancedness for *small* agents. That is, if two equal agents' claims satisfy the condition in conditional full compensation, then both agents receive the same amount (i.e., their equal claim). Nevertheless, conditional full compensation does not imply balancedness in general. For instance, in the two-agent, the implication does not hold. But, if we combine conditional full compensation together with composition down, then we get the following result. The proof is in Appendix B. [8]

Proposition 4.1. *In the two-agent case, conditional full compensation and composition down together imply balancedness.*

Theorems 4.1 and 4.2 are based on two types of properties, robustness on one hand (composition down and consistency) and fairness (linked claim-resource monotonicity and balancedness respectively) on the other. It would be interesting to determine the family of *robust* rules, that is, those rules satisfying simultaneously composition down and consistency. This is still an open question in the literature. Nevertheless, in claims problems with indivisibilities, as it is in any other rationing context, fairness is a crucial point. Fairness properties are, on the other hand, always desirable as well. Consider conditional full compensation as a notion of fairness. We characterize the rules satisfying conditional full compensation, composition down and consistency. In fact, as next theorem shows, such a family of rules is precisely the M-down methods.

We present now our main result.

Theorem 4.3. *A rule F satisfies conditional full compensation, composition down, and consistency if and only if there exists a monotonic standard of comparison $\sigma \in \Sigma^M$ such that $F = D^\sigma$.*

5 From the indivisibility to the continuum

In Theorems 4.2 and 4.3 above we obtained characterization results for the family of M-down methods. Some of those characterizations have analogous counterparts in characterization results of the continuous constrained equal awards (*cea*) rule.[9] Actually, the relationship between those M-down methods and the constrained equal awards rule is

[7] This property was formulated by Herrero and Villar (2002) under the name of "sustainability".

[8] Proposition 4.1 is parallel to the one presented by Herrero and Villar (2001), but, unlike them, in this case we show the relation among the properties without the requirement of continuity.

[9] Under the assumption that the estate were completely divisible, one of the most widely studied rules is the so-called constrained equal awards rule. The idea is equality in gains, adjusting, if it is necessary, to ensure that no agent receives more than his claim.

Constrained equal awards rule, *cea*: For each $e \in \mathbb{C}$, selects the unique vector $cea(e) = \min\{c, \lambda\}$ for some $\lambda \in \mathbb{R}$ such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

stronger. Any M-down method can be interpreted as a discrete version of the constrained equal awards rule. In this section we further explore the relationship between the family of M-down methods and the *cea* rule. Two types of results are obtained. On one hand, we see that, for any problem, the allocations prescribed by the constrained equal awards rule coincides with the ex-ante expectations of the agents under the application of M-down methods, if all plausible monotonic standard of comparison are equally likely. On the other hand, the allocations prescribed by any M-down method converges to the allocation recommended by the constrained equal awards rule when the size of the indivisibilities goes to zero. These results are presented below. Proofs are relegated to Appendix B.

Proposition 5.1. *Let $e \in \mathbb{C}$. Let $\Sigma^M_{(N,c)}$ denote the subset of Σ^M of the different standards of comparison involved in the problem e.*[10] *Then*

$$\frac{1}{\left|\Sigma^M_{(N,c)}\right|} \sum_{\sigma \in \Sigma^M_{(N,c)}} D^\sigma(e) = cea(e)$$

We face know the question of making the size of the indivisibilities smaller and smaller. As a way of example, assume that we are distributing secretaries to the different Departments of the University, and consider the possibility of the hours of work of the secretary to be distributed between two different Departments. That is, it is possible to have a fraction of a secretary's time. The amount of secretaries to distribute is the same but the size of the indivisibility is reduced to half its previous size. Let $\sigma \in \Sigma^M$, and consider an M-down method associated to σ, D^σ. The next result says that, if the size of the units the estate comes in is getting smaller and smaller, in the limit, the allocation prescribed by D^σ coincides with the allocation recommended by the constrained equal awards rule.

Proposition 5.2. *Let $(N, E, c) \in \mathbb{C}$ and let $k \in \mathbb{Z}_{++}$. Then, for each $\sigma \in \Sigma^M$,*

$$\lim_{k \to \infty} \frac{1}{k} D^\sigma(N, kE, kc) = cea(N, E, c)$$

6 Final Remarks

In this paper we have considered claims problems with indivisibilities, that is, problems in which the estate, the claims and the allocations are expressed in integer units. Moulin and Stong (2002) (Theorem 4.1) use linked claim-resource monotonicity, composition down, and consistency to characterize the down methods. Adding balancedness we characterize the M-down methods (Theorem 4.2). From all these four properties the most demanding one is linked claim-resource monotonicity. It requires an increase in the claim and in the estate, and both by the same amount. These three requirements seem very unlikely to be met. We propose to use another fairness property (conditional full compensation) requiring that only agents responsible of having a claims problem should be rationed. If we do that, we obtain that only M-down methods fulfil conditional full compensation, composition down and consistency. Theorem 4.3 states that both balancedness and linked claim-resource monotonicity in Theorem 4.2 we can replaced by conditional full compensation. Interestingly, we show that any M-down method can be interpreted as a discrete version of the *constrained equal awards* rule.

[10]In Σ^M we consider all possible orders over $\mathbb{N} \times \mathbb{Z}_{++}$. Notice that, for a given claims problem e, no all of them rank the pairs (i, c_i) involved in e in different ways. $\Sigma^M_{(N,c)}$ denotes precisely the subset of those different orders.

Considering as a reference point the study done in the paper, it is plausible to make an alternative analysis by using the idea of *duality*. Two rules are called dual rules if one of them allocates the awards in the same way the other allocates the losses. In this sense, and for each standard of comparison $\sigma \in \Sigma$, U^σ and D^σ are dual. The same idea can be applied to the properties, thus, two properties, $\mathcal{P}$ and $\mathcal{P}^*$, are dual if whenever a rule satisfies $\mathcal{P}$ then the dual rule satisfies $\mathcal{P}^*$. Therefore, considering the dual properties of the ones presented in this work, and taking into account that the M-down methods and M-up methods are dual, characterizations results can be obtained for the application of up methods with a monotonic standard of comparison.

The rules shown here have been defined using two ingredients: a standard of comparison and the way we use this standard. Alternatively to the methods proposed here, standards of comparison can be used in a different way. Consider a monotonic standard of comparison defined over $\mathbb{N} \times \mathbb{R}_+$. Identify each agent with her claim. Give one unit of the estate to the agent corresponding to the pair with the highest priority according to σ. Divide by two this agent's claim. Identify the agent corresponding to the pair with the highest priority, according to σ, and give to her one unit. Now, if she is the agent whose claim was divided in the previous stage, then divide now the original claim by three; if she is not, divide her original claim by two. In general, once you have identified the pair with the highest priority, divide her original claim by $k + 1$ if it was divided by k in a previous stage. Repeat this process until the estate runs out. Allocations obtained by the aforementioned procedure coincide with those coming from the D'Hont rule, used in most of the European elections.

Appendix A. On the tightness of characterizations results.

The characterizations in Theorems 4.2 and 4.3 are tight. We here prove the independence of the properties. For that purpose we define the three rules below.

Example 6.1. We define the rule $G^{\succ}$ as follows: Let $\succ: \mathbb{N} \longrightarrow \mathbb{Z}_{++}$ be an order defined over the set of potential agents. We start by dividing the estate among the agents with the lowest claims, attempting to give to each of them the same amount. If this is not possible, because of the indivisibility, then their allocations will differ by one unit. The agents with the highest priority according to $\succ$ are those who receive the extra unit. Then, if there is still some estate left, we divide it equally (again, respecting the weak equal treatment of equals principle) among agents with the second lowest claim. We continue the process until the estate runs out. Formally, let $e \in \mathbb{C}$, let $\mu^t(c) = t.th \min_{j \in N}\{c_j\}$, $M^t(c) = \{j \in N : \mu^t(c) = c_j\}$, $m^t(c) = |M^t(c)|$. Then for each $i \in M^k(c)$ let us define $\nu^k(c) = \sum_{s<k} m^s(c)\mu^s(c)$, then

$$G_i(e) = \begin{cases} 0 & \text{if } 0 \leq E \leq \nu^k(c) \\ \left\lfloor \frac{E-\nu^k(c)}{m^k(c)} \right\rfloor & \text{if } \nu^k(c) \leq E \leq \nu^{k+1}(c) \text{ and } i \notin Q^k_{E'}(c) \\ \left\lfloor \frac{E-\nu^k(c)}{m^k(c)} \right\rfloor + 1 & \text{if } \nu^k(c) \leq E \leq \nu^{k+1}(c) \text{ and } i \in Q^k_{E'}(c) \\ c_i & \text{otherwise} \end{cases}$$

where $Q^k_{E'}(c)$ is the subset of $M^k(c)$ involving the E' agents in $M^k(c)$ with the highest priority according to $\succ$, defining $E' = E - \nu^k(c) - \sum_{j \in M^k(c)} \left\lfloor \frac{E-\nu^k(c)}{m^k(c)} \right\rfloor$.

Example 6.2. We define the rule $R^{\succ}$ as $R^{\succ}(N, E, c) = c - G^{\succ^{-1}}(N, L(e), c)$, where $G^{\succ^{-1}}$ is the rule defined in Example 6.1 with the reverse order.

Example 6.3. We define the rule, F, as follows. Let $\sigma_1, \sigma_2 \in \Sigma^M$ be two different monotone standards such that $\sigma_1(i, x) < \sigma_1(i+1, x)$ and $\sigma_2(i+1, x) < \sigma_2(i, x)$. Then, we define the solution $F^{(\sigma_1, \sigma_2)}$ as

$$F^{(\sigma_1,\sigma_2)}(N, E, c) = \begin{cases} D^{\sigma_1}(N, E, c) & \text{if } |N| = 2 \\ D^{\sigma_2}(N, E, c) & \text{otherwise} \end{cases}$$

Example 6.4. We define the rule $H = D^{\sigma}$, where $\sigma \in \Sigma$ is such that $\sigma(i, .) < \sigma(2, .) < \sigma(3, .) < \ldots$.

Next table shows the properties appearing in Theorems 4.2 and 4.3 are independent.

Property	$G^{\succ}$	$R^{\succ}$	$F^{(\sigma_1,\sigma_2)}$	H
Balancedness	Y	Y	Y	N
Linked claim-resource monotonicity	Y	N	Y	Y
Conditional full compensation	Y	N	Y	N
Composition down	N	Y	Y	Y
Consistency	Y	Y	N	Y
Converse consistency	Y	Y	N	Y

Table 1: Independence of properties.

Appendix B. Proofs

Proof of Proposition 4.1

Let $e \in \mathbb{C}$ such that $N = \{i, j\}$ and $c = (c_i, c_j)$, where $c_i = c_j$. Let us suppose that the result is not true for some value of the estate E. Let $x = F(N, E, c)$, it happens that $x_i = F_i(N, E, c) < F_j(N, E, c) - 1 = x_j - 1$ $(x = (x_i, x_j))$. Note that in this case $x_j \neq 0$ (otherwise $x_i < 0$). Let $E' = 2x_i$, then, by *conditional full compensation*, $F(N, E', x) = (x_i, x_i)$. By *composition down*,

$$F(N, E', c) = F(N, E', F(N, E, c)) = F(N, E', x) = (x_i, x_i).$$

Let $\overline{E} \in [E', E]$, by *resource monotonicity* (implied by *composition down*), $(x_i, x_i) = F(N, E', c) \leq F(N, \overline{E}, c) \leq F(N, E, c) = (x_i, x_j)$, hence $F(N, \overline{E}, c) = (x_i, \overline{E} - x_i)$. Let $E_1 \geq E$ such that $F_i(N, E_1, c) = x_i$ and for all $\widetilde{E} > E_1$, $F_i(N, \widetilde{E}, c) > x_i$. Let us take, in particular, $\widetilde{E} = E_1 + 1$. Let $(\widetilde{x}_i, \widetilde{x}_j) = F(N, E_1 + 1, c)$ and $E_2 = 2\widetilde{x}_i$. Then

- By *conditional full compensation*, $F(N, E_2, \widetilde{x}) = (\widetilde{x}_i, \widetilde{x}_i)$. By *composition down*, $F(N, E_2, c) = (\widetilde{x}_i, \widetilde{x}_i)$.
- *Resource monotonicity* together with the fact that $\widetilde{x}_i > x_i$, we obtain that $\widetilde{x}_i = x_i + 1$. Then, $E_2 \in [E', E]$, since $E_2 = 2\widetilde{x}_i > 2x_i = E'$ and $E_2 = 2\widetilde{x}_i = 2(x_i + 1) = (x_i + 1) + (x_i + 1) < x_i + x_j + 1 = E + 1$, i.e., $E_2 \leq E$. By the reasoning above, $F(N, E_2, c) = (x_i, E_2 - x_i)$.

Taking into account both facts,we rearch a contradiction. Therefore $|x_i - x_j| \leq 1$, and hence F satisfies *balancedness*.

Proof of Theorem 4.2

It is straightforward to check that any M-down method satisfies all the four properties. Conversely, by Theorem 4.2 we know that only up methods satisfy *linked claim-resource monotonicity*, *composition down* and *consistency*. We will show that if D^σ is *balanced* then $\sigma \in \Sigma^M$. Let us suppose that this is not true and there exists $\sigma \in \Sigma \setminus \Sigma^M$ such that D^σ is *balanced*. Since $\sigma \notin \Sigma^M$, there exist $\{i, j\} \in N$ and $x, y \in \mathbb{Z}_{++}$ such that $x > y$ and $\sigma(j, y) < \sigma(i, x)$. By *consistency*, it is enough to reach a contradiction in the two-agent case. By definition of standard of comparison, we have that $\sigma(j, x) < \sigma(j, y) < \sigma(i, x)$. Consider now the problem $e = (\{i, j\}, 2x - 2, (x, x))$; then $D^\sigma(e) = (x, x - 2)$ violating *balancedness*. We reach in this way a contradiction and, therefore, $\sigma \in \Sigma^M$.

Proof of Theorem 4.3

It is easy to check that all M-down methods satisfy the three properties. Conversely, let us suppose that F is a rule satisfying *conditional full compensation*, *composition down*, and *consistency*. First we define a monotonic standard of comparison $\sigma \in \Sigma^M$. Afterwards we show that F coincides with D^σ.

Step 1. *Definition of the monotonic standard of comparison.* Let $\sigma \in \Sigma^M$ be defined as follows

$$x > y \Rightarrow \sigma(i,x) < \sigma(j,y)$$
$$x = y \Rightarrow [\sigma(i,x) < \sigma(j,y) \Leftrightarrow F_i(\{i,j\}, 2x-1, (x,y)) = x-1].$$

It is straightforward to see that σ is complete and antisymmetric. Let us show that σ is transitive. Suppose that there exist $\{i,j,k\} \subseteq N$ such that $\sigma(i,x) < \sigma(j,y)$, $\sigma(j,y) < \sigma(k,z)$, but $\sigma(i,x) > \sigma(k,z)$. By construction, this can only happen when $x = y = z$. By definition of σ, in such a case, $F_i(\{i,j\}, 2x-1, (x,y)) = x-1$, $F_j(\{j,k\}, 2x-1, (x,z)) = x-1$ and $F_k(\{k,i\}, 2x-1, (z,x)) = x-1$. Consider the problem $(\{i,j,k\}, 3x-2, (x,y,z))$. It only admits three possible allocations: $(x-1, x-1, x)$, $(x-1, x, x-1)$ and $(x, x-1, x-1)$. Suppose that $F(\{i,j,k\}, 3x-2, (x,y,z)) = (x-1, x-1, x)$, by *consistency*, $F_i(\{i,k\}, 2x-1, (x,z)) = x-1$, achieving in this way a contradiction with $F_k(\{i,k\}, 2x-1, (x,z)) = x-1$. An analogous argument is applied if $F(\{i,j,k\}, 3x-2, (x,y,z)) = (x-1, x, x-1)$, or if $F(\{i,j,k\}, 3x-2, (x,y,z)) = (x, x-1, x-1)$. Therefore $\sigma(i,x) < \sigma(k,z)$, and then σ is transitive.

Step 2. *Let us prove now that $F = D^\sigma$.* It is easy to check that D^σ satisfies *resource monotonicity* and *consistency*. Then, it satisfies *converse consistency* in application of Lemma 3.2. Since F is *consistent* and D^σ is *conversely consistent*, in application of Lemma 3.1, it is enough to consider the two-agent case. We also know by Proposition 4.1 that in such a case F fulfils *balancedness*. We make the proof in several steps. Let us consider the problem $(S,E,c) \in \mathbb{C}$ where $S = \{i,j\}$. Without loss of generality we can assume that $c_i \leq c_j$. We analyze the following cases:

Case 1. If $c_i = c_j = x$ and E is even. Then, by *balancedness*, $F(S,E,c) = \left(\frac{E}{2}, \frac{E}{2}\right) = U^\sigma(S,E,c)$.

Case 2. If $c_i = c_j = x$ and E is odd ($E = 2\lambda + 1$ for some $\lambda \in \mathbb{Z}$). Then, by *composition down*, $F(S,E,(x,x)) = F(S, 2\lambda+1, F(S, 2\lambda+2, (x,x)))$. By definition of σ, and applying *balancedness* in $(S, 2\lambda+2, (x,x))$, we conclude that

$$\begin{aligned} F(S,E,c) &= F(S, 2\lambda+1, (\lambda+1, \lambda+1)) \\ &= D^\sigma(S, 2\lambda+1, (\lambda+1, \lambda+1)) \\ &= D^\sigma(S,E,c) \end{aligned}$$

Case 3. If $E \geq 2c_i$. By *conditional full compensation*, $F(S,E,c) = (c_i, E-c_i) = D^\sigma(S,E,c)$.

Case 4. If $E < 2c_i$. By *composition down*, $F(S,E,c) = F(S,E,F(S,2c_i,c))$. Applying the arguments of Cases 1, 2 and 3, we have that $F(S,E,c) = D^\sigma(S,E,(c_i,c_i)) = D^\sigma(S,E,(c_i,c_i)) = D^\sigma(S,E,c)$.

Then, F coincides with D^σ in the two agents case, and therefore they also coincide in general.

Proof of Proposition 5.1

On one hand, we know that the constrained equal awards rule is *conversely consistent* (see Chun (1999)). On the other hand, it is easy to check that the M-down methods are *consistent*. Then, the average given by the left hand side in the formula is also consistent (see Thomson (2004)). By the Elevator Lemma (Lemma 3.1) it is enough to consider the two-agent case. But it is straightforward that in this case, both the constrained equal awards rule and the average coincide. As a result, they are equal in general.

Proof of Proposition 5.2

Let $\sigma \in \Sigma^M$. Let $(N, E, c) \in \mathbb{C}$. Let $k \in \mathbb{Z}_{++}$, then $(N, kE, kc) \in \mathbb{C}$. Let d_k be the distance between $\frac{1}{k}D^\sigma(N, kE, kc)$ and $cea(N, E, c)$:

$$d_k = \left\| \frac{1}{k} D^\sigma(N, kE, kc) - cea(N, E, c) \right\|_\infty .$$

We show that d_k goes to cero as k goes to infinity. We can manipulate d_k as follows:

$$\begin{aligned} d_k &= \max_{i\in N} \left| \tfrac{1}{k} D_i^\sigma(N, kE, kc) - cea_i(N, E, c) \right| \\ &= \max_{i\in N} \left| \tfrac{1}{k} \left(D_i^\sigma(N, kE, kc) - cea_i(N, kE, kc) \right) + \tfrac{1}{k} \left(cea_i(N, kE, kc) - cea_i(N, E, c) \right) \right| \\ &\leq \max_{i\in N} \tfrac{1}{k} \left| D_i^\sigma(N, kE, kc) - cea_i(N, kE, kc) \right| + \left| \tfrac{1}{k} cea_i(N, kE, kc) - cea_i(N, E, c) \right| \end{aligned}$$

On one hand, the constrained equal awards rule satisfies homogeneity.[11] Because of that $\left| \frac{1}{k} cea_i(N, kE, kc) - cea_i(N, E, c) \right| = 0$. On the other hand, notice that $|D_i^\sigma(N, kE, kc) - cea_i(N, kE, kc)| \leq 1$. Therefore, $d_k \leq \frac{1}{k}$. To conclude the proof it is enough to take limits when k goes to infinity.

[11] This property says that $cea(N, \lambda E, \lambda c) = \lambda cea(N, E, c)$ for all $\lambda \in \mathbb{R}$.

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