INTERGENERATIONAL EQUITY AND THE DISCOUNT RATE FOR COST-BENEFIT ANALYSIS

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Abstract. Current Office of Management and Budget (OMB) guidelines use the interest rate as a basis for the discount rate, and have nothing to say about an intergenerationally fair discount rate. We derive this discount rate by differentiating a social welfare function with respect to perturbations in individual endowments (which induce perturbations of equilibria) in an overlapping generations model with exogenous growth. A traditional utilitarian approach leads to too high values, and in a wide range, while Relative Utilitarianism implies it equals the growth rate of real per-capita consumption, independent of the interest rate.

The differentiation is based on a novel method, applicable to arbitrary policy-variations, and that reveals a deep and very general property of exogenous growth models.

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1. Introduction

Many public policy decisions — whether about a housing project, about managing extraction of natural resources, about pension reform, etc. — involve trade-offs of economic costs and benefits, that are spread over time. Crucial, then, is the choice of an appropriate discount rate, i.e., how to translate benefits and costs into present (consumption) terms (either explicitly stated as part of the regulatory principles, or implicitly embedded in a specific policy analysis). This choice should, clearly, be based on well-defined normative principles, i.e., has to be objective and justified, for policy-makers as well as their consultants. In this paper we use Relative Utilitarianism to derive such an intergenerationally fair discount rate in an overlapping generations model.

Our purpose here is analytical: taking existing practices (say, summarised in the OMB Circulars) as given, we suggest a way to think about the underlying principles behind these practices, and to translate abstract "equity" requirements into concrete terms. To derive the discount rate we compute the impact on social welfare of a small change in consumption endowments.

Circular A-4 of the U.S. Office of Management and Budget (2003) mandates that all executive agencies and establishments conduct a "regulatory analysis" for any new proposal, and more specifically (pp. 33–36), a cost-benefit analysis, at the rates of both 3% and 7%. Both rates are rationalised there as "the interest rate": the first one relative to private savings, the second one relative to capital formation and/or displacement, i.e., as the gross return on capital.

The OMB circular does refer explicitly to the requirement of equity vis-à-vis of future generations, and acknowledges it by requiring, for projects with substantial long-term impact, a further analysis at a "lower but positive" discount rate (p. 36), but more specific suggestions are hard to find.[1] This is the question we want to address.

The issue of discounting and, more broadly, intergenerational justice, has been controversial in the literature since, probably, Sidgwick (1874).[2] Ramsey (1928) (p. 543) presents discounting future utility ('enjoyments') as a "practice which is ethically indefensible and arises

[1]Other practitioners share this view, e.g.: "Morally speaking, there is no difference between current and future risk. Theories which, for example, attempt to discount effects on human health in twenty years to the extent that they are equivalent to only one-tenth of present-day effects in cost-benefit considerations are not acceptable."Wildi, Appel, Buser, Dermange, Eckhardt, Hufschmied, and Keusen (2000)

[2]"How far we are to consider the interests of posterity when they seem to conflict with those of existing human beings? It seems, however, clear that the time at which a man exists cannot affect the value of his happiness from a universal point of view; and that the interests of posterity must concern a Utilitarian as much as those of his contemporaries, except in so far as the effect of his actions on posterity — and even the existence of human beings to be affected — must necessarily be more uncertain." (p. 414)

merely from the weakness of the imagination". He then suggests a way to overcome 'technical' difficulties of constructing a discount-free utilitarian social welfare criterion (based on the difference between actual and 'bliss' level of utility), later referred to as the "Ramsey criterion." Discounting utilities, or 'social impatience,' was axiomatised by Koopmans (1960). A growing literature in social choice and welfare economics is concerned with incorporating intergenerational justice principles in a social welfare criterion (see, e.g., d'Aspremont (2006); also Asheim, Mitra, and Tungodden (2006) show existence of welfare functions satisfying some of Koopmans' (1960) postulates and principles of intergenerational equity, in particular, Chichilnisky's (1996) axioms of 'sustainable development'); in addition several contributions aim at characterising ethically acceptable (just) allocations over time (e.g., Asheim (1991), Fleurbaey and Michel (2003)).

To adequately tackle questions of intergenerational equity we suggest to use Relative Utilitarianism, a welfare criterion introduced in Dhillon and Mertens (1999), that allows for a meaningful comparison of well-being across individuals born at different times and faced with different consumption choices and different economic environments.[3] It explicitly requires equal treatment of individuals of different generations in its anonymity axiom. The importance of using explicit criteria for cost-benefit analysis was stressed by Drèze and Stern (1987), distinguishing this approach from that examining "potential improvements"[4] stemming from a project. Formulating a social welfare function, they argue, provides greater transparency to the cost-benefit analysis, assures consistency of related choices and avoids a special preference for inaction.[5]

We focus on "small projects", viewed as "a disturbance to the economy, displacing it from some initial equilibrium to a new one"(Bell and Devarajan (1983), pp. 457–8). This linearisation is essential to cost-benefit analysis itself, both in order to be able to speak of costs and benefits, rather than welfare differences, and in order to be able to conduct a separate cost-benefit analysis for each project, rather than having to do an overall welfare optimisation over all conceivable combinations of projects by all branches of the government. This means, projects are evaluated via 'shadow prices', and the discount rate is

[3]Relative Utilitarianism is discussed in more detail in section 1.2.

[4]See Mishan (1976) for an in-depth discussion of "potential Pareto improvements" (traced back to Pigou (1932)) and their application to cost-benefit analysis. For a more recent overview of cost-benefit criteria see Coate (2000).

[5]"... a fundamental shortcoming of evaluation criteria based on Pareto improvements, whether actual or potential, is that, unless they are taken to imply that Pareto-improving changes are the only acceptable ones (a view which we regard as extremely unappealing and which attaches undue weight to the status quo), they provide no decision criterion for projects which cannot lead to Pareto improvements. It is difficult to overcome this problem without accepting the need to specify a social welfare function which embodies more definite judgements."(p.49)

the shadow price for tomorrow's goods in terms of today's. Bell and Devarajan raised a concern that the shadow prices might not be well-defined if the corresponding policy is not fully specified. One way to avoid this is to translate the effect of a public project into its consumption equivalent for individuals. Viewing public projects this way, we have no reason to introduce public goods into the model, which makes the analysis more transparent. Moreover, this representation is closer to the practical guidance for conducting cost-benefit analysis suggesting that the impact of a public project be monetised (cf. OMB's Circular A-4). Thus, the relevant shadow price becomes the marginal social value created by an additional unit of consumption.

It is not uncommon to use prevailing prices to represent the shadow prices, and then it is the interest rate that has to be used as discount rate. The corresponding welfare criterion is very specific:

The status-quo is a given competitive equilibrium. Construct a social welfare function (SWF), W, as a weighted sum of individual utilities, $\sum_n \lambda_n u_n$, the weights being chosen such as to equalise the individual marginal utilities of consumption *at the given equilibrium*, so $\lambda_n \nabla u_n = \mu p$,[6] for the equilibrium price system p and some $\mu > 0$.[7]

Viewing projects as small perturbations of individual endowments, $\delta\omega_n$, we are interested in the induced variation of social welfare:

$$\delta W = \sum \lambda_n \delta u_n = \sum \lambda_n \langle \nabla u_n, \delta c_n \rangle = \mu \langle p, \sum \delta c_n \rangle = \mu \langle p, \sum \delta \omega_n \rangle$$

since $\langle p, \delta y \rangle = 0$, where y is the equilibrium production.[8,9]

Thus, with those specific weights, the prevailing prices are, indeed, the relevant shadow prices, reflecting the relative impact of the endowments on social welfare. In a dynamic interpretation, where goods become dated goods, the equilibrium price system includes, in particular, the interest rate, as the price of tomorrow's money in terms of today's.

One rationale for that approach is that cost-benefit analysis is to be carried out in a quite decentralised way by different government agencies, project by project. So the only way to ensure some coherence, and to ensure that each one stays within its area of competence, is to assume that the others do their job correctly — and in particular, that redistribution policy (typical competency of the legislature) is optimal, so that transfers are welfare-neutral at the margin, and hence the above

[6]This condition is implied, for example, by one of Samuelson's (1954) optimality conditions, see his condition (3).

[7]Equivalently, assuming, e.g., concave utility functions, one can deduce from the First Welfare Theorem the existence of utility weights such that the given equilibrium maximises the corresponding weighted sum of utilities over all feasible allocations; this yields the same weights.

[8]$\langle x, y \rangle$ denotes the inner product of two vectors x and y.

[9]Since we omit for simplicity public goods and externalities from our formal model, the $\delta\omega_n$ are assumed to include, in addition to the direct effect of the project, also the compensating variation (in goods) for the different external effects.

weights. In this paper we are, however, interested in the implications of intergenerational equity; hence we cannot assume that the prevailing interest rate is the correct discount rate. Thus we must depart from the above weights,[10] and use a SWF that explicitly embodies this concept of intergenerational equity.

In the next section we try to do this, in a toy-model, using the most traditional form of utilitarianism in applied policy analysis.

1.1. **A simple computation using the traditional methodology.** Let us start with a very simple model of an economy, in which individuals live for just one period, enjoying consumption $c_t > 0$ during their lifetime at t. Individual preferences over (lifetime) consumption are represented by a constant relative risk aversion utility function with coefficient, $\rho > 0$, so that $u(c) = c^{1-\rho}/(1-\rho)$; and suppose the economy is on a balanced growth path with per-capita consumption growing exponentially at rate $\gamma > 0$. Consider a policy that involves a variation in aggregate consumption of δC_t at each future date t and that is to be evaluated at time 0. The status-quo per-capita consumption at time t is $c_0 e^{\gamma t}$, where c_0 is the initial (time 0) per-capita consumption. Taking a traditional utilitarian criterion ($W = \sum_t e^{-\beta t} N_t u(c_t)$, where N_t is the number of agents at time t) as a guide for evaluating this policy, the net (social) benefit equals

$$\begin{array}{cl} & \sum_t e^{-\beta t} N_t \left[u\left(c_0 e^{\gamma t} + \frac{\delta C_t}{N_t}\right) - u\left(c_0 e^{\gamma t}\right)\right] \\ = & \sum_t e^{-\beta t} N_t u'\left(c_0 e^{\gamma t}\right) \cdot \frac{\delta C_t}{N_t} = \sum_t c_0^{-\rho} e^{-(\rho\gamma+\beta)t} \delta C_t \end{array}$$

This means that future consumption is discounted at the rate $\rho\gamma + \beta$ under this criterion. Even if we follow Sidgwick (1874) and Ramsey (1928) and set $\beta = 0$, to write explicitly that we want to treat future generations equally, the magnitude of the suggested discount rate, $\rho\gamma$, is far above any rates applicable in practice, and the estimated values have an extremely wide range, as the next subsection demonstrates.

1.2. **Orders of Magnitude for the Discount Rate.** To estimate γ one may use a measure of growth of real per-capita GDP. Based on the data from the Bureau of Economic Analysis, over the past 70 years the average in the U.S. is estimated to be around 2–2.5% per annum (with averages over various decades since 1950 ranging from 3% to 1.8%).

In the above model, individuals live for one period, so ρ does not reflect individual intertemporal preferences. Consistency with Harsanyi's axiomatisation(s) of such additive SWFs forces one to interpret u as the individual's von Neumann-Morgenstern utility function, and hence ρ as his coefficient of relative risk aversion. One of the most recent overviews compiling various (micro) estimates of the risk aversion coefficients is

[10] As mentioned before, any such departure will make transfers non welfare-neutral, and hence will imply that aggregation (i.e., that everything depends only on $\sum \delta\omega_n$) is no longer possible. We will deal with that difficulty in sect. 4.2.

contained in Einav and Cohen (2005). Remarkable is both the range as well as the magnitude of the suggested values, ranging from single- to three-digit values. They measure relative risk aversion coefficients from individual-level data on car insurance and annual income, obtaining *two-digit* estimates. Clearly, with a discount factor of this magnitude a cost-benefit analysis will select only very short-sighted policies. This remains true even with more conservative estimates, like, say, derived by Drèze (1981) ($\rho \sim 12$–15), or like those which seem accepted as corresponding to "representative" (instead of individual) behaviour in financial markets — say 3, leading to $\rho\gamma \sim 6$–7%, way too high.

In sum, it is impossible to view the traditional methodology described above as a correct interpretation of "treating future generations equally" — which is exactly what the SWF tried to do, by using $\beta = 0$.

1.3. **Discount Rate under Relative Utilitarianism.** Since the traditional methodology failed so badly, producing unreasonably high discount factors within a wide range, let us now look at Relative Utilitarianism, introduced in Dhillon and Mertens (1999).

The axiomatisation consists basically of applying Arrow's axioms to preferences over lotteries, after "surgically removing" from them everything which is clearly objectionable — i.e., which anyone would expect a good social welfare functional to violate: the implications that variations in the intensity of preference of x over y don't matter.

After this removal, one can add anonymity (implying here also that individuals of different generations are treated equally) to obtain an axiomatisation of a unique social welfare functional,[11] relative utilitarianism, that takes for each individual's preferences the von Neumann-Morgenstern representation having minimum 0 and maximum 1 over the feasible set, and sums those to obtain a representative of the corresponding social preferences.

It is stressed in that paper that this dependence on the feasible set implies that in actual use it should be applied with some universal feasible set, to quote "all alternatives that are feasible and just". In particular, in the present situation, the feasible set should consist not only of the "baseline" and the different proposals under consideration, but of all policies and policy-changes that might be considered by any agency of the government.

In (exogenous) growth models, the rate of growth is unaffected by any policy variable: policies affect only the height of the growth path, which, in the simple setup described in subsection 1.1, translates into multiplying per-capita consumption by some constant along the growth path. Therefore, the set of feasible policies at time t can be viewed as a range of induced per-capita consumption levels between $(1-\eta)c_0e^{\gamma t}$ and $(1+\zeta)c_0e^{\gamma t}$ for some constants η and ζ. Applying relative utilitarianism

[11] The axiomatisation assumes a finite number of agents.

to the simple model, we have to normalise individual utility $u(c_t)$ on the set of feasible policies:

$$v(c_0 e^{\gamma t} + \delta c_t) = \frac{u(c_0 e^{\gamma t} + \delta c_t)}{u\big((1+\zeta)c_0 e^{\gamma t}\big) - u\big((1-\eta)c_0 e^{\gamma t}\big)}$$

i.e., we divide by

$$\frac{\rho - 1}{c_0^{\rho-1}}\left[\frac{-1}{(1+\zeta)^{(\rho-1)}} + \frac{1}{(1-\eta)^{(\rho-1)}}\right]e^{(1-\rho)\gamma t} \quad \sim e^{(1-\rho)\gamma t}$$

So the variation of our SWF becomes

$$\sum_t e^{(\rho-1)\gamma t}\delta C_t u'\left(c_0 e^{\gamma t}\right) \sim \sum_t e^{(\rho-1)\gamma t} e^{-\rho\gamma t}\delta C_t = \sum_t e^{-\gamma t}\delta C_t$$

This implies that the previous discount rate of $\rho\gamma$ becomes now simply γ, 2–$2\frac{1}{2}$ %, right in the ball-park of "positive and $< 3\%$".

One could argue that the example is not representative; in particular, since individuals live only one period they have no incentive to save, so there can be no capital accumulation and growth. In a real model where there is growth and savings, there is also an interest rate, which individuals would use to smooth the shock over their lifetime, so one would expect the result to be driven back to the interest rate.

We will nevertheless show that the result, as well as that of sect. 1.1, does remain valid in the much more general framework of next section. Section 3 describes the solution concepts to which the main statement is applicable, and section 4 deals with the construction and interpretation of relative utilitarian welfare functions in an OLG context. The main result appears then in section 5 (and 6.1), and is discussed and given further interpretations in section 6. Section 7 concludes.

2. The model

We use a general-equilibrium overlapping generations model, cast in an exogenous growth framework. The main assumptions that we impose on the economy — homogeneity of utility functions with respect to (streams of) consumption, and constant returns to scale in production — are there to allow for a balanced growth path.[12]

2.1. **The Consumption Sector.**

2.1.1. *Population Dynamics.* Time is continuous, ranging from $-\infty$ to $+\infty$. There are several types of individuals. An individual of type τ lives up to age T_τ. The population dynamics are fully specified by non-decreasing right-continuous functions $P_{\tau,\tau'}$, defined on $[0, T_\tau]$ with $P_{\tau,\tau'}(s)$ being the number of children of type τ' an individual of type τ has at age s, and by saying that we are looking at a corresponding

[12]See King, Plosser, and Rebelo (2002) and Arrow and Kurz (1970) for a similar discussion.

invariant distribution.[13] But as long as we do not introduce bequest motives, it is only this distribution that matters. It is such that, at time t, the number of individuals of type τ in age-group $(s, s + ds)$ $(0 \leq s < T_\tau)$ is given by $N^\tau e^{\nu(t-s)}ds$, where N^τ is population of type τ born at time 0. So, population grows at rate $\nu > 0$, keeping the proportion of each age group of each type constant.

2.1.2. *Preferences and Endowments.* At each instant of his life, s, an individual of type τ born at time x consumes non-negative quantities of n goods, $c^{\tau,x}(s) \in \mathbb{R}^n_+$ and allocates fractions of his time to h types of labour, $z^{\tau,x}(s) \in \mathbb{R}^h_+$.[14]

His preferences over integrable life-time consumption-streams in $\mathbb{R}^{n+h}$ are derived from a utility function U^τ (e.g., increasing in the goods, decreasing in labour, concave, differentiable). He has no bequest motive.[15]

For balanced growth to be at all possible, we assume U^τ to be homogeneous, say of degree $1-\rho^\tau$, in the n streams of consumption-goods.[16]

Endowments are 0 — except for the "endowment of time", which is unity at every instant (24h/day). This imposes an instantaneous constraint on the individual feasible set requiring the sum of fractions of time devoted to all possible occupations to be always less than unity.

In what follows, a policy will be associated with a perturbation of endowments of consumption goods, $(\delta\omega)_i$ for $i = 1 \ldots n$ — cf footnote 9.

2.2. **Production.**

2.2.1. *Instantaneous production.* Instantaneous production is described by a closed convex cone $Y \subset \mathbb{R}^{h+m+n+m}$, $t \in \mathbb{R}$, describing feasible production plans transforming $h + m$ inputs (h types of (effective) labour,

[13] We keep everything deterministic here, just to avoid having to discuss irrelevant insurance markets for idiosyncratic risks.

[14] Sometimes we will use the notation $c^\tau(s,t)$ and $z^\tau(s,t)$ to stand for consumption and labour of an individual of type τ who is of age s at time t, so that $x = t - s$.

[15] We index consumption streams by age, in $[0, T_\tau]$, so all individuals of the same type have the same consumption set, $C[0, T_\tau]$, and utility function, independently of their birth-date.

[16] This model "contains" the one used by Arrow and Kurz (1970) (mentioned in footnote 31), which has been widely used to evaluate public investments in the literature since then. To see this assume all individuals live for a fixed period of time (unity), and an individual born at time t has a life-time utility of the form

$$U_t(c.) = \int_t^{t+1} e^{-\alpha(s-t)} u(c_s) ds$$

where c_t is consumption per head at date t (assumed independent of age for simplicity), and α is the individual time preference. Assume also that population grows exponentially at a rate ν. Then, aggregating over all individuals (integrate over t from $-\infty$ to $+\infty$) when discounting their expected utilities at birth at a rate β, one gets the Arrow-Kurz criterion:

$$W \equiv \int_{-\infty}^{\infty} N_t e^{-\beta t} U_t(c.) dt = \int_{-\infty}^{\infty} N_t e^{-\beta t} \int_t^{t+1} e^{-\alpha(s-t)} u(c_s) ds dt$$

$$= N_0 \int_{-\infty}^{\infty} e^{-\alpha s} u(c_s) \int_{s-1}^{s} e^{(\alpha+\nu-\beta)t} dt = M \int_{-\infty}^{\infty} e^{-\beta t} N_t u(c_t) dt, \text{ where } M \equiv \int_0^1 e^{(\alpha+\nu-\beta)x} dx$$

$L(t) \in \mathbb{R}^h_+$, and m types of capital) into n consumption goods and m investment goods. Assume no free lunch, $Y \cap \mathbb{R}^n_+ = \{0\}$.

Individuals supply labour (time) to the firms, and their productivity changes with time and age. The amount of effective labour of type i received at time t by a production firm from an individual of type τ and of age s is $e^{\gamma t}\varepsilon_i^\tau(s) z_i^\tau(s,t)$, where $\varepsilon_i^\tau(s)$ is this individual's life-cycle 'productivity' (in occupation i),[17] and where γ is (labour-enhancing) technological progress. Recall $z_i^\tau(s,t)$ is the amount of labour (time) supplied by an individual (of type τ) born at time $t-s$.

Thus, (exogenous) growth in this model is driven by a steady increase in labour productivity.

2.2.2. *Capital accumulation.* There are m capital goods K^i $(i = 1 \dots m)$, each with its corresponding investment good I^i, depreciation rate δ_i, and capital-accumulation equation $\frac{dK^i(t)}{dt} = I^i(t) - \delta^i K^i(t)$,[18] together with the "initial condition" that $\limsup_{t\to-\infty} e^{-(\gamma+\nu)t} K^i(t) < \infty$.

The rest of this subsection is devoted to verifying that the production set is well-defined (proofs in appendix).

The capital accumulation condition and the initial condition have the following implication:

Lemma 1. *$K^i(t) = e^{-\delta^i t}\int_{-\infty}^t e^{\delta^i s} I^i(s)ds$ for all t, where the integral is a Lebesgue integral.*

To ensure bounded production possibilities, capital cannot reproduce itself ("rabbit economy").[19] Lemma 1 ensures that $K^i(\cdot)$ is uniquely determined by $I^i(\cdot)$. However it might not be sufficient to guarantee that any investment policy (e.g., I is a function of current K instead of time) has a well-determined outcome, without either using the full strength of the "initial condition" (lemma 2 below), or slightly reinforcing the assumption that capital cannot reproduce itself (lemma 3).

Lemma 2. *Assume Y is such that no investment good can be produced without some form of labour input. Assume $R \equiv \gamma + \nu + \delta > 0$. Then the set of all feasible functions $K^i(t)$ and $I^i(t)$ is bounded above by $\bar{K}e^{(\gamma+\nu)t}$ for some $\bar{K}$.*

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[17] For example, in the textbook OLG models going back to Samuelson (1958) ε would be 1 during the first half of life and 0 after.

[18] Assumed to hold a.e., and implying that K_t^i is assumed locally a Perron primitive and I_t^i locally Perron-integrable.

[19] For instance, assume a single good, a single type of labour, a CES production function $(AK^\alpha + BL^\alpha)^{1/\alpha}$, and a policy where all agents work full-time and consume nothing (e.g., in order to get an upper bound on capital and investment). Assume also $A^{1/\alpha} \geq R$ with $R = \gamma + \nu + \delta$. Note that $L_t = L_0 \exp(\gamma + \nu)t$, so for $D = BL_0^\alpha$, $K'(t) = (AK^\alpha(t) + De^{(\gamma+\nu)t})^{1/\alpha} - \delta K(t)$; or with $x(t) = K(t)e^{-(\gamma+\nu)t}$, $x'(t) = (Ax^\alpha(t) + D)^{1/\alpha} - Rx(t) \geq D^{1/\alpha} > 0$. Since $x(t) \geq 0$, there is no solution, i.e., the upper bound of $K(t)$ is infinity. And even if $B = 0$, the solutions are $x(t) = Ce^{(A^{1/\alpha}-R)t}$, with $C \geq 0$ arbitrarily large, so $K(t)$ is unbounded in this case too.

In fact, at least with a slightly stronger condition on Y, the formula of lemma 1 suffices, without the initial condition:

Lemma 3. *Assume* $\exists\varepsilon > 0, A, B\colon (-L, -K, C, I) \in Y \implies \|I\| \leq A\|L\| + B\|K\|^{1-\varepsilon}\|L\|^{\varepsilon}$. *Then the conclusions of lemma 2 hold, assuming just lemma 1, without the need for the "initial condition".*

3. Equilibria (Solution Concepts)

In addition to the classical Arrow-Debreu equilibrium concept, there are other possible solution concepts for this economy, to which our theorem is applicable too.

3.1. Time Invariance.

The economy we have described possesses a convenient time-invariance property that will prove to be useful later. We consider the effect on the economy (i.e., the description of the population, the feasible consumption and production plans, and individual preferences thereon) of shifting the origin of time by h.

Definition 4. *The transformation* $\mathbb{T}_h$ *of the economy ('time-shift by h')*

1. *shifts all consumption, production and endowment vectors (both goods and labour) forward in time by* h*,*
2. *multiplies all non-labour individual quantities (endowments of goods, allocations of goods) in the economy by* $\exp(\gamma h)$*,*
3. *multiplies the aggregate quantities of population and labour in the economy by* $\exp(\nu h)$*.*[20]

Now we claim that the economies we consider are time-invariant in the sense of this transformation:

Lemma 5. *Each* $\mathbb{T}_h$ *is an automorphism of the model:*[21]

- *it maps feasible production plans in a 1-to-1 way onto feasible production plans.*
- *it maps the preferences of each consumer between different consumption bundles (goods and labour) to the preferences of his image, born time h later. And his initial endowment is mapped as well to the initial endowment of his image.*

Remark 6. $\mathbb{T}_h$ *induces a map from allocations in the initial economy to the allocations in the image economy.*

Definition 7. *A (set-valued) solution concept is time-invariant if time-shifts* $\mathbb{T}_h$ *map solutions to solutions of the image economy.*

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[20] And hence aggregate quantities of all non-labour goods (consumption, capital, investment) are multiplied by $\exp((\gamma+\nu)h)$.

[21] Hence, if endowments are invariant under $\mathbb{T}_h$, i.e., if the endowment of goods of an agent of type τ born at time t is of the form $\omega^{\tau}\exp(\gamma t)$ (in particular, 0), then $\mathbb{T}_h$ is even an automorphism of the economy.

3.2. **Examples of Time-Invariant Solution Concepts.** Next, let us consider several examples of time-invariant solution concepts. First we discuss Arrow-Debreu equilibrium, and then briefly mention a couple of other examples: an adaptation of Diamond's (1965) equilibrium to this framework, and a 'social planner' solution, allocating goods to maximise a 'time invariant' objective, e.g., the specific Relative Utilitarian criterion that we use to evaluate welfare perturbations.

3.2.1. *Arrow-Debreu Equilibrium.* To describe Arrow-Debreu equilibria for the our economy we have to define profits of a firm. It is convenient to think of two types of firms: a single firm that handles the instantaneous production and has Y as technology,[22] and one firm per capital good that handles the corresponding investment and has the capital accumulation equation as technology.

Given $p_c(t) \in \mathbb{R}^n$, $p_I(t) \in \mathbb{R}^m$, the equilibrium prices for consumption and investment goods, and $p_k \in \mathbb{R}^m$, $p_l \in \mathbb{R}^h$, the equilibrium rental rates for capital and labour, the *production firm*[23] chooses the amount of inputs to rent from the investment firms (aggregate capital, $K(t) \in \mathbb{R}^m_+$) and consumers (aggregate efficient labour, $L(t) \in \mathbb{R}^m_+$) as well as outputs of final (aggregate consumption, $C(t) \in \mathbb{R}^n_+$ and aggregate investment, $I(t) \in \mathbb{R}^m_+$) goods to maximise its profits, Π_C,

$$\langle p_c(t), C(t)\rangle + \langle p_I(t), I(t)\rangle - \langle p_k(t), K(t)\rangle - \langle p_l(t), L(t)\rangle,$$
$$(-L(t), -K(t), C(t), I(t)) \in Y$$

which are zero.

The *investment firms* choose a time-path of investment and rent out their capital (uniquely determined by lemma 1) to the production firm.

An investment firm i owns capital $K^i(t) = e^{-\delta^i t}\int_{-\infty}^t e^{\delta^i s} I^i(s)ds$ of type i and chooses an investment policy $I^i(\cdot)$ to maximise its profits

$$\Pi_I^i(I(\cdot)) \equiv \int_{-\infty}^{+\infty} I^i(t)[-p_I^i(t) + \int_0^{+\infty} e^{-\delta^i s} p_k^i(t+s)ds]dt$$

which should be zero.[24] This condition implies

$$p_I^i(t) = \int_0^{\infty} e^{-\delta^i s} p_k^i(t+s)ds \tag{1}$$

[22] A choice of production plan at time t involves no implications for profits of the firm at other dates, so the profit maximisation problem of this firm is static.

[23] One could introduce many production firms that have access to the technology described by Y without changing the results. Indeed, the composition of the industry is irrelevant as long as the total production set is preserved.

[24] We don't investigate here the important question as to under what conditions (on I, p_I, p_k) the order of integration can be changed. The definition is written this way to lead as easily as possible to a definition of equilibrium.

Finally, we have to define the life-time budget constraint of an individual of type τ born at time x:

$$\int_0^{T_\tau} \langle p_z(s+x), z^{\tau,x}(s)\rangle - \langle p_c(s+x), c^{\tau,x}(s)\rangle \, ds = 0 \tag{2}$$

$$z^{\tau,x}(s), c^{\tau,x}(s) \geq 0, \quad \langle 1, z^{\tau,x}(s)\rangle \leq 1 \tag{3}$$

Clearly, the price for efficient unit of labour, $p_l(t)$, is proportional to the price of labour time, $p_z(t)$, at each instant t: $p_z(t) = e^{\gamma t}\langle \varepsilon^\tau(s), p_l(t)\rangle$.

After the above definitions, the definition of an Arrow-Debreu equilibrium is standard.

3.2.2. *Diamond Equilibrium.* One could also reproduce an equilibrium concept introduced by Diamond (1965) for this model. There are no investment firms, consumers hold the capital stock of different types and rent it out to the production firm. As in Arrow-Debreu equilibrium, consumers can also lend to each other (IOU's) — e.g., if they must borrow when young. So, the value of the total net savings of the consumers at each point in time equals the total value of the accumulated capital.

3.2.3. *Selection.* Observe that the above equilibrium concepts are typically multi-valued, so to get from them a single-valued time-invariant solution concept, as our main result below (theorem 9) requires, one has to make a selection in a neighbourhood of the given balanced growth equilibrium — cf. footnote 30 below for this.

3.2.4. *Maximising Welfare.* Maximising welfare — where the utilities can be discounted, but must be normalised as in Relative Utilitarianism, cf. section 4.1 below[25] — is also a time-invariant solution concept.[26]

One could then maximise, for perturbed endowments too, this same discounted sum of normalised utilities (where the normalisation includes the subtraction of the utility level on the status-quo path), giving thus another example of time-invariant solution concept (and typically single-valued this time).

3.2.5. *Unanticipated shocks.* All the above deal with fully anticipated shocks. One can, for the same concepts, consider the polar case, where all contracts have already been signed for the unperturbed economy, so the effective initial endowment that gets perturbed is the final allocation for the unperturbed economy with that solution concept (and it is from that perturbed endowment that individuals re-trade). This being

[25]Except of course if all individuals' relative risk aversion coefficients (i.e., degrees of homogeneity) are the same, then this is irrelevant. But else the relative weight on the types with higher risk aversion would, without the normalisation, effectively go down to 0 as the economy grows.

[26]Since however time varies from $-\infty$ to $+\infty$, it is not immediately obvious that a maximum exists, but arguably reasonable social welfare functions (discount rates — e.g., ν) should ensure this existence. Cf. sect. 6.2 for further discussion.

a balanced path, by the assumption of the theorem, it suffices then to observe that the theorem remains applicable as is to economies with such initial endowments, the time-invariance property being preserved.

4. The Relative Utilitarian Welfare Function

4.1. **The Set of Alternatives.** To formulate the social welfare function we need the feasible set, and the simplistic formulation used in section 1.3 is no longer adequate in view of the multiple goods, and several types of consumers. Ideally it should be defined in the space of policies, but since one of our aims is to prove that our result is completely independent of it, we will define it as the corresponding set in the space of (final — i.e., after all equilibrium readjustments) allocations.

The set of available allocations should be *time-invariant*, i.e., it should be mapped to itself by any time-shift $\mathbb{T}_h$.

So, the time invariance is here to capture the previous idea that policies affect only the height of the growth path — while leaving the geometry of the feasible set completely arbitrary in all other respects.

Further, an obvious implication of the justice requirement on the feasible set is that each individual's utility is bounded below.

4.2. **The distribution of costs and benefits.** As observed in the introduction (cf. footnote 10), in our set up, the effect of an arbitrary perturbation of endowments on welfare cannot depend only on the aggregate perturbation, it must depend on its distribution too. To cast nevertheless our result in the familiar framework of cost-benefit analysis, we will assume that the individual perturbations are themselves a (linear) function of the aggregate perturbation, and more specifically, to avoid re-distributive effects as much as possible, that the aggregate perturbation is distributed in a fixed way across age-groups and types.[27] The generalisation of our main result in section 6.1 below is anyway independent of any such restriction; it applies in particular to completely arbitrary perturbations of endowments.

Let thus $\vartheta^\tau(s)$ be some integrable function, with $\vartheta^\tau(s) = 0$ for $s < 0$ and $s > T_\tau$, and with $\sum_\tau \int_{-\infty}^{+\infty} \vartheta^\tau(s)ds = 1$. Then, the perturbed endowment of an individual of type τ who was born at time y is related to the aggregate perturbation $\Omega(t)$ in the following way:[28]

$$\omega_y^\tau(t) = \vartheta^\tau(t-y)\frac{\Omega\,(t)}{N^\tau e^{\nu y}}$$

[27] "fixed" means independent both of the date and of the aggregate perturbation (as a function as well as as a value).

[28] Recall, the population (within each type) grows at a constant rate ν, so the number of people of type τ who are born per unit of time at time y is $N^\tau e^{\nu y}$ with N^τ being that number at time 0.

5. The Main Statement

To demonstrate the result, we need to evaluate, at a balanced growth equilibrium, the effect on social welfare of a small perturbation Ω of aggregate endowment. The perturbation will affect equilibrium allocations, which, in turn, alter individual well-being and social welfare. Thus, we must compute the differential of the map from perturbed endowments to welfare and prove that whenever it exists it is of the form $\int \langle q, \Omega(t)\rangle e^{-\gamma t} dt$ for some $q \in \mathbb{R}^n$ — i.e., that the discount rate equals γ.

Technically, to make the main statement as strong as possible, we should use the weakest notion of differential, i.e., that of Gateaux.[29] We must also specify the space of perturbations and its topology; we will use the space K, defined below, because then the statement implies the same statement for about any other space of perturbations, since K embeds continuously as a dense subspace in about any other space.

We follow Schwartz (1957-59) and Gelfand and Shilov (1959) in defining K and its dual, the space K^* of "generalised functions":

Definition 8. *K is the space of infinitely differentiable functions with compact support, and a sequence of functions $\varphi_i \in K$ converges to zero if $\exists h \in \mathbb{R}$: $|x| \geq h \Longrightarrow \varphi_i(x) = 0$ for all n, and φ_n and all its successive derivatives converge uniformly to zero.*

K^* is the space of linear functionals ψ on K s.t. $\psi(\varphi_i) \to 0$ whenever $\varphi_i \to 0$ in K.

The chief economic meaning of $\Omega \in K^n$ is that we perturb endowments only over a bounded interval of time. Note that the status-quo (zero endowment) point also belongs to this space, so we can view the social welfare function W as being defined on K^n.

Next step is to define precisely the map from endowments to social welfare, given a time-invariant solution concept. Consider a single-valued time-invariant solution concept ψ, that maps consumption endowments, $\Omega \in K^n$ to final allocations.[30] Assume that its domain, D,

[29] A function f from a subset D of a topological vector space X to $\mathbb{R}$ is Gateaux-differentiable at zero, if $\forall x \in X$ the set $\{t \in \mathbb{R} \mid tx \in D\}$ is a neighbourhood of zero in $\mathbb{R}$, say V_x, and if $t \mapsto f(tx)$ is differentiable at $t = 0$, say with derivative d_x, and if $x \mapsto d_x$ is a continuous linear functional on X.

[30] Given a perturbation $\Omega(t)$ of consumption endowments, it is true that several equilibria might emerge. If dealing with a solution concept that does not guarantee local uniqueness, we choose e.g. out of those the one closest to the initial stable growth path in terms of the L_p norm $\sum_i \|\ln p_i(t) - \ln p_i^{\Omega}(t)\|_p$, where $p(t)$ is the price vector at time t prevailing at the initial equilibrium and $p^{\Omega}(t)$ is the price vector of a perturbed economy: though the price system does not necessarily fully specify an equilibrium, it does specify the individual utility levels, which is all we need. The effect of the logarithms is to make the distance independent of price normalisation, hence to induce a distance between equilibria (or: between price-rays): for any multiple of p_i the minimum, over all multiples of p_i^{Ω} (clearly there is at most one such multiple where the value is finite, when $p < \infty$), will be achieved at the

contains zero, which corresponds to the economy we described, in which individuals are born with no consumption endowments. As the solution concept is time-invariant, $\psi(0)$ describes a balanced growth path. Define the social welfare function W of relative utilitarianism by subtracting from each individual's normalised utility function its value at $\psi(0)$. (Thus, a constant is subtracted from each of the individual utilities to assure that welfare is well-defined on the growth path). Denote by $\aleph'$ the subset of the space $\aleph$ of allocations where W is well defined (i.e., the integral converges). This set, for example, might include allocations that are not 'too different' from those on the balanced growth path $\psi(0)$, e.g., those that deviate from it over a bounded interval of time. Let us focus on the subset of consumption endowments, D', for which W is well defined: $D' = \psi^{-1}(\aleph')$, and note that $0 \in D'$, i.e., the status-quo belongs to this set, as by construction the welfare function is zero at the initial balanced growth path $\psi(0)$. Finally, we define the map from endowments to social welfare corresponding to the chosen solution concept ψ: W_ψ is the composite map $W \circ \psi$ from D' to $\mathbb{R}$.

Now the main result can be stated in the following succinct form:[31]

Theorem 9. *Let ψ be a point-valued time-invariant solution concept. For the relative utilitarian welfare function W, the Gateaux-differential of W_ψ at 0, if it exists, equals $\int \langle q, \Omega(t)\rangle e^{-\gamma t} dt$ for some $q \in \mathbb{R}^n$.*

This implies that the discount rate is the growth rate γ of per capita output. One interpretation is tempting: the cost of consumption in terms of inputs becomes cheaper with time, due to the enhancement of labour productivity. Individual productivity grows at rate γ, so it is exactly at this rate that we have to discount consumption in order to value labour time equally. See section 6.3 below for another interpretation.

6. Discussion of the Main Result

6.1. **A policy re-interpretation.**

Real-life policies rarely involve direct consumption transfers (changes in endowments). The model can be re-interpreted to incorporate more realistic policies as follows. Assume a set[32] of basic policies. Let a policy be a specification of such a basic policy as a function of time. Assume that shifting a policy forward in time by h transforms its effect on the economy (through the

corresponding multiple, and the value of the minimum is independent of this multiple, and remains the same when permuting the roles of p_i and p_i^Ω. Finally, because of the L_p norms (i.e., Lebesgue measure being shift-invariant), the selection will be time-invariant. Obviously there will remain to show that there is some equilibrium at a finite distance, and that locally the minimum is achieved at a unique point.

[31] A similar result could be shown in the traditional set-up, provided (the multidimensional analog of) risk-aversion, ρ^τ, is independent of the type τ — giving then a discount factor of $\rho\gamma$, and hence showing the robustness of our conclusions from the mini-model in the introduction.

[32] More precisely, a manifold, to make differentiability meaningful.

solution concept) by $\mathbb{T}_h$[33]. Then the result still holds, in the sense that, at a given status-quo stationary policy π^*, the welfare effect of a small policy variation $\delta\pi_t \in K$ is given by $\int e^{\nu t}\langle q, \delta\pi_t\rangle dt$.[34]

6.2. **On non-vacuity.** The theorem relies on differentiability of the map from endowments to welfare. Indeterminacy is known to plague some classes of OLG models;[35] hence one would need to show that this problem is avoided in our case — in particular, that a policy change generates predictable changes in the economy — or use fn. 30. To demonstrate the non-vacuity of the statement, one has to show that (1) the solution concept is non-empty valued;[36] (2) for some time-invariant single-valued selection, the map from endowments to allocations is differentiable; (3) the map from allocations to welfare is differentiable. Verifying each of the requirements (even in a model with fully specified preferences and technology) is non-trivial, and lies beyond the scope of this paper, but will be dealt with in subsequent research.

6.3. **The Value of a Human Life.** We show now that not only $\rho\gamma$ is not of the right order of magnitude as compared to γ, but even that the former formula is conceptually wrong, and the latter exactly correct.

The value of life, according to any criteria [e.g., each of the four in Mishan's (1971) introduction, or even judicial criteria in assessing damages], is proportional to the individual's life-time income, or to average life-time income at his time: anyway, proportional to $e^{\gamma t}$ in

[33] In particular, any constant policy leads to some balanced growth path. So basic policies might be for example linear taxes — or non-linear (sales or income) tax-schedules indexed by average income.

[34] q is typically a vector of 'small' dimension (that of the set of basic policies), so its computation is *much* easier than following the equilibrium dynamics of the model for an arbitrary policy. E.g., one might try changing one policy variable at a time to some close-by stationary value, and compute the resulting balanced growth path.

[35] See Geanakoplos and Brown (1985), Geanakoplos and Polemarchakis (1991).

[36] For example, for a time-invariant social welfare function (a discounted sum of normalised utilities), one might expect that, if a maximum exists, it is achieved at some balanced growth path. It should thus suffice to maximise the welfare of any fixed generation over all feasible balanced paths (leading thus to a natural generalisation of the 'golden rule' paths) — and then to show that, for reasonable discount rates, when the utility levels on that path are subtracted from each individual's utility function, the social welfare is indeed maximised. But this criterion to which we were led — to maximise for any fixed generation over all balanced paths — is completely independent of the discount rate we started with! This clearly suggests there might very well be no 'reasonable' discount rates (for utilities) beyond the obvious candidate, ν, the rate of growth of the population (this being the only one to weigh equally the past and the future). Observe that this leads to a discount rate of $\nu + \gamma$ on real consumption, i.e., the interest rate, in the framework of golden rule equilibria. Clearly the above heuristics need confirmation by a full proof, but in that case, they might conceivably form the basis for a critique of our criterion of intergenerational fairness in the framework of economic models with growing population.

an exogenous growth model. Further, one should note that if our full-fledged model above were extended such as to allow for variable life-spans — so, individual consumption-sets are of the type $\cup_T C[0, T]$ —, then this conclusion would also formally follow from the model, given the homogeneity assumption on individual utilities (which is forced upon us to get balanced growth solutions).

Hence, if we want 1 human life one generation down the road to count as much as 1 now, we must discount future consumption *exactly* by $e^{-\gamma t}$.

6.4. **What makes the traditional Utilitarian approach fare so badly?** Observe first that the essence of the difficulties we encountered has nothing to do with time; e.g., the index t in our first ("toy") model can be reinterpreted as referring to different islands, populated with N_t individuals living from the coconut harvest — neither people nor coconuts being transportable. How to evaluate the welfare effects of new techniques to slightly influence global weather conditions, with differential effects across the islands?

Now, in its utilitarian interpretation (fn. 31), we have a very pure model: all agents of the same type are assumed to be biological and psychological clones of each other, which would show completely identical reactions to identical stimuli; if an agent born at time t were moved at his birth to a world identical to the world at t', he would in all circumstances express exactly the same reactions as agents of his type born at time t' do. Further all agents are assumed to satisfy full von Neumann-Morgenstern rationality, so that u^τ is really the correct utilitarian utility function of those agents over all risky prospects,[37] and is assumed to be expressed in common utility units across types.[38]

[37]So, the practicability or not of the utilitarian prescript plays no role here.

[38]To be fully precise, we also assume in the model that our individuals are perfect egoists, i.e., that those utilities correctly describe our individuals both in the sense of tastes and of values, in Harsanyi's terminology, or, in Sen's (2000, p. 64) terminology, as well in the sense of pleasure as in that of a representation of choice. The latter in each case is needed because we deduce individual demand functions from them, and the former presumably as to give meaning to the sum of utilities as a measure of social welfare. Interpretations as "fulfilled desires" (loc. cit.) would on the other hand probably not square well with the need in economics to interpret utility as *expected utility at birth*, because of the need to explain choices such as between different careers (more formally, "life-time consumption plans"), and probably also because of obvious incentive problems if e.g. society were to have as policy to bail out anybody who went to the casino and lost his fortune. There are probably two other different interpretations of "fulfilled desires": the first one being that of "utility" as the integral of a flow of instantaneous "felicities", in the terminology of Arrow and Kurz (1970) – i.e., an assumption of additive separability, which we cannot admit as a sufficiently general representation of preferences, but all our arguments here would remain valid even with that additional restriction —, and the other that of utility as an expectation of utilities at terminal nodes — we *did* assume our utilities were also von Neumann-Morgenstern utilities.

And, as stressed in the previous paragraph, our argument concerning the value of a human life is a valid argument in that ideal world too.

Even in this ideal world, we encountered two different difficulties. First, even with a single type, using a zero discount rate for utilities — which is the true utilitarian social welfare function — led to conclusions that did not correspond to the common intuition of what it meant to treat future generations equally, both in the sense of the magnitude of the implied discount factor and in that of treating human lives equally.

The second was that, when types differ, all the weight of the welfare function gets concentrated on the types with lowest risk-aversion as the economy grows (this is the root of the aggregation difficulty mentioned in fn. 31), with all the unpleasant consequences one can imagine.

The surprise was that the same fix was needed to cure all those problems: relative utilitarianism can be viewed here as rescaling future utilities with a *negative, type-dependent* "discount" rate $(1-\rho)\gamma$.[39]

Thus, what went so spectacularly wrong with the traditional approach was to assume — as everywhere in applied policy analysis — that all agents of the same type (i.e., perfect clones of each other) share the same utility function, independent of their environment; "…the arbitrary assumption that if two persons have the same demand function, then they must get the same utility level from a given commodity basket…is, of course, totally illegitimate" (Sen 2000).[40]

[39]How can this be re-interpreted in the standard utilitarian framework? To quote e.g., Sen (2000), p. 63, "the utility of a member of a group can depend on variables other than his or her own income and incomes of others within this group (for example, it may be influenced by the incomes of others outside this group)." Thus, utility is no longer an individual characteristic, but depends also on the environment — i.e., in our case, on the production set Y_t, or on the feasible set at the given date. The value that an individual attaches to consuming a fixed bundle depends on the generation he belongs to; indeed, an average earner nowadays might have felt blissfully happy with his current consumption back in the days of Great Depression.

So, utility functions, like those in RU, that depend both on individual characteristics and on the society where the individual lives are completely kosher for an utilitarian. Remains only to understand the utilitarian interpretation of our specific utility weights $e^{(\rho-1)\gamma t}$. This goes beyond the scope of this paper; to give just a couple of hints: one interpretation would be that individuals look at *their* feasible (and just) set, and discriminate *therein* a more or less fixed number of utility levels. Or: that their aspirations are based on this set, and that utility stems basically from a comparison with aspirations.

[40]Or, closer to our vNM framework: assume we have 2 poor individuals, say one in the US and one in India, which have CRRA vNM preferences for money, with the same risk-aversion coefficient. It is nonsense to infer from this that they have the same utility function for money at the current exchange rate, on the grounds that the latter makes the 2 currencies essentially perfect substitutes.

This objection is specific to "demand functions", i.e., to utility functions as defined on a space of personal consequences. Even Harsanyi himself argues at times that when two individuals have the same preferences, it is reasonable to assign them

To avoid this "arbitrary assumption" and to do correctly a full utilitarian analysis (cf. fn. 39), that would lead to the right scalings, seems hopelessly complex in practice, until one observes that one can equivalently use directly RU, which is simply the straight application of Bentham's "everybody to count for one, nobody for more than one" (whether this "one" refers to human lives, or, in a voting interpretation of RU, to "one man, one vote").

7. Conclusions

We show that, under the relative utilitarian criterion, which incorporates equal treatment of different generations, the appropriate discount rate is the growth rate of real per-capita income. The conclusion is true under any 'time-invariant' solution concept, and, more importantly, the discount rate is independent of a particular equilibrium (and the associated prices) that the economy is in. The per-capita growth rate represents the 'true' shadow cost of (consumption-generating) resources today in terms of those in the future, so the prescribed discount rate is based solely on the fundamentals of the economy.

Let us stress that the result is independent of individual impatience; the model does not even require time-separable preferences. Crucial assumptions are those needed for stable growth to be feasible, i.e., (1) homogeneity of individual utility functions (over life-time consumption streams, for fixed life-time streams of labour activities), and (2) constant returns to scale in production — plus, in addition, the differentiability of the social welfare function.

The next step will be to prove that the main statement is non-vacuous, as it might be, e.g., in case of indeterminacy. Based on our work in progress we can conjecture that the required differentiability is not a very restrictive assumption.

Appendix A

Proof of lemma 1. Continuity yields that K_t^i is bounded on any interval $(-\infty, t_0)$. The production technology implies a similar upper bound for I_t^i. In particular I_t^i is locally-integrable and thus K_t^i locally absolutely continuous. Letting $M_t = e^{\delta^i t} K_t^i$, the differential equation becomes $M_t' = e^{\delta^i t} I_t^i$, hence, by the local absolute continuity, $M_t = M_0 +$

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the same utilities (principle of insufficient reason): "If all individuals' personal preferences were identical, then we could ascribe the same utility function to all individuals" (1977 p. 58). But that argument is for preferences and utilities over the (common) set of alternatives, while here — and in most applications — it is about preferences and utilities over an individual set of personal consequences. Even the sets are not directly comparable, since goods are indexed by date and location — and there is no economic reason to use the same physical units at different times or places.

This is also why relative utilitarianism fares much better, by deriving those different weights from a normalisation over *the common set of alternatives.*

$\int_0^t e^{\delta^i s} I_s^i ds$. Therefore $K_t^i \geq 0$ yields $\int_{-T}^0 e^{\delta^i s} I_s^i ds \leq M_0 \,\forall T$, and $K_t^i \leq \bar{K} e^{(\gamma+\nu)t}$ for $t \leq 0$ yields $\int_{-T}^0 e^{\delta^i s} I_s^i ds \geq M_0 - \bar{K} e^{-(\gamma+\nu+\delta^i)T}$ for $T \geq 0$. In particular, the (locally integrable, as just seen) function $h(t) = e^{\delta^i t} I_t^i$ is such that $\int_{-T}^0 h(s)ds$ converges to M_0 when $T \to \infty$. Since our upper bound for I_t^i implies $h(t) \leq \bar{h} e^{(\gamma+\nu+\delta^i)t}$ for $t \leq 0$, we conclude that $h(t)$ is (absolutely) integrable on $(-\infty, t)$ for all t, and thus $K_t^i = e^{-\delta^i t} \int_{-\infty}^t e^{\delta^i s} I_s^i ds$ for all t, where the integral is a Lebesgue integral. □

Proof of lemma 2. Clearly, we can set consumption to zero and assume that all agents work full-time. Fix a vector $L_0 \in \mathbb{R}^h$ such that any feasible vector of labour inputs $L(t) \leq L_0 e^{(\gamma+\nu)t}$. Enlarge the instantaneous production cone Y by allowing all investment goods to be perfect substitutes for each other and the same for the capital goods. Let $F\colon \mathbb{R}_+^2 \to \mathbb{R}_+\colon (K, l) \mapsto \sup\{\sum_i I^i \mid \exists K^i \geq 0, \sum_i K^i \leq K, \big(-lL_0, -(K^i)_i, 0, (I^i)_i\big) \in Y\}$. The supremum is achieved, else with bounded inputs unbounded outputs would be feasible and, as Y is convex and closed, the same would be true for zero inputs, thus contradicting the assumption $Y \cap \mathbb{R}_+^n = \{0\}$. In particular, $F(K, l)$ is finite. Clearly, F is positively homogeneous of degree one, concave and continuous. Further, by the assumptions of the lemma, $F(K, 0) = 0$.

Let us, finally, improve the possibilities of capital accumulation by lowering each δ^i to $\delta = \min_i \delta^i$. The capital accumulation equations become then $K'(t) = F(K(t), e^{(\gamma+\nu)t}) - \delta K(t)$. Let $x(t) \equiv K(t)e^{-(\gamma+\nu)t}$, and $f(x) \equiv F(x, 1)$ — so $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and concave. Then the differential equation becomes $x'(t) = f(x(t)) - Rx(t)$. As $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ (because $F(1, 0) = 0$ and continuity), there is $\bar{x} \geq 0$ such that $f(x) - Rx \leq -1$ for $x \geq \bar{x}$.

Let now $y(t) = e^{-(\gamma+\nu)t} \sum_i K_t^i$ along some feasible path in the original economy: a fortiori $y(t) \geq \bar{x}$ implies $y'(t) \leq -1$. Since, by the initial condition, $\exists \bar{y}\colon y(t) \leq \bar{y}$ for $t < 0$, it follows that for all t, $y(t) \leq \bar{x}$. Hence our bound on each K_t^i, which themselves imply (via Y) a similar upper bound for the I_t^i. □

Proof of lemma 3. All norms on $\mathbb{R}^n$ being equivalent, we can assume the ℓ_1 norm in the statement. The right hand member of the inequality is then concave, and we can proceed as in the proof of lemma 2: now $f(x) = A + Bx^{1-\varepsilon}$, and, since, as seen above, if $x(t) > \bar{x}$ then $x(s)$ must have been decreasing (and hence $x(s) > \bar{x}$) for all $s \leq t$, we can assume $A = 0$, by majorising f on $[\bar{x}, +\infty]$ by another such function (and if necessary increasing $\bar{x}$ to an appropriate value for that new function). Thus we have to show that over all feasible paths (k_t, i_t) ($= e^{-(\gamma+\nu)t)}(K_t, I_t)$) the k_t are uniformly bounded. And feasibility means $i_t \leq Bk_t^{1-\varepsilon}$ and $e^{Rt}k_t = \int_{-\infty}^t e^{Rs} i_s ds$ (and $k_t \geq 0$, $e^{Rs}i_s$ integrable on $[-\infty, t]$). I.e., letting $y_t = \int_{-\infty}^t e^{Rs} i_s ds$, we have $y_{-\infty} = 0$, $k_t = e^{-Rt} y_t$,

$i_t = e^{-Rt} y'_t$, so our inequality becomes $y'_t \leq B e^{\varepsilon R t} y_t^{1-\varepsilon}$, i.e., $\frac{dy_t^\varepsilon}{de^{\varepsilon R t}} \leq \frac{B}{R}$. Since $y_{-\infty} = 0$, this yields $y_t^\varepsilon \leq \frac{B}{R} e^{\varepsilon R t}$, i.e., $k_t \leq \left(\frac{B}{R}\right)^{1/\varepsilon}$. □

Proof of lemma 5. The second part is obvious: for the endowments, it holds by definition of the transformation, and for the preferences, it follows because all agents of the same type have the same utility function over their consumption set, which is homogeneous in the goods: so multiplying the "goods-component" by a constant just multiplies to whole utility function by a constant, and hence doesn't change preferences.

For the first part, note that the capital-accumulation equations are not affected, since they are linear and homogeneous in the aggregate goods. Remains to check for the "instantaneous production cone" Y that it too is preserved by the transformation. Assume thus for some t a vector $(-L, y)$ in Y_t — i.e., $(-\exp(\gamma t) L, y)$ in Y — before the transformation — where the coordinates of $y = (-K, C, I)$ are all aggregate consumption and investment outputs and capital inputs, and those of L are the aggregate labour input. Then, after the transformation, this vector becomes $[-\exp(\nu h) L, \exp\left((\gamma+\nu)h\right) y]$, and we must show it belongs to Y_{t+h} — i.e., that $[-\exp(\gamma(t+h))\exp(\nu h) L, \exp((\gamma+\nu)h) y]$ belongs to Y. Since this vector equals $\exp((\gamma+\nu)h)\,[-\exp(\gamma t) L, y]$, this follows straight from the fact that Y is a cone. □

Proof of theorem 9. By definition of a Gateaux differential,

$$\begin{aligned} DW\left(\Omega^0\right) &= \lim_{\varepsilon \to 0} \frac{\delta_\varepsilon W\left(\Omega^0\right)}{\varepsilon}, \\ \delta_\varepsilon W\left(\Omega^0\right) &= W\left(\Omega^0 + \varepsilon \Omega\right) - W\left(\Omega^0\right) \end{aligned}$$

By assumption,

$$DW(0) = \langle \Omega, \mu \rangle$$

where $\mu \in (K^*)^n$, i.e., the differential at $\Omega^0 = 0$ is linear in Ω. It is sufficient for what follows to describe $\delta_\varepsilon W\left(\Omega^0\right)$, i.e., the change in the social welfare function caused by the perturbation of endowments, which amounts to subtracting a constant from each agent's utility, the utility on the baseline, so the criterion of interest is the difference δW.

To construct δW let us first normalise life-time utilities. Recall the set of available allocations is time-invariant. We have to compute w_t^τ, the difference between the sup and the inf over this set of the utility of an agent of type τ born at time t. By time-invariance, the set of consumption and labour allocations of this agent equals that for an agent of the same type born at time 0, except for rescaling the consumption component by $e^{\gamma t}$. Therefore, by the homogeneity of U^τ of degree $1 - \rho^\tau$ with respect to consumption, $w_t^\tau = e^{(1-\rho^\tau)\gamma t} w_0^\tau$. Let $w^\tau \equiv (w_0^\tau)^{-1}$. Then we get for the normalised utility $U_t^{*\tau}$ (that enters the social welfare function)

$$U_t^{*\tau} = e^{(\rho^\tau - 1)\gamma t} w^\tau U^\tau$$

So the social welfare function takes the following form

$$\delta W(\cdot) \equiv \int_{-\infty}^{\infty} \sum_{\tau} N_t^{\tau} \left(\delta U_t^{*\tau}\right) dt \tag{4}$$

Let us define $V_t^{\tau} : \Omega_t \mapsto \mathbb{R}$ to be the utility level of individual of type τ born at time t, under an equilibrium with the perturbed endowments.

$$\begin{aligned} W\left(\Omega^0\right) &= \sum_{\tau} w^{\tau} W^{\tau}\left(\Omega^0\right) \\ W^{\tau}\left(\Omega^0\right) &\equiv \int_{-\infty}^{\infty} N_t^{\tau} e^{(\rho^{\tau}-1)\gamma t} V_t^{\tau}\left(\Omega_t^0\right) dt \end{aligned}$$

Consider now the perturbation $\tilde{\Omega}_t$, where

$$\tilde{\Omega}_{t+h} = e^{(\gamma+\nu)h}\Omega_t \tag{5}$$

By lemma 5, the corresponding "response" of the system is obtained from the response to Ω_t by delaying everything by h, multiplying all aggregate quantities of goods by $e^{(\gamma+\nu)h}$, and all per-capita quantities by $e^{\gamma h}$, and correspondingly for prices.

Hence, for utilities, by their homogeneity property in goods $1, \ldots, n$,

$$V_{t+h}^{\tau}\left(\tilde{\Omega}\right) = e^{(1-\rho^{\tau})\gamma h} V_t^{\tau}(\Omega)$$

and, in particular, when $\tilde{\Omega} = \Omega = 0$,

$$V_{t+h}^{\tau}\left(\tilde{\Omega}\right) - V_{t+h}^{\tau}(0) = e^{(1-\rho^{\tau})\gamma h}\left(V_t^{\tau}(\Omega) - V_t^{\tau}(0)\right)$$

Therefore,

$$\begin{aligned} & W^{\tau}\left(\tilde{\Omega}\right) - W^{\tau}(0) = \\ & \quad = \int_{-\infty}^{+\infty} N_0^{\tau} e^{\nu(t+h)} e^{(\rho^{\tau}-1)\gamma(t+h)} \left[V_{t+h}^{\tau}\left(\tilde{\Omega}\right) - V_{t+h}^{\tau}(0)\right] d(t+h) \\ & \quad = \int_{-\infty}^{+\infty} N_0^{\tau} e^{\nu(t+h)+(1-\rho^{\tau})\gamma h} e^{(\rho^{\tau}-1)\gamma(t+h)} \left[V_t^{\tau}(\Omega) - V_t^{\tau}(0)\right] dt \\ & = e^{\nu h} \int_{-\infty}^{+\infty} N_t^{\tau} e^{(\rho^{\tau}-1)\gamma t} \left[V_t^{\tau}(\Omega) - V_t^{\tau}(0)\right] dt = e^{\nu h}\left[W^{\tau}(\Omega) - W^{\tau}(0)\right] \end{aligned}$$

(i.e., the factor $(1-\rho^{\tau})\gamma$ drops out). As a consequence, the total change in welfare is

$$\begin{aligned} W\left(\tilde{\Omega}\right) - W(0) &= \sum_{\tau} w^{\tau}\left(W^{\tau}\left(\tilde{\Omega}\right) - W^{\tau}(0)\right) \\ &= e^{\nu h}\left[W(\Omega) - W(0)\right] \end{aligned}$$

Therefore, by the definition of the derivative,

$$\begin{aligned} \langle \tilde{\Omega}, \mu \rangle &= \lim_{\varepsilon \to 0} \frac{W\left(\varepsilon\tilde{\Omega}\right) - W(0)}{\varepsilon} = \\ &= e^{\nu h} \lim_{\varepsilon \to 0} \frac{W(\varepsilon\Omega) - W(0)}{\varepsilon} = e^{\nu h}\langle \Omega, \mu \rangle \end{aligned} \tag{6}$$

Define $S_h(\xi)\colon t \mapsto \xi(t+h)$. By (5)

$$\tilde{\Omega} = e^{(\gamma+\nu)h} S_h \Omega$$

Combining with (6), we get

$$e^{(\gamma+\nu)h} \langle S_h(\Omega), \mu \rangle = \langle \tilde{\Omega}, \mu \rangle = e^{\nu h} \langle \Omega, \mu \rangle$$

hence the following holds for all $h \in \mathbb{R}$ and all perturbations $\Omega \in K^n$:

$$\langle \Omega - e^{\gamma h} S_h(\Omega), \mu \rangle = 0$$

for $\mu \in (K^*)^n$. Dividing by h and taking the limit as $h \to 0$, we get

$$\langle \gamma\Omega - (\Omega)', \mu \rangle = 0$$

The definition of the derivative of a generalised function,

$$\langle \mu', f \rangle = -\langle \mu, f' \rangle, \; f \in K, \; \mu \in K^*$$

yields then

$$\langle \gamma\mu + \mu', \Omega \rangle = 0, \; \forall \Omega \in K^n$$

so we have to solve the differential equation $\gamma\mu + \mu' = 0$, which, by lemma (10), has $\mu = qe^{-\gamma t}$ for some $q \in \mathbb{R}^n$ as only solutions, so,

$$DW = e^{-\gamma t} \langle q, \Omega \rangle = \int_{-\infty}^{+\infty} e^{-\gamma t} \langle q, \Omega_t \rangle dt, \; \forall \Omega \in K^n.$$

□

Lemma 10. *Consider a homogeneous differential equation of the form*

$$y' = \lambda y, \tag{7}$$

for a given constant λ. Then every solution of that system in the class K^* of generalised functions is of the form

$$y = Ce^{\lambda t}, \; C \in \mathbb{R}$$

i.e., is a "classical solution".[41]

Proof. From (7) we have that for any $\varphi \in K$, $\langle y', \varphi \rangle = \lambda \langle y, \varphi \rangle$; by definition of the derivative of a distribution this implies $\langle y, -\varphi' \rangle = \lambda \langle y, \varphi \rangle$, and so $\langle y, \lambda\varphi + \varphi' \rangle = 0$. Let $K_\lambda = \{\psi \in K | \int_{-\infty}^{\infty} e^{\lambda t} \psi(t)\, dt = 0\}$. Observe that $\forall \psi \in K_\lambda \exists \varphi \in K\colon \psi = \lambda\varphi + \varphi'$: take $\varphi(t) = \int_{-\infty}^{t} e^{\lambda s} \psi(s) ds$ (the converse is true as well, but we won't use it). So $y = 0$ on K_λ.

Note that any $\varphi \in K$ can be represented in the form $\varphi = \psi + c\varphi_0$, where $\psi \in K_\lambda$, c is a constant and $\varphi_0 \in K \setminus K_\lambda$ is fixed: choose $c = \frac{\int_{-\infty}^{\infty} e^{\lambda t} \varphi(t) dt}{\int_{-\infty}^{\infty} e^{\lambda t} \varphi_0(t) dt}$, then $\psi = \varphi - c\varphi_0 \in K_\lambda$.

Thus $\langle y, \varphi \rangle = c \langle y, \varphi_0 \rangle$, so, letting the constant $C = \frac{\langle y, \varphi_0 \rangle}{\int_{-\infty}^{\infty} e^{\lambda t} \varphi_0(t) dt}$, we get $\langle y, \varphi \rangle = C \int_{-\infty}^{\infty} e^{\lambda t} \varphi(t)\, dt$, $\forall \varphi \in K$, i.e., $y = Ce^{\lambda t}$. □

[41] Gelfand and Shilov (1959, p. 53); included to make the argument self-contained.

References

Arrow, K. J., and M. Kurz (1970): *Public Investment, the Rate of Return, and Optimal Fiscal Policy*. The Johns Hopkins Press, Baltimore and London.

Asheim, G. B. (1991): "Unjust intergenerational allocations.," *Journal of Economic Theory*, 54, 350–371.

Asheim, G. B., T. Mitra, and B. Tungodden (2006): "Sustainable recursive social welfare functions," mimeo.

Bell, C., and S. Devarajan (1983): "Shadow prices for project evaluation under alternative macroeconomic specifications," *Quarterly Journal of Economics*, 98, 454–477.

Bergstrom, T. C. (2006): "Benefit-Cost in a Benevolent Society," *The American Economic Review*, 96(1), 339–351.

Bliss, R. R., and N. Panigirtzoglou (2004): "Option-Implied Risk Aversion Estimates," *Journal of Finance*, LIX(1), 407–420.

Chichilnisky, G. (1996): "An axiomatic approach to sustainable development," *Social Choice and Welfare*, 13, 231–257.

Coate, S. (2000): "An Efficiency Approach to the Evaluation of Policy Changes," *Economic Journal*, 110(463), 437–455.

d'Aspremont, C. (2006): "Formal Welfarism and Intergenerational Equity," in *Intergenerational Equity and Sustainability, Conference Proceedings of the IEA Roundtable Meeting on Intergenerational Equity*, ed. by J. Roemer, and K. Suzumura. To Appear.

Dhillon, A., and J.-F. Mertens (1999): "Relative Utilitarianism," *Econometrica*, 67(3), 471–498.

Diamond, P. (1965): "National Debt in a Neoclassical Growth Model," *American Economic Review*, LV(5), 1126–1150, Part 1.

Drèze, J., and N. Stern (1987): "The Theory of Cost-Benefit Analysis," in *Handbook of Public Economics*, ed. by A. J. Auerbach, and M. Feldstein, vol. II, chap. 14. Elsevier Science Publishers.

Drèze, J. H. (1981): "Inferring Risk Tolerance from Deductibles in Insurance Contracts," *The Geneva Papers on Risk and Insurance*, pp. 48–52.

Einav, L., and A. Cohen (2005): "Estimating Risk Preferences from Deductible Choice," NBER WP 11461.

Erosa, A., and M. Gervais (2001): "Optimal Taxation in Infinitely-Lived Agent and Overlapping Generations Models: A Review," *Federal Reserve Bank of Richmond Economic Quarterly*, 87(2), 23–44.

Fleurbaey, M., and P. Michel (2003): "Intertemporal equity and the extension of the Ramsey criterion," *Journal of Mathematical Economics*, 39, 777–802.

Geanakoplos, J. D., and D. J. Brown (1985): "Comparative Statics and Local Indeterminacy in OLG Economies: An Application of the Multiplicative Ergodic Theorem," Cowles Discussion Paper 773.

Geanakoplos, J. D., and H. M. Polemarchakis (1991): "Overlapping Generations," in *Handbook of Mathematical Economics*, ed. by W. Hildenbrand, and H. Sonnenschein, vol. 4, chap. 35, pp. 1899–1960. Elsevier.

Gelfand, I., and G. Shilov (1959): *Obobshennuye funkzii i deistviya nad nimi*. Fizmatgiz, Moscow, 2 edn.

HARSANYI, J. C. (1977): *Rational Behavior and Bargaining Equilibrium in Games and Social Situations*. Cambridge University Press, Cambridge.

HICKS, J. R. (1939): "Foundations of welfare economics," *The Economic Journal*, 49(196), 696–712.

——— (1942): "Consumers' Surplus and Index-Numbers," *The Review of Economic Studies*, 9(2), 126–137.

KALDOR, N. (1989): "Welfare propositions and interpersonal comparisons of utility," *Economic Journal*, 49, 549–552.

KING, R. G., C. I. PLOSSER, AND S. T. REBELO (2002): "Production, Growth and Business Cycles: Technical Appendix," *Computational Economics*, 20(1-2), 87–116.

KOOPMANS, T. C. (1960): "Stationary ordinal utility and impatience," *Econometrica*, 28, 287–309.

KOTLIKOFF, L. J. (1998): "The A-K Model — Its Past, Present, and Future," NBER WP 6684.

——— (2002): "Generational Policy," in *Handbook of Public Economics*, ed. by A. J. Auerbach, and M. Feldstein, vol. 4, chap. 27, pp. 1873–1932. Elsevier.

MISHAN, E. J. (1971): "Evaluation of Life and Limb: A Theoretical Approach," *The Journal of Political Economy*, 79(4), 687–705.

——— (1976): *Cost-benefit analysis*. Praeger Publishers.

PIGOU, A. C. (1932): *The Economics of Welfare*. Macmillan, 4th edn.

RAMSEY, F. P. (1928): "A Mathematical Theory of Saving," *The Economic Journal*, 38(152), 543–559.

SAMUELSON, P. A. (1954): "The Pure Theory of Public Expenditure," *The Review of Economics and Statistics*, 36(4), 387–389.

——— (1958): "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *The Journal of Political Economy*, 66(6), 467–482.

SCHWARTZ, L. (1957-59): *Théorie des distributions*, vol. I and II. Hermann, Paris.

SEN, A. (2000): "Social Justice and the Distribution of Income," in *Handbook of Income Distribution*, ed. by A. B. Atkinson, and F. Bourguignon, vol. I, chap. 1, pp. 59–85. Elsevier.

SIDGWICK, H. (1874): *The Methods of Ethics*. Macmillan and Company, London.

U.S. OFFICE OF MANAGEMENT AND BUDGET (2003): "OMB Circular A-4: Regulatory Analysis," .

WILDI, W., D. APPEL, M. BUSER, F. DERMANGE, A. ECKHARDT, P. HUFSCHMIED, AND H.-R. KEUSEN (2000): "Disposal Concepts for Radioactive Waste," written on behalf of the Federal Department for the Environment, Transport, Energy and Communication of Switzerland.