CORE DISCUSSION PAPER
2007/18

Indirect Estimation of Elliptical Stable Distributions

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February 2007

Abstract

We present an indirect estimation approach for elliptical stable distributions which relies on the use of a multivariate t distribution as auxiliary model. This distribution is also elliptical and we show that its parameters have a one-to-one relationship with those of the elliptical stable, therefore making the proposed indirect approach especially suitable. Standard asymptotic properties are also shown and we analyze the finite sample behavior of the estimators via a comprehensive Monte Carlo study. An application to 27 emerging markets stock indexes concludes the paper.

Keywords: Stable, elliptical, high dimension, multivariate, Indirect Inference.

JEL classification: C13, C15, G11.

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Work supported in part by the European Community's Human Potential Programme under contract HPRN-CT-2002-00232, [MICFINMA]. With the usual disclaimer, we are grateful to Claudia Czado, Marc Hallin, Alexander McNeil, Stefan Mittnik, John Nolan, Gabriel Frahm, Davy Paindaveine, Marc Paolella, Eric Renault and Sebastien Van Bellegem for useful comments that improve the paper. Research assistantship by Delphine Cassart and David de Antonio is highly appreciated.
This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

1 Introduction

The modelling of multivariate financial time series has emerged as a pole of attention among researchers and practitioners. Typically, the assumptions underlying multivariate time series modelling refer to the specification of the first two moments and of the distribution from which the data is assumed to be generated. As for the moments, researchers have mainly focussed on conditional moments and have proposed the use of VAR and GARCH type of models, such as BEKK, CCC and DCC (cf. Bauwens, Laurent and Rombouts (2006)). As for the distribution, many financial models rely on the multivariate Gaussian distribution as a building block - for instance, the classical CAPM, factor models or the Black and Scholes option pricing equation. The reason behind this choice is twofold. On one side the presence of the central limit theorem in a sense justifies the insurgence of the Gaussian distribution whenever the phenomenon of interest can be thought of as the aggregation of a large number of micro-contributions. On the other side the fact that the Gaussian family of distributions has a number of useful properties, which are very helpful in establishing theoretical results. However, using multivariate Gaussian distributions has a major shortcoming: the tails of the distribution are seldom able to accommodate for extreme gains and losses that are frequently observed on financial markets. Some alternatives have been proposed in the literature. The multivariate Student's t and its skewed version (cf. Bauwens and Laurent (2005)) are two of them. However, although they provide a clear improvement in the fit of the distribution, the Student's t has the clear shortcoming of not being closed under summation, which makes the derivation of theoretical results much more cumbersome. An alternative is the use of copulas (cf. Patton (2004) and references therein), which circumvents the choice of the multivariate distribution by the appropriate choice of the marginals and the copula function. Yet, their main advantage also turns out to be a serious shortcoming, as the cornerstone of the method lies precisely the correct specification of a number of marginal densities which is likely to be large.

Among other possible heavy-tailed alternatives, the multivariate stable distribution (cf. Samorodnitsky ad Taqqu (1994)) has a special role. It originates from a generalization of the central limit theorem in which the assumption on the finiteness of the variance of the components is replaced by a much less restrictive one concerning a somewhat regular tail behavior (cf. Ibragimov and Linnik (1971)). As a consequence, stable distributions enjoy many of the properties of the Gaussian, including closeness under summation, and a number of theoretical results in asset allocation and option pricing are available (cf. Fama (1965a) and (1965b), Ortobelli, Huber and Schwartz (2002), Ortobelli and Rachev (2005), McCulloch (2003) and the survey by Bradley and Taqqu (2001)).

Notwithstanding the appealing properties of the stable distribution, estimation has always been challenging as they are defined via the characteristic function and the density function cannot in general be expressed in a closed form. Several techniques have been put forward for the estimation of univariate distributions, which work fairly well (cf. the survey in Garcia, Renault and Veredas (2006) and references therein). Roughly, these techniques can be divided in four groups. The first, and most used, is the group of characteristic function methods; the second are the quantile based methods; the third are maximum likelihood techniques and the last are simulation-based methods.

At the multivariate level characteristic function methods are not operational for dimension, say, higher than three. Quantile methods are neither applicable as the concept of quantile itself is not clear-cut in a multidimensional perspective. As for maximum likelihood, it is a complex issue even in the univariate case, due to the absence of the density function in closed form, and this carries over and amplifies at the multivariate level. In fact, most of the available results refer solely to the estimation of the so called spectral measure, that is, a measure that contains information on the scale and skewness of the process.[1] Furthermore, two approaches has been taken on the estimation of the spectral measure. The first is based on the multivariate characteristic function (Nolan, Panorska and McCulloch, 2001, and Pivato and Seco, 2003). Though theoretically possible, it is not clear how to make these methods operational for the estimation of high dimension processes. The second approach is based on one dimensional projections of the multivariate process (Nolan, Panorska and McCulloch, 2001; Rachev and Xin, 1993; and Cheng and Rachev, 1995). The only paper, to our knowledge, that estimates all the parameters of the multivariate stable distribution is Nolan (2005) who extends above mentioned results based on projections to the location and tail index.

These estimation difficulties have hindered their widespread in applied works and call for the use of the last group of methods: simulation based methods. Since random numbers from stable distributions can be obtained straightforwardly, simulation-based methods such as the Indirect Inference of Gourieroux, Monfort and Renault (1993) and Efficient Method of Moments -EMM hereafter- of Smith (1993) and Gallant and Tauchen (1996) are especially appealing. These two methods will be refereed in the sequel as indirect estimation methods. In the univariate case, indirect approaches have been proposed independently by Garcia, Renault and Veredas (2006) and Lombardi and Calzolari (2006). In this paper we move a step forward, considering an indirect approach to the estimation of elliptical stable distributions.

Elliptical stable distributions, ESD hereafter, are nested into the class of elliptical distributions, introduced by Kelker (1970).[2] This class is particularly compelling as it contains important laws, some of them already mentioned (Gaussian, Student's t and ESD), and it possesses many of the attractive properties as the Gaussian and stable. For instance, they are invariant to affine transformations, marginal and conditional distributions are also elliptical and they are closed under convolution. The fact that the elliptical class of distributions includes the Student's t and the ESD suggests that an indirect estimation approach could be fruitful. According to the indirect methods, an auxiliary model, easy to estimate, replaces the model of interest, and simulations performed under the latter are then used to correct for bias. The fact that the model of interest and the auxiliary model belong to the same family and share the same structure is helpful in establishing the asymptotic properties, as the parameters have a one-to-one relationship.

Standard asymptotic theory of indirect estimation can be applied, as the information content on the parameters of the Student's t is sufficient to identify the parameters of the ESD and the score of the Student's t distribution is asymptotically Gaussian. However, in finite sample asymptotics do not apply. The problem, highlighted in Garcia, Renault and Veredas (2006) and Lombardi and Calzolari (2006), is that as the tail index of the stable distribution approaches two, and hence the distribution approaches the Gaussian, the degrees of freedom of the Student's t appears to be spuriously attracted by infinity. While it should not be a problem asymptotically, it entails important estimation difficulties with finite samples. To avoid it, we constraint the degrees of freedom to remain below an upper bound, therefore resorting to the constrained indirect estimation of Calzolari, Fiorentini and Sentana (2004).

A comprehensive Monte Carlo study for six different values of the tail index, different specifications of the scatter matrix and for two and five dimensions shows that the finite sample properties of the estimates are reasonably good, unbiased in virtually all cases and

[1] As pointed out in Pivato and Seco (2003), the spectral measure should be called Feldheim measure, after Feldheim (1937), and not spectral as it is unrelated with any other "spectral measures" currently existent in statistics.

[2] See Cambanis, Huang and Simons (1981) and Fang, Kotz and Ng (1990) for further references. A nice, short and concise survey is chapter one of Frahm's PhD thesis (2004).

with root mean square errors that decrease with the number of indirect optimizations. The empirical application is on weekly Morgan Stanley Corporate Indexes (MSCI) of 27 emerging markets. We estimate the ESD on standardized residuals, filtered by a GARCH(1,1) model, and we show that the tail index is below two and the estimated scatter matrix mimics the empirical correlation matrix.

The plan of the paper is as follows. Section 2 introduces elliptical distributions and, in particular, the ESD and Student's t. Section 3 presents the indirect estimation methods and prove its asymptotic properties in our setting as well as the one-to-one relationship between the parameters of the two distributions. A detailed simulation study highlights the small sample properties of the estimators in Section 4. Next, we illustrate the method to 27 emerging markets indexes and Section 6 concludes and gives directions for further research.

2 Elliptical Distributions

A k dimensional random vector $\mathbf{X}$ is elliptically distributed if

$$\mathbf{X} =^d \boldsymbol{\mu} + \mathcal{R}\boldsymbol{\Lambda}\mathbf{U}^{(k)},$$

where $\boldsymbol{\mu}$ is a k dimensional vector of location parameters, $\boldsymbol{\Lambda}$ is a $k \times k$ full rank arbitrary matrix of scale parameters and $\mathbf{U}^{(k)}$ is a k dimensional random vector uniformly distributed in the unit sphere with $k-1$ dimensions

$$\mathcal{S}^{k-1} = \left\{ \mathbf{x} \in \mathbb{R}^k : \|\mathbf{x}\|_2 = 1 \right\}.$$

$\mathcal{R}$ is the so called generating variate of $\mathbf{X}$. It is a nonnegative random variable stochastically independent of $\mathbf{U}^{(k)}$. The starting point in the construction of an elliptically distributed random variable is $\mathbf{U}^{(k)}$, which is radial. It is premultiplied by $\boldsymbol{\Lambda}$, such that $\boldsymbol{\Lambda}\mathbf{U}^{(k)}$ is not radial anymore but rather elliptical, with the generating variate $\mathcal{R}$ giving the thickness, or thinness, of the tails of $\mathcal{R}\boldsymbol{\Lambda}\mathbf{U}^{(k)}$. The vector $\boldsymbol{\mu}$ shifts the location of the density. If $\boldsymbol{\Lambda}$ equals the identity matrix, the density of $\mathbf{X}$ remains radial. $\boldsymbol{\Lambda}$ is a matrix such that $\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}'$ is a positive definite matrix of rank k and $\boldsymbol{\Sigma}$ is called the dispersion or scatter matrix of $\mathbf{X}$. We are ultimately interested in $\boldsymbol{\Sigma}$, though elliptical distributions are expressed in terms of $\boldsymbol{\Lambda}$. In fact, the decomposition of $\boldsymbol{\Sigma}$ in terms of $\boldsymbol{\Lambda}$ is itself irrelevant as $\boldsymbol{\Lambda}$ is not identified.[3]

Some important densities are embedded in the class of elliptical distributions: Gaussian, Student's t and ESD among others.[4] We obtain a Gaussian distribution if $\mathcal{R} = \sqrt{\chi^2_k}$. Similarly, the Student's t is obtained if $\mathcal{R} = \sqrt{\nu\chi^2_k/\chi^2_\nu}$ where χ^2_k and χ^2_ν are stochastically independent. Finally, we obtain an ESD if $\mathcal{R} = \sqrt{\chi^2_k}\sqrt{S_{\alpha/2}}$, where $S_{\alpha/2}$ is a positive, and hence totally skewed to the right, $\alpha/2$ stable distributed random variable and χ^2_k and $S_{\alpha/2}$ are stochastically independent.

From these examples it is evident the close connection between the Student's t and ESD. The location and scale parameters play the same role in both distributions. The tail parameter, either α or ν, enters in both cases through the generating variate. This leads to the intuitive

[3]Indeed, let $\mathbf{T}$ be an orthonormal matrix then $\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}\mathbf{T}\mathbf{T}'\boldsymbol{\Lambda}' = \boldsymbol{\Lambda}^*\boldsymbol{\Lambda}^{*'}$ and therefore $\boldsymbol{\Lambda}$ and $\boldsymbol{\Lambda}^*$ generate the same scatter matrix.

[4]Hereafter we will skip the term multivariate. Nonetheless, the reader should always keep in mind that $\mathcal{R}$ is a random variable and $\mathbf{U}^{(k)}$ is a random vector, thus $\mathbf{X}$ is a random vector as well.

idea, proven in the next section, that if the true data generating process is stable but the assumed distribution is Student's t, a change in the location of the elliptical stable process will lead to a change in the location in the Student's t, and likewise for a change in the scale.

The class of elliptical distributions possesses a number of useful properties, among which we highlight closeness to affine transformations, conditional and marginal distributions being also elliptical and closeness to aggregation (cf. Fang, Kotz and Ng (1990) for further details.). As for the last property, it is worth remarking that the sum of *i.i.d.* elliptically distributed random vectors remains elliptical in the sense that the resulting distribution belongs to the elliptical class, but it is not necessarily the same type as that of their addends. For instance, the sum of two Student's t distributions is elliptical but not Student's t. By contrast, the sum of two Gaussians (elliptical stables) remains Gaussian (elliptical stable).

Another important property of the elliptical distributions is that the density function can be expressed in terms of the density function of the generating variate. More precisely, the *p.d.f.* of $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \sqrt{|\boldsymbol{\Sigma}^{-1}|}g_{\mathcal{R}}\left((\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right) \tag{1}$$

where $|\cdot|$ denotes the determinant,

$$g_{\mathcal{R}}(t) = \frac{\Gamma\left(\frac{k}{2}\right)}{(2\pi)^{k/2}}\sqrt{t}^{-(k-1)}f_{\mathcal{R}}(\sqrt{t})$$

and $f_{\mathcal{R}}$ is the *p.d.f.* of the generate variate. For instance in the Student's t case

$$f_{\mathcal{R}}(t) = \frac{2t}{k}f_F\left(\frac{t^2}{k}\right)$$

where f_F represents the *p.d.f.* of a $F_{k,\nu}$ distributed random variable and hence $f_{\mathcal{R}}(t)$ is the *p.d.f.* of the random variable $\sqrt{kF_{k,\nu}}$. After some arrangements the *p.d.f.* of $\mathbf{X}$ takes the form

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{\Gamma\left(\frac{k+\nu}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{|\boldsymbol{\Sigma}^{-1}|}{(\nu\pi)^k}\right)^{1/2}\left(1+\frac{(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})}{\nu}\right)^{-\frac{k+\nu}{2}}, \tag{2}$$

which is a Student's t density with ν degrees of freedom and $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the location and scale parameters. Unfortunately, it does not exist an equivalent closed form expression of (2) for the ESD. However, the fact that the Student's t and the ESD are elliptical paves the road to the use of indirect methods, where the Student's t density (2) is used for estimating indirectly the ESD.

3 Indirect Estimation

Let $\mathbf{X}$ be a sample of T *i.i.d.* copies from an ESD. Given (1), its log likelihood is[5]

$$\ln \ell^\star(\boldsymbol{\theta},\mathbf{x}) = \frac{1}{2}\ln|\boldsymbol{\Sigma}^{-1}| + \ln g_{\mathcal{R}}\left((\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right),$$

where $\boldsymbol{\theta} = (\alpha,\boldsymbol{\Sigma},\boldsymbol{\mu}) \in \boldsymbol{\Theta} =]0,2[\times\mathbb{R}^{k\times k}_{++}\times\mathbb{R}^k$, α is the tail index, $\boldsymbol{\Sigma}$ a $k\times k$ positive definite scatter matrix, $\boldsymbol{\mu}$ a $k\times 1$ location parameter vector and $g_{\mathcal{R}}$ is the generating variate of

[5]From this section on, we will slightly change the notation. Because we will use the Student's t and the elliptical stable distributions, we will denote differently to the location and scatter parameters of each law.

$\sqrt{\chi^2_k}\sqrt{S_{\alpha/2}}$. This is the *model of interest*. However, as previously noted, this log likelihood has not a closed-form expression and has difficult tractability.[6] Instead, we assume, mistakenly but on purpose, that $\mathbf{X}$ follows a Student's t distribution with density (2) and therefore we can easily maximize the log likelihood

$$\ln \tilde{\ell}(\boldsymbol{\zeta}, \mathbf{x}) = \ln \frac{\Gamma\left(\frac{\nu+k}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} + \frac{1}{2}\ln\left(\frac{|\boldsymbol{\Psi}^{-1}|}{(\nu\pi)^k}\right) - \frac{k+\nu}{2}\ln\left(1 + \frac{\left((\mathbf{x}-\boldsymbol{\delta})'\boldsymbol{\Psi}^{-1}(\mathbf{x}-\boldsymbol{\delta})\right)}{\nu}\right)$$

where $\boldsymbol{\zeta} = (\nu, \boldsymbol{\Psi}, \boldsymbol{\delta}) \in \mathbf{Z} =]0, \infty[\times \mathbb{R}^{k\times k}_{++} \times \mathbb{R}^k$, ν is the tail index, $\boldsymbol{\Psi}$ a $k \times k$ positive definite scatter matrix and $\boldsymbol{\delta}$ a $k \times 1$ location parameter vector. This is the *auxiliary model*.[7] Since this model is misspecified, the estimators that maximize the above log-likelihood, $\hat{\boldsymbol{\zeta}}(\mathbf{x})$, are not necessarily consistent. The central idea of indirect methods is to exploit simulations under the model of interest to correct for the asymptotic bias.

Let $\mathbf{X}_s(\boldsymbol{\theta})$, $s = 1, \ldots, S$, be S simulated sample of T *i.i.d.* copies from an ESD and for a given arbitrary parameter vector $\boldsymbol{\theta}$. And let

$$\hat{\boldsymbol{\zeta}}_s(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\zeta}\in\mathbf{Z}} \ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x}_s(\boldsymbol{\theta}))$$

be the maximum likelihood estimator of the Student's t distribution.[8] Furthermore let

$$\hat{\boldsymbol{\zeta}}_S(\boldsymbol{\theta}) = \frac{1}{S}\sum_{s=1}^{S} \hat{\boldsymbol{\zeta}}_s(\boldsymbol{\theta}).$$

The Indirect Inference estimates, $\hat{\boldsymbol{\theta}}(\mathbf{x})$, are the values for which the following distance is minimized

$$\left[\hat{\boldsymbol{\zeta}}(\mathbf{x}) - \hat{\boldsymbol{\zeta}}_S(\boldsymbol{\theta})\right] \boldsymbol{\Omega} \left[\hat{\boldsymbol{\zeta}}(\mathbf{x}) - \hat{\boldsymbol{\zeta}}_S(\boldsymbol{\theta})\right]$$

where $\boldsymbol{\Omega}$ is a symmetric nonnegative matrix defining the metric.[9] Alternatively, EMM considers directly the score of the Student's t

$$\sum_{t=1}^{T} \frac{\partial \ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x})}{\partial \boldsymbol{\zeta}}.$$

The EMM estimates, $\check{\boldsymbol{\theta}}(\mathbf{x})$, are the values for which the following distance is minimized

$$\left\{\sum_{s=1}^{S} \frac{\partial \ln \tilde{\ell}\left(\boldsymbol{\zeta}; \mathbf{x}_s(\boldsymbol{\theta})\right)}{\partial \boldsymbol{\zeta}}\right\}' \boldsymbol{\Upsilon} \left\{\sum_{s=1}^{S} \frac{\partial \ln \tilde{\ell}\left(\boldsymbol{\zeta}; \mathbf{x}_s(\boldsymbol{\theta})\right)}{\partial \boldsymbol{\zeta}}\right\}$$

[6] It should be emphasized that difficult does not imply impossible. Nolan (2005) shows how to compute the log-likelihood numerically.

[7] Admittedly, the Student's t is not the only *good* candidate for the auxiliary model. For instance, the symmetric generalized hyperbolic distribution is also appropriate. The choice of the Student's t motivated by Demarta and McNeil (2005), Frahm, Junker and Szimayer (2005) and Frahm (2006) whom suggest that this distribution as a reference model for elliptically contoured distributions.

[8] In actual facts, it is not strictly necessary to work with standard maximum likelihood, and more efficient algorithms such as the EM (Meng and van Dijk, 1995) can be employed. However, as we will point out later, we will follow a gradient based approach according to which the estimation of the auxiliary model is required only once and it is instead crucial the availability of an analytic version of the gradient; for this reason we believe that ML is more appropriate in this context.

[9] Typically, the optimal matrix is the inverse of the product of the scores.

where $\boldsymbol{\Upsilon}$ is a symmetric nonnegative definite matrix. Gourieroux, Monfort and Renault (1993) shown that, choosing appropriately the weighting matrices, the two methods are asymptotically equivalent and hence $\check{\boldsymbol{\theta}}(\mathbf{x}) - \hat{\boldsymbol{\theta}}(\mathbf{x}) \rightarrow 0$ as $T \rightarrow \infty$.

In order to identify $\boldsymbol{\theta}$ it is needed that the dimension of $\boldsymbol{\zeta}$ is at least as big as that of $\boldsymbol{\theta}$. If both dimensions are equal, as it is the case of the elliptical distributions considered in this article, $\check{\boldsymbol{\theta}}(\mathbf{x})$ does not depend of $\boldsymbol{\Upsilon}$ and one can choose the Indirect Inference or the EMM estimators that suits the best for the practical problem to be analyzed. For instance, EMM is especially useful when an analytic expression for the gradient of the auxiliary model is available, since it allows us to avoid the numerical optimization routines in the estimation of the auxiliary model.[10]

The asymptotic behavior of the log-likelihood of the auxiliary model is

$$\lim_{T\rightarrow\infty} \frac{1}{T} \ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x}_s(\boldsymbol{\theta})) = E_\theta \left[\ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x}_s(\boldsymbol{\theta})) \right]$$

and the solution of the maximization problem is

$$\mathbf{b}(\boldsymbol{\theta}) = \arg\max_{\boldsymbol{\zeta}\in Z} E_{\boldsymbol{\theta}} \left[\ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x}_s(\boldsymbol{\theta})) \right],$$

that is $\hat{\boldsymbol{\zeta}}_S(\boldsymbol{\theta})$ is a consistent estimator of $\mathbf{b}(\boldsymbol{\theta})$, the *binding function* that maps the parameters space of the true model onto the parameter space of the auxiliary model. The indirect estimator of $\boldsymbol{\theta}$ is thus based on the evaluation of the binding function at the true optimum $\boldsymbol{\theta}_0$. $\mathbf{b}(\boldsymbol{\theta})$ defines the pseudo-true value of the Student's t parameters when the true probability distribution is the ESD. The fact that the model of interest and the auxiliary model belong to the same family of elliptical distributions allows us to devise a one-to-one relationship between the binding function and $\boldsymbol{\theta}$. Intuitively, the location parameters are the same for both distributions and the difference in the tail behavior between the two generating variates is exclusively given by α and ν. Hence, $\boldsymbol{\Psi}$ is very informative for estimating $\boldsymbol{\Sigma}$. Following Garcia, Renault and Veredas (2006), we denote

$$\begin{aligned} \mathbf{b}_1(\alpha, \boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \nu \\ \mathbf{b}_2(\alpha, \boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \boldsymbol{\delta} \\ \mathbf{b}_3(\alpha, \boldsymbol{\mu}, \boldsymbol{\Sigma}) &= \boldsymbol{\Psi} \end{aligned}$$

The following proposition proves that the relationship is one to one.

Proposition 1 Let $\mathbf{a} \in \mathbb{R}^k$ be a k dimensional vector and $\boldsymbol{\Delta} \in \mathbb{R}^{k\times k}$ an arbitrary matrix of full rank. Then for any k dimensional vector $\boldsymbol{\mu} \in \mathbb{R}^k$ and scatter matrix $\boldsymbol{\Sigma}$ such that $\boldsymbol{\Sigma} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}'$, $\boldsymbol{\Lambda} \in \mathbb{R}^{k\times k}$,

$$\begin{aligned} \mathbf{b}_1(\alpha, \boldsymbol{\mu} + \mathbf{a}, \boldsymbol{\Delta}\boldsymbol{\Sigma}\boldsymbol{\Delta}') &= \nu \\ \mathbf{b}_2(\alpha, \boldsymbol{\mu} + \mathbf{a}, \boldsymbol{\Sigma}) &= \boldsymbol{\delta} + \mathbf{a} \\ \mathbf{b}_3(\alpha, \boldsymbol{\mu}, \boldsymbol{\Delta}\boldsymbol{\Sigma}\boldsymbol{\Delta}') &= \boldsymbol{\Delta}\boldsymbol{\Psi}\boldsymbol{\Delta}' \end{aligned}$$

that is $\boldsymbol{\delta} \leftrightarrow \boldsymbol{\mu}$, $\boldsymbol{\Psi} \leftrightarrow \boldsymbol{\Sigma}$ and $\alpha \leftrightarrow \nu$.

[10] Because they are equivalent, hereafter we will use them indistinguishably though we will favour the EMM estimator for reasons that will become clear at the end of this section.

Proof Let an elliptically distributed random vector

$$\mathbf{X} =^d \boldsymbol{\mu} + \mathcal{R}^S \boldsymbol{\Lambda} \mathbf{U}^{(k)},$$

where $\mathcal{R}^S = \sqrt{\chi_k^2}\sqrt{S_{\alpha/2}}$. Its characteristic function corresponds to

$$\varphi_{\mathbf{X}^S}(t) = \exp(it'\boldsymbol{\mu}) \int_0^\infty \varphi_{\mathbf{U}}(r^2 t' \boldsymbol{\Sigma} t) dF_{\mathcal{R}^S}(r)$$

where $\varphi_{\mathbf{U}}(\cdot)$ is the characteristic function of $\mathbf{U}^{(k)}$ and $F_{\mathcal{R}^S}$ is the cdf of $\sqrt{\chi_k^2}\sqrt{S_{\alpha/2}}$. Consider instead the integration with respect to $F_{\mathcal{R}^{St}}$, the cdf of $\sqrt{\nu\chi_k^2/\chi_\nu^2}$

$$\varphi_{\mathbf{X}^{St}}(t) = \exp(it'\boldsymbol{\mu}) \int_0^\infty \varphi_{\mathbf{U}}(r^2 t' \boldsymbol{\Sigma} t) dF_{\mathcal{R}^{St}}(r)$$

A change in the location and the scale $\mathbf{Y} := \mathbf{a} + \boldsymbol{\Delta}\mathbf{X}$, corresponds to

$$\begin{aligned} \varphi_{\mathbf{Y}^S}(t) &= E[\exp(it'(\mathbf{a} + \boldsymbol{\Lambda}\mathbf{X}))] \\ & \exp(it'(\boldsymbol{\mu} + \mathbf{a})) \int_0^\infty \varphi_d(r^2 t' \boldsymbol{\Delta\Sigma\Delta}' t) dF_{\mathcal{R}^S}(r), \end{aligned}$$

and as well to

$$\begin{aligned} \varphi_{\mathbf{Y}^{St}}(t) &= E[\exp(it'(\mathbf{a} + \boldsymbol{\Lambda}\mathbf{X}))] \\ & \exp(it'(\boldsymbol{\mu} + \mathbf{a})) \int_0^\infty \varphi_d(r^2 t' \boldsymbol{\Delta\Sigma\Delta}' t) dF_{\mathcal{R}^{St}}(r) \end{aligned}$$

which are the characteristic functions of the elliptically distributed random vectors, $\mathbf{Y}^{St} =^d (\boldsymbol{\mu} + \mathbf{a}) + \mathcal{R}^{St}\boldsymbol{\Delta\Lambda}\mathbf{U}^{(k)}$ and $\mathbf{Y}^S =^d (\boldsymbol{\mu} + \mathbf{a}) + \mathcal{R}^S\boldsymbol{\Delta\Lambda}\mathbf{U}^{(k)}$, with identical location and scatter matrices.□

This means that a change in the location only affects the location parameter and a scale change only affects the scatter matrix. Moreover, the generating variate of the affinely transformed vector remains the same. Therefore even if we estimate with $\mathcal{R} = \sqrt{\nu\chi_k^2/\chi_\nu^2}$, the affine transformation does not affect the tail index. In other words, the location and scale parameters of the Student's t carry over exclusively information on the location and scale parameters of the ESD respectively: $\boldsymbol{\delta} \leftrightarrow \boldsymbol{\mu}$ and $\boldsymbol{\Psi} \leftrightarrow \boldsymbol{\Sigma}$. Hence the tail index is not modified by location and scale changes but by the stability index: $\alpha \leftrightarrow \nu$.

Under some regularity conditions -see Appendix- the EMM estimator $\check{\boldsymbol{\theta}}(\mathbf{x})$ is consistent for fixed S and $T \to \infty$. Furthermore, $\check{\boldsymbol{\theta}}(\mathbf{x})$ is asymptotically Gaussian for fixed S and $T \to \infty$

$$\sqrt{T}(\check{\boldsymbol{\theta}}(\mathbf{x}) - \boldsymbol{\theta}_0) \xrightarrow{d} \mathcal{N}(0, \mathcal{W}(S, \boldsymbol{\Upsilon}))$$

where

$$\mathcal{W}(S,\Upsilon) = \left(1 + \frac{1}{S}\right) \left[\frac{\partial^2 E_{\boldsymbol{\theta}}[\ln \tilde{\ell}(\boldsymbol{\zeta}, \mathbf{x}(\boldsymbol{\theta}))]}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\theta}'}' \Upsilon^* \frac{\partial^2 E_{\boldsymbol{\theta}}[\ln \tilde{\ell}(\zeta, \mathbf{x}(\boldsymbol{\theta}))]}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\theta}'} \right]^{-1} \quad (3)$$

and

$$\boldsymbol{\Upsilon}^* = \lim_{T\to\infty} Var \left\{ \sqrt{T} \frac{\partial \ln \tilde{\ell}\,[\mathbf{b}(\boldsymbol{\theta}); \mathbf{x}]}{\partial \boldsymbol{\zeta}} \right\}.$$

A consistent estimator for $\check{\mathcal{W}}$ is[11]

$$\check{\mathcal{W}}(S, \boldsymbol{\Upsilon}) = \left(1 + \frac{1}{S}\right) \left[\frac{\partial^2 \ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x})}{\partial \boldsymbol{\theta} \partial \boldsymbol{\zeta}'}' \check{\Upsilon}^* \frac{\partial^2 \ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x})}{\partial \boldsymbol{\theta}' \partial \boldsymbol{\zeta}} \right]^{-1}$$

where

$$\check{\Upsilon}^* = \frac{1}{T} \left[\left. \frac{\partial \ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x})'}{\partial \boldsymbol{\zeta}} \right|_{\boldsymbol{\zeta} = \hat{\boldsymbol{\zeta}}} \left. \frac{\partial \ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x})}{\partial \boldsymbol{\zeta}'} \right|_{\boldsymbol{\zeta} = \hat{\boldsymbol{\zeta}}} \right].$$

Indirect estimators are asymptotically well behaved because the information content on the parameters of the Student's t is sufficient to identify the parameters of the ESD and the score of the Student's t distribution is asymptotically Gaussian. However, in finite samples the information content in ν is not sufficient to identify α as it approaches to 2 because $\check{\nu}$ tends to infinity. This is a finite sample artifact, as asymptotically tends to its true value, and to avoid it we constrain ν to remain below an upper bound $\bar{\nu}$. Let

$$\hat{\boldsymbol{\beta}}(\mathbf{x}) = \arg \max_{\boldsymbol{\beta} \in \mathbf{Z} \times \mathbb{R}} \ln \tilde{\ell}(\boldsymbol{\zeta}; \mathbf{x}(\boldsymbol{\theta})) + (\nu - \bar{\nu})\rho$$

be the constrained estimator of the Student's t distribution that satisfies the inequality restriction plus the slackness restriction $(\nu - \bar{\nu})\rho = 0$. The parameter set is $\boldsymbol{\beta} = (\boldsymbol{\zeta}, \rho) \in \mathbf{Z} \times \mathbb{R}$ and $\bar{\nu}$ is the upper bound for ν. Equivalently for $\hat{\boldsymbol{\beta}}(\boldsymbol{\theta})$.

Calzolari, Fiorentini and Sentana (2004) have shown that the EMM estimator under constraints is analogous to that derived by Gallant and Tauchen (1996) and the weighting matrix remains the same. The reason why theoretical results for EMM remain unchanged is that the score is taken with respect to $\boldsymbol{\zeta}$ and not with respect to ρ and hence $\boldsymbol{\theta}$ remains exactly identified. However, results change for Indirect Inference as the multiplier ρ is also minimized and therefore $\boldsymbol{\theta}$ is overidentified and an optimal weighting matrix is needed. Furthermore, this optimal matrix takes a complicated form as it accounts for the inequalities and the slackness condition. All in one, the inclusion of constraints in the Student's t distribution does not change standard EMM estimation while it does in Indirect Inference. For this reason we choose the former method for the Monte Carlo study and the empirical illustration.

4 Monte Carlo study

In order to proceed with the Monte Carlo study and to indirectly estimate the stable parameters, we need to produce random samples from an ESD. In the first part of this section we explain how to simulate while in the second we explain the Monte Carlo study and its results.

Simulating from an ESD is fairly simple. This is due to the fact that the tail index appears only in the generating variate, which is univariate. In other words, in order to simulate random numbers from an ESD it suffices to be able to simulate from its univariate counterpart -using, for instance, the Chambers, Mallows and Stuck (1976) method. The ESD can be rewritten as

$$\mathbf{X} =^d \boldsymbol{\mu} + \sqrt{S_{\alpha/2}} \mathbf{G} \tag{4}$$

[11]The following expressions are specific to the *i.i.d.* case. The general expressions for serial dependence can be found in Appendix 2 of Gourieroux, Monfort and Renault (1993).

where $\mathbf{G} = \sqrt{\chi_k^2}\mathbf{\Lambda}\mathbf{U}^{(k)} \sim \mathcal{N}(0, \mathbf{\Sigma})$. Therefore to simulate $\mathbf{X}$ we only need to simulate from a multivariate Gaussian density and from a univariate stable density. More precisely, if

$$A \sim S_{\alpha/2}\left(\left(\cos\frac{\pi\alpha}{4}\right)^{2/\alpha}, 1, 0\right)$$

and $\mathbf{G} \sim \mathcal{N}(0, \mathbf{\Sigma})$ independent of A then $A^{1/2}\mathbf{G} \sim S_\alpha(\Sigma, 0, \mu)$. Notice that if α approaches 2, then $\sqrt{S_{\alpha/2}}\sqrt{\chi_k^2} \to \sqrt{\chi_k^2}$. This is a counter intuitive result as A has a location parameter 0 and a scale that equals 0 for $\alpha = 2$. However, A converges to a Dirac delta measure. To see this, take the Laplace transform of $\mathbf{X}$ defined in (4)

$$E(\exp(-\gamma A)) = \exp(-\gamma^{\alpha/2})$$

As $\alpha \to 2$, $\exp(-\gamma^{\alpha/2}) \to \exp(-\gamma)$, which is the Laplace transform of a Dirac delta function. That is, $\sqrt{S_{\alpha/2}}$ converges in distribution to a degenerate random variable with value 1. Because an elliptical stable random vector can be viewed a scale mixture of normal random vector, it is also referred as sub Gaussian random vectors.

We use this procedure for the Monte Carlo study. To check the finite sample properties of the estimator, we conduct a thorough Monte Carlo study for a number of combinations of values for α and $\mathbf{\Sigma}$, while setting the location parameter $\boldsymbol{\mu}$ equal to zero. The upper bound $\bar{\nu}$ is set to 100. Note that whether the bound is higher or lower is not important (cf Garcia, Renault and Veredas (2006)) as what matters is that the estimated degrees of freedom are not attracted by infinity. We simulate 100 draws of 500 observations for a grid of parameter values, two different dimensions (2 and 5) and two different values of the indirect draws (S equal to 1 and 5). The stability index α takes values $0.7, 1.1, 1.5, 1.7, 1.9$ and 1.95. As for the scatter matrices, we choose four specifications. The first is the identity, corresponding to the spherical case. The second is diagonal with different elements: $diag(\mathbf{\Sigma}) = (0.5, 2)$ for dimension 2 and $diag(\mathbf{\Sigma}) = (0.25, 0.5, 1, 2, 4)$ for dimension 5. In the third scenario the scatter matrix has the same diagonal elements as in the previous case but with off-diagonal elements $\sigma_{12} = 0.2$ for dimension 2 and

$$\mathbf{\Sigma} = \begin{pmatrix} 0.25 & 0.25 & 0.4 & 0 & 0 \\ 0.25 & 0.5 & 0.4 & 0 & 0 \\ 0.4 & 0.4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2.55 \\ 0 & 0 & 0 & 2.55 & 4 \end{pmatrix}$$

for dimension 5. In the last case, the diagonal elements are as in the previous case and off-diagonal elements $\sigma_{12} = 0.9$ for dimension 2 and

$$\mathbf{\Sigma} = \begin{pmatrix} 0.25 & 0.25 & 0.4 & -0.4 & -0.9 \\ 0.25 & 0.5 & 0.4 & -0.5 & -1 \\ 0.4 & 0.4 & 1 & -1.1 & -1.6 \\ -0.4 & -0.5 & -1.1 & 2 & 2.55 \\ -0.9 & -1 & -1.6 & 2.55 & 4 \end{pmatrix}$$

for dimension 5.

The last three cases for dimension 2, the off-diagonal can be seen, loosely speaking, as correlations (or more precisely standardized covariations) as one of the diagonal elements is the inverse of the other. Therefore we consider a case where the two random variables are uncorrelated, another where they are weakly correlated and a last one where they are strongly correlated. Likewise, for the 5-dimensional case, the last three cases have some meaning in terms of standardized covariations. In the second case all random variables are uncorrelated. In the third they are positively block correlated while in the last case they are positively block correlated and negatively off-block correlated.[12].

Tables 1 and 2 show the results for dimension 2 while Tables 3-6 show the results for dimension 5. Due to space constraints we only show the median and the root mean square error (RMSE).[13] In general, except for some pathological cases, results indicate that estimators are unbiased. Furthermore, small biases present when $S = 1$ are corrected when $S = 5$. Admittedly, a closer look to higher moments -not shown here- such as skewness and kurtosis, reveals that they are not Gaussian but this is not surprising as this is a finite sample exercise. Nonetheless, the density of the estimators are, in some sense, well behaved. Figure 1 shows the kernel densities for one of the cases in dimension 2 and $S = 5$. Though not Gaussian, they do not present remarkable skewness or kurtosis, which are positive indications given that the density is for only 100 estimates.

In most of the cases, particularly for dimension 2, the RMSE for $S = 5$ are lower than for $S = 1$, indicating that, as the theory predicts, the higher the number of indirect draws, the lower the variance of the estimators -see (3). However, the exception is the location parameter vector, $\boldsymbol{\mu}$, for which results in terms of RMSE are not conclusive. Nonetheless, clearly the RMSE of the scatter matrix parameters and the tail index reduces substantially when $S = 5$.

Last, as a further check we have also partly carried out the Monte Carlo study with a multivariate skewed-t distribution (cf. Bauwens and Laurent (2005) and Azzalini and Capitanio (2003)) as auxiliary model. In this case we don't remain anymore within the elliptical class and, furthermore, we need an appropriate weighting matrix that is given by the inverse of the product of the scores of the auxiliary model. Simulation results did not change significantly with respect to the Student's t case, suggesting that the skewness parameter does not convey significant information for estimating an elliptical distribution.

5 Illustration

We illustrate the method with an application to 27 MSCI (Morgan Stanley Composite Index) emerging markets indexes. The MSCI indexes are free float-adjusted market capitalization indexes that are designed to measure equity market performance. The emerging markets areas and countries we consider are: East Asia (Philippines, Sri Lanka, Pakistan, China, South Korea, India, Indonesia, Russia, Thailand, Taiwan and Malaysia), Eastern Europe (Poland, Czech Republic and Hungary), South America (Mexico, Colombia, Chile, Argentina, Brazil, Venezuela and Peru), Africa (South Africa, Egypt and Morocco) and Middle East (Israel, Turkey and Jordan).

[12]In fact, the off diagonal values have been chosen such that the standardized covariation matrices are, approximately,

$$\boldsymbol{\Xi} = \begin{pmatrix} 1 & 0.7 & 0.8 & 0 & 0 \\ 0.7 & 1 & 0.6 & 0 & 0 \\ 0.8 & 0.6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.9 \\ 0 & 0 & 0 & 0.9 & 1 \end{pmatrix} \text{ and } \boldsymbol{\Xi} = \begin{pmatrix} 1 & 0.7 & 0.8 & -0.6 & -0.9 \\ 0.7 & 1 & 0.6 & -0.5 & -0.7 \\ 0.8 & 0.6 & 1 & -0.8 & -0.8 \\ -0.6 & -0.5 & -0.8 & 1 & 0.9 \\ -0.9 & -0.7 & -0.8 & 0.9 & 1 \end{pmatrix}$$

for the third and fourth specifications respectively. These cases may correspond to two portfolios. One with a risk-free and a risky set of assets, with positive covariation in between but not within. The second has two risky sets of assets where one set is hedging the other, and hence the negative covariation

[13]More detailed results, including mean, standard deviation, maximum and minimum are available upon request.

[FIGURE ONE ABOUT HERE]

We use weekly prices, in USD, between April 2001 to April 2006, hence 261 observations per country. Figure 2 shows the indexes for a sample of six countries: 2 East Asian and 1 for the other areas. Volatility behaviour is very heterogeneous. Some countries display strong clustering, like Jordan and Czech Republic, while other present large deviations but not clusters, like Malaysia and Sri Lanka. It is known (Ghose and Kroner, 1995) that heavy tails generated by GARCH effects can be mistakenly interpreted as evidence in favour of stable distributions. To safeguard against this, we consider standardized GARCH(1,1) residuals such that the remaining heteroskedasticity is not due to dynamic conditional volatility.

Figure 3 shows the standardized residuals for the same indexes as in previous figure. The volatility clustering has disappeared, as it is clearly visible for Czech Republic and Jordan. However, they do not appear to be Gaussian. The kurtosis coefficients range from 3.56 for Mexico to 7.93 for Sri Lanka, meaning that a fat-tailed distribution can be an appropriate choice. Figure 4 shows a heat map of the empirical correlations of the standardized residuals. Clusters by geographical areas are very clear. For instance Eastern European and Latin American countries are very related. Others, like Israel (the second from the upper right) and, surprisingly, China (the fourth from the left bottom), are not correlated at all with any other country.

Admittedly, the application is subject to certain critiques. First the tail index α is the same for all countries. We estimated univariate stable distributions for each country and, as expected, the tail indexes are not constant across countries. They vary from 1.5 to 2. Yet this shortcoming is in fact applicable to any multivariate distribution like the Student's t, skewed-t or the Gaussian, in which case is even worse as the tail index is fixed to 2. Second, data are skewed yet we do not allow for asymmetries. Last we consider constant covariation, which may not be the case for the MSCI indexes. Despite all these shortcomings, this estimation exercise is purely illustrative and an application in a dynamic and asymmetric context is beyond the scope of the paper; though it is an interesting research avenue, as explained in the conclusions.

[FIGURE TWO ABOUT HERE]

[FIGURE THREE ABOUT HERE]

The estimated tail index is 1.75, implying thicker tails than in the Gaussian case. The estimated degrees of freedom for the Student's t is 7.19, which produces a mismatch in the existence of moments with respect to the stable distribution. Figure 5 shows the estimates for the location vector. They all vary around zero, which makes sense as the standardized residuals have mean equal to zero. As for the scatter matrix, we do not show all the results, because of space limitations.[14] Instead, we present the estimated covariations for the six countries above considered, which are the solid lines in Figure 6 while the dotted lines are the empirical covariances. The estimated covariations are very close to the empirical ones in all cases. We may compare then with the estimate covariances matrices of the Student's t, plotted in Figure 7. Clearly the estimated scatter matrix of the Student's t distribution has a worst fit than the estimated scatter matrix of the stable distribution. In fact, one can observe that, for Student's t, the estimated covariations are higher than one, which is not sensible as residuals are standardized.

[14]Detailed results are of course available upon request.

[FIGURE FOUR ABOUT HERE]

[FIGURE FIVE ABOUT HERE]

[FIGURE SIX ABOUT HERE]

6 Conclusion

In this paper we propose indirect estimation methods for multivariate elliptical stable distributions. Theory shows that the Student's t distribution is an adequate auxiliary model and standard asymptotics of indirect methods apply. A Monte Carlo study points out that even for small samples, 500 observations, estimates have reasonable properties. Finally an empirical study further illustrates the proposed method.

Further research can take several directions. First, an obvious generalization is to allow for skewness. The reader may be tempted to extend the indirect methods to asymmetric stable distributions. However, we do not think that this may be a sensible decision as in the asymmetric case, skewness and scatter are merged into the so called spectral measure, which takes very complicated forms. Furthermore, simulation from an asymmetric multivariate stable distributions turns out to be difficult hindering the use of indirect inference estimation methods, as precisely these methods are appropriate when simulating from the model of interest is straightforward.[15] An alternative is to use a recent method proposed by Nolan (2005), which is based on projections parameter functions. Another alternative is the use of generalized elliptical distributions (cf. Frahm (2004)), which share many of the properties of the elliptical distributions.

Another direction of further research is the extension to a time series context. In particular, allow the location vector and the scatter matrix to be time varying. For instance VAR and multivariate GARCH type of models are natural choices. On these grounds one has to be careful on the way VAR and GARCH models are defined as the inexistance of moments of orders higher that α entails some difficulties. For instance univariate GARCH models under stable distribution has been analyzed by Mittnik, Paolella and Rachev (2002). This extension seems appropriate, given that most of the economic processes are time dependent. Furthermore, from a theoretical perspective it is feasible as Gourieroux, Monfort and Renault (1993) and Calzolari, Fiorentini and Sentana (2004) do not assume *i.i.d.* returns in their analysis.

A Assumptions

C1. $\mathbf{X}$ is strictly stationary and ergodic.

C2. $\ln \tilde{\ell}(\boldsymbol{\zeta}, \mathbf{x})$ is twice continuously differentiable with respect to $\boldsymbol{\zeta}$.

C3. $E_{\boldsymbol{\theta}}[\ln \tilde{\ell}(\boldsymbol{\zeta}, \mathbf{x})]$ is twice continuously differentiable with respect to $\boldsymbol{\theta}$ and $\boldsymbol{\zeta}$ and has an unique maximum.

C4. $\mathbf{b}(\boldsymbol{\theta})$ is unique

[15] Nonetheless, it is worth noticing that simulation of multivariate stable distributions is possible -Modarres and Nolan (1994).

C5. $\mathbf{b}(\boldsymbol{\theta})$ and $E_{\boldsymbol{\theta}}[\ln \tilde{\ell}(\boldsymbol{\zeta}, \mathbf{x})]$ admit the unique solution $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ and are continuously differentiable at $\boldsymbol{\theta}$.

C6.

$$\frac{\partial^2 \ln \tilde{\ell}(\boldsymbol{\zeta}, \mathbf{x})}{\partial \boldsymbol{\zeta} \partial \boldsymbol{\zeta}} - \mathcal{H}_0 = o_p(1)$$
$$\sqrt{T} E_{\boldsymbol{\theta}}[\ln \tilde{\ell}(\boldsymbol{\zeta}, \mathbf{x})] \rightarrow^d \mathcal{N}(0, \mathcal{I}_0)$$

C7. The asymptotic covariance between the gradients of two units s_1 and s_2 of the simulated sample is constant.

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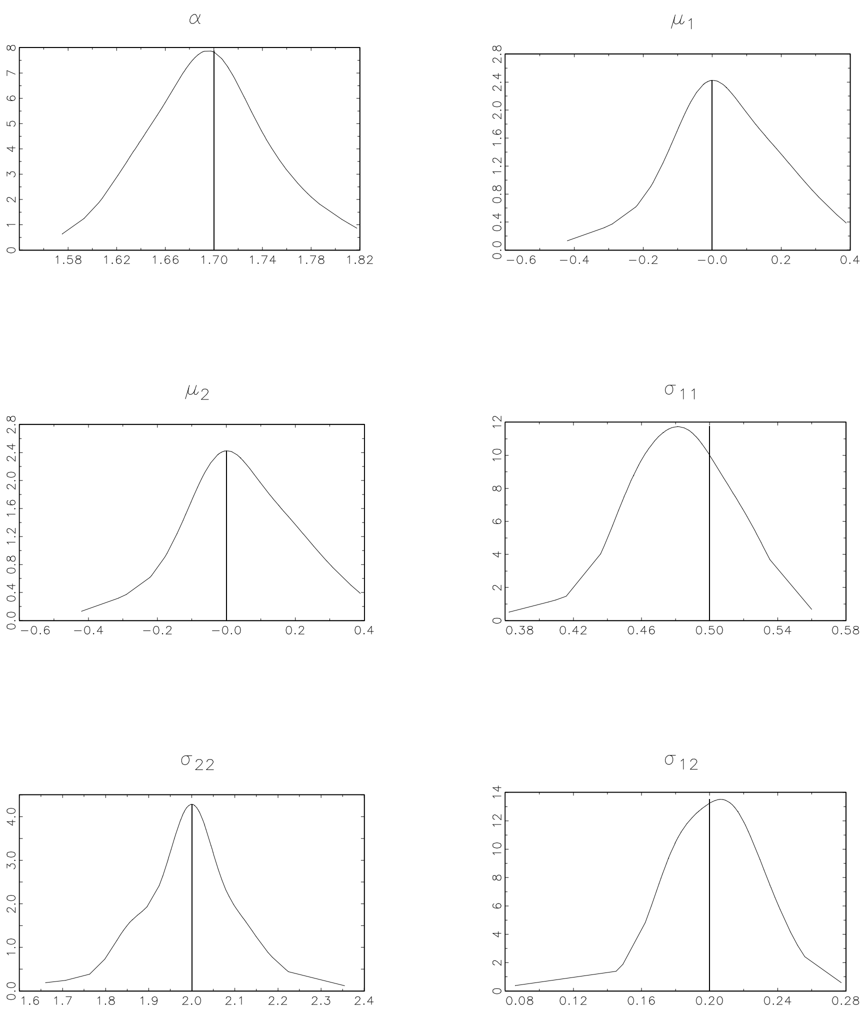

Figure 1: Densities of the estimated parameters of one the Monte Carlo cases for dimension 2.

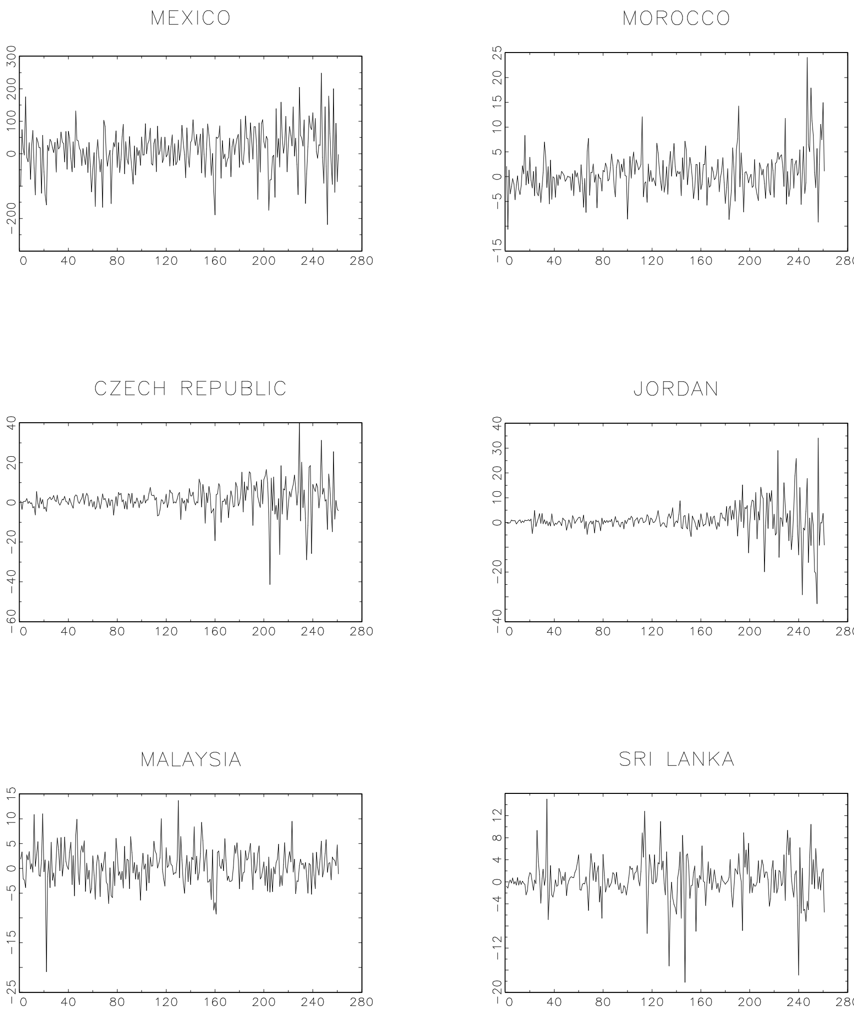

Figure 2: MSCI indexes for a selection of countries.

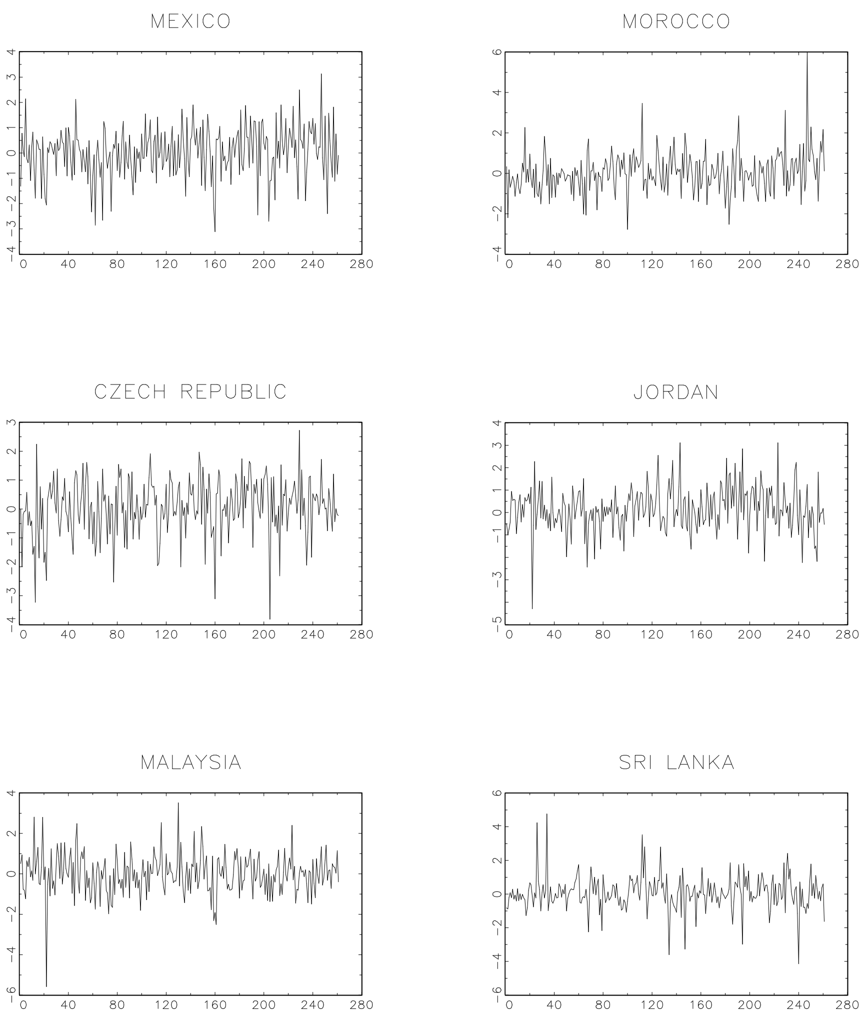

Figure 3: Standardized GARCH(1,1) residuals .

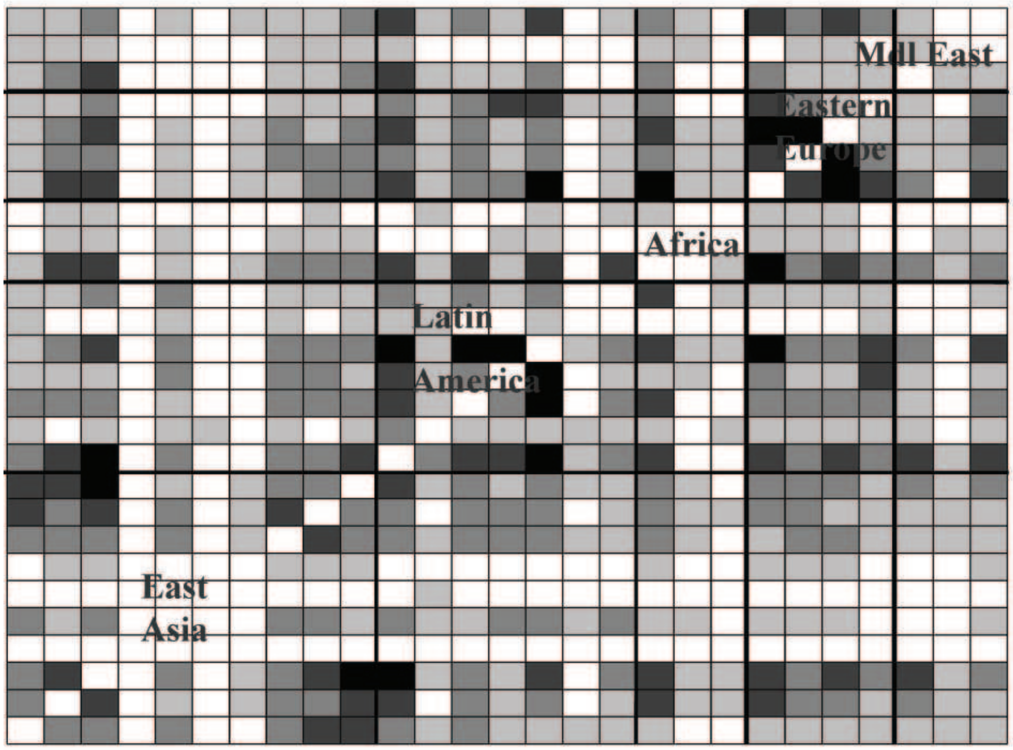

Figure 4: Heat map of the empirical correlations of standardized residuals. The darker (lighter) the higher (lower) the empirical correlations. For representation purposes main diagonal has been replaces by zeros, and hence the white.

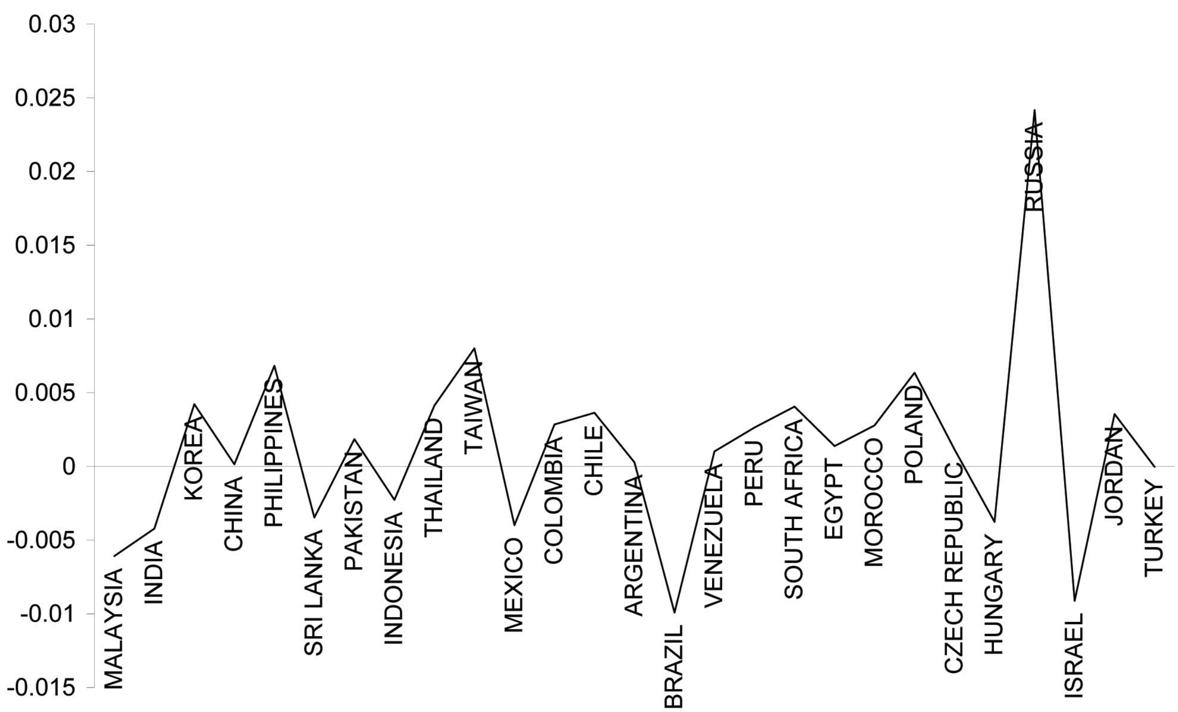

Figure 5: Estimated location parameters

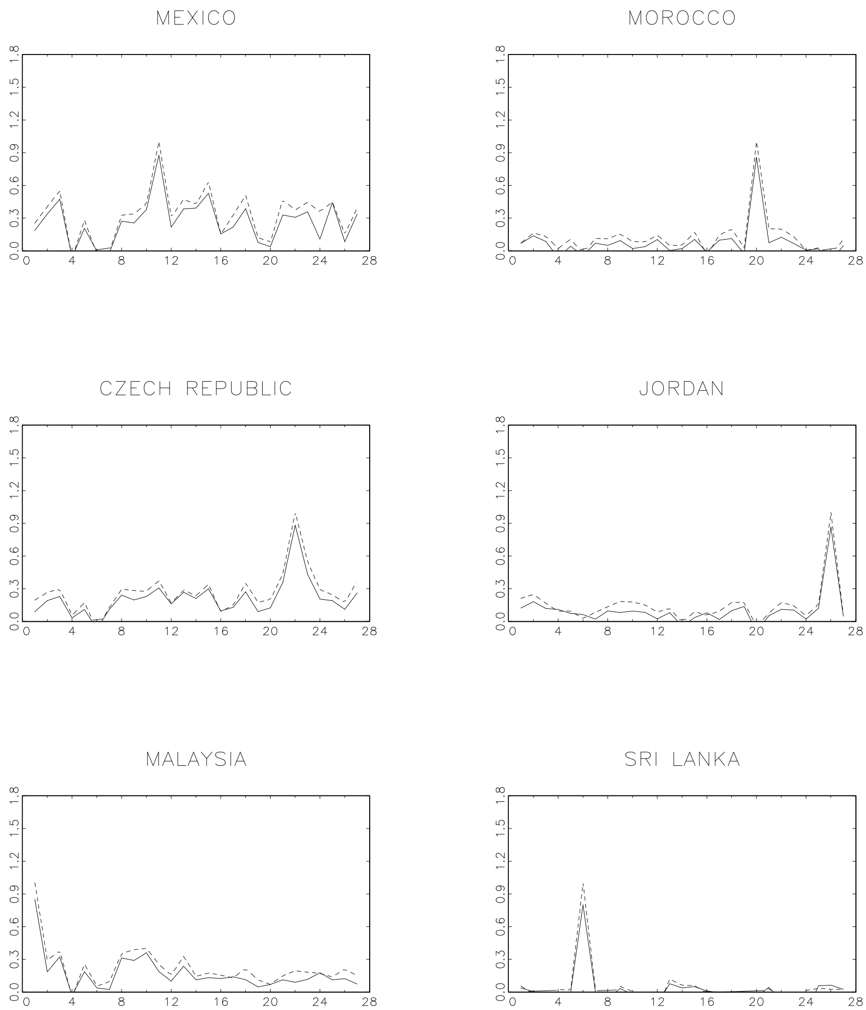

Figure 6: Results for the stable distribution: Estimated covariations (solid line) and empirical correlations (dotted line) of standardized residuals for six countries.

Figure 7: Results for the Student's t distribution: Estimated covariations (solid line) and empirical correlations (dotted line) of standardized residuals for six countries.

Table 1: Simulation Results with $k = 2$ and $S = 1$

α	True	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95
	Median	0.72	1.13	1.71	1.90	1.95	0.71	1.12	1.72	1.89	1.95	0.73	1.13	1.70	1.89	1.95	0.70	1.10	1.74	1.88	1.94
	RMSE	0.11	0.17	0.07	0.03	0.02	0.05	0.16	0.04	0.04	0.02	0.07	0.10	0.06	0.03	0.01	0.04	0.08	0.08	0.06	0.03
σ_{11}	True	1	1	1	1	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
	Median	0.97	0.94	0.96	1.00	1.03	0.48	0.48	0.44	0.51	0.51	0.49	0.47	0.48	0.50	0.52	0.50	0.49	0.49	0.51	0.50
	RMSE	0.20	0.13	0.10	0.10	0.16	0.08	0.07	0.17	0.08	0.06	0.06	0.05	0.06	0.05	0.06	0.02	0.03	0.09	0.27	0.05
σ_{22}	True	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
	Median	0.98	0.95	0.97	1.02	1.01	2.00	1.94	1.92	2.01	2.01	2.00	1.91	1.94	2.03	2.03	2.00	1.99	1.83	2.00	2.01
	RMSE	0.20	0.14	0.11	0.11	0.13	0.19	0.32	0.15	0.23	0.10	0.22	0.16	0.15	0.21	0.08	0.04	0.13	0.27	0.16	0.13
σ_{12}	True	0	0	0	0	0	0	0	0	0	0	0.2	0.2	0.2	0.2	0.2	0.9	0.9	0.9	0.9	0.9
	Median	-0.06	-0.05	-0.05	-0.04	-0.04	-0.04	-0.04	-0.04	-0.04	-0.02	0.16	0.15	0.15	0.16	0.17	0.90	0.89	0.78	0.90	0.90
	RMSE	0.11	0.08	0.07	0.06	0.07	0.08	0.18	0.05	0.06	0.05	0.09	0.07	0.06	0.06	0.07	0.03	0.06	0.15	0.10	0.09
μ_1	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	0.36	1.19	3.45	2.64	1.09	0.40	-0.32	0.54	2.05	0.86	-0.02	-0.28	1.41	4.46	3.46	0.00	-0.16	1.62	0.53	0.98
	RMSE	6.76	7.35	8.81	11.0	12.0	3.31	15.6	4.40	8.94	9.89	3.57	3.76	4.83	11.0	9.62	1.78	3.83	8.40	12.4	10.0
μ_2	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	1.86	1.70	0.56	0.25	0.65	0.17	0.00	0.86	1.72	0.48	0.08	1.86	0.03	9.00	4.14	0.00	0.00	1.12	0.25	0.01
	RMSE	5.83	6.00	9.42	8.21	12.8	6.89	59.1	16.5	24.9	10.4	6.29	12.1	13.9	27.9	26.6	3.47	7.51	7.56	26.6	21.7

Entries are the true parameter, the median of the 100 replications of 500 observations and the Root Mean Square Error (RMSE). For the ease of exposition the median and RMSE of the location parameters have been multiplied by 100.

Table 2: Simulation Results with $k = 2$ and $S = 5$

α	True	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95
	Median	0.71	1.11	1.68	1.89	1.94	0.70	1.11	1.69	1.89	1.95	0.70	1.10	1.70	1.89	1.95	0.70	1.10	1.70	1.90	1.95
	RMSE	0.06	0.06	0.06	0.04	0.03	0.04	0.06	0.06	0.04	0.02	0.05	0.07	0.05	0.04	0.02	0.03	0.04	0.04	0.05	0.01
σ_{11}	True	1	1	1	1	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
	Median	0.99	0.97	0.97	1.01	1.01	0.49	0.48	0.48	0.50	0.50	0.49	0.48	0.48	0.50	0.51	0.50	0.50	0.49	0.50	0.50
	RMSE	0.16	0.08	0.08	0.11	0.18	0.05	0.04	0.04	0.08	0.05	0.05	0.05	0.04	0.05	0.05	0.01	0.01	0.02	0.11	0.05
σ_{22}	True	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
	Median	1.00	1.00	1.01	1.03	1.04	2.00	2.00	2.00	2.04	2.02	2.00	2.00	2.00	2.02	2.01	2.00	2.00	2.00	2.02	2.01
	RMSE	0.11	0.06	0.07	0.15	0.21	0.12	0.13	0.15	0.23	0.16	0.12	0.16	0.11	0.23	0.23	0.01	0.04	0.06	0.19	0.12
σ_{12}	True	0	0	0	0	0	0	0	0	0	0	0.2	0.2	0.2	0.2	0.2	0.9	0.9	0.9	0.9	0.9
	Median	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.20	0.20	0.20	0.21	0.21	0.90	0.90	0.90	0.91	0.90
	RMSE	0.09	0.03	0.03	0.04	0.07	0.05	0.04	0.02	0.04	0.04	0.06	0.03	0.03	0.04	0.05	0.01	0.02	0.03	0.12	0.06
μ_1	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	1.48	2.93	1.86	0.21	0.31	0.82	0.98	2.78	4.22	0.77	0.92	0.95	2.45	2.64	-0.25	0.00	0.23	0.35	0.03	0.59
	RMSE	8.67	8.49	9.52	8.58	13.4	2.55	3.25	5.18	8.02	9.23	4.26	3.58	5.00	8.10	9.93	1.31	3.74	5.83	7.35	10.5
μ_2	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	1.78	1.54	1.87	0.00	0.07	0.00	4.90	1.58	0.34	0.22	0.02	1.30	1.56	-0.12	-0.08	0.00	0.05	0.43	0.02	-0.05
	RMSE	7.37	5.82	9.36	10.6	9.46	5.12	13.5	14.9	20.0	12.5	7.61	12.6	16.6	17.0	15.67	2.35	7.68	11.7	15.3	21.9

See legend Table 1

Table 3: Simulation Results with $k = 5$ and $S = 1$, cont.

α	True	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95
	Median	0.72	1.13	1.68	1.89	1.94	0.70	1.10	1.69	1.89	1.94	0.70	1.10	1.70	1.88	1.95	0.70	1.10	1.70	1.89	1.95
	RMSE	0.12	0.10	0.05	0.05	0.01	0.11	0.08	0.05	0.02	0.05	0.16	0.02	0.03	0.01	0.01	0.04	0.03	0.04	0.02	0.01
σ_{11}	True	1	1	1	1	1	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
	Median	1.00	1.01	1.01	1.01	1.03	0.25	0.26	0.25	0.26	0.26	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
	RMSE	0.06	0.08	0.08	0.07	0.06	0.04	0.02	0.02	0.03	0.05	0.05	0.02	0.02	0.02	0.01	0.03	0.03	0.02	0.03	0.01
σ_{22}	True	1	1	1	1	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
	Median	1.00	1.01	1.00	1.02	1.01	0.50	0.50	0.50	0.51	0.51	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
	RMSE	0.05	0.11	0.07	0.07	0.07	0.05	0.05	0.04	0.03	0.04	0.06	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.07	0.02
σ_{33}	True	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	Median	1.00	1.00	1.00	1.02	1.03	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
	RMSE	0.06	0.09	0.08	0.07	0.07	0.04	0.06	0.05	0.05	0.04	0.05	0.01	0.02	0.04	0.01	0.02	0.02	0.03	0.02	0.01
σ_{44}	True	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
	Median	1.00	1.00	0.99	1.01	1.01	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
	RMSE	0.07	0.09	0.09	0.07	0.06	0.03	0.09	0.10	0.06	0.04	0.05	0.02	0.02	0.04	0.02	0.03	0.03	0.05	0.03	0.02
σ_{55}	True	1	1	1	1	1	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
	Median	1.00	0.99	0.99	1.00	1.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.01	4.00	4.00	4.00	4.01	4.01
	RMSE	0.07	0.10	0.09	0.08	0.05	0.02	0.16	0.14	0.08	0.07	0.03	0.01	0.02	0.07	0.02	0.02	0.02	0.09	0.07	0.02
σ_{12}	True	0	0	0	0	0	0	0	0	0	0	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
	Median	-0.01	-0.05	-0.04	-0.03	-0.02	0.00	-0.02	-0.02	-0.01	-0.01	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
	RMSE	0.07	0.06	0.05	0.04	0.04	0.05	0.03	0.02	0.02	0.03	0.07	0.02	0.01	0.03	0.01	0.05	0.03	0.02	0.01	0.01
σ_{13}	True	0	0	0	0	0	0	0	0	0	0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
	Median	0.00	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40
	RMSE	0.02	0.06	0.04	0.03	0.03	0.08	0.02	0.02	0.03	0.02	0.07	0.02	0.01	0.06	0.02	0.03	0.03	0.02	0.03	0.02
σ_{23}	True	0	0	0	0	0	0	0	0	0	0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
	Median	0.01	0.02	0.02	0.01	0.00	0.00	0.00	0.01	0.00	0.00	0.40	0.40	0.40	0.40	0.39	0.40	0.40	0.40	0.39	0.39
	RMSE	0.03	0.11	0.10	0.08	0.04	0.06	0.02	0.02	0.02	0.02	0.08	0.03	0.02	0.02	0.02	0.03	0.04	0.03	0.02	0.02
σ_{14}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.4	-0.4	-0.4	-0.4	-0.4
	Median	-0.02	-0.01	-0.03	-0.03	-0.01	0.00	-0.01	-0.02	-0.02	0.00	0.00	0.00	0.00	-0.00	-0.40	-0.40	-0.40	-0.40	-0.40	-0.40
	RMSE	0.06	0.08	0.09	0.06	0.04	0.06	0.03	0.04	0.03	0.04	0.07	0.02	0.02	0.02	0.03	0.02	0.04	0.02	0.03	0.03
σ_{24}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	-0.5	-0.5	-0.5	-0.5
	Median	-0.01	-0.03	-0.02	0.00	0.01	0.00	-0.01	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	-0.49	-0.50	-0.50	-0.50	-0.49	-0.49
	RMSE	0.03	0.14	0.09	0.06	0.03	0.07	0.03	0.03	0.03	0.04	0.10	0.02	0.01	0.02	0.02	0.02	0.02	0.03	0.03	0.02

See legend Table 1

Table 4: Simulation Results with $k = 5$ and $S = 1$

σ_{34}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.1	-1.1	-1.1	-1.1	-1.1
	Median	0.01	0.02	0.02	0.02	0.02	0.00	0.00	0.01	0.01	0.00	0.00	0.00	0.00	-0.00	-1.10	-1.10	-1.10	-1.10	-1.10	-1.10
	RMSE	0.04	0.10	0.10	0.07	0.05	0.06	0.03	0.03	0.04	0.03	0.05	0.01	0.01	0.03	0.01	0.03	0.03	0.03	0.02	0.01
σ_{15}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.9	-0.9	-0.9	-0.9	-0.9
	Median	-0.02	-0.06	-0.05	-0.05	-0.04	0.00	-0.01	-0.03	-0.03	-0.02	0.00	0.00	0.00	-0.02	-0.91	-0.90	-0.90	-0.91	-0.91	-0.91
	RMSE	0.05	0.11	0.10	0.07	0.06	0.05	0.04	0.05	0.05	0.03	0.07	0.02	0.02	0.03	0.03	0.03	0.03	0.03	0.05	0.03
σ_{25}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1
	Median	-0.01	-0.01	-0.01	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.00	-1.00	-1.00	-1.00	-1.00	-1.00
	RMSE	0.06	0.11	0.08	0.04	0.04	0.04	0.05	0.05	0.05	0.03	0.07	0.02	0.01	0.04	0.01	0.03	0.02	0.04	0.02	0.01
σ_{35}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.6	-1.6	-1.6	-1.6	-1.6	
	Median	0.00	0.01	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.60	-1.60	-1.60	-1.60	-1.61	-1.60
	RMSE	0.06	0.05	0.03	0.04	0.04	0.05	0.05	0.06	0.05	0.03	0.04	0.02	0.02	0.05	0.02	0.02	0.06	0.04	0.02	0.02
σ_{45}	True	0	0	0	0	0	0	0	0	0	0	2.55	2.55	2.55	2.55	2.55	2.55	2.55	2.55	2.55	2.55
	Median	0.00	-0.01	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.01	2.55	2.55	2.55	2.54	2.54	2.55	2.55	2.55	2.54	2.54
	RMSE	0.06	0.04	0.04	0.04	0.04	0.06	0.06	0.06	0.04	0.06	0.05	0.01	0.01	0.03	0.02	0.02	0.03	0.06	0.02	0.02
μ_1	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	0.63	0.49	0.80	0.11	0.28	0.06	0.10	0.73	1.84	0.40	0.00	0.02	0.00	0.06	-0.11	0.00	0.06	0.13	-0.03	-0.11
	RMSE	3.62	6.68	7.37	6.50	8.90	2.48	1.50	2.61	3.82	6.49	2.58	1.61	1.96	4.12	1.69	2.01	2.41	4.98	1.84	1.69
μ_2	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	0.72	1.65	1.13	-0.01	0.40	0.36	1.30	1.10	1.06	0.65	0.00	0.00	0.03	0.19	0.19	0.00	0.03	0.03	0.18	0.19
	RMSE	3.86	6.96	9.27	5.05	6.82	3.23	3.11	4.25	5.22	6.04	1.92	1.30	3.11	3.77	1.47	1.25	2.52	10.02	3.89	1.47
μ_3	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	0.00	-0.18	1.38	0.01	-0.04	0.22	0.19	0.53	0.44	0.27	0.00	0.00	0.01	0.00	0.81	0.00	-0.04	0.05	0.53	0.81
	RMSE	3.75	9.17	7.36	4.78	4.30	4.35	4.07	6.00	5.41	9.00	2.31	2.21	3.73	5.67	2.25	1.16	2.16	6.92	1.88	2.25
μ_4	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	0.81	1.25	0.44	-0.35	-0.21	0.00	0.98	1.07	0.43	0.00	0.00	0.03	0.00	-0.08	0.25	0.00	0.00	-0.04	0.16	0.25
	RMSE	3.53	9.71	6.94	4.89	6.57	5.04	8.08	11.97	7.64	17.28	3.94	2.14	4.75	2.34	1.46	1.40	2.01	8.13	1.59	1.46
μ_5	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
	Median	0.81	1.69	0.60	0.22	0.38	0.02	0.02	0.90	0.09	0.42	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.11	0.13	0.07
	RMSE	2.98	10.13	10.22	5.88	5.60	4.02	8.51	23.29	22.07	21.72	1.95	1.63	7.09	2.34	0.97	0.99	2.81	11.88	0.95	0.97

See legend Table 1

Table 5: Simulation Results with $k = 5$ and $S = 5$, cont.

α	True	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95	0.7	1.1	1.7	1.9	1.95
	Median	0.71	1.10	1.69	1.89	1.94	0.71	0.98	1.69	1.89	1.95	0.70	1.10	1.70	1.89	1.94	0.70	1.10	1.70	1.90	1.95
	RMSE	0.11	0.04	0.05	0.03	0.06	0.06	0.24	0.03	0.01	0.01	0.01	0.02	0.03	0.02	0.01	0.04	0.02	0.05	0.02	0.01
σ_{11}	True	1	1	1	1	1	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
	Median	1.00	1.00	0.98	1.00	1.02	0.25	0.24	0.26	0.25	0.28	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
	RMSE	0.08	0.08	0.06	0.07	0.08	0.03	0.03	0.06	0.05	0.07	0.01	0.01	0.02	0.03	0.01	0.02	0.02	0.02	0.06	0.01
σ_{22}	True	1	1	1	1	1	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5
	Median	1.01	1.00	1.00	1.03	1.04	0.50	0.50	0.53	0.51	0.51	0.50	0.50	0.51	0.50	0.50	0.50	0.50	0.50	0.51	0.50
	RMSE	0.07	0.07	0.07	0.07	0.09	0.06	0.04	0.08	0.04	0.03	0.02	0.02	0.03	0.03	0.01	0.02	0.01	0.02	0.07	0.01
σ_{33}	True	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
	Median	1.00	1.00	1.00	1.01	1.02	1.00	1.00	1.02	1.02	1.01	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	0.99	1.00
	RMSE	0.08	0.08	0.05	0.08	0.07	0.05	0.06	0.05	0.05	0.05	0.02	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.06	0.01
σ_{44}	True	1	1	1	1	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
	Median	1.00	1.01	1.01	1.04	1.02	2.00	2.00	2.05	2.02	2.01	2.00	2.01	2.01	2.00	2.00	2.00	2.00	2.00	1.98	2.00
	RMSE	0.08	0.09	0.07	0.09	0.07	0.02	0.09	0.10	0.07	0.06	0.02	0.02	0.04	0.04	0.03	0.03	0.02	0.04	0.13	0.02
σ_{55}	True	1	1	1	1	1	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
	Median	1.00	0.99	0.97	1.01	1.02	4.00	4.00	4.00	4.00	4.00	4.00	3.99	4.00	4.00	4.00	4.00	4.00	4.00	4.00	4.01
	RMSE	0.08	0.09	0.10	0.07	0.07	0.04	0.16	0.06	0.04	0.05	0.01	0.02	0.03	0.04	0.01	0.02	0.01	0.07	0.08	0.02
σ_{12}	True	0	0	0	0	0	0	0	0	0	0	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25
	Median	0.00	0.01	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.25	0.40
	RMSE	0.08	0.04	0.03	0.03	0.03	0.04	0.02	0.02	0.01	0.02	0.01	0.01	0.01	0.01	0.01	0.03	0.02	0.02	0.03	0.01
σ_{13}	True	0	0	0	0	0	0	0	0	0	0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
	Median	-0.01	-0.01	-0.01	0.00	0.00	0.00	0.00	-0.01	-0.01	0.00	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.39	0.40
	RMSE	0.10	0.05	0.03	0.03	0.03	0.04	0.03	0.02	0.02	0.02	0.01	0.01	0.01	0.02	0.01	0.04	0.03	0.03	0.04	0.01
σ_{23}	True	0	0	0	0	0	0	0	0	0	0	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4
	Median	0.00	0.00	0.00	0.01	0.01	0.00	-0.01	0.02	0.01	0.01	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40	0.40
	RMSE	0.09	0.08	0.07	0.06	0.05	0.05	0.02	0.02	0.02	0.02	0.01	0.02	0.02	0.02	0.02	0.06	0.02	0.02	0.05	0.01
σ_{14}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.4	-0.4	-0.4	-0.4	-0.4
	Median	0.00	-0.01	0.00	-0.01	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.40	-0.40	-0.40	-0.41	-0.40
	RMSE	0.10	0.07	0.07	0.07	0.05	0.06	0.03	0.01	0.01	0.02	0.01	0.01	0.01	0.02	0.02	0.05	0.02	0.03	0.04	0.01
σ_{24}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.5	-0.5	-0.5	-0.5	-0.5
	Median	0.01	0.01	0.02	0.02	0.02	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.50	-0.50	-0.50	-0.51	-0.50
	RMSE	0.11	0.06	0.06	0.06	0.05	0.06	0.05	0.02	0.02	0.02	0.01	0.01	0.01	0.03	0.01	0.02	0.02	0.02	0.05	0.01

See legend Table 1

Table 6: Simulation Results with $k = 5$ and $S = 5$

σ_{34}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.1	-1.1	-1.1	-1.1	-1.1
	Median	0.00	0.00	0.00	-0.01	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-1.10	-1.10	-1.10	-1.12	-1.10
	RMSE	0.11	0.08	0.06	0.07	0.04	0.07	0.04	0.03	0.03	0.03	0.01	0.02	0.02	0.03	0.03	0.04	0.02	0.02	0.08	0.01
σ_{15}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-0.9	-0.9	-0.9	-0.9	-0.9
	Median	0.00	0.03	0.03	0.02	0.01	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-0.90	-0.90	-0.90	-0.90	-0.90
	RMSE	0.09	0.08	0.07	0.08	0.04	0.08	0.04	0.02	0.02	0.02	0.01	0.01	0.02	0.03	0.02	0.03	0.04	0.03	0.04	0.01
σ_{25}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-1	-1	-1
	Median	0.00	-0.02	-0.02	-0.01	0.00	0.00	0.00	0.00	-0.01	0.00	0.00	0.00	0.00	-0.01	-0.01	-1.00	-1.00	-1.00	-1.00	-1.01
	RMSE	0.08	0.07	0.06	0.06	0.05	0.05	0.09	0.03	0.02	0.02	0.01	0.01	0.02	0.03	0.03	0.03	0.01	0.03	0.04	0.02
σ_{35}	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1.6	-1.6	-1.6	-1.6	-1.6
	Median	0.00	0.00	-0.01	0.00	-0.01	0.00	0.00	-0.03	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	-1.60	-1.60	-1.60	-1.61	-1.60
	RMSE	0.10	0.03	0.03	0.09	0.03	0.06	0.07	0.05	0.04	0.03	0.02	0.02	0.03	0.05	0.04	0.03	0.02	0.02	0.07	0.01
σ_{45}	True	0	0	0	0	0	0	0	0	0	0	2.55	2.55	2.55	2.55	2.55	2.55	2.55	2.55	2.55	2.55
	Median	0.00	0.00	0.01	0.01	0.01	0.00	0.01	0.00	0.01	0.00	2.55	2.55	2.55	2.55	2.55	2.55	2.55	2.55	-2.55	2.54
	RMSE	0.07	0.04	0.03	0.03	0.04	0.04	0.06	0.05	0.07	0.06	0.02	0.03	0.02	0.02	0.03	0.05	0.02	0.04	0.11	0.02
μ_1	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	0.88	0.78	1.46	0.52	0.01	0.63	0.37	0.95	0.38	-0.04	0.11	0.01	0.22	0.16	0.28	0.09	0.01	0.14	-0.26	-0.03
	RMSE	3.99	5.12	8.28	9.15	6.24	1.76	1.76	2.28	2.05	4.22	0.89	1.27	2.11	3.05	3.59	2.04	1.96	4.23	6.65	1.91
μ_2	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	0.74	0.15	1.18	-0.19	-0.11	0.00	0.24	-0.19	0.30	0.29	-0.11	0.00	-0.05	0.07	0.10	0.08	-0.02	-0.13	-0.94	0.35
	RMSE	3.51	4.53	8.06	3.87	7.51	2.93	2.72	3.58	4.66	5.77	1.36	1.80	3.32	4.68	5.56	1.68	2.81	6.16	8.31	2.96
μ_3	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	-0.13	-0.37	-1.06	-0.19	-0.18	-0.05	-0.32	-0.08	0.05	0.24	0.04	-0.02	-0.31	0.01	-0.06	0.00	0.00	-0.03	-1.55	0.14
	RMSE	3.10	5.11	7.31	6.74	6.45	3.98	4.91	5.56	5.35	7.73	1.79	2.72	3.66	6.22	6.78	2.92	2.56	6.41	14.69	3.02
μ_4	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	1.32	1.41	3.18	0.17	0.22	0.16	2.96	0.19	0.20	0.21	0.30	0.71	0.54	-0.23	-0.02	0.18	0.11	0.16	-0.87	0.05
	RMSE	4.23	5.72	10.94	6.97	11.02	5.14	10.16	7.47	6.98	16.06	1.31	1.99	6.69	2.14	2.78	2.06	1.61	7.10	11.91	2.08
μ_5	True	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	Median	0.08	-0.32	-0.23	-0.14	-0.18	-0.12	-0.04	-0.30	0.07	-0.01	-0.14	-0.36	-0.34	0.16	0.09	-0.14	-0.04	-0.03	-0.82	0.07
	RMSE	4.04	8.76	7.91	7.22	7.50	2.87	8.80	8.31	2.24	12.45	1.04	2.42	9.46	1.49	2.10	1.27	2.25	10.74	10.30	1.43

See legend Table 1