“Almost” subsidy-free spatial pricing in a multi-dimensional setting*

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Abstract

Consider a population of citizens uniformly spread over the entire plane, that faces a problem of locating public facilities to be used by its members. The cost of every facility is financed by its users, who also face an idiosyncratic private access cost to the facility. We assume that the facilities' cost is independent of location and access costs are linear with respect to the Euclidean distance. We show that an external intervention that covers 0.19% of the facility cost is sufficient to guarantee *secession-proofness* or *no cross-subsidization*, where no group of individuals is charged more than its stand alone cost incurred if it had acted on its own. Moreover, we demonstrate that in this case the Rawlsian access pricing is the only secession-proof allocation.

Keywords: Secession-proofness, optimal jurisdictions, Rawlsian allocation, hexagonal partition, cross-subsidization.

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1 Introduction

Consider a society that faces a problem of locating one or several public facilities (or public projects as in Mas-Colell, 1980) to serve its members. The facilities, say libraries, are to be located on the plane and could be visited by citizens at some private "transportation" cost related to distance between their residence and the facility they are assigned to. Assuming that setting up and operating a facility entails a fixed set-up and operational cost, the following problems arise:

- how many facilities should be built;
- where to locate the facilities;
- how to assign citizens to the facilities;
- how to allocate the facilities costs (in the form of access fees) to citizens-users.

In this paper we examine the case where

- the demand for use of services is uniformly distributed over the plane, independently of the cost of services;

- the cost of setting up a facility is independent of location;

- access cost is proportional to Euclidean distance.

We assume that for any number and location of facilities, assignment of users to facilities and access fees, all citizens-users enjoy a "free entry" option: *any group can build a new facility for their own benefit at the standard fixed cost, and locate it at will.* A threat of free entry leads us imperatively to impose the "secession-proofness" or "core" property: at equilibrium, no group of users should be able to benefit by seceding from the proposed arrangement to set up and operate its own facility.[1] The secession-proofness also can be considered as a "no cross-subsidization" condition where no group of users is required to contribute more than its stand-alone cost. In other words, the equilibrium cost allocation should ensure the voluntary participation of any group of citizens.

The secession-proofness immediately yields the total cost minimization requirement: *the society should minimize the total burden of setting up and operating all facilities and of the aggregate access*

[1] Since a geographical area served by a public facility can be identified as a political jurisdiction, the secession-proofness could then be viewed as a requirement of political stability.

fees of all citizens. This requirement, in turn, leads to two simple but important observations: (i) since the facilities costs are location-independent, every group of citizens assigned to the same facility should place it at the "central" location that minimizes the aggregate access costs of that group;[2] (ii) every citizen should be assigned to the facility closest to her residence. These observations allow us to reduce the societal problem described above to finding a partition of the society into disjoint groups of users, called *jurisdictions*, that, under (i) and (ii), satisfies the total cost minimization requirement. Any such partition will be called *efficient*.

The characterization of efficient partitions in this geometric setting is a well-documented problem in mathematics. An efficient partition consists of identical regular hexagons,[3] whose size is calculated as a function of the ratio of fixed costs per facility to access costs per unit of distance. However, the area over which total costs per user are minimized is not a hexagon but a *disk*! Since the plane cannot be partitioned into disks, this helps to explain the first result of this paper that demonstrates that the set of secession-proof allocations is empty. This simply means that it is impossible to allocate facilities' cost over hexagons in an efficient partition so as to rule out a threat of secession by all disk-shaped jurisdictions.

The non-existence of secession-proof allocations implies that the stability can be ensured only at some cost. We consider the situation where an external source is willing to finance a fraction δ of the total cost incurred by jurisdictions if they follow the prescribed agreement. Suppose that the total costs (set-up plus operation plus access fees) for the jurisdiction-to-be at the proposed equilibrium are subject to the discount factor $1-\delta$, whereas forming a new jurisdiction to set up and operate an independent facility requires a full non-subsidized cost. Then the allocation will be δ-secession-proof if the savings reaped by the seceding jurisdiction fall short of the subsidy obtained by members of that jurisdiction at the proposed access fee allocation.

We then turn to the examination of the minimal subsidy that can rectify stability failure. By using Fubini's theorem, we demonstrate that the set of δ-secession-proof allocations is non-empty if and only if the value of the subsidy, δ is no less than the threshold $\delta^* \approx 0.0019$. This value is

[2]It is easy to see that such a location is uniquely defined for every group of citizens.

[3]See discussion below.

actually determined by the ratio $1-\delta^*$ of total per capita costs to users in an optimal disk and in a optimal hexagon. A tiny value of δ^* (less than 0.2%) lends credence to the δ-secession-proofness as a stability concept.

The second result of the paper is the characterization of the δ^*-secession-proof allocations. It is provided by the so-called Rawls principle that requires the minimization of the total cost incurred by the least privileged citizen-user and produces the complete equalization of total cost for all citizens-users. A transparent characterization of that principle requires to subject access to a fee that declines linearly with the residence-to-facility distance and to adjust access fees so that operators of the facilities break even.

The Rawls principle defines uniquely the δ^*-secession-proof allocation. (For higher values of δ, the set of δ-secession-proof allocations contains other allocations as well.) The Rawlsian policies are often advocated on the basis of *justice* considerations, whereas our result offers a *stability* argument in support of the Rawls principle.

Related Literature. To the best of our knowledge, the related literature in economics deals almost exclusively with the uni-dimensional case. Cremer, de Kerchove and Thisse (1985), Alesina and Spolaore (1997) examine the existence of secession-proof allocations in the case where the population is uniformly spread over a *bounded* interval and the unique cost share rule available for each jurisdiction is the *equal-share* scheme, according to which all citizens in the same jurisdiction are subject to an identical access fee. Casella (2001) studies a model where individuals are uniformly distributed over the circle. Le Breton and Weber (2003) and Haimanko, Le Breton and Weber (2004) address the existence and characterization of secession-proof access fee allocations in the case of general distributions. Bogomolnaia et al. (2005a,b) examine the issue of secession-proof allocations under various notions of stability. Drèze, Le Breton and Weber (2005) prove that the Rawlsian distribution is the unique secession-proof allocation in the case where the population is uniformly spread over the entire real line. Thus, in the uni-dimensional setting δ^* is equal to zero and there is no gap between efficiency and optimality so that an efficient partition consists of optimal-size intervals. Notice also that our set-up is that of "horizontal differentiation" where individuals display distinct preferences over geographical locations of public facilities. This is in contrast to the "vertical differentiation" framework, where individuals exhibit identical preferences over quantity or quality attributes of public projects.[4]

The paper is organized as follows. The next section contains the model and introduces the needed definitions. The main results, whose proofs are relegated to the Appendix, are stated in Section 3.

2 The Model

We consider a society with a continuum of individuals that has to determine a partition into multiple groups (jurisdictions). Each jurisdiction has to be assigned a public facility accessible to its members and an allocation of access fees to share the facility cost among them. The facilities will be located in a multi-dimensional space:

Assumption A.1 — Multidimensionality: The space of facilities' locations is the two-dimensional Euclidean space $X = \Re^2$.

Citizens have idiosyncratic preferences over possible facilities they could be assigned to. We assume that for every individual the access cost is represented by the Euclidean distance from her residence to the facility in her jurisdiction:

Assumption A.2 — Euclidean access costs: For every individual located at $l = (l_1, l_2) \in X$, her accession cost to every $t = (t_1, t_2) \in X$ is given by

$$||t - l|| = \sqrt{|t_1 - l_1|^2 + |t_2 - l_2|^2}. \tag{1}$$

This formalization allows us to identify an individual with her location and to characterize the society by the distribution of individuals' locations. We assume that the citizens are uniformly distributed over the entire space X:

[4]See e.g., Westhoff (1977), Guesnerie and Oddou (1981, 1988), Wooders (1978, 1980), Guesnerie (1995), Weber and Zamir (1985), Greenberg and Weber (1986), Jéhiel and Scotchmer (1997, 2001), Konishi, Le Breton and Weber (1998).

Assumption A.3 — Uniform distribution: The citizens' distribution is given by the two-dimensional Lebesgue measure[5] λ over $\Re^2$.

The area of a measurable[6] set S will be denoted by $\lambda(S)$, i.e., $\lambda(S) = \int_S dt$. In what follows, the null-measured sets with $\lambda(S) = 0$ will be disregarded, so that the qualification "up to a null-set" should be added to almost all our results.

In our set-up, every jurisdiction is a measurable bounded subset of X with positive measure. The collection of such sets will be denoted by $\mathcal{M}(X)$. We assume that the cost of each facility is fixed:

Assumption A.4 — Facility cost: The cost of every facility is given by the positive value g.[7]

We now formally introduce the notion of a partition of a measurable subset $S \subset X$:

Definition 2.1: A partition P of a (possibly infinite-measured) set S is a jurisdiction structure that consists of sets from $\mathcal{M}(X)$ which are "almost" pairwise disjoint: $\lambda(T \bigcap T') = 0$ for all $T \neq T'$ in P, and whose union covers the entire set S: $\bigcup_{T \in P} T = S$. The set of partitions of S is denoted by $\mathcal{P}(S)$. Obviously, if the measure of S is infinite, then every $P \in \mathcal{P}(S)$ consists of an infinite number of jurisdictions.

Now let us turn to the determination of facility choices. For each $S \in \mathcal{M}(X)$ and a location $l \in X$ we denote by $D(S, l)$ the value of total access cost in S (with respect to location l):

$$D(S, l) = \int_S ||t - l|| dt. \tag{2}$$

In what follows, the efficiency requires that the facility location in each jurisdiction S would minimize the total access cost of its members. That is, each jurisdiction S is assigned to the facility located at m, which is a solution of the following problem:

$$D(S, m) = \min_{l \in X} D(S, l). \tag{3}$$

[5]See Halmos (1950), p. 153.

[6]A subset of X is measurable if its intersection with every measurable subset of a finite measure is measurable; hence, we allow for infinite-measured measurable subsets.

[7]This assumption can be easily generalized as to incorporate a variable operational cost proportional to the jurisdiction size. That is, we may assume that the facility cost $f(S)$ assigned to jurisdiction S is given by $f(S) = g + \alpha\lambda(S)$, where α is a positive constant.

The value of this problem is called "$MAT(S)$" (Minimal Aggregate Transportation cost of the set S).[8] A solution to (3) is called a *central location* of T.[9] We use the following lemma:

Lemma 2.2: For every jurisdiction $S \in \mathcal{M}(X)$, the central location, denoted by $m(S)$, is unique.[10]

Lemma 2.2 resolves the issue of an efficient facility location choice for every jurisdiction. Thus, we denote by $D(S)$ the aggregate access cost of members of S:

$$D(S) = D(S, m(S)). \tag{4}$$

Every measurable set $S \subset X$ can be partitioned into several jurisdictions. We define the stand alone average total cost[11] in S as the minimum over all possible partitions P of S:

$$K(S) = \inf_{P} \frac{\sum_{T \in P} [D(T) + g]}{\lambda(S)}. \tag{5}$$

We have

Definition 2.3: A partition P is S-efficient if it is a solution to (5). An X-efficient partition will be simply called an efficient partition.

From now on, we will focus our analysis on efficient partitions. The characterization of efficient partitions in our geometric setting is a well documented problem in mathematics. The qualitative result (re)discovered by many authors states that there is a unique "shape" of efficient partitions which consists of identical regular hexagons.[12] We have:

Result 2.4: Partition P is efficient if and only if it is comprised of identical regular hexagons, whose stand-alone value is minimal among all regular hexagons.

[8] A solution to this problem exists. Indeed, the integral in (2) is continuous in l, and for $l \to \infty$ the value of the program goes to $+\infty$.

[9] Note that in the unidimensional setting, for every bounded set T, a location is central if and only if it is a median of T.

[10] This result is essentially multi-dimensional: obviously, in the unidimensional setting a set may have a continuum of central points.

[11] Since the total cost for an infinite-measured set is infinite, in this case we will take a limit of the sequence of sets that uniformly approach S.

[12] See Fejes Toth (1953), Haimovich and Magnanti (1988), Morgan and Bolton (2002), as well as Christaller (1933), Lösch (1954), Bollobas and Stern (1972), and Stern (1972) in the economic geography context.

The size of hexagons in efficient partitions obviously depends upon the value of the fixed component of facility costs: the smaller the cost, the smaller are jurisdictions in an efficient partition. The size of "efficient" hexagons is explicitly derived in the Appendix.

Let us turn to the examination of accession fees. In every potential jurisdiction $S \in \mathcal{M}(X)$, a sharing rule y^S is a measurable function on S that specifies accession fees of members of S, if this jurisdiction forms. We impose the budget-balancedness condition:

Assumption A.5 — Budget balancedness: The accession fees of members of S cover the costs of the facility:

$$\int_S y^S(t)dt = g. \tag{6}$$

Now consider sharing rules defined on the entire set X. Given an efficient partition P, this amounts to an assignment of a sharing rule to every hexagon H in P. It would be natural to consider the notion of a *consistent sharing rule*. Indeed, since the whole plane is partitioned into identical (hexagonal) jurisdictions, it makes sense to demand that the individuals in identical locations within different jurisdictions bear the same costs. We impose a weak form of consistency that requires that any two individuals t and t' in two different jurisdictions H and H', whose location is identical with respect to their corresponding central points $m(H)$ and $m(H')$, are assigned the same accession fees.[13] Formally,

Assumption A.6 — Consistent sharing rule: For every efficient partition P, every two different (hexagonal) jurisdictions $H, H' \in P$ and every two individuals $t \in H, t' \in H'$ satisfying $t - m(S) = t' - m(S')$, we have $y^H(t) = y^{H'}(t')$.

The sharing rule y associated with an efficient partition P determines the following cost allocation c^P that identifies the total cost for every individual $t \in X$

$$c^P(t) = y(t) + ||t - m(H^t)||, \tag{7}$$

where $H^t \in P$ is the hexagon in P (with the center at $m(H^t)$) that contains t.

[13]The assumption can be dispensed with at the cost of substantially lengthening our proof.

The cost sharing rule chosen by the society satisfies a requirement of *voluntary participation* when no group of individuals should contribute more than the cost incurred if it had acted on its own. Thus, the formation of jurisdictions and the allocation of accession fees within each of them rules out the emergence of a potentially seceding group that can benefit all its members. Formally,

Definition 2.5: Let an efficient partition P and a cost allocation c^P be given. A set $S \in \mathcal{M}(X)$ is *prone to secession* if

$$c^P(S) = \frac{1}{\lambda(S)} \int\limits_S c(t)dt > K(S). \tag{8}$$

A cost allocation c^P is *secession-proof* if no set $S \in \mathcal{M}(X)$ is prone to secession. The set of secession-proof cost allocations on X will be denoted by $\mathcal{A}^P$.

The next definition introduces the allocations that satisfy the *Rawls principle* that requires the minimization of the total cost of the most disadvantaged individual in each jurisdiction. It implies the *cost equalization* across the entire society:

Definition 2.6: A cost allocation r^P is called *Rawlsian* if the value $r^P(t)$ is constant within each $H \in P$, and, hence, on X. That is, for every $t, t' \in X$ we have $r^P(t) = r^P(t')$.

3 Results

To state the main results of the paper, let us fix one of the (fully equivalent to each other) efficient partitions, say P^*. Then we can use the notation $c(t)$, $r(t)$, $\mathcal{A}$ instead of $c^P(t)$, $r^P(t)$, $\mathcal{A}^P$, for cost allocations, the Rawlsian allocation and the set of secession-proof allocations, respectively, that are associated with the partition P^* into optimal-sized hexagons.

First, we demonstrate that under our assumptions, a secession-proof allocation fails to exist.

Proposition 3.1: Suppose that assumptions A.1-A.6 hold. Then the set of secession-proof allocations $\mathcal{A}$ is empty.

In absence of secession-proof allocations, we will turn to the search for a solution which is the "closest" to be secession-proof. For instance, we may assume that there is a fixed per capita secession cost for any subgroup $S \subset X$; alternatively, one can consider government intervention to subsidize a certain fraction of the total cost of every citizen to prevent the formation of groups prone to secession. Both approaches are essentially equivalent and yield the following definition of δ*-secession-proofness*:

Definition 3.2: Let $\delta > 0$ be given. A cost allocation c is δ*-secession-proof* if for all $S \in \mathcal{M}(X)$ the following inequality holds:

$$(1-\delta)c(S) \leq K(S). \tag{9}$$

The set of δ-secession-proof allocations for X will be denoted by $\mathcal{A}(\delta)$.

In other words, if individuals follow the prescribed agreement, then the δ-part of their total cost is covered "from outside". If, however, a jurisdiction wants to secede, then its members will have to bear costs on their own.[14]

Definition 3.2 relaxes the constraints which determine secession-proof allocations and, obviously, if δ is sufficiently large, the set $\mathcal{A}(\delta)$ is nonempty. Moreover, if $\mathcal{A}(\delta)$ is nonempty for some δ, it is also the case for all $\delta' \geq \delta$. Thus, we obtain the existence of a threshold value δ^*:

$$\delta^* = \inf\{\delta > 0 | \ \mathcal{A}(\delta) \neq \emptyset\}. \tag{10}$$

It will be shown that the set $\mathcal{A}(\delta^*)$ is itself nonempty. The value δ^* therefore can represent the *cost of stability*, which is the minimal per-capita subsidy which sustains secession-proofness. We can now state our main result:

Proposition 3.3: Under Assumptions A.1-A.6,

(i) $\delta^* \approx 0.0019$;

(ii) The set $\mathcal{A}(\delta^*)$ is a singleton which consists of the Rawlsian allocation.

That is, the cost of stability δ^* is very small. Moreover, the only δ^*-secession-proof allocation is Rawlsian.

[14] Note that alternatively we could have considered the situation where the outside subsidy is applied to the facility costs alone rather than to total costs, including the facility and access costs. This change would require only a slight modification of the notion of a δ-secession-proof allocation in (9) and the subsequent adjustment of the threshold value of δ^* defined below. Otherwise, the main results would not be affected.

The statement of this proposition requires some explanation. Consider a hexagon H, which is an element of an efficient partition. Obviously, this hexagon is not optimal in terms of per capita cost of its members and the value $K(H)$ exceeds

$$\min_{S \in \mathcal{M}(X)} K(S). \tag{11}$$

In fact, no hexagon represents a solution for (11), and, unsurprisingly, jurisdictions with the minimal per capita total cost are disks. Denote by $K(B)$ the value of the problem in (11). We then show that the cost of stability δ^* is given by

$$\delta^* = 1 - \frac{K(B)}{K(H)}, \tag{12}$$

which is obviously positive since $K(B) < K(H)$. Thus, the cost gap between an efficient hexagon and an optimal disk necessitates government intervention and subsidization of efficient partitions. It is important to point out that this feature does not appear in the uni-dimensional setting where efficient and optimal jurisdictions are intervals of the same size and the stability cost is zero (see Drèze, Le Breton and Weber (2005)).

Let us now provide an intuition for the result that the δ^*-secession-proofness yields only the Rawlsian allocation. Note that the Rawlsian allocation assigns to every individual in X the total cost of $K(H)$ derived from an optimal-sized hexagon. Consider an arbitrary δ^*-secession-proof allocation $c(\cdot)$ and estimate the number of individuals whose cost is "substantially" below the level $K(H)$. First, one can show that, under c the total costs over every optimal-sized disk should be the same and equal to $K(B)$, or else there would exist a disk-shaped potential jurisdiction whose members' total burden exceeds their stand-alone cost. Furthermore, take a disk B with the optimal radius l^* which contains a disk B' with the same center and a smaller radius $l^* - \gamma$, where γ is a small positive number. It turns out that the number of individuals in the ring $R = B \setminus B'$ whose total cost is substantially below $K(H)$ is "negligible" and is represented by the second degree term with respect to γ. Finally, we fix a rectangular in X and show that it can be covered by the rings of the type R and the number of such rings in the cover is of the order $\frac{1}{\gamma}$. Since R could be chosen arbitrarily thin, i.e., γ could be made sufficiently small, we conclude that the measure of the citizens whose cost is less than the society average is zero. Thus, c must satisfy the Rawls principle.

4 Appendix

Proof of Lemma 2.2: Let $S \in \mathcal{M}(X)$ be given and assume that S has two different central points, m and m'. Let L be the straight line connecting m and m'. Denote $S' = S \setminus L$ and $\bar{m} = \frac{m+m'}{2}$. Obviously m and m' are central points of S' as well and $D(S) = D(S')$. Then for every $t \in S'$ we have

$$\frac{1}{2}\left(||t-m|| + ||t-m'||\right) > ||t-\bar{m}|| \tag{13}$$

and, since $\lambda(S) = \lambda(S') > 0$, this implies that

$$\int\limits_{S'} ||t-\bar{m}||dt < \frac{1}{2}\left(\int\limits_{S'} ||t-m||dt + \int\limits_{S'} ||t-m'||dt\right). \tag{14}$$

However, by (2) and (3), the right-hand side of (14) is equal to $D(S) = D(S')$, a contradiction to m and m' being central points of S'. $\square$

Before proceeding with the proof of Propositions 3.1 and 3.3, we need a notation to state some preliminary results.

Lemma A.1: A set S is a solution of (11) if and only if S is a disk of the radius l^*, where the value of l^* is given by

$$l^* = \left(\frac{3g}{\pi}\right)^{\frac{1}{3}} \approx 0.985 g^{\frac{1}{3}}. \tag{15}$$

Moreover, the per capita cost, $K(B)$, in such a disk is equal to l^*.

Denote by B_a^l the disk with the center at $a \in X$ and the radius $l > 0$. The disk of the optimal size l^* and center a will be referred to as simply B_a. Denote the disk of radius $l > 0$ with the center at $m(S)$ by B^l.

Proof: Take a set S that solves (11). It is easy to see that there exist two nonnegative numbers l_1, l_2 with $0 \leq l_1 \leq l_2 < \infty$ such that both $B^{l_1} \setminus S$ and $S \setminus B^{l_2}$ are null-sets, and two sets, $B^l \setminus S$ and $S \setminus B^l$, have a positive measure for all $l \in (l_1, l_2)$. We claim that $l_1 = l_2$, i.e., $S = B^{l_1} = B^{l_2}$.

If not, take $l_3 = (2l_1 + l_2)/3$ and $l_4 = (l_1 + 2l_2)/3$. Then both $\lambda(S \setminus B^{l_4})$ and $\lambda(B^{l_3} \setminus S)$ are positive numbers. Shift a positive mass of individuals from $S \setminus B^{l_4}$ to $B^{l_3} \setminus S$ to guarantee that the newly created set $\tilde{S}$ has the same measure as S. However,

$$D(\tilde{S}) = \int\limits_{\tilde{S}} ||p - m(\tilde{S})||dp \leq \int\limits_{\tilde{S}} ||p - m(S)||dp < D(S), \tag{16}$$

a contradiction to S being a solution of (11).

It is left to derive l^* and $K(B)$. Notice that for every disk B^l, the total access cost $D(B^l) = \frac{2\pi l^3}{3}$. Since the area of B^l is πl^2, the average cost within B^l is $K(B^l) = \frac{g}{\pi l^2} + \frac{2l}{3}$. It is straightforward to verify that the last expression attains its minimum at

$$l^* = \left(\frac{3g}{\pi}\right)^{\frac{1}{3}}, \tag{17}$$

yielding the minimal average cost $K(B) = l^*$.□

We will utilize the lemma that evaluates the average cost of jurisdictions that are "close" to optimal disks:

Lemma A.2: Let $\gamma > 0$ be sufficiently small and the set S be located between two disks with the same center, $B_a^{l^*-\gamma}$ and $B_a^{l^*}$, i.e., $B_a^{l^*-\gamma} \subset S \subset B_a^{l^*}$. Then $K(S)$, the aggregate average cost over S, differs from the aggregate average cost over optimal disk $K(B)$ only in the second order term:

$$K(S) < l^* + \frac{4}{l^*}\gamma^2. \tag{18}$$

Proof: Let $\tilde{S} = S \bigcap \left(B_a^{l^*} \setminus B_a^{l^*-\gamma}\right)$. In our derivations below we take into account that the total access cost within S (weakly) increases if we replace the $m(S)$ by a, and that the distance between any point in $\tilde{S}$ to a is bounded from above by l^*. Denote $z = \frac{3}{\pi}\lambda(\tilde{S})$. By utilizing (17) we have:

$$\begin{gathered} K(S) = \frac{g + D(S)}{\lambda(S)} \leq \frac{g + \int\limits_S ||a - t||dt}{\lambda(S)} \leq \frac{g + D(B_a^{l^*-\gamma}) + l^*\lambda(\tilde{S})}{\lambda(B_a^{l^*-\gamma}) + \lambda(\tilde{S})} = \\ \frac{(l^*)^3 + 2(l^* - \gamma)^3 + zl^*}{3(l^* - \gamma)^2 + z} < \frac{3(l^*)^3 - 6(l^*)^2\gamma + 6l^*\gamma^2 + zl^*}{3(l^* - \gamma)^2 + z} = \\ \frac{3l^*(l^* - \gamma)^2 + zl^*}{3(l^* - \gamma)^2 + z} + \frac{3l^*\gamma^2}{3(l^* - \gamma)^2 + z} < l^* + \frac{3l^*\gamma^2}{3\left(l^*/2\right)^2} = l^* + \frac{4}{l^*}\gamma^2, \end{gathered} \tag{19}$$

as for γ small enough $l^* - \gamma > \dfrac{1}{2} l^*$.□

Lemma A.3: Let H be a hexagon in an efficient partition. The per capita cost over H is given by

$$K(H) = \frac{\sqrt{3}}{2}\left(\frac{2}{3} + \ln\sqrt{3}\right)^{\frac{2}{3}} g^{\frac{1}{3}} \approx g^{\frac{1}{3}}. \tag{20}$$

Proof: Consider a regular hexagon H_l, where l denotes the distance between the center $m(H_l)$ and a midpoint of its side. The total access cost in H_l is

$$\begin{aligned} D(H_l) &= 12 \int_0^l \int_0^{\frac{x}{\sqrt{3}}} \sqrt{x^2+y^2}\,dxdy = 6\int_0^l \left[y\sqrt{x^2+y^2} + x^2 \ln\left(y + \sqrt{x^2+y^2}\right)\right]_0^{\frac{x}{\sqrt{3}}} dx \\ &= 6\int_0^l \left[\tfrac{x}{\sqrt{3}}\sqrt{x^2+\tfrac{x^2}{3}} + x^2 \ln\left(\tfrac{x}{\sqrt{3}} + \sqrt{x^2+\tfrac{x^2}{3}}\right) - x^2 \ln x\right] dx \\ &= 6\int_0^l x^2 \left[\tfrac{2}{3} + \ln\sqrt{3}\right] dx = 2l^3\left[\tfrac{2}{3} + \ln\sqrt{3}\right]. \end{aligned} \tag{21}$$

Since the area of H_l is $2\sqrt{3}l^2$, the average cost per citizen in jurisdiction H_l is given by

$$K(H_l) = \frac{g}{2\sqrt{3}l^2} + \frac{l}{\sqrt{3}}\left[\frac{2}{3} + \ln\sqrt{3}\right], \tag{22}$$

which attains its minimum at the efficient hexagon $H_{\tilde{l}}$, where

$$\tilde{l} = \left(\frac{2}{3} + \ln\sqrt{3}\right)^{-\frac{1}{3}} g^{\frac{1}{3}}. \tag{23}$$

By substituting the expression for $\tilde{l}$ into (22), it is easy to verify that the value $K(H) = K(H_{\tilde{l}})$, which represents the average cost over the whole plane X under an efficient partition, coincides with K_H, determined by (20). □

Take the efficient partition P^* of X. For every positive integer N, consider a subset G_N of P^* that consists of N^2 adjacent hexagons (see Figure 1). Let the sequence $\{G_N\}_{N=1,\ldots,\infty}$ be nested, i.e., each G_N is imbedded into G_{N+2} "symmetrically", such that the set $G_{N+2} \setminus G_N$ is a "hexagonal ring" comprised of $4N+4$ regular hexagons (see Figure 1b for $N=2$).

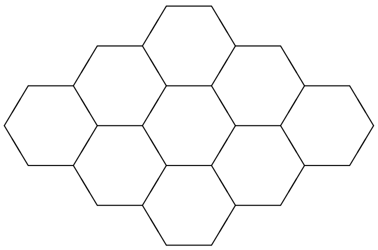

Figure 1a. Hexagonal rectangular
G_N for $N = 3$

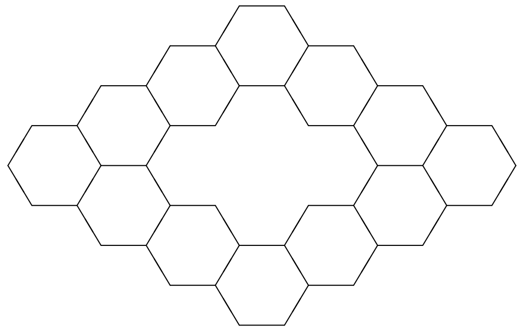

Figure 1b. Hexagonal ring
$G_{N+2} \setminus G_N$ for $N = 2$

Figure 1. Hexagonal rectangular and hexagonal ring

We have the following result:

Lemma A.4: For every $a \in G_N$, the disk B_a is contained in G_{N+2}.

Proof: Denote by $\bar{l}$ the side of a hexagon in partition P^*. Since the minimal width of the hexagonal ring F_N is equal to $\bar{l}$, it suffice to demonstrate that $\bar{l} > l^*$. Note that $\bar{l} = \frac{2}{\sqrt{3}}\tilde{l}$, where $\tilde{l}$ is the distance between the center of the efficient hexagon and a middle point of one of its sides, the length of which has been derived in (23). Thus,

$$\bar{l} = \frac{2}{\sqrt{3}} \left(\frac{2}{3} + \ln \sqrt{3} \right)^{-\frac{1}{3}} g^{\frac{1}{3}}, \tag{24}$$

which, by (15), exceeds the value l^*. $\square$

Let the efficient partition P^* be endowed with the sharing rule y, that generates cost allocation c, and H is a (hexagonal) jurisdiction in P^*. Denote by λ^H the Lebesgue measure of H and by λ^B the Lebesgue measure of an optimal disk.

For every $a \in X$ denote by the value $\varphi(a)$ the aggregated cost incurred by the members of the disk B_a:

$$\varphi(a) = c(B_a) = \int\limits_{B_a} c(t)dt \tag{25}$$

Define $\bar{\varphi}$ as the aggregated cost incurred by the allocation c on all disks of optimal size whose centers belong to the hexagon H:

$$\bar{\varphi} := \int\limits_H \varphi(a)da. \tag{26}$$

Note that, due to the consistency assumption A.6, the value $\bar{\varphi}$ is invariant to a choice of a hexagon in P^*. We need the following result:

Lemma A.5:

$$\bar{\varphi} = I, \quad \text{where} \quad I := \lambda^B \int\limits_H c(t)dt. \tag{27}$$

Proof: Define the function $\Psi(a,t)$ on $G_N \times G_{N+2} \subset \Re^4$ by

$$\Psi(a,t) = \begin{cases} c(t), & \text{if} \quad t \in B_a; \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

We will integrate the function $\Psi(a,t)$ over the set $G_N \times G_{N+2}$. According to Fubini's theorem (Halmos (1950), p.148), two different orders of integration yield the same result. First, we integrate with respect to t and then to a. By (25) and (26), and using Lemma A.4 we have

$$\int\limits_{G_N} \left[\int\limits_{G_{N+2}} \Psi(a,t)dt \right] da = \int\limits_{G_N} \left[\int\limits_{B_a} c(t)dt \right] da = \int\limits_{G_N} \varphi(a)da = N^2 \int\limits_H \varphi(a)da = N^2\bar{\varphi}. \tag{29}$$

Before integrating in the reverse order, note that the following duality property

$$\{a | t \in B_a\} \equiv B_t \tag{30}$$

holds for every $t \in X$. This is a simple consequence of the symmetry of the distance $||t-p||$ as a function of two arguments, and the circle B_t being the set of points p for which $||p-t|| = ||t-p|| \le l^*$. Take a point $t \in G_{N-2}$. By Lemma A.4, $B_t \subset G_N$, and

$$\int\limits_{G_N} \Psi(a,t)da = \int\limits_{B_t} c(t)da = c(t) \int\limits_{B_t} da = \lambda^B c(t). \tag{31}$$

We have:

$$\int\limits_{G_{N+2}} \left[\int\limits_{G_N} \Phi(a,t)da \right] dt = \int\limits_{G_{N-2}} \left[\int\limits_{G_N} \Phi(a,t)da \right] dt + L_N, \tag{32}$$

where

$$L_N := \int\limits_{G_{N+2}\setminus G_{N-2}} \left[\int\limits_{G_N} \Phi(a,t)da \right]. \tag{33}$$

By using (31), the first term in (32) can be presented as:

$$\int\limits_{G_{N-2}} \left[\int\limits_{G_N} \Phi(a,t)da \right] dt = \int\limits_{G_{N-2}} \lambda^B c(t)dt = (N-2)^2 I. \tag{34}$$

Fubini's theorem allows us to rewrite (32) as

$$N^2\bar{\varphi} = (N-2)^2 I + L_N = N^2 I + L_N - 4(N-1)I. \tag{35}$$

Let us estimate the absolute value of the last two terms. Since for any $t \in G_{N+2}$, hence, for any $t \in G_{N+2} \setminus G_{N-2}$, we have that $\int\limits_{G_N} \Phi(a,t)da = \int\limits_{G_N \bigcap B_t} c(t)da \le \int\limits_{B_t} c(t)da = \lambda^B c(t)$, it follows that

$$|L_N - 4(N-1)I| \le |L_N| + 4(N-1)I \le 4(N-1)I + \int\limits_{G_{N+2}\setminus G_{N-2}} \lambda^B c(t)dt = (12N-4)I < 12NI. \tag{36}$$

Thus,

$$|N^2\bar{\varphi} - N^2 I| \le 12NI, \quad \text{or} \quad |\bar{\varphi} - I| \le \frac{12I}{N}. \tag{37}$$

Since N can be made arbitrarily large, it immediately yields the desired equality $\bar{\varphi} = I$.□

Proof of Proposition 3.1: It is a corollary of Proposition 3.3.

Proof of Proposition 3.3: Let us show first that

$$\delta^* = 1 - \frac{K(B)}{K(H)}, \tag{38}$$

which, by Lemmas A.1 and A.3, can be calculated as

$$\delta^* = 1 - \frac{2}{\pi^{\frac{1}{3}} 3^{\frac{1}{6}} (\frac{2}{3} + \ln\sqrt{3})^{\frac{2}{3}}} \approx 0.0019. \tag{39}$$

We will demonstrate that the set of δ-secession-proof allocations is empty if and only if $\delta < \delta^*$.

Consider a δ-secession-proof allocation c. The budget balancedness assumption A.5 implies that the value of I, determined by (27), is equal to $\lambda^B\lambda^H K(H)$, and by Lemma A.4. so is the value of $\bar{\varphi}$.

Hence, there exists $a \in H$ such that $\varphi(a) \geq \lambda^B K(H)$. On the other hand, the stand alone aggregate cost in B_a is $\lambda^B K(B)$. Since c is δ-secession-proof, Definition 3.2 implies that $(1-\delta)\lambda^B K(H) \leq \lambda^B K(B)$, or $\delta \geq 1 - \dfrac{K(B)}{K(H)}$.

Let us show that Rawlsian allocation is δ-secession-proof whenever $\delta \geq \delta^*$. Indeed, since $r(t) = K(H)$ for every $t \in X$, then for $S = B^{l^*}$ — an optimal disk we observe that $(1-\delta)K(H) \leq K(B)$. Now, for any $S \in \mathcal{M}(X)$ we have $K(S) \geq K(B)$ and therefore $(1-\delta)K(H) \leq K(B) \leq K(S)$.

To complete the proof of the proposition, it remains to demonstrate that the Rawlsian allocation (which assigns to every individual in X the total cost of $K(H)$) is the only δ^*-secession-proof. For this end, consider an arbitrary δ^*-secession-proof allocation $c(\cdot)$ and estimate the number of individuals whose access fee is "substantially" below the level $K(H)$.

Take a positive number $\varepsilon > 0$. Consider first an arbitrary ring $B_a \setminus B_a^{l^*-\gamma}$ and evaluate the measure of individuals t whose cost $c(t)$ satisfies $c(t) < K(H) - \varepsilon$. Denote this set by U, and consider the set $S = B_a \setminus U$, for which, by Lemma A.2, we have $K(S) < l^* + \dfrac{4}{l^*}\gamma^2$. On the other hand,

$$c(S) = c(B^{l^*}) - c(U) \geq \lambda^B K(H) - \lambda(U)K(H) + \lambda(U)\varepsilon = \lambda(S)K(H) + \lambda(U)\varepsilon. \tag{40}$$

The δ^*-secession-proofness of $c(\cdot)$ implies that the average per capita cost in group S, adjusted by $1-\delta^*$, does not exceed its stand-alone value, $K(S)$:

$$(1-\delta^*)\frac{c(S)}{\lambda(S)} = (1-\delta^*)(K(H) + \frac{\lambda(U)}{\lambda(S)}\varepsilon) \leq K(S) < l^* + \frac{4}{l^*}\gamma^2. \tag{41}$$

Since $K(B) = l^* = (1-\delta^*)K(H)$, we have:

$$\lambda(U) \leq \frac{4\lambda(S)}{l^*(1-\delta^*)\varepsilon}\gamma^2 < \frac{4\pi(l^*)^2}{l^*(1-\delta^*)\varepsilon}\gamma^2 = W\gamma^2, \tag{42}$$

where W is a constant independent of γ.

Now consider the rectangular Q with sides of $2l^*$ and l^* centered at the origin. For any small positive number γ, denote by $R[i,\gamma]$ the ring $B_{p_i} \setminus B_{p_i}^{l^*-\gamma}$ centered at the point $p_i = (i\gamma, 0)$, where i is any (positive or negative) integer. For large enough positive integer N we have the following inclusion:

$$Q \subset \bigcup_{i=-N}^{N} R\left[i, \frac{l^*}{N}\right]. \tag{43}$$

Indeed, it is easy to see that $\forall x \in Q$ there exist at least one i such that $x \in B_{p_i}$, and at least one j such that $x \notin B_{p_j}$. Hence, there exist such i and $j = i \pm 1$ that the two statements

$$x \in B_{p_i}; \quad x \notin B_{p_j} \tag{44}$$

hold simultaneously. As obviously $B_{p_i}^{l^*-\gamma} \subset B_{p_j}$ for $j = i \pm 1$, we have that $x \in B_{p_i} \setminus B_{p_i}^{l^*-\gamma} = R[i,\gamma]$, with $\gamma = \dfrac{l^*}{N}$.

For $i = -N, \ldots, -1, 0, 1, \ldots, N$, denote by U and U_i the sets of individuals in Q and $R\left[i, \dfrac{l^*}{N}\right]$, respectively, who incur the cost less than $K(H) - \varepsilon$ under the allocation c. By utilizing (42), we have $\lambda(U_i) \leq W\dfrac{(l^*)^2}{N^2}$. Thus, since $U \subset \bigcup_{i=-N}^{N} U_i$, we have

$$\lambda(U) \leq (2N+1)W\frac{(l^*)^2}{N^2} < \frac{3}{N}W(l^*)^2. \tag{45}$$

Since N can be chosen arbitrarily large, (45) implies that $\lambda(U) = 0$. Note that this argument actually implies that for any rectangular with the sides of $2l^*$ and l^*, the Lebesgue measure of the set of individuals who incur the cost less than $l^* - \varepsilon$ under the allocation c has the zero measure.

Finally, consider an arbitrary hexagon H in the efficient partition P^*. It is contained in the union of several rectangulars with the sides of $2l^*$ and l^*. Hence, the measure of the set of individuals in H whose cost is less than $K(H) - \varepsilon$ is zero. But the set of individuals in H who contribute less than $K(H)$ is the union of the sets in H whose members contribute less than $K(H) - 1/n$ for $n = 1, 2, \ldots$, and as the countable union of null-sets, this set has zero measure. Hence, the budget balancedness implies that set of those incurring the cost higher than $K(H)$ has zero measure as well. Finally, Assumption A.6 guarantees that every $t \in X$ contributes $K(H)$, implying that the only δ^*-secession-proof allocation is Rawlsian.□

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