CORE DISCUSSION PAPER
9777

AUCTIONING PUBLIC GOODS TO GROUPS OF AGENTS

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October 1997

Abstract

A profit-maximizing auctioneer can provide a public good to at most one of a number of groups of agents. The groups may have non-empty intersections. Each group member has a private value for the good being provided to the group. We investigate an auction mechanism where the auctioneer provides the good to the group with the highest sum of the agents' bids, only if this sum exceeds a minimum price declared previously by the auctioneer. For the one-group two-bidder case with private values drawn from a uniform distribution we characterize the continuously differentiable symmetric equilibrium bidding functions for the agents, and find the optimal minimum price for the auctioneer when such functions are used by the bidders. We also examine another interesting family of equilibrium bidding functions for this case, with a discrete number of possible bids, and show the relation (in the limit) to the differentiable bidding functions.

Keywords: public goods, auctions, externalities.

JEL Classifications: **C00, C72, D44, D82, H41**.

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Version 1.32, 01/10/97 (first version, 6/96).
We are grateful to Sergio Currarini, Eric van Damme, Françoise Forges, Jacob Glazer, John McMillan, David Schmeidler, and especially Dov Samet for helpful discussions.
This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsability is assumed by the authors.

1 Introduction

We consider a number of groups of agents competing for a prize which can be allocated to at most one of the groups. The groups are not necessarily disjoint. The prize is a public good for members of a group; each can enjoy the prize as if she were the sole owner. The motivation for this problem is the allocation of a license to use a certain range of frequencies for local radio broadcasts. Until recently, such licenses in the US were allocated by the Federal Communications Commission, without cost to the licensees. Coase (1959) advocated that such licenses should be subject to competitive bidding, a view taken also in Meckling (1968). However, such procedures have only recently been started (see McMillan, 1994).

We model the following situation. Instead of providing a license for a frequency range to a single user, it can be provided to a group of broadcasters who are geographically located in such a way that their broadcasts do not interfere with each other. We can describe the situation as a graph. The nodes of the graph are the potential broadcasters, and an arc connects any two broadcasters that cannot broadcast simultaneously without interference. A license to broadcast can therefore be assigned to any independent set of this graph and the license is a public good for members of such a set. This model was studied in Lerner (1996), where a noncooperative solution concept was used to divide the license among the maximal independent sets of nodes. In the case dealt with by Lerner, no payment is extracted from the broadcasters, and the solution is a function only of the structure of the groups.

We make the following informational assumptions on the situation. Each agent (broadcaster) has a private value for the good being assigned to each group (this value is zero for groups that the agent does not belong to). We assume that each agent does not know the valuations of other agents for the prize to be awarded to different groups. They know only the probability distribution from which these valuations were drawn. The other main assumption is that the provider of the prize is a profit maximizer. If the provider knew the private values, she could of course offer the prize to the group with the highest sum of values, obtaining an efficient outcome while extracting their entire surplus. However, since the values are not known to the provider, other methods must be used, and efficiency is lost in the quest for profit maximization[1].

We examine an auction situation with the following procedure. The provider states a minimum price at which the good will be provided. Every agent bids (simultaneously) how much she is willing to pay for each group to be assigned the object[2]. If the group with the maximum sum of bids has bid a sum exceeding the minimum price (ties are broken randomly), then the good is provided to the group, and each agent pays her bid for that group.

The informational asymmetries of our assumptions make the equilibrium analysis of this situation mathematically complex even for relatively simple situations. Since this paper is a first attempt to solve such models, we will make many simplifying assumptions[3]. Even with these assumptions the problem is not trivial. We describe a general model, but most of our analysis is restricted to one of the simplest cases possible – two agents who belong to one group. Each agent has a private value independently drawn from a uniform distribution over the unit interval. We thus sidestep the problems of multiple and non-disjoint groups. The possibility of free riding remains, as each agent would prefer the other to bear the cost of bidding the minimum price.

Our model is related both to the theory of public goods and to that of auctions. However, it seems that it has not been previously addressed by either stream of literature. Auctions with incomplete information where the object is awarded according to the sum of bids are a novelty; similarly the provision (or non-provision) of a public good with the aim of profit maximizing replacing the quest for efficiency.

There are many related works containing some aspects of our model. None of them deal with non-empty intersections of groups[4], and most do not assume incomplete information or profit-maximizing providers. We first address the public goods literature. Much is known about provision of public goods to agents with private values, and the consequent problem of free riding. This problem was first mentioned in Samuelson (1954). Other theoretical papers on free riding are Olson (1965), Stigler (1974), Brubaker (1975) and Cornes and Sandler (1984). Some experimental works on free riding are Schneider

[1]We elaborate on the incompatibility of efficiency and profit maximizing in Section 6.

[2]An agent will of course bid zero for any group to which she does not belong.

[3]These are similar to the assumptions made in Landsberger et al (1996), who investigate a novel auction situation and find technical difficulties even in the two-bidder case.

[4]Which we include in the model but do not solve. One such situation is discussed in Section 6.

and Pommerehne (1981), Marwell and Ames (1981), Isaac, Walker and Thomas (1984) and Isaac, McCue and Plott (1985). A mechanism such as the Groves-Ledyard (1977) mechanism can elicit truthful declarations of private values and therefore achieve efficient allocations, but has a number of limitations. The most problematic of these is that to achieve budget balance the Groves-Ledyard strategies depend on the actions of other players, thus necessitating much information about the preferences of others. As they note, their mechanism is not practical to implement. For our assumption that the provider wishes to maximize her expected income, and does not care directly about efficiency, this class of mechanism is completely unsuitable.

The rent-seeking literature contains a number of related works. In Baik (1993) and in Katz, Nitzan and Rosenberg (1991), groups compete to win group-specific *public* goods. In Nitzan (1991) groups compete to win group-specific *private* goods. These three papers differ from our analysis as the valuations of the outcomes are common knowledge among all bidders, and bids are paid even by the losers (rent seeking). Loehman, Quesnel and Babb (1996) analyze such a model in an experimental situation.

As in standard auctions, the bidders in our setting must take into account (for each group to which they belong) that the bids of other members of the group also count in determining whether the group will get the good. However, contrary to the situation of auctions of private goods, if other members of one of a bidder's groups make higher bids, this increases the bidder's chances of profiting from the good through this group. The externalities the good provides are thus positive. Auctions with negative externalities have been dealt with by Jehiel, Moldovanu and Stacchetti (1994) and by Jehiel and Moldovanu (1996). Auctions with negative externalities are simpler to analyze, since there is no problem of free riding. McAffee and McMillan (1992) model collusion in bidding cartels, which can be viewed as a case of auctions with positive externalities, as the profits from collusion are distributed among the cartel members. The bidders in their model can reach efficient outcomes, but they require punishments to enforce collusive agreements, and the collusion decreases the auctioneer's profits.

Even with our simplifying restrictions (two bidders and one group) we cannot provide a unique prediction or prescription of a solution. We describe an outcome in which the provider chooses a specific minimum price and the agents then use symmetric equilibrium functions (from private values to bids). These equilibrium bidding strategies are drawn from a family of continuously differentiable functions and are uniquely determined by the minimum price. These functions are in general quite complicated, but at the optimal minimum price for the provider the function reduces to a simple linear function of the private value[5]. We justify this emphasis on differentiable bidding strategies in Section 6. We also examine other symmetric equilibrium bidding functions, including a family of functions with discrete bids (a reasonable assumption when there is a smallest monetary unit), which tend to the differentiable functions when the number of possible bids tends to infinity.

The structure of the paper is as follows. In Section 2 we present the model. In Section 3 we present the conditions for Nash equilibria of the auction situation, and give properties of such equilibria for the one-group two-bidder case. Sections 4 characterizes differentiable symmetric equilibrium bidding functions, and Section 5 analyzes a family of discrete ones. We conclude with final remarks in Section 6.

2 The Model

We assume a public good which can be provided by a provider in at most one of a number of forms $J = \{1, 2, \ldots, k\}$ to a group of agents $N = \{1, 2, \ldots, n\}$. For the broadcasting license example, the forms are the maximal independent sets of the graph. Each agent i has a private value v_i^j for each form of the good j which is independently drawn from a distribution F_i^j over $[0, \overline{v}]$. For the case of the broadcasting license, if agent i belongs to group j, then v_i^j has positive probability of being strictly positive. We denote by v_i the vector $(v_i^j)_{j \in J}$

The procedure of the auction is as follows: At stage 1, the provider of the good, the auctioneer, states a minimum price $a \in I\!R_+$. Then, at stage 2, after hearing the value of a, each agent i submits a bid $b_i^j \in [0, \overline{v}]$ (this restriction is without loss of generality, as any higher bid is dominated by a bid in this range) for each group S^j. The bids are made simultaneously. If $\max_{j \in J} \sum_{i \in N} b_i^j \geq a$ then the good is provided in the form j, where j

[5]Notably, such multiplicative functions are commonly used in analyses of offshore oil lease auctions, as in Dougherty and Nozaki (1975). However, they may not be optimal for these situations, as Engelbrecht-Wigggans (1978) contains a case of disequilibrium of such strategies in a federal offshore lease sale.

is the form with the maximal sum of bids (ties are broken randomly). If the sum of bids for form j is less than a for all forms $j \in J$, then the good is not provided.

The payoffs for the players, given their private values, are as follows: The auctioneer receives a payoff of $\sum_{i \in N} b_i^j$ if the good is provided in form j, and zero if the good is not provided. Each agent i receives a payoff of $v_i^j - b_i^j$ if the good is provided in form j, and zero if the good is not provided. We assume risk neutrality of the agents.

A strategy profile of the auction procedure consists of a value a that the auctioneer declares, and a bidding function for each bidder i for each form j, as a function of the declared value a and the player's private value vector. This bidding function is of the form $b_i^j : I\!R_+ \times [0, \overline{v}]^J \longrightarrow [0, \overline{v}]$ which specifies his bid for providing the object in form j for any value of a and any private value vector v_i. Denote by b_i the vector of bidding functions $(b_i^j)_{j \in J}$.

3 Nash Equilibria of the Auction

Any predicted (or prescribed) outcome should be a Nash equilibrium, as otherwise at least one player could gain from a unilateral deviation. Such a player, being aware of the prediction, would not follow it and the prediction or prescription would not be useful. Therefore, we are interested only in strategy profiles that are Nash equilibria.

Denote by S_j the event that the good is provided in form j.

The conditions for a profile $(a, (b_i)_{i \in N})$ to be a Nash equilibrium are the following:

1. Each bidder i's bidding function vector b_i maximizes his expected profit for every possibility of his private value vector (for the given a and bidding functions of the other players). Formally (we denote by b_{-i} and v_{-i} the bidding functions and private values, respectively, of the other agents),

$$\sum_{j \in J} Pr(S_j | a, b_{-i}, b_i(a, v_i))(v_i^j - b_i^j(a, v_i)) \geq \sum_{j \in J} Pr(S_j | a, b_{-i}, \hat{b}_i)(v_i^j - \hat{b}_i^j) \qquad (1)$$

for all $i \in N$, for all $v_i \in [0, \overline{v}]^J$, and for all $\hat{b}_i \in [0, \overline{v}]^J$. The probabilities in the equation are the expected probabilities as viewed by agent i with respect to v_{-i}.

2. The auctioneer's declaration of a maximizes her expected profit, given the bidding functions of the bidders. Formally,

$$\sum_{j\in J} Pr(S_j|a,(b_i)_{i\in N})E(\sum_{i\in N} b_i^j(a,v_i)|S_j) \geq$$
$$\sum_{j\in J} Pr(S_j|\hat{a},(b_i)_{i\in N})E(\sum_{i\in N} b_i^j(\hat{a},v_i)|S_j) \tag{2}$$

for all $\hat{a} \in I\!R_+$. The probabilities are the expected probabilities with respect to the distribution of the private values.

Note that $\sum_{j\in J} Pr(S_j|a,(b_i)_{i\in N})$ may be less than 1, if there is a positive probability that no form j generates bids summing to at least a. This could occur even in equilibrium.

From this point, we make some simplifying assumptions which will enable us to continue with a more tractable model. As mentioned in the introduction, even with these assumptions the problem is not trivial. We assume that there are two agents, there is one form of the public good, and the question is therefore whether or not it will be provided and at what cost to the agents. We assume that the private values of the good to the agents are independently uniformly distributed over the interval $[0,1]$. These assumptions are fixed for the remainder of the paper[6].

A strategy profile for this case (two bidders) is a triple (a, b_1, b_2), where

$$b_i : I\!R_+ \times [0,1] \longrightarrow [0,1]$$

is agent i's bidding function, and $b_i(a, v_i)$ denotes agent i's bid for the auctioneer's declaration of a and his private value v_i.

A strategy profile (a, b_1, b_2) is a Nash equilibrium if for each agent i and each private value v_i, the bid $b_i(a, v_i)$ is optimal, i.e.

$$b_i(a,v_i) \in \arg\max_x (v_i - x)Pr(b_{-i}(a,v_{-i}) + x \geq a),$$

and $a \in \arg\max_{a' \in R_+} \Pi_0(a', b_1, b_2)$, where

$$\Pi_0(a,b_1,b_2) = \int_0^1 \int_0^1 (b_1(a,v_1) + b_2(a,v_2)) 1_{\{b_1(a,v_1)+b_2(a,v_2)\geq a\}} dv_2 dv_1.$$

[6]Except in Section 6, where we discuss a two-group case with a non-empty intersection.

A remark on the notation. The bidding functions are used after a is announced. Therefore, we first concentrate on the properties of the bidding functions for a given, fixed a, and abuse notation by referring to $b_i(v_i)$ instead of $b_i(a, v_i)$. We later return to the question of finding the optimal value of a, when given a family of bidding functions, one for each possible value of a. Note that even if $b_1(a, v_1)$ and $b_2(a, v_2)$ are in equilibrium only for one fixed a^*, then there exists a pair of bidding functions b'_1 and b'_2 such that (a^*, b'_1, b'_2) is an equilibrium profile, by defining

$$b'_i(a, v_i) = \begin{cases} b_i(a, v_i) & \text{if } a = a^* \\ 0 & \text{otherwise} \end{cases} . \tag{3}$$

The following lemmas show some general properties that equilibrium bidding profiles must satisfy. The first lemma deals with cases where a bidder cannot make a positive profit with any bid, given her private value and the other bidder's bidding function.

Lemma 1 *Assume that* (a, b_1, b_2) *is an equilibrium, and* b_1, b_2 *are nondecreasing. If* $b_2(1) < a$ *then for* $0 \leq v \leq a - b_2(1)$*, any bid in the interval* $[0, a - b_2(1))$ *is a best response for agent 1.*

Proof: A player with v in the above range can never receive a positive payoff. Any bid in the interval $[0, a - b_2(1))$ will give him a payoff of zero, and any higher bid will give an expected payoff of at most zero. Therefore any bid in this interval is a best response. ■(Lemma 1)

Denote by

$$P_w(x, b) = \int_0^1 1_{\{b(v) \geq a - x\}} dv$$

the probability of winning with a bid of x if the other player is using the bidding function b.

The next lemma shows that equilibrium bidding functions must be non-decreasing. The restriction that the following lemma applies only for private values at which it is possible to receive a positive payoff is implied by Lemma 1.

Lemma 2 *If* (a, b_1, b_2) *is an equilibrium, and for some* x_1 *and* x_2 *such that* $x_1 < x_2$ *we have* $P_w(b_1(x_1), b_2) > 0$ *and* $P_w(b_1(x_2), b_2) > 0$*, then* $b_1(x_1) \leq b_1(x_2)$*.*

Proof: Denote $p_1 = P_w(b_1(x_1), b_2)$ and $p_2 = P_w(b_1(x_2), b_2)$. Since $p_1 > 0$ it is true that $b_1(x_1) \leq x_1$, otherwise the functions are not in equilibrium. From the equilibrium assumption we have

$$p_1 \cdot (x_1 - b_1(x_1)) \geq p_2 \cdot (x_1 - b_1(x_2)) \tag{4}$$

and

$$p_2 \cdot (x_2 - b_1(x_2)) \geq p_1 \cdot (x_2 - b_1(x_1)). \tag{5}$$

Multiplying both sides of (4) by $(x_2 - b_1(x_1))$ (which is positive since $x_2 > x_1 \geq b_1(x_1)$), we get

$$p_1 \cdot (x_1 - b_1(x_1))(x_2 - b_1(x_1)) \geq p_2 \cdot (x_1 - b_1(x_2))(x_2 - b_1(x_1)). \tag{6}$$

Multiplying both sides of (5) by $(x_1 - b_1(x_1))$, which is non-negative, we get

$$p_2 \cdot (x_2 - b_1(x_2))(x_1 - b_1(x_1)) \geq p_1 \cdot (x_2 - b_1(x_1))(x_1 - b_1(x_1)). \tag{7}$$

Combining (6) and (7), dividing by $p_2 > 0$, multiplying out and cancelling equal terms gives us:

$$b_1(x_2)(x_2 - x_1) \geq b_1(x_1)(x_2 - x_1), \tag{8}$$

and as $x_2 > x_1$ the conclusion of the lemma holds. ■(Lemma 2)

We are especially interested in the case of symmetric equilibria, where the two agents use the same bidding function. For a fixed value of a, a bidding function $b : [0, 1] \longrightarrow [0, a]$ is in equilibrium if for each player i, for any private value v_i, bidding $b(v_i)$ is a best response (in expectation) to the other player using the same function b. Formally, b is a symmetric equilibrium bidding function, given a, if

$$b(v) \in \arg\max_{x \in [0,a]} (v - x) Pr(b(v_{-i}) + x \geq a) \quad \forall v \in [0, 1], \text{ for } i \in \{1, 2\}. \tag{9}$$

Denote $d = \lim_{v \to 1-0} b(v)$. The following lemmas hold for symmetric equilibria. The first one states that in a symmetric equilibrium, any bid higher than $a - d$ has a positive probability of winning. Thus, the interval not "covered" by Lemma 1 is $[a - d, 1]$, and this is the interval where Condition 9 is non-trivial.

Lemma 3 *If (a, b, b) is an equilibrium profile, then for all $\varepsilon > 0$, $P_w(a - d + \varepsilon, b) > 0$.*

Proof: If for all $v \in [0, 1]$, $b(v) \le d - \varepsilon$, then using Lemma 2, $\lim_{v \to 1} b(v) < d$, contradicting the definition. Therefore, there exists $v_0 < 1$ such that $b(v_0) > d - \varepsilon$, and, using Lemma 2 again, $b(v) > d - \varepsilon$ for all $v \in [v_0, 1]$. This implies that $P_w(a - d + \varepsilon, b) \ge 1 - v_0 > 0$. ■(Lemma 3)

Lemma 4 states that for any private value above $a - d$, an equilibrium bid will be *strictly* lower than the private value.

Lemma 4 *If (a, b, b) is an equilibrium profile, then for all $v \in (a - d, 1]$, $b(v) < v$.*

Proof: For any such v, bidding a value greater to or equal to v gives a non-positive expected profit, so such a bid is dominated by bidding $\frac{v+(a-d)}{2}$, which gives positive gain, with positive probability (from Lemma 3). Therefore, $b(v) < v$ for any $v \in (a - d, 1]$. ■(Lemma 4)

We are now ready for the final lemma, which allows us to conclude that if a symmetric equilibrium bidding function is continuous, then it is strictly increasing for values of v in $(a - d, 1]$.

Lemma 5 *Assume (a, b, b) is an equilibrium profile. If there exist $x'' > x'$ such that $y = b(x'') = b(x') > a - d$, then there exists $\varepsilon > 0$ such that $b(v) \notin [a - y - \varepsilon, a - y)$ $\forall v \in [0, 1]$.*

Proof: Assume the claim is not true, i.e. there exist such x, x' and y, but no such ε. Then there exists a sequence $\{t_n\}_{n=1}^{\infty}$ and a sequence $\{v_n\}_{n=1}^{\infty}$, such that $b(v_n) = t_n$ for all n, $t_n < a - y$ for all n, and $\lim_{n \to \infty} t_n = a - y$. Define, for all n, $b^{-1}(t_n) = \inf\{x | b(x) = t_n\}$.

For all n, we have

$$(v_n - t_n)(1 - x'') \ge (v_n - t_n) P_w(t_n, b), \tag{10}$$

since $t_n < a - y$ implies $P_w(t_n, b) \le 1 - x''$, and

$$(v_n - t_n)P_w(t_n, b) \ge (v_n - t_n)(1 - x'), \tag{11}$$

since b is in equilibrium. Combining (10) and (11) we get

$$(v_n - t_n)(1 - x'') \ge (v_n - t_n)(1 - x'), \tag{12}$$

which implies

$$(a - y) - t_n \ge v_n(x'' - x') + (a - y)x' - t_n x''. \tag{13}$$

Taking the limit $n \to \infty$, we have

$$0 \ge (x'' - x')(\lim_{n\to\infty} v_n - (a - y)), \tag{14}$$

and since $x'' > x'$, this implies

$$\lim_{n\to\infty} b^{-1}(t_n) \le a - y. \tag{15}$$

Since b is increasing, so is b^{-1}, and the bid at the limit is at least as high as the private value there, in contradiction to Lemma 4. ■(Lemma 5)

We have thus shown that for symmetric equilibrium bidding functions, existence of a plateau in the "relevant" area (above $a - d$) implies existence of a jump in the function. Therefore, if such a bidding function is continuous (has no jumps), it is strictly increasing for values in the range $[a - d, 1]$.

4 Differentiable Symmetric Equilibria

In this section we characterize the Nash equilibria for the two agents, given the minimum selling price a announced by the auctioneer, with the restriction that the two agents use the same bidding function, and the bidding function is differentiable over its entire range. This assumption of differentiability can actually be considerably weakened, and assuming continuity allows us to derive differentiability as a result. Given a continuous symmetric equilibrium bidding function, we can divide the interval of possible private values $[0,1]$ into two subintervals. Denoting as before $d = \lim_{v \to 1-0} b(v)$, the first interval is $[0, a-d]$, in which no bid can give expected positive profit (assuming the other player uses b). In the interval $(a-d, 1]$ a positive profit is possible, and is indeed achieved with b. A symmetric equilibrium bidding function b that is continuous in the interval $(a-d, 1]$ is also differentiable on $(a-d, 1)$.[7] Thus, for the functions dealt with in Section 4.1 (the case $b(1) = a$), it is enough to assume that the function is continuous over $[0,1]$ to get differentiability on $(0,1)$, and for those of Section 4.2 (the case $b(1) = d < a$), continuity over $(a-d, 1]$ implies differentiability over $(a-d, 1)$. Combining this result with Lemma 5 we have that a continuous symmetric equilibrium function satisfying $b(1) = a$ is both strictly increasing and differentiable on $(0,1)$, and an analogous result holds on $(a - b(1), 1)$ when $b(1) < a$.

4.1 Differentiable Symmetric Equilibria with $b(1) = a$

We consider first the case where $b(1) = a$ (the case $b(1) < a$ will be dealt with in Section 4.2). Assuming that b is differentiable, it is strictly increasing from Lemma 5. Therefore, there exists a unique inverse $b^{-1} : [0, a] \longrightarrow [0, 1]$. We start with Condition (9), which is a necessary condition for any symmetrical equilibrium.

Note that since v_i is uniformly distributed over the unit interval, $Pr(b(v_{-i}) + x \geq a)$ $= Pr(v_{-i} \geq b^{-1}(a-x)) = 1 - b^{-1}(a-x)$, hence differentiable with respect to x. Since we assume that b is differentiable, a necessary condition for Condition (9) to hold is that the derivative of the right hand side of the condition is equal to zero for any x in the argmax. Equating this derivative to zero gives

$$b^{-1\prime}(a-x) = \frac{1 - b^{-1}(a-x)}{v - x}. \tag{16}$$

and since $x = b(v)$ must satisfy this for equilibrium, it must satisfy

$$b^{-1\prime}(a - b(v)) = \frac{1 - b^{-1}(a - b(v))}{v - b(v)} \tag{17}$$

[7] A proof can be obtained from the authors.

for all $v \in (0,1)$. Substituting $t = a - b(v)$ and therefore $v = b^{-1}(a-t)$ we have the following necessary condition:

$$b^{-1\prime}(t) = \frac{1 - b^{-1}(t)}{b^{-1}(a-t) - (a-t)} \tag{18}$$

for $t \in (0,a)$.

Thus, replacing b^{-1} by c, we seek a function c which is differentiable on $(0,a)$ and continuous on $[0,a]$, satisfying

$$c'(t) = \frac{1 - c(t)}{c(a-t) - (a-t)} \tag{19}$$

for all $t \in (0,a)$, and $c(0) = 0$.

We now give the main theorem, which contains the characterization of all symmetric equilibrium bidding functions that are differentiable and satisfy $b(1) = a$.

Theorem 1 *Equation (19) has a solution if and only if* $0 < a < 1$. *The solution is unique, it is strictly increasing and satisfies* $0 \le c(t) \le c(a) = 1$ *on* $[0,a]$. *Moreover one has that* c *is continuously differentiable on* $[0,a)$ *and is continuously differentiable at* a *if and only if* $a \le \frac{1}{2}$. *The solution can be explicitly given by*

$$c(t) = 1 + \frac{1-a}{a - 1 - t - t^{1/a}(a-t)^{1-1/a}} \tag{20}$$

The proof of this theorem requires a thorough knowledge of calculus, and non-mathematically inclined readers are encouraged to skip the proof on a first reading.

Proof: We start with the "only if"-part. Assume c is a solution of (19) and let $m := \frac{a}{2}$. For $t \in [0,m]$ define

$$u_1(t) := c(m-t) - m + t, \quad u_2(t) := c(m+t) - m - t. \tag{21}$$

Then

$$u_1'(t) = 1 - c'(m-t) = 1 - \frac{1 - c(m-t)}{c(m+t) - m - t} = 1 + \frac{u_1(t) + m - t - 1}{u_2(t)} \tag{22}$$

$$u_2'(t) = -1 + c'(m+t) = -1 + \frac{1 - c(m+t)}{c(m-t) - m + t} = -1 - \frac{u_2(t) + m + t - 1}{u_1(t)} \tag{23}$$

and $u_0 := u_1(0) = c(m) - m = u_2(0)$, $u_1(m) = 0$. Next observe that

$$u_1'(t)u_2(t) + u_1(t)u_2'(t) = -2t \quad \forall t \in [0, m]. \tag{24}$$

Integrating (24), we have $u_1(t)u_2(t) = K - t^2$. Substituting $t = 0$ gives $K = u_0^2$, therefore $u_1(t)u_2(t) = u_0^2 - t^2$. Since $0 = u_1(m)u_2(m) = u_0^2 - m^2$, we see that $|u_0| = m$. In particular, $u_2(t) \neq 0$ on $[0, m)$ and

$$\frac{1}{u_2(t)} = \frac{u_1(t)}{m^2 - t^2} \quad \text{for } 0 \leq t < m$$

and by (22) we obtain a Riccati type equation for u_1

$$u_1'(t) = 1 + \frac{u_1(t)(u_1(t) + \alpha - t)}{m^2 - t^2} \tag{25}$$

with $\alpha = m - 1$ and initial condition $u_1(0) = u_0$ and $u_1(m) = 0$. To solve (25) we first perform a "velocity transform" by $w(t) := u_1(\phi(t))$, $\phi(0) = 0$ such that $\phi'(t) = m^2 - \phi(t)^2$. Hence $\phi(t) = m \tanh(mt)$, $\phi^{-1}(x) = \frac{1}{m} arctanh(\frac{x}{m})$ and

$$w'(t) = \phi'(t)u_1'(\phi(t)) = u_1(\phi(t))^2 + (\alpha - \phi(t))u_1(\phi(t)) + \phi'(t) \tag{26}$$

The transform $y(t) = \exp\left(-\int_0^t w(s)ds\right)$ now yields

$$y'(t) = -w(t)y(t), \quad y''(t) = -(w'(t) - w^2(t))y(t) \tag{27}$$

Combining (26) and (27) one obtains

$$\begin{aligned} y''(t) &= -(w'(t) - w^2(t))y(t) = && (28)\\ &= ((\alpha - \phi(t))u_1(\phi(t)) + \phi'(t))\frac{y'(t)}{w(t)} = && (29)\\ &= (\alpha - \phi(t))y'(t) - \phi'(t)y(t) = && (30)\\ &= \alpha y'(t) - (\phi(t)y(t))' && (31) \end{aligned}$$

and so, integrating,

$$y'(t) = \alpha y(t) - \phi(t)y(t) + c_0, \tag{32}$$

with $c_0 = y'(0) - \alpha y(0) + \phi(0)y(0) = -u_0 - \alpha$, since $y(0) = 1$ by definition. Equation (32) is a linear differential equation of first order. It is solved using the variation-of-parameter formula

$$y(t) = \exp\left(\alpha t - \int_0^t \phi(s)ds\right) + c_0 \int_0^t \exp\left(\alpha(t-s) - \int_s^t \phi(\sigma)d\sigma\right)ds,$$

since the solution of the homogenous equation (i.e. for the case $c_0 = 0$) is given by $\exp\left(\alpha t - \int_0^t \phi(s)ds\right)$. Now since

$$\int_s^t \phi(\sigma)d\sigma = \int_s^t m\tanh(m\,\sigma)\;d\sigma \quad = \quad \ln(\frac{\cosh(m\,t)}{\cosh(m\,s)})$$

and

$$\int_0^t \exp\left(\alpha(t-s) - \int_s^t \phi(\sigma)d\sigma\right)ds = \frac{\mathrm{e}^{\alpha t}}{\cosh(m\,t)}\int_0^t \mathrm{e}^{-\alpha s}\cosh(m\,s)\;ds \tag{33}$$

$$= \int_0^t \mathrm{e}^{\alpha t}\mathrm{e}^{-\alpha s}\frac{\cosh(m\,s)}{\cosh(m\,t)}\;ds \tag{34}$$

$$= \frac{\mathrm{e}^{\alpha t}}{\cosh(m\,t)}\left(\frac{\mathrm{e}^{-\alpha t}}{m^2-\alpha^2}\left[\alpha\cosh(m\,t) + m\sinh(m\,t)\right] - \frac{\alpha}{m^2-\alpha^2}\right) \tag{35}$$

$$= \frac{\alpha\mathrm{e}^{\alpha t}}{(\alpha^2-m^2)\cosh(m\,t)} - \frac{\alpha}{\alpha^2-m^2} - \frac{m\tanh(m\,t)}{\alpha^2-m^2}. \tag{36}$$

Using the fact that $\int \tanh x\;dx = \ln\cosh x$ to show that

$$\exp\left(\alpha t - \int_0^t \phi(s)ds\right) = \frac{\mathrm{e}^{\alpha t}}{\cosh mt} \tag{37}$$

and (33)-(36), we obtain

$$y(t) = \frac{m^2 + u_0\alpha}{m^2-\alpha^2}\frac{e^{\alpha t}}{\cosh(mt)} - \frac{m(u_0+\alpha)}{m^2-\alpha^2}\tanh(mt) - \frac{\alpha(u_0+\alpha)}{m^2-\alpha^2}.$$

Now, using the definition of w, (27) and (32) gives

$$u_1(t) = w(\phi^{-1}(t)) = -\frac{y'(\phi^{-1}(t))}{y(\phi^{-1}(t))} = \phi(\phi^{-1}(t)) - \alpha - \frac{c_0}{y(\phi^{-1}(t))}.$$

Since the equation

$$\operatorname{arctanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \tag{38}$$

and the definition of ϕ^{-1} imply that

$$e^{\alpha\phi^{-1}(t)} = \left(\frac{m+t}{m-t}\right)^{\frac{\alpha}{2m}}, \tag{39}$$

and since the equation

$$\cosh x = \frac{1}{\sqrt{1-\tanh(x)^2}} \tag{40}$$

implies that

$$\cosh(\operatorname{arctanh}(\frac{t}{m})) = \frac{m}{\sqrt{m^2-t^2}}, \tag{41}$$

therefore, if $0 \le t < m$ and $u_0 + \alpha \ne 0$, then

$$\begin{aligned} u_1(t) &= t - \alpha - \frac{c_0}{y(\phi^{-1}(t))} = t - \alpha - \frac{m^2-\alpha^2}{t+\alpha-\frac{m^2+\alpha u_0}{m(\alpha+u_0)}(m+t)^{\frac{1}{2}+\frac{\alpha}{2m}}(m-t)^{\frac{1}{2}-\frac{\alpha}{2m}}} \\ &= t - \frac{a}{2} + 1 - \frac{a-1}{\frac{a}{2}-1+t-\operatorname{sgn}(u_0)(\frac{a}{2}+t)^{1-1/a}(\frac{a}{2}-t)^{1/a}} \end{aligned}$$

using $\alpha = \frac{a}{2} - 1$ and $|u_0| = \frac{a}{2}$ for the last equation. If $2m = a = 1$ we have $u_1(t) = t - \alpha = t - m + 1$. In this case, however, $u_1(m) = 1$, contradicting the condition $u_1(m) = 0$. Hence $a \ne 1$.

Now for u_2 we obtain instead of (25) the following Riccati equation

$$u_2'(t) = -1 - \frac{u_2(t)(u_2(t)+\alpha+t)}{m^2-t^2} \tag{42}$$

Since $u_2(0) = u_1(0)$, it can be verified that $u_2(t) = u_1(-t)$ gives the unique solution of (42) if $0 \le t < m$.

Now, since c solves (19), u_1 and u_2 are defined on all of $[0, m)$. This implies that the denominator of u_1,

$$\eta(t) := \frac{a}{2} - 1 - t - \mathrm{sgn}(u_0)(\frac{a}{2} + t)^{1-1/a}(\frac{a}{2} - t)^{1/a}$$

does not equal zero in $(-\frac{a}{2}, \frac{a}{2})$. Now

$$\eta(m) = a - 1 \quad \text{and} \quad \lim_{t \downarrow -m} \eta(t) = \begin{cases} -\mathrm{sgn}(u_0)\,\infty & , \quad \text{if } a < 1, \\ -1 & , \quad \text{if } a > 1, \end{cases}$$

so from the mean value theorem we rule out the case $a > 1$ and the case $a < 1$, $u_0 = -m$, as these cases would cause η to be zero at some point in $(-\frac{a}{2}, \frac{a}{2})$. Hence we obtain

$$c(t) = 1 + \frac{1-a}{a - 1 - t - t^{1/a}(a-t)^{1-1/a}},$$

which shows that c is uniquely determined.

On the other hand, one immediately checks that c given by (20) satisfies the equation (19) and so the "if"-part is proven, too.

One can verify that c is strictly increasing and continuously differentiable on $[0, a)$. Finally,

$$c'(t) = (1-a)\frac{(a-t)^{1/a} + (1-t)\,t^{\frac{1}{a}-1}}{\left((a-1-t)\,(a-t)^{\frac{1}{2a}} - t^{1/a}\,(a-t)^{1-\frac{1}{2a}}\right)^2},$$

which shows that $\lim_{t \uparrow a} c'(t)$ exists if and only if $a \leq \frac{1}{2}$. By continuity c is (left-) differentiable at a if and only if $a \leq \frac{1}{2}$. ■(Theorem 1)

Examples of differentiable symmetric equilibrium bidding functions, with $b(1) = a$, are given in Figure 1. Note that for $a = \frac{1}{2}$, the differentiable symmetric equilibrium bidding function is $b(v) = \frac{v}{2}$, a simple linear function of the private value. Such a multiplicative bidding function is extremely easy to use. The following calculations show that this is not the only advantage of having $a = \frac{1}{2}$: it is also the *optimal* choice by the auctioneer, if the bidders use the differentiable bidding functions characterized by Theorem 1.

We now investigate the expected payoff the auctioneer receives as a function of the announced price a. If player 1 and player 2 use bidding functions $b_1(a, \cdot)$ and $b_2(a, \cdot)$

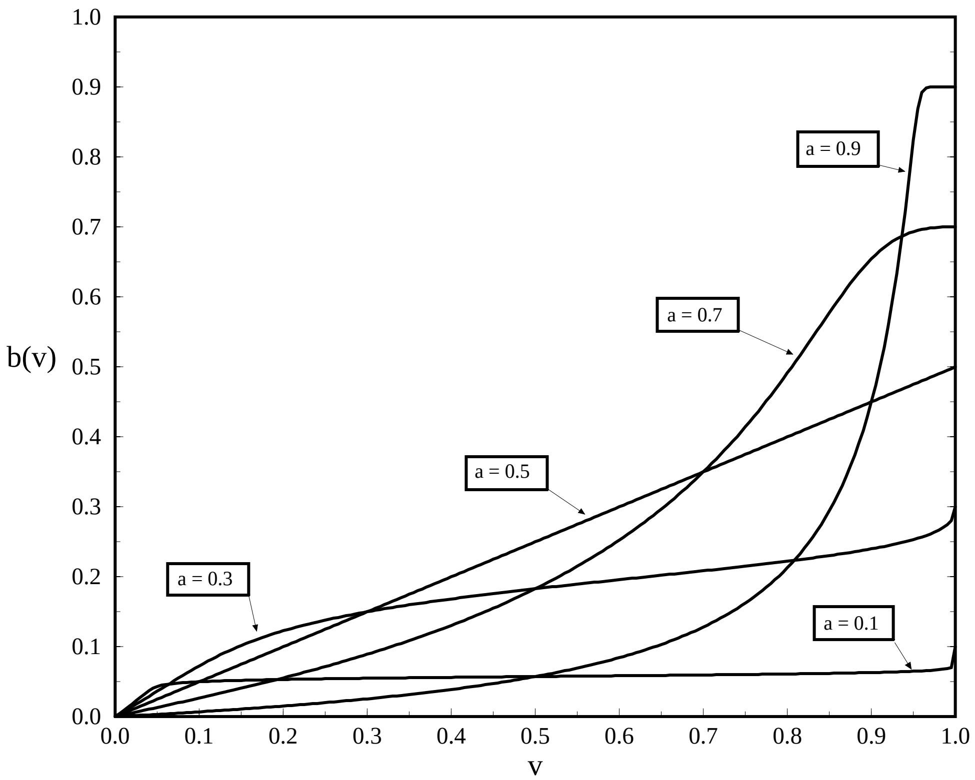

Figure 1: Differentiable symmetric equilibrium bidding functions for five different values of a.

respectively, then the payoff function (relating the private values of the players to the auctioneer's payoff) for the auctioneer reads as follows

$$\begin{aligned} \Pi_0 : [0,1] \times [0,1] &\to \mathbf{R} \\ (v_1, v_2) &\mapsto \begin{cases} b_1(a,v_1) + b_2(a,v_2) & , \text{ if } b_1(a,v_1) + b_2(a,v_2) \geq a \\ 0 & , \text{ else.} \end{cases} \end{aligned}$$

Put $c_i(a,\cdot) := (b_i(a,\cdot))^{-1}$, $(i = 1,2)$, then the expected value for the auctioneer's payoff is

$$\begin{aligned} \mathbf{E}\Pi_0 &= \int_0^1 \int_0^1 (b_1(a,v_1) + b_2(a,v_2)) \cdot 1_{\{(v_1,v_2) | b_1(a,v_1) + b_2(a,v_2) \geq a\}}(v_1, v_2)\, dv_2\, dv_1 \\ &= \int_0^1 \int_{a - b_1(a,v_1)}^{a} (b_1(a,v_1) + t_2)\, \partial_2 c_2(a,t_2)\, dt_2\, dv_1 \end{aligned}$$

$$= \int_0^a \Big(t_1\,(c_2(a,a) - c_2(a,a-t_1))$$

$$+ \int_{a-t_1}^a t_2\,\partial_2 c_2(a,t_2)\;dt_2\Big)\,\partial_2 c_1(a,t_1)\;dt_1. \tag{43}$$

(Here ∂_2 denotes the partial derivative w.r.t. the second variable.) For the case of an equilibrium bidding function, where $c = c_1(a,\cdot) = c_2(a,\cdot)$ this becomes

$$\begin{aligned}
\mathbf{E}\Pi_0 &= c(a)\int_0^a t\,c'(t)\,dt - \int_0^a t\,c(a-t)\,c'(t)\,dt + \int_0^a\int_{a-t_1}^a t_2\,c'(t_2)\,c'(t_1)\,dt_2\,dt_1 \\
&= 2\,a\,c(a)^2 - a\,c(0)\,c(a) - c(a)\int_0^a c(t)\,dt - a\int_0^a c(a-t)\,c'(t)\,dt \\
&\quad - \int_0^a\int_{a-t_2}^a c'(t_1)\,dt_1 c(t_2)\,dt_2 \\
&= 2\,a\,c(a)^2 - a\,c(0)\,c(a) - 2\,c(a)\int_0^a c(t)\,dt - a\int_0^a c(a-t)\,c'(t)\,dt \\
&\quad + \int_0^a c(a-t)\,c(t)\,dt
\end{aligned}$$

using partial integration and by interchanging the integrals in the last step.

Now for the equilibrium bidding function obtained in Theorem 1 we have $c(0) = 0$, $c(a) = 1$ and from the differential equation (19)

$$\int_0^a c(a-t)c'(t)\,dt = \int_0^a \left((a-t)c'(t) + 1 - c(t)\right)\,dt = a \tag{44}$$

where we again have performed partial integration. This yields

$$\mathbf{E}\Pi_0 = (2-a)a - \int_0^a (2 - c(a-t))c(t)\,dt. \tag{45}$$

Now we use the special structure of the equilibrium bidding function (20) by writing $c(t) = 1 + (1-a)c_0(t)$ and obtain

$$\begin{aligned}
\int_0^a (2 - c(a-t))c(t)\,dt &= a - (1-a)\int_0^a c_0(a-t)\,dt + (1-a)\int_0^a c_0(t)\,dt \\
&\quad - (1-a)^2\int_0^a c_0(a-t)\,c_0(t)\,dt.
\end{aligned}$$

Observe that the first and the second integral on the right hand side are equal and that the integrand of the third one is symmetric around $\frac{a}{2}$ (which is basically of numerical interest), we thus obtain

$$\mathbf{E}\Pi_0 = a(1-a) + (1-a)^2 2\int_0^{\frac{a}{2}} c_0(a-t)c_0(t)\,dt \tag{46}$$

Numerical evaluation shows that $\mathbf{E}\Pi_0$ assumes its maximum at $a = \frac{1}{2}$ with a value of $\frac{1}{3}$. Note that for this case the bidding functions of the bidders are the simple linear functions $b(v) = \frac{v}{2}$. The graph of the expected payoff of the auctioneer as a function of a is given in Figure 2.

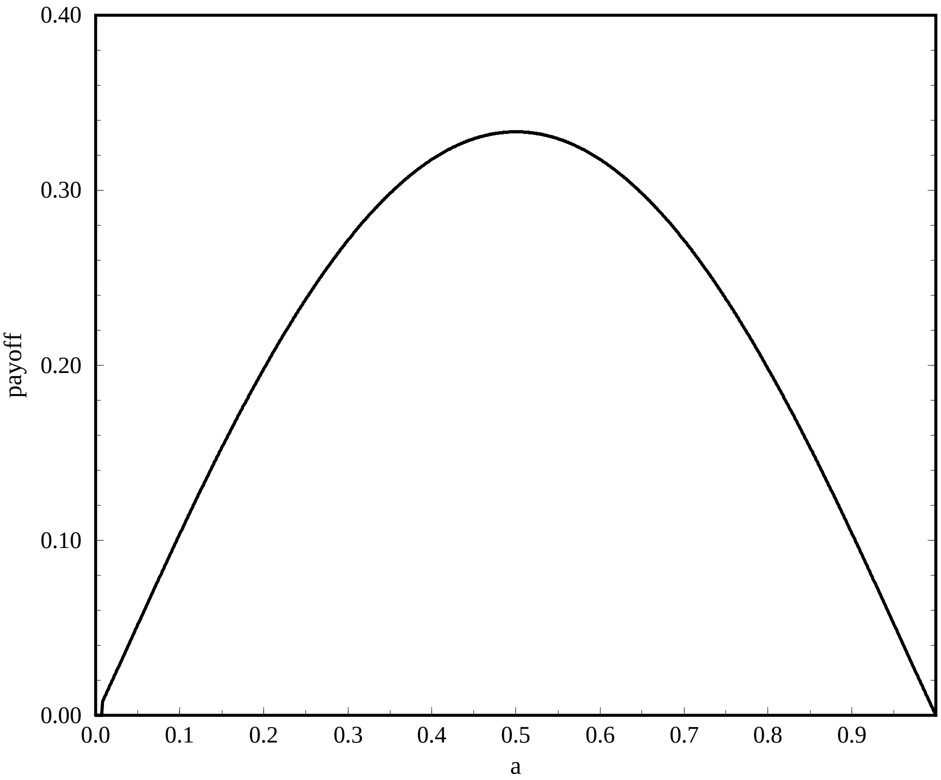

Figure 2: The auctioneer's expected profit as a function of a, when the bidders use the differentiable symmetric equilibrium bidding function with $b(1) = a$.

Next we compute the expected payoff each agent receives for different values of the announced price a, if they both use the symmetrical differentiable bidding function derived in this section. In general, the payoff function for each agent is

$$\Pi_1 : [0,1] \times [0,1] \;\to\; \mathbf{R} \tag{47}$$

$$(v_1, v_2) \;\mapsto\; \begin{cases} v_1 - b_1(a, v_1) & , \ \text{if } b_1(a,v_1) + b_2(a,v_2) \geq a \\ 0 & , \ \text{else.} \end{cases} \tag{48}$$

Hence the expected payoff for our case is

$$\mathbf{E}\Pi_1 = \int_0^1 \int_0^1 (v_1 - b_1(a, v_1)) \cdot 1_{\{(v_1,v_2)|b_1(a,v_1)+b_2(a,v_2)\geq a\}} \, dv_1 \, dv_2 \tag{49}$$

$$= \int_0^a \int_{a-t_2}^a (c_1(a,t_1) - t_1)\, c_1'(a,t_1)\, c_2'(a,t_2)\, dt_1\, dt_2 \tag{50}$$

$$\begin{aligned} &= \int_0^a \frac{1}{2}(c_1(a,a)^2 - c_1(a,a-t)^2)\, c_2'(a,t)\, dt \\ &\quad - \int_0^a (a\, c_1(a,a) - (a-t)\, c_1(a,a-t))\; c_2'(a,t)\, dt \\ &\quad + \int_0^a \int_{a-t_2}^a c_1(a,t_1)\, dt_1\, c_2'(a,t_2)\, dt_2 \end{aligned} \tag{51}$$

$$\begin{aligned} &= \frac{1}{2} - \frac{1}{2}\int_0^a c_1(a,a-t)^2\, c_2'(a,t)\, dt - a + \int_0^a (a-t)\, c_1(a,a-t)\, c_2'(a,t)\, dt \\ &\quad + \int_0^a \int_{a-t_2}^a c_1(a,t_1)\, dt_1\, c_2'(a,t_2)\, dt_2 \end{aligned} \tag{52}$$

where c_i is b_i^{-1}, and we used also $c_i(a,a) = 1, c_i(a,0) = 0$. Now we put $c := c_1(a,\cdot) = c_2(a,\cdot)$, use (44), and

$$\int_0^a t\, c(a-t)\, c'(t)\, dt = -\int_0^a (c(a-t) - t\, c'(a-t))\, c(t)\, dt \tag{53}$$

$$= -\int_0^a (c(a-t)\, c(t)\, dt + a^2 - \int_0^a t\, c(a-t)\, c'(t)\, dt \tag{54}$$

(by partial integration and (44)), which yields

$$\int_0^a t\, c(a-t)\, c'(t)\, dt = -\frac{1}{2}\int_0^a c(a-t)\, c(t)\, dt + \frac{1}{2}a^2. \tag{55}$$

This yields

$$\mathbf{E}\Pi_1 = \frac{1}{2} - \frac{1}{2}\int_0^a c(a-t)(c(a-t)\, c'(t) + c(t))\, dt - a + \frac{1}{2}a^2 + \int_0^a c(t)\, dt \tag{56}$$

(where we have performed partial integration on the fourth term and changed the order of integration of the fifth term in (52)). Now from the differential equation

$$c(a-t)c'(t)+c(t)=(a-t)\,c'(t)+1 \tag{57}$$

we obtain by (53)

$$\int_0^a c(a-t)(c(a-t)\,c'(t)+c(t))\,dt=\frac{a^2}{2}+\frac{1}{2}\int_0^a c(a-t)\,c(t)\,dt+\int_0^a c(t)\,dt, \tag{58}$$

which finally shows

$$\mathbf{E}\Pi_1=\frac{1}{2}-a+\frac{a^2}{4}-\frac{1}{4}\int_0^a c(a-t)\,c(t)\,dt+\frac{1}{2}\int_0^a c(t)\,dt \tag{59}$$

A graph of the expected payoff for each agent when both use the symmetrical differentiable bidding function is given in Figure 3.

When the auctioneer chooses $a=0.5$, the bidders have an (ex-ante) expected profit of $\frac{1}{6}$, using their equilibrium bidding functions $b(v)=\frac{v}{2}$.

4.2 Differentiable Symmetric Equilibria with $\mathbf{b(1)<a}$

In this section we seek symmetric equilibrium bidding functions when $\frac{a}{2}<b(1)<a$ (since if $b(1)<\frac{a}{2}$ the good is never provided, we do not need to investigate this case for characterization of the symmetric equilibrium bidding functions). Denote $d=b(1)$. We seek functions that are differentiable. Thus, they are strictly increasing for values in the range $[a-b(1),1]$ (from Lemma 5).

For b to be such a symmetric equilibrium bidding function, it must satisfy (9) for all $v\in[0,1]$. Since any bid of less than $a-d$ gives probability 0 of the good being provided, for $v\in[0,a-d)$ no bid gives a positive expected profit, and therefore bids for such v will be between 0 and $[a-d]$. For any higher value of v, i.e. $v\in[a-d,1)$, Equation (16) must be satisfied. Since b is strictly increasing, it has a unique inverse. Denoting the inverse by c, we derive the requirement that Equation (19) is satisfied for $t\in(a-d,d)$, by c which is differentiable on $(a-d,d)$, continuous on $[a-d,d]$, and that $c(a-d)=a-d, c(d)=1$.

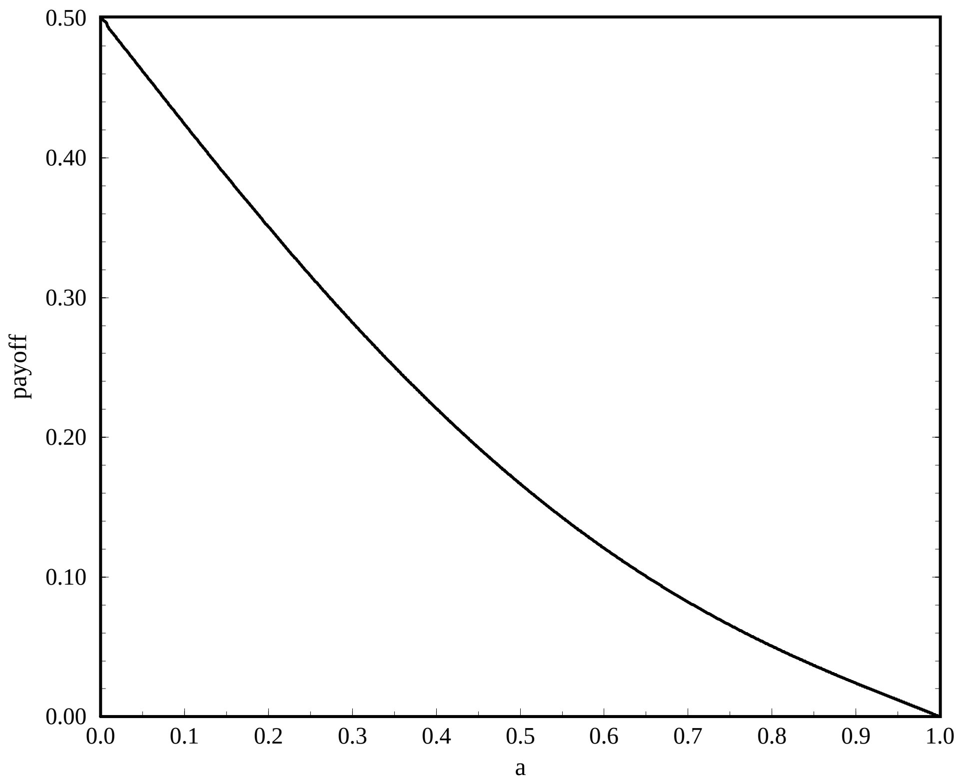

Figure 3: Agents' expected payoffs when using the differentiable symmetric equilibrium bidding function with $b(1) = a$.

Using the same method as in Theorem 1, it can be shown that the unique function c satisfying the above requirements is given by

$$c(t) = 1 + \frac{1 - a + ad - d^2}{a - 1 - t - (d - t)^{\frac{d-1}{2d-a}}(d - a + t)^{\frac{d-a+1}{2d-a}}}. \tag{60}$$

This gives us (by inversion) a function b from $[a - d, 1]$ to $[a - d, d]$. Note that if $d = a$, then (60) reduces to (20). We now need to determine possible values of b for private values in $[0, a - d)$. From Lemma 1 we know that for this range, $b(v) \leq a - d$. The binding constraint for equilibrium is that such a bid should not cause the other bidder to gain more by bidding more than d when he has a private value of 1. This constraint is satisfied if

$$(1 - (a - b(v)))(1 - v) \leq (1 - d)(1 - a + d) \quad \forall v \in [0, a - d], \tag{61}$$

which is equivalent to

$$b(v) \leq \frac{d(a-d)+v(1-a)}{1-v} \quad \forall v \in [0, a-d]. \tag{62}$$

Noting that $b(0) = 0$ and $b(a-d) = a-d$ satisfy (62), and that the left derivative of b at $a-d$ is bounded, it is obvious that many differentiable functions can be found that will satisfy (62) and will have a right derivative at $a-d$ equal to the left derivative of b there, giving us, together with the inverse of c for values of v in $[a-d, d]$, a differentiable symmetric equilibrium bidding function.

To summarize, we have

Theorem 2 *For any auctioneer's declaration $a \in (0,1)$, if $b : [0,1] \longrightarrow [0,a]$ is a differentiable function, then b is a symmetric equilibrium bidding function if and only if (denoting $d = b(1)$ and $c = b^{-1}$):*

1. *$\frac{a}{2} \leq d \leq a$.*
2. *For $t \in [a-d, d]$, c satisfies (60) and $c(a-d) = a-d$.*
3. *b satisfies (62), is differentiable for such values, and the right derivative at $(a-d)$ is equal to the left derivative of c^{-1} there.*

5 Step Equilibria

In actual auctions, bids are not real numbers but amounts of money. Each bid must be an integer multiple of a smallest unit. This motivates the approach in this section. We now examine a family of non-differentiable equilibria parameterized by the number of discontinuities. The limit, as the number of discontinuities goes to infinity, is a differentiable bidding function as described in the previous section. Therefore, as long as the basic unit of money is small enough, the results of the previous section are approximately valid. A step-function equilibria has bidding functions of the following form for $a \leq 1$

and integer $s \geq 1$

$$b^s(v) = \begin{cases} 0 & \text{if } 0 \leq v < c_1 \\ \frac{1}{s} \cdot a & \text{if } c_1 \leq v < c_2 \\ \vdots & \vdots \\ \frac{k}{s} \cdot a & \text{if } c_k \leq v < c_{k+1} \\ \vdots & \vdots \\ a & \text{if } c_s \leq v \leq 1 \end{cases} \tag{63}$$

Necessary conditions for such a function to be in equilibrium are that when the true value is equal to c_k for some $1 \leq k \leq s$ the agent is indifferent between bidding $\frac{(k-1)a}{s}$ and bidding $\frac{ka}{s}$. Formally, defining $c_0 = 0$,

$$\left(c_k - \frac{(k-1)a}{s}\right)(1 - c_{s-k+1}) = \left(c_k - \frac{ka}{s}\right)(1 - c_{s-k}) \quad 1 \leq k \leq s \tag{64}$$

The following theorem will show that if Equation (64) is satisfied for $0 = c_0 < c_1 < \ldots < c_k < \ldots < c_s \leq 1$, then b^s is an equilibrium bidding function.

Theorem 3 *If Equation (64) is satisfied for* $0 = c_0 < c_1 < \ldots < c_k < \ldots < c_s \leq 1$*, then* $b^s(v)$ *given by (63) is an equilibrium bidding function.*

Proof: Assume $c_k \leq v \leq c_{k+1}$. Thus, $b^s(v) = \frac{ka}{s}$. It is obvious that under the assumption that the other agent is using the bidding function b^s, any bid which is not a multiple of $\frac{a}{s}$ is dominated by a bid that is such a multiple. We need to show that the expected payoff from bidding $b^s(v)$ is at least as good as the expected gain from bidding any other multiple of $\frac{a}{s}$. The proof will be by induction. First we show that $\frac{ka}{s}$ is no worse an action than $\frac{(k-1)a}{s}$, i.e.

$$\left(v - \frac{(k-1)a}{s}\right)(1 - c_{s-k+1}) \leq \left(v - \frac{ka}{s}\right)(1 - c_{s-k}).$$

This is equivalent to

$$(v - c_k)(1 - c_{s-k+1}) + \left(c_k - \frac{(k-1)a}{s}\right)(1 - c_{s-k+1}) \leq$$

$$(v - c_k)(1 - c_{s-k}) + \left(c_k - \frac{ka}{s}\right)(1 - c_{s-k}).$$

The second terms of each side are equal from Equation (64), so we need to show that $(1 - c_{s-k+1}) \leq (1 - c_{s-k})$, which is true from the assumption that $c_i \leq c_j$ for $i < j$.

For the induction step we assume that bidding $\frac{ka}{s}$ is no worse than bidding $\frac{(k-m)a}{s}$ for $m \geq 1$ and show that it is no worse than bidding $\frac{(k-m-1)a}{s}$. From Equation (64) it is true that

$$(1 - c_{s-k+m}) = \frac{\left(c_{k-m+1} - \frac{(k-m-1)a}{s}\right)(1 - c_{s-k+m+1})}{\left(c_{k-m+1} - \frac{(k-m)a}{s}\right)}. \tag{65}$$

Our induction assumption is that

$$\left(v - \frac{ka}{s}\right)(1 - c_{s-k}) \geq \left(v - \frac{(k-m)a}{s}\right)(1 - c_{s-k+m}),$$

and we need to show that

$$\left(v - \frac{(k-m)a}{s}\right)(1 - c_{s-k+m}) \geq \left(v - \frac{(k-m-1)a}{s}\right)(1 - c_{s-k+m+1}).$$

This is equivalent (substituting from Equation (64)) to

$$\frac{\left(v - \frac{(k-m)a}{s}\right)\left(c_{k-m+1} - \frac{(k-m-1)a}{s}\right)(1 - c_{s-k+m+1})}{\left(c_{k-m+1} - \frac{(k-m)a}{s}\right)} \geq$$

$$\left(v - \frac{(k-m-1)a}{s}\right)(1 - c_{s-k+m+1}).$$

After simplifying, we are left with $v \geq c_{k-m+1}$ which is true from our assumptions. Thus we have shown that bidding $\frac{ka}{s}$ is no worse than anything smaller. Similarly, it can be shown that this bid is no worse than anything larger. ■(Theorem 3)

The following lemma shows that for a step bidding function that is a symmetric equilibrium, with an even number of steps s, the value of $c_{s/2}$ must be equal to a. Using this fact and a recursive algorithm, a computer program enabled us to calculate the values of the other c_i's for any even s. We also calculated equilibria for low odd s's by using brute force in solving Equation (64) for such s's, with the constraints that the c_i's are increasing and between 0 and 1.

Lemma 6 *If (a, b^s, b^s), with b^s given by (63), is a Nash equilibrium, and if s is even, then $c_{s/2} = a$.*

Proof: If (a, b^s, b^s) is an equilibrium, then Equation (64) holds. Defining $u_k = c_k - \frac{k}{s} \cdot a$ for $1 \leq k \leq s$, Equation (64) is equivalent to

$$u_k(u_{s-k+1} - u_{s-k}) = \frac{a}{s}\left(-u_k + 1 - \frac{s-k+1}{s} \cdot a - u_{s-k+1}\right), \tag{66}$$

hence

$$u_k u_{s-k} - u_{k-1} u_{s-k+1} = (u_k - u_{k-1})u_{s-k+1} - (u_{s-k+1} - u_{s-k})u_k = \tag{67}$$

$$\frac{a^2}{s^2}(s - 2k + 1). \tag{68}$$

If $s = 2n$ then

$$u_n^2 = u_0 u_2 n + \sum_{k=1}^{n} u_k u_{2n-k} - u_{k-1} u_{2n-k+1} = \tag{69}$$

$$0 + \sum_{k=1}^{n} \frac{a^2}{s^2}(s - 2k + 1) = \tag{70}$$

$$\frac{a^2}{4}, \tag{71}$$

so $(c_{s/2} - \frac{a}{2})^2 = \frac{a^2}{4}$ and therefore $c_{s/2} = a$. ∎(Lemma 6)

It is interesting to look at the limiting case when the steps are regular and the number of steps becomes large. As s goes to infinity, replacing $k\frac{a}{s}$ by t, c_k by $c(t)$, and $\frac{a}{s}$ by ε, (64) becomes

$$(c(t) - (t - \varepsilon))(1 - c(a - t + \varepsilon)) = (c(t) - t)(1 - c(a - t)). \tag{72}$$

Rearranging, we have

$$\frac{c(a - t + \varepsilon) - c(a - t)}{\varepsilon} = \frac{1 - c(a - t + \varepsilon)}{c(t) - t}. \tag{73}$$

Taking the limit as ε goes to zero, we get

$$c'(a-t) = \frac{1-c(a-t)}{c(t)-t}, \tag{74}$$

which is equivalent to (19), the relation for the differentiable case when $b(1) = a$. This case was solved analytically in Section 4.1.

A similar analysis can be given, with the appropriate modifications, for the case where $\frac{a}{2} < b(1) < a$, the case dealt with in Section 4.2.

6 Concluding Remarks

We conclude with some remarks and directions for future research.

1. There are a number of justifications for our emphasis on differentiable bidding functions. Many analyses of standard auctions assumed that equilibrium bidding functions were differentiable to derive their results. For example see Rothkopf (1969), Wilson (1969), Oren and Williams (1975), Wilson (1977), Rothkopf (1977) and Reece (1978). For more details about auction theory see McAfee and McMillan (1987) and Milgrom (1987). Even though in our model there do exist symmetric equilibria with non-differentiable bidding functions, we emphasize the case of differentiable functions. Chatterjee and Samuelson (1983) call such functions "well-behaved", and justify the emphasis placed on them with a number of arguments.

2. When the good is a public good for a group of participants, it seems reasonable in an auction setting to allow the whole group to pay their bids, as once the good is provided it benefits all members of the group[8]. This is the basis for the auction mechanism that we use, in which the good is provided to the group with the highest sum of bids (if this sum exceeds a minimum price declared by the auctioneer).

[8]In Bliss and Nalebuff's (1984) model of public good provision, the public good can be paid for only by one agent (in equilibrium the one that values it most).

3. Efficiency and profit maximizing are incompatible if we are restricted to incentive compatible schemes (which is a reasonable assumption in the auction setup), and the distribution of the private values for each agent includes zero. If the announced minimum price a is greater than zero (which must hold for profit maximization), then in any case where the sum of the private values is less than a the good will definitely not be provided in any equilibrium, which is not an efficient outcome.

4. We now give an example of a case where there is more than one group of agents, and the groups have a non-empty intersection. Consider the case where there are three agents and two groups. The agents are $\{1,2,3\}$ and the groups are $\{1,3\}$ and $\{2,3\}$. Each agent has a private value independently drawn from the uniform distribution over the unit interval. For agent 3, this is her value for both the groups to which she belongs. For this example, the auctioneer can make a positive profit even by announcing $a = 0$, as there is competition between agents 1 and 2. However, to induce agent 3 to make a positive bid, the auctioneer must announce a positive value for a, possibly leading to an inefficient outcome (and in some equilibria agent 3 always bids zero even for positive values of a). There appears to be scope for future research into situations like this, even though the mathematical complexity increases considerably.

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