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VARIATIONAL DISCRETIZATION OF LAVRENTIEV-REGULARIZED STATE CONSTRAINED ELLIPTIC CONTROL PROBLEMS

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Abstract. In the present work, we apply a variational discretization proposed by the first author in [14] to Lavrentiev-regularized state constrained elliptic control problems. We extend the results of [18] and prove weak convergence of the adjoint states and multipliers of the regularized problems to their counterparts of the original problem. Further, we prove error estimates for finite element discretizations of the regularized problem and investigate the overall error imposed by the finite element discretization of the regularized problem compared to the continuous solution of the original problem. Finally we present numerical results which confirm our analytical findings.

Key words. Optimal control of elliptic equations, quadratic programming, pointwise state constraints, mixed constraints, Lavrentiev regularization

AMS subject classifications. 49K20, 49N10, 49M20

1. Introduction. In the present work, we apply variational discretization proposed by the first author in [14] to Lavrentiev-regularized state-constrained elliptic control problems. Let $\Omega \subset \mathbb{R}^n (n = 2, 3)$ denote an open, bounded domain with $C^{0,1}$-boundary Γ. As model problem, we consider for states $y \in Y := H^1(\Omega) \cap C(\bar{\Omega})$ and controls $u \in L^2(\Omega)$

$$
\text{(P)} \quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2} \displaystyle\int_\Omega |y - y_d|^2 \, dx + \dfrac{\alpha}{2} \int_\Omega u^2 \, dx \\ \text{subject to} & y = S\,u \text{ and } y(x) \le y_c(x) \text{ a.e. in } \Omega, \end{cases}
$$

where $y_d \in L^2(\Omega)$, $y_c \in C(\bar{\Omega})$ denote given functions, and $S : L^2(\Omega) \to Y$ denotes the control-to-state mapping, i.e. the solution operator of the Neumann problem

$$
-\Delta\, y + y = u \text{ in } \Omega \text{ and } \partial_n y = 0 \text{ on } \Gamma.
$$

Associated to (P) is the Lavrentiev-regularized control problem

$$
(\mathrm{P}_\lambda) \quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2} \displaystyle\int_\Omega |y - y_d|^2 \, dx + \dfrac{\alpha}{2} \int_\Omega u^2 \, dx \\ \text{subject to} & y = S\,u \text{ and } \lambda u(x) + y(x) \le y_c(x) \text{ a.e. in } \Omega, \end{cases}
$$

where $\lambda > 0$ denotes the regularization parameter. Since the constraints in (P) and (P_λ), respectively, define closed convex sets, both problems admit unique solutions (y^*, u^*) and $(\bar{y}_\lambda, \bar{u}_\lambda)$.

The numerical treatment of problem (P) causes difficulties through the presence of the pointwise state constraints, since the corresponding Lagrange multiplier in general

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only represents a regular Borel measure (see Casas [7] or Alibert and Raymond [1]). In [18], Rösch, Tröltzsch, and the second author propose to circumvent these difficulties through approximating problem (P) by the family of problems (P_λ) ($\lambda > 0$). Among other things, they prove convergence of $(\bar y_\lambda, \bar u_\lambda) \to (y^*, u^*)$ in $L^2(\Omega)$ for $\lambda \to 0$. Furthermore, they show that the Lagrange multiplier associated to the mixed control–state constraint in (P_λ) is an L^2-function for every $\lambda > 0$. The development of numerical approaches to tackle problem (P) is ongoing [3, 17, 19]. An excellent overview can be found in [12, 13], where also further references are given.

Numerical analysis for problem (P) is presented by the first author and Deckelnick in [9]. Among other things, they prove convergence of finite element approximations to the control and to the state of order $1-\varepsilon$ in two-dimensions, and of order $1/2-\varepsilon$ in three dimensions, in L^2 and H^1, respectively. In [16], the second author obtains the same convergence order for piecewise constant approximations of the controls, and also extends these results to problems with additional box constraints on the control, compare also [11]. A general framework for numerical analysis of problems with pointwise state together with general constraints on the control is presented by Deckelnick and the first author in [10].

In the present paper, we extend the results of [18] for problem (P_λ) and prove weak convergence of the adjoint states p_λ in L^2 for λ tending to zero. Moreover, weak-$*$ convergence of the multipliers μ_λ in $C(\bar\Omega)^*$ to their counterparts of problem (P) for $\lambda \downarrow 0$ is shown. Based on these results, we prove error estimates for variational discrete approximations to problem (P_λ). More precisely, in Theorem 3.8, we show

$$\|\bar u_\lambda - \bar u_{\lambda,h}\| + \|\bar y_\lambda - \bar y_{\lambda,h}\|_{H^1} \le C h^{1-\frac{n}{4}}, \tag{1.1}$$

and

$$\|\bar u_\lambda - \bar u_{\lambda,h}\| + \|\bar y_\lambda - \bar y_{\lambda,h}\|_{H^1} \le C\,\frac{1}{\lambda^2}\left(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\right) \tag{1.2}$$

is proven in Theorem 3.5. Here, $n = 2, 3$ denotes the space dimension and C is a generic positive constant independent of the finite element grid size h and of λ. To prove the first estimate we adapt the techniques developed in [10] for the analysis of the limit problem (P). The key idea of the proof of the second estimate consists in the fact that the substitution

$$v(x) = \lambda\, u(x) + y(x) \tag{1.3}$$

transforms (P_λ) into the purely control constrained optimal control problem

$$(\mathrm{PV}) \quad \begin{cases} \text{minimize} & \tilde J(y,v) := \dfrac{1}{2}\,\|y - y_d\|^2 + \dfrac{\alpha}{2\,\lambda^2}\,\|v - y\|^2 \\ \text{subject to} & -\Delta\, y + c_\lambda\, y = \dfrac{1}{\lambda}\, v \quad \text{in } \Omega \\ & \qquad \partial_n\, y = 0 \qquad \text{on } \Gamma \\ \quad \text{and} & v(x) \le y_c(x) \text{ a.e. in } \Omega. \end{cases}$$

Here, $c_\lambda := 1 + 1/\lambda$. Since (PV) is a purely control-constrained problem, it admits a unique Lagrange multiplier in $L^2(\Omega)$ associated to the inequality constraint. Moreover, the discretization techniques developed in [14] are directly applicable to

(PV) which is of major importance for the implementation of a semi–smooth Newton method for the numerical solution of (PV) and (P_λ), respectively. Furthermore, we also relate the finite element solution $(\bar y_{\lambda,h}, \bar u_{\lambda,h})$ to (y^*, u^*), i.e. the solution of the original purely state-constrained problem (P). Under the additional assumption that the solutions u_λ of (P_λ) are uniformly bounded in $L^\infty(\Omega)$, it follows by combining a result of [19] with (1.1) that

$$\|u^* - \bar u_{\lambda,h}\| \le C\Big(\sqrt{\lambda} + \max\{h\,|\log(h)|, h^{2-n/2}\}\Big), \tag{1.4}$$

while its combination with (1.2) implies

$$\|u^* - \bar u_{\lambda,h}\| \le C\,\Big(\sqrt{\lambda} + \frac{1}{\lambda^2}\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)\Big). \tag{1.5}$$

In view of (1.4) and (1.5), the overall error consists of two different contributions: one arising from the regularization and another one caused by the discretization. Moreover, from (1.5), we deduce that both error contributions behave contrarily with respect to λ (cf. Remark 3.7) which is also confirmed by our numerical findings (see Section 4). Hence, the optimal value of λ for a given mesh size h is larger than zero, and (1.4) indicates that the coupling $\lambda \sim h^2$ in case of $n = 2$ and $\lambda \sim h$ in three dimensions might be optimal (see Remark 3.10). Indeed, this result is also confirmed by our numerical observations.

The paper is organized as follows. In Section 2 we prove that, beside control and state, also the adjoint state and the Lagrange multipliers converge in some weaker sense to the solution of the original problem. Section 3 addresses the error analysis for the regularized problems and investigates how to couple λ and h. In Section 4, the numerical example is presented.

1.1. Notation. Throughout this article, we use the following notation. Given an open, bounded set $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, we denote by $(.\,,\,.)$ the natural inner product of in $L^2(\Omega)$. The corresponding norm is denoted by $\|.\|$. Moreover, for the dual pairing between $C(\bar\Omega)$ and $C(\bar\Omega)^*$, we write $\langle .\,,\,.\rangle$.

2. Weak convergence of the Lagrange multipliers. In the present section we prove convergence of the adjoint states and of the Lagrange multipliers of problem (P_λ) to their counterparts of problem (P). For this purpose it is convenient to introduce the reduced objective functional by $f(u) = J(S\,u, u)$ and the Lagrange functional $\mathcal{L} : L^2(\Omega) \times C(\bar\Omega)^* \to \mathbb{R}$ by

$$\mathcal{L}(u, \mu) := f(u) + \langle S\,u - y_c\,,\, \mu\rangle.$$

Lagrange multipliers associated to the state constraint in (P) then are defined as follows:

Definition 2.1. *Let u^* denote the solution of* (P). *Then, $\mu \in C(\bar\Omega)^*$ is called Lagrange multiplier, if it satisfies the following conditions:*

$$\frac{\partial \mathcal{L}}{\partial u}(u^*, \mu) = f'(u^*) + S^*\mu = 0 \tag{2.1}$$

$$\langle S\,u^* - y_c\,,\, \mu\rangle = 0 \tag{2.2}$$

$$\langle y\,,\, \mu\rangle \ge 0 \quad \forall\, y \in C(\bar\Omega)^+, \tag{2.3}$$

where $C(\bar\Omega)^+$ *is defined by* $C(\bar\Omega)^+ = \{y \in C(\bar\Omega) \,|\, y(x) \geq 0 \ \forall\, x \in \bar\Omega\}$.

By means of the generalized Karush-Kuhn-Tucker theory, it can be proven that, under a certain Slater condition, problem (P) admits a Lagrange multiplier in $C^*(\bar\Omega)$ that satisfies the conditions in Definition 2.1 (see for instance Casas [7] or Alibert and Raymond [1]). This Slater condition in the present setting is equivalent to the existence of a $\hat u \in L^2(\Omega)$ with $(S\,\hat u)(x) < y_c(x)$ for all $x \in \bar\Omega$. Due to the special structure of the state equation, this is trivially fulfilled in our case, since every constant k with $k < y_c(x)$ everywhere in $\bar\Omega$, satisfies $(S\,k)(x) \equiv k < y_c(x)$ for all $x \in \bar\Omega$. Next, define $G : L^2(\Omega) \to L^2(\Omega)$ by the operator that arises if one considers the control-to-state operator as an operator with range in $L^2(\Omega)$, and set $p^* = G^*(G\,u^* - y_d) + S^*\,\mu$ such that $p^* \in L^2(\Omega)$. Casas [7] and Alibert and Raymond [1] proved that p^* is the unique very weak solution of

$$\begin{aligned} -\Delta\, p^* + p^* &= y^* - y_d + \mu|_\Omega \quad && \text{in } \Omega \\ \partial_n\, p^* &= \mu|_\Gamma && \text{on } \Gamma, \end{aligned} \tag{2.4}$$

that belongs to $W^{1,s}(\Omega)$, $1 < s < n/(n-1)$. With the definition of p^*, (2.1) is equivalent to

$$p^* + \alpha\, u^* = 0, \tag{2.5}$$

which implies in turn $u^* \in W^{1,s}(\Omega), 1 < s < n/(n-1)$. Notice that, together with the state equation and the pointwise state constraint, (2.2), (2.3), (2.4), and (2.5) are equivalent to the following optimality system

$$\left.\begin{aligned} &\begin{aligned} -\Delta\, y^* + y^* &= u^* \quad \text{in } \Omega \qquad & -\Delta\, p^* + p^* &= y^* - y_d + \mu_\Omega \quad \text{in } \Omega \\ \partial_n\, y^* &= 0 \quad\ \ \text{on } \Gamma & \partial_n\, p^* &= \mu_\Gamma \qquad\qquad\quad \text{on } \Gamma \end{aligned} \\ &\qquad\qquad \alpha\, u^* + p^* = 0 \\ &\int_{\bar\Omega} (y^* - y_c)\, d\mu = 0\,, \quad y^*(x) \leq y_c(x) \quad \forall\, x \in \bar\Omega \\ &\qquad \int_{\bar\Omega} y\, d\mu \geq 0 \quad \forall\, y \in C(\bar\Omega)^+, \end{aligned}\right\} \tag{2.6}$$

where μ_Ω and μ_Γ denote the restrictions of μ on Ω and Γ, respectively (cf. also [7] and [1]).

Based on the first-order necessary conditions for the auxiliary problem (PV) that was introduced in the introduction, it is straightforward to derive the optimality system for (P_λ). The latter is given by

$$\left.\begin{aligned} &\begin{aligned} -\Delta\, \bar y + \bar y &= \bar u \quad \text{in } \Omega \qquad & -\Delta\, p + p &= \bar y - y_d + \mu \quad \text{in } \Omega \\ \partial_n\, \bar y &= 0 \quad \text{on } \Gamma & \partial_n\, p &= 0 \qquad\qquad\quad \text{on } \Gamma \end{aligned} \\ &\quad \alpha\, \bar u(x) + p(x) + \lambda\, \mu(x) = 0 \quad \text{a.e. in } \Omega \\ &\qquad (\mu\,,\, \lambda\, \bar u + \bar y - y_c) = 0 \\ &\qquad \mu(x) \geq 0 \quad \text{a.e. in } \Omega \\ &\quad \lambda\, \bar u(x) \,+\, \bar y(x) \leq y_c(x) \quad \text{a.e. in } \Omega, \end{aligned}\right\} \tag{2.7}$$

where $(\bar{y}, \bar{u})$ denotes the unique optimal solution to (P_λ). Now, let us consider a sequence of positive real numbers $\{\lambda_n\}$ tending to zero for $n \to \infty$. The associated regularized problems are denoted by (P_n) and their solutions will be referred to as $(\bar{y}_n, \bar{u}_n) \in Y \times L^2(\Omega)$ with an adjoint state $p_n \in Y$ and Lagrange multiplier $\mu_n \in L^2(\Omega)$. In [18] and [17], it is proven that the control and the state converge strongly to the solution of (P), i.e.

$$\bar{u}_n \to u^* \quad \text{in } L^2(\Omega), \qquad \bar{y}_n \to y^* \quad \text{in } Y. \tag{2.8}$$

In the following, we establish corresponding convergence results for μ_n and p_n. It is clear that one cannot expect a result similar to (2.8) for μ_n as the multiplier in the limit is only an element of $C^*(\bar{\Omega})$. We start with the following lemma.

Lemma 2.2. *The sequence of Lagrange multipliers associated to the mixed constraint in* (P_n)*, denoted by* $\{\mu_n\}$*, is uniformly bounded in* $L^1(\Omega)$*.*

Proof. The variational formulation of the adjoint equation is given by

$$\int_\Omega \nabla p_n \cdot \nabla w \, dx + \int_\Omega p_n \, w \, dx = \int_\Omega (\bar{y}_n - y_d + \mu_n) w \, dx \quad \forall\, w \in H^1(\Omega).$$

If we insert $w \equiv 1$ as test function, then

$$\int_\Omega \mu_n \, dx = \int_\Omega (p_n - \bar{y}_n + y_d) \, dx = \int_\Omega (-\alpha \, \bar{u}_n - \lambda_n \, \mu_n - \bar{y}_n + y_d) \, dx$$

follows due to the gradient equation in (2.7). Together with the positivity of the Lagrange multiplier, this implies

$$\|\mu_n\|_{L^1(\Omega)} \le (1 + \lambda_n) \, \|\mu_n\|_{L^1(\Omega)} \le \alpha \, \|\bar{u}_n\| + \|\bar{y}_n\| + \|y_d\| \le C_\mu$$

with a constant C_μ independent of n since the optimality of $(\bar{y}_n, \bar{u}_n)$ implies their uniform boundedness in $L^2(\Omega)$. ☐

Lemma 2.3. *The sequence of Lagrange multipliers* $\{\mu_n\}$ *converges weakly-$*$ in* $C(\bar{\Omega})^*$ *to a weak-$*$ limit* $\tilde{\mu} \in C(\bar{\Omega})^*$*, i.e.*

$$\int_\Omega \mu_n \, w \, dx \to \langle w \,, \, \tilde{\mu} \rangle \quad \forall\, w \in C(\bar{\Omega})$$

Proof. First, let us identify the function $\mu_n \in L^2(\Omega)$ with an element $\tilde{\mu}_n$ in $C(\bar{\Omega})^*$ by defining

$$\langle w \,, \, \tilde{\mu}_n \rangle = \int_{\bar{\Omega}} w \, d\tilde{\mu}_n := \int_\Omega w \, \mu_n \, dx \quad \forall\, w \in C(\bar{\Omega}).$$

Using Lemma 2.2, we obtain

$$\|\tilde{\mu}_n\|_{C(\bar{\Omega})^*} = \sup_{\substack{g \in C(\bar{\Omega}) \\ g \neq 0}} \frac{|\langle g \,, \, \tilde{\mu}_n \rangle|}{\|g\|_{C(\bar{\Omega})}} = \sup_{\substack{g \in C(\bar{\Omega}) \\ g \neq 0}} \frac{\left| \int_\Omega g \, \mu_n \, dx \right|}{\|g\|_{C(\bar{\Omega})}} \le \|\mu_n\|_{L^1(\Omega)} \le C_\mu,$$

i.e. the uniform boundedness of $\{\tilde\mu_n\}$ in $C(\bar\Omega)^*$. Hence, since the closed unit ball in $C(\bar\Omega)^*$ is weakly-$*$ compact, we are allowed to select a subsequence, converging weakly-$*$ in $C(\bar\Omega)^*$ to a weak limit denoted by $\tilde\mu$. Because everything what follows is also valid for any other weakly-$*$ converging subsequence, a known argument yields the assertion. ∎

Based on the previous lemma, we are now in the position to discuss the convergence of $\{p_n\}$. We will see that it converges weakly in $L^2(\Omega)$ which is also important for the finite element error analysis in the subsequent section (see the proof of Lemma 3.4 below).

Lemma 2.4. *The sequence of adjoint states associated to* (P$_n$)*, denoted by* $\{p_n\}$*, converges weakly in* $L^2(\Omega)$ *to the solution of*

$$\begin{aligned} -\Delta\, p + p &= y^* - y_d + \tilde\mu|_\Omega && \text{in } \Omega\\ \partial_n\, p &= \tilde\mu|_\Gamma && \text{on } \Gamma, \end{aligned} \tag{2.9}$$

which is denoted by $\tilde p$ *in all what follows.*

Proof. Using again the identification of $\mu_n \in L^2(\Omega)$ with $\tilde\mu_n \in C(\bar\Omega)^*$, one obtains for a fixed, but arbitrary $w \in L^2(\Omega)$

$$\begin{aligned} (w\,,\,p_n) &= \big(w\,,\,G^*(\bar y_n - y_d + \mu_n)\big)\\ &= \big(w\,,\,G^*(\bar y_n - y_d)\big) + (w\,,\,S^*\,\tilde\mu_n)\\ &= (G\,w\,,\,\bar y_n - y_d) + \langle S\,w\,,\,\mu_n\rangle \;\to\; (G\,w\,,\,y^* - y_d) + \langle S\,w\,,\,\tilde\mu\rangle\\ &\qquad\qquad = \big(w\,,\,G^*(y^* - y_d) + S^*\,\tilde\mu\big) = (w\,,\,\tilde p), \end{aligned}$$

where we used Lemma 2.3 and $\bar y_n \to y^*$ in $L^2(\Omega)$. Since $w \in L^2(\Omega)$ was chosen arbitrarily, this is equivalent to $p_n \rightharpoonup \tilde p$. ∎

Next, it is shown that the weak-$*$ limit $\tilde\mu$ indeed represents a Lagrange multiplier for problem (P).

Theorem 2.5. *The sequence of Lagrange multipliers associated to the regularized pointwise state constraints in* (P$_\lambda$)*, denoted by* $\{\mu_n\}$*, converges weakly-$*$ in* $C(\bar\Omega)^*$ *to* $\tilde\mu$ *if* $n \to \infty$*. Moreover, the weak-$*$ limit* $\tilde\mu$ *is a Lagrange multiplier for the unregularized problem* (P) *according to Definition 2.1.*

Proof. The weak-$*$ convergence is stated in Lemma 2.3. It remains to show that the weak-$*$ limit satisfies the conditions in Definition 2.1, i.e. (2.1)–(2.3). Using Lemma 2.3, the positivity of $\tilde\mu$ is straightforward to show: the positivity property of μ_n in (2.7) implies

$$\int_\Omega \mu_n\, w\, dx \ge 0 \quad \forall\, w \in C(\bar\Omega)^+$$

with $C(\bar\Omega)^+$ as defined in Definition 2.1. Hence for every fixed, but arbitrary $w \in C(\bar\Omega)^+$, Lemma 2.3 yields

$$0 \le \int_\Omega \mu_n\, w\, dx \to \langle w\,,\,\tilde\mu\rangle$$

and thus (2.3).

To verify (2.1), we multiply the gradient equation in (2.7) with a fixed but arbitrary function $w \in C(\bar\Omega)$ and integrate over Ω:

$$\int_\Omega (\alpha\,\bar u_n + p_n)\,w\,dx + \lambda_n \int_\Omega \mu_n\,w\,dx = 0 \quad \forall\, w \in C(\bar\Omega). \tag{2.10}$$

In view of Lemma 2.3, we have $\int_\Omega \mu_n\,w\,dx \to \langle w\,,\,\tilde\mu\rangle$, and hence

$$\lambda_n \int_\Omega \mu_n\,w\,dx \to 0, \tag{2.11}$$

for every fixed, but arbitrary $w \in C(\bar\Omega)$, because of $\lambda_n \to 0$ for $n \to \infty$. Due to $\bar u_n \to u^*$ in $L^2(\Omega)$ and $p_n \rightharpoonup \tilde p$ in $L^2(\Omega)$, (2.11) implies for (2.10), when passing to the limit,

$$0 = \int_\Omega (\alpha\,\bar u_n + p_n)\,w\,dx + \lambda_n \int_\Omega \mu_n\,w\,dx \to \int_\Omega (\alpha\,u^* + \tilde p)\,w\,dx \quad \forall\, w \in C(\bar\Omega),$$

and hence, $\alpha\,u^* + \tilde p = 0$, where $\tilde p$ solves (2.9). However, as already stated in context of (2.5), this equation is equivalent to (2.1) in Definition 2.1, i.e. $f'(u^*) + S^*\tilde\mu = 0$.

It remains to prove the complementary slackness condition (2.2). The slackness conditions in (2.7) read

$$\int_\Omega \lambda_n\,\mu_n\,\bar u_n\,dx + \int_\Omega (\bar y_n - y_c)\mu_n\,dx = 0,$$

where the second addend converges to $\langle y^* - y_c\,,\,\tilde\mu\rangle$ thanks to Lemma 2.3 and $\bar y_n \to y^*$ in Y. Notice that one can of course not apply (2.11) to the first addend since $\{\bar u_n\}$ does clearly not converge in $C(\bar\Omega)$. However, the gradient equation in (2.7) implies

$$\int_\Omega \lambda_n\,\mu_n\,\bar u_n\,dx = -\int_\Omega \bar u_n\,(\alpha\,\bar u_n + p_n)\,dx \to 0,$$

due to $\bar u_n \to u^*$ in $L^2(\Omega)$ and $(\alpha\,\bar u_n + p_n) \rightharpoonup (\alpha\,u^* + \tilde p) = 0$ in $L^2(\Omega)$ as derived above. Therefore, we obtain

$$\langle y^* - y_c\,,\,\tilde\mu\rangle = 0,$$

which is equivalent to (2.2). $\square$

Remark 2.6. *In view of Lemma 2.5, $\tilde p$ is clearly an adjoint state associated to the original problem.*

2.1. The homogeneous Dirichlet case. Similarly to (P), one can discuss an analogous optimal control problem with homogeneous Dirichlet boundary conditions, i.e.

$$(\mathrm{Q})\quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2}\displaystyle\int_\Omega |y - y_d|^2\,dx + \dfrac{\alpha}{2}\displaystyle\int_\Omega u^2\,dx \\ \text{subject to} & -\Delta\,y = u \quad \text{in } \Omega \\ & \qquad y = 0 \quad \text{on } \Gamma \\ \qquad\text{and} & y(x) \le y_c(x) \quad \text{a.e. in } \Omega. \end{cases}$$

As will be seen subsequently, the weak-$*$ convergence of the Lagrange multipliers associated to the pointwise state constraints in (Q) can be proven similarly to the theory above. The main difference is the uniform $L^1(\Omega)$-boundedness of the multipliers which is established by Lemma 2.8 above. It is well known that the state equation in (Q) admits a unique solution y in the state space $Y := H_0^1(\Omega) \cap C(\bar{\Omega})$ for every $u \in L^2(\Omega)$. Again, we denote the associated control-to-state operator with range in $C(\bar{\Omega})$ by S and with range in $L^2(\Omega)$ by G. In view of the homogeneous Dirichlet boundary conditions, problem (Q) only is meaningful if $y_c(x) \geq 0$ everywhere on Γ. To satisfy the Slater condition for (Q), we further have to require $y_c(x) > 0$ for all $x \in \Gamma$. The Slater condition then reads

Assumption 2.7. *There exists a $\hat{u} \in L^2(\Omega)$ such that*

$$(S\,\hat{u})(x) < y_c(x) \quad \textit{for all } x \in \bar{\Omega}.$$

Notice that this condition need not be automatically fulfilled as in case of (P). However, if for instance $y_c(x) > 0$ everywhere in $\bar{\Omega}$, then the Slater condition is satisfied with $\hat{u} \equiv 0$. Based on Assumption 2.7, one can verify that the optimal control u^* satisfies the following optimality system (cf. for instance Casas [6]):

$$\left.\begin{aligned} &\begin{aligned} -\Delta\, y^* &= u^* \quad \text{in } \Omega & \qquad -\Delta\, p^* &= y^* - y_d + \mu \quad \text{in } \Omega\\ y^* &= 0 \quad\ \ \text{on } \Gamma & p^* &= 0 \qquad\qquad\quad\ \ \text{on } \Gamma \end{aligned}\\ &\qquad\qquad \alpha\, u^* + p^* = 0\\ &\int_{\bar{\Omega}} (y^* - y_c)\, d\mu = 0\,, \quad y^*(x) \leq y_c(x) \quad \forall\, x \in \bar{\Omega}\\ &\qquad \int_{\bar{\Omega}} y\, d\mu \geq 0 \quad \forall\, y \in C(\bar{\Omega})^+, \end{aligned}\right\} \tag{2.12}$$

where the Lagrange multiplier μ is again an element of $C(\bar{\Omega})^*$. In [6], it is shown that the adjoint equation admits a solution $p^* \in W^{1,s}$, $1 \leq s < n/(n-1)$. Notice that the adjoint equation exhibits homogeneous Dirichlet boundary conditions, i.e. the multiplier does not generate a measure on Γ. This is due to the fact that the singular part of μ is concentrated on the boundary of the active set which was proven by Bergounioux and Kunisch in [4]. Hence, thanks to the Slater condition which ensures that the state constraint is inactive on the boundary, we have $\mu_\Gamma = 0$ (see also [6]).

As above, we introduce the regularized counterpart of (Q) by

$$(\mathrm{Q}_\lambda) \quad \begin{cases} \text{minimize} & J(y,u) := \dfrac{1}{2}\,\|y - y_d\|^2 + \dfrac{\alpha}{2}\,\|u\|^2\\ \text{subject to} & -\Delta\, y = u \quad \text{in } \Omega\\ & \qquad y = 0 \quad \text{on } \Gamma\\ \text{and} & \lambda\, u(x) + y(x) \leq y_c(x) \quad \text{a.e. in } \Omega. \end{cases}$$

By the same arguments as in case of (P_λ), this problem exhibits a regular Lagrange multiplier in $L^2(\Omega)$. Similarly to (2.7), the optimality system, satisfied by the unique optimal solution $(\bar y, \bar u)$, is given by

$$\left.\begin{array}{c}
\begin{array}{rlrl}
-\Delta\,\bar y = \bar u & \text{in } \Omega \qquad & -\Delta\, p = \bar y - y_d + \mu & \text{in } \Omega \\
\bar y = 0 & \text{on } \Gamma & p = 0 & \text{on } \Gamma
\end{array}\\
\alpha\,\bar u(x) + p(x) + \lambda\,\mu(x) = 0 \quad \text{a.e. in } \Omega\\
(\mu\,,\,\lambda\,\bar u + \bar y - y_c) = 0\\
\mu(x) \geq 0 \quad \text{a.e. in } \Omega\\
\lambda\,\bar u(x) \,+\, \bar y(x) \leq y_c(x) \quad \text{a.e. in } \Omega.
\end{array}\right\} \tag{2.13}$$

As in the section above, we consider a sequence of regularization parameters tending to zero, i.e. $\{\lambda_n\}$ with $\lambda_n \to 0$ for $n \to \infty$. The associated regularized control problems as well as their solutions and the corresponding adjoint states and Lagrange multipliers are again referred to by the subscript n. It is easy to see that the analysis in [17] that yields the strong convergence of $\bar u_n$ to u^* in $L^2(\Omega)$ and $\bar y_n$ to y^* in Y, respectively, can be adapted to the case with homogeneous Dirichlet boundary conditions. To be more precise, the theory in [17] is mainly based on the fact that $G : L^2(\Omega) \to L^2(\Omega)$ is compact and self adjoint, which is clearly also fulfilled in case of (Q). For the adjoint state and the Lagrange multiplier, we derive a result analogous to Lemma 2.5 and Remark 2.6. We again start with the boundedness of the multipliers that follows from the Slater condition in assumption 2.7.

Lemma 2.8. *Under Assumption 2.7, the sequence of Lagrange multipliers $\{\mu_n\}$ is uniformly bounded in $L^1(\Omega)$.*

Proof. Together with the maximum principle for the state equation, Assumption 2.7 yields the existence of a function $u_0 \in L^2(\Omega)$ with $u_0(x) \leq 0$ a.e. in Ω and $(S\,u_0)(x) < y_c(x)$ for all $x \in \bar\Omega$. Thus, there is a $\tau > 0$ such that, for all $\lambda \geq 0$,

$$\lambda\,u_0(x) + (S\,u_0)(x) \leq y_c(x) - \tau \quad \text{a.e. in } \Omega, \tag{2.14}$$

i.e. u_0 is a Slater point for the regularized problem (Q_λ), $\lambda \geq 0$. Next, let us define an auxiliary sequence $\{\hat u_n\}$ by

$$\hat u_n = u_0 - \bar u_n.$$

Together with (2.14), this definition immediately implies

$$-\big(\lambda_n \hat u_n(x) + (S\,\hat u_n)(x)\big) \geq \tau + \lambda_n \bar u_n(x) + (S\,\bar u_n)(x) - b(x) \quad \text{a.e. in } \Omega. \tag{2.15}$$

The gradient equation in (2.13) is equivalent to

$$\int_\Omega (\alpha\,\bar u_n + G^*(G\bar u_n - y_d + \mu_n) + \lambda_n \mu_n) u\,dx = 0 \quad \text{for all } u \in L^2(\Omega).$$

If we now choose $u = \hat u$, we obtain

$$\int_\Omega -(\lambda_n\,\hat u_n + G\,\hat u_n)\mu_n\,dx = \int_\Omega (\alpha\,\bar u_n + G^*(G\bar u_n - y_d))\hat u\,dx.$$

Together with the complementary slackness condition, i.e.

$$\int_\Omega (\lambda_n \bar u_n + G\,\bar u_n - b)\mu_n\,dx = 0$$

and (2.15), this gives in turn

$$\int_\Omega \tau\, \mu_n\, dx \leq \Big((\alpha + \|G\|^2) \|\bar u_n\| + \|G\|\, \|y_d\| \Big) \big(\|u_0\| + \|\bar u_n\| \big).$$

Due to the uniform boundedness of $\{\bar u_n\}$ in $L^2(\Omega)$ that follows from the optimality of $\bar u_n$, this and the positivity property of μ_n imply the assertion. ∎

For the rest of the proof, we can proceed as in case of the homogeneous Neumann boundary conditions, since the underlying analysis does not depend on the particular structure of the state equation. In this way, one obtains the following result:

THEOREM 2.9. *Suppose that Assumption 2.7 holds true and let $\{\mu_n\}$ denote the sequence of Lagrange multipliers associated to the regularized pointwise state constraints in* (Q_λ)*, while $\{p_n\}$ is the sequence of adjoint states. Then*

$$\mu_n \overset{*}{\rightharpoonup} \tilde\mu \quad \textit{in } C(\bar\Omega)^* \qquad \textit{and} \qquad p_n \rightharpoonup \tilde p \quad \textit{in } L^2(\Omega)$$

hold true, where $\tilde\mu \in C(\bar\Omega)^*$ is a Lagrange multiplier for (Q) *in the sense of Definition 2.1 and $\tilde p \in W^{1,s}(\Omega)$, $1 \leq s < n/(n-1)$, solves the adjoint equation in* (2.12) *with $\tilde\mu$ on the right-hand side.*

Now, we turn to the impact of the Lavrentiev regularization on the numerical treatment of state-constrained optimal control problems. To be more precise, we discuss the variational discretization of the regularized problem in the spirit of [14]. The analysis is carried out for problem (P), i.e. the problem with homogeneous Neumann boundary conditions. Nevertheless, it is easy to verify that the same arguments apply in case of (Q) such that the error estimates in Theorem 3.5 and Theorem 3.8 also hold for homogeneous Dirichlet boundary conditions.

3. Error analysis for the regularized problem.

In the following, we discuss a variational discretization of problem (P_λ) according to the approach proposed in [14]. To that end, let us introduce a family of regular triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω, i.e. $\bar\Omega = \bigcup_{T\in\mathcal{T}_h} \bar T$. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $R(T)$, where $\rho(T)$ denotes the diameter of the set T and $R(T)$ is the diameter of the largest ball contained in T. The mesh size of $\mathcal{T}_h$ is defined by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. For the upcoming error analysis, we have to require some additional conditions on $\mathcal{T}_h$ and the domain.

ASSUMPTION 3.1. *The domain Ω is a open bounded and convex subset of $\mathbb{R}^n$, $n = 2, 3$ and its boundary Γ is a polygon ($n = 2$) or a polyhedron ($n = 3$). Moreover, there exist two positive constants ρ and R such that*

$$\frac{\rho(T)}{R(T)} \leq R\,, \quad \frac{h}{\rho(T)} \leq \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$. Furthermore, the regularization parameter is bounded from above by by a constant $\lambda_{\max} < \infty$.

Notice that the last assumption on the values for λ is not really restrictive, since our aim is to approximate the original state-constrained problem (P).

For domains satisfying Assumption 3.1, one finds the following result (cf. for instance Dauge [8]):

Lemma 3.2. *Suppose that Ω fulfills the condition in Assumption 3.1 and let f be a given function in $L^2(\Omega)$, while z solves*

$$\begin{aligned} -\Delta z + z &= f \quad \text{in } \Omega \\ \partial_n z &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{3.1}$$

Then, $z \in H^2(\Omega)$ and the estimate

$$\|z\|_{H^2(\Omega)} \le c\,\|f\|$$

holds true with a constant c independent of f and h.

The finite element approximation for (3.1) is given by

$$(\nabla z_h\,,\,\nabla w_h) + (z_h\,,\,w_h) = (f\,,\,w_h) \quad \forall\, w_h \in W_h,$$

where W_h denotes the space of linear finite elements, i.e. $W_h = \{w \in C(\bar{\Omega}) \mid w|_T \in \mathcal{P}_1\ \forall\, T \in \mathcal{T}_h\}$. Standard finite element error analysis yields

$$\|z - z_h\| \le c\,h^2\,\|f\| \tag{3.2}$$

$$\|z - z_h\|_{L^\infty(\Omega)} \le c\,h^{2-n/2}\,\|f\|, \tag{3.3}$$

where, as above, n denotes the spatial dimension. Applying the finite element approximation to the state equation, we arrive at the variational discrete version of (P_λ), which is then given by

$$(\mathrm{P}_{\lambda,h}) \quad \begin{cases} \displaystyle\min_{u\in L^2(\Omega)} & J(y_h,u) := \dfrac{1}{2}\,\|y_h - y_d\|^2 + \dfrac{\alpha}{2}\,\|u\|^2 \\ \text{subject to} & (\nabla y_h\,,\,\nabla w_h) + (y_h\,,\,w_h) = (u\,,\,w_h) \quad \forall\, w_h \in W_h \\ \text{and} & \lambda\,u(x) + y_h(x) \le y_c(x) \text{ a.e. in } \Omega. \end{cases}$$

Note that we do not discretize the control u. The optimal solution of $(\mathrm{P}_{\lambda,h})$ is denoted by $\bar{u}_h$ while, as above, $\bar{u}$ denotes the solution of (P_λ) in all what follows. Notice that in general $\bar{u}_h \notin W_h$. In the following, we derive two different error estimates for $\|\bar{u} - \bar{u}_h\|$ the first one depending on λ whereas the second one is uniform in λ.

3.1. An error estimate for fixed λ. The overall error analysis of this section is based on a consideration of the transformed problem (PV) with the auxiliary control v. Based on $(\mathrm{P}_{\lambda,h})$ and the transformation formula (1.3), which reads

$$v = \lambda\,u + y_h \tag{3.4}$$

n the discrete setting, we obtain for the variational discrete counterpart of (PV)

$$(\mathrm{PV}_h) \quad \begin{cases} \displaystyle\min_{v\in L^2(\Omega)} & \tilde{J}(y_h,v) := \dfrac{1}{2}\,\|y_h - y_d\|^2 + \dfrac{\alpha}{2\,\lambda^2}\,\|v - y_h\|^2 \\ \text{subject to} & (\nabla y_h\,,\,\nabla w_h) + c_\lambda\,(y_h\,,\,w_h) = \dfrac{1}{\lambda}\,(v\,,\,w_h) \quad \forall\, w_h \in W_h \\ \text{and} & v(x) \le y_c(x) \text{ a.e. in } \Omega. \end{cases}$$

The optimal solution of (PV_h) is denoted by $\bar v_h$ and again, in general $\bar v_h \notin W_h$. Notice that (PV_h) coincides with the variational discretization of purely control-constrained problems following the approach of [14]. Now let us turn to the optimality conditions for (PV) and (PV_h). In a standard way, one deduces

$$(\nabla \bar y\,,\, \nabla w) + c_\lambda\,(\bar y\,,\, w) = (\bar v\,,\, w) \quad \forall\, w \in H^1(\Omega) \tag{3.5}$$

$$(\nabla p\,,\, \nabla w) + c_\lambda\,(p\,,\, w) = \big(\bar y - y_d + \frac{\alpha}{\lambda^2}\,(\bar y - \bar v)\,,\, w\big) \quad \forall\, w \in H^1(\Omega) \tag{3.6}$$

$$\big(\bar v - \bar y + \frac{\lambda}{\alpha}\, p\,,\, v - \bar v\big) \geq 0 \quad \forall\, v \in V_{ad} \tag{3.7}$$

with $V_{ad} := \{v \in L^2(\Omega)\,|\, v(x) \leq y_c(x) \text{ a.e. in } \Omega\}$. For the optimality system of (PV_h), we find

$$(\nabla \bar y_h\,,\, \nabla w_h) + c_\lambda\,(\bar y_h\,,\, w_h) = (\bar v_h\,,\, w_h) \quad \forall\, w_h \in W_h \tag{3.8}$$

$$(\nabla p_h\,,\, \nabla w_h) + c_\lambda\,(p_h\,,\, w_h) = \big(\bar y_h - y_d + \frac{\alpha}{\lambda^2}\,(\bar y_h - \bar v_h)\,,\, w_h\big) \quad \forall\, w_h \in W_h \tag{3.9}$$

$$\big(\bar v_h - \bar y_h + \frac{\lambda}{\alpha}\, p_h\,,\, v - \bar v_h\big) \geq 0 \quad \forall\, v \in V_{ad}. \tag{3.10}$$

Due to the variational discrete approach, the solution $\bar v$ of (PV) is feasible for (PV_h) and therefore, we are allowed to insert $\bar v$ in the variational inequality (3.10). On the other hand, we insert $\bar v_h$ in (3.7). Adding both inequalities then yields

$$\big(\bar v - \bar v_h - (\bar y - \bar y_h) + \frac{\lambda}{\alpha}\,(p - p_h)\,,\, \bar v_h - \bar v\big) \geq 0,$$

which in turn gives

$$\begin{aligned} 0 \leq -\|\bar v - \bar v_h\|^2 &+ \big(y_h(\bar v) - \bar y\,,\, \bar v_h - \bar v\big) + \frac{\lambda}{\alpha}\,\big(p - p^h(\bar v)\,,\, \bar v_h - \bar v\big) \\ &+ \underbrace{\frac{\lambda}{\alpha}\,\big(p^h(\bar v) - p_h(\bar v)\,,\, \bar v_h - \bar v\big)}_{=:\,A} \\ &+ \underbrace{\big(\bar y_h - y_h(\bar v)\,,\, \bar v_h - \bar v\big) + \frac{\lambda}{\alpha}\,\big(p_h(\bar v) - p_h\,,\, \bar v_h - \bar v\big)}_{=:\,B} \end{aligned} \tag{3.11}$$

Here, the notation $y(v)$ with an arbitrary $v \in L^2(\Omega)$ corresponds to the solution of

$$(\nabla y\,,\, \nabla w) + c_\lambda\,(y\,,\, w) = \frac{1}{\lambda}\,(v\,,\, w) \quad \forall\, w \in H^1(\Omega), \tag{3.12}$$

while $y_h(v)$ solves

$$(\nabla y_h\,,\, \nabla w_h) + c_\lambda\,(y_h\,,\, w_h) = \frac{1}{\lambda}\,(v\,,\, w_h) \quad \forall\, w_h \in W_h. \tag{3.13}$$

Moreover, $p^h(v)$ is defined as solution of

$$(\nabla p^h\,,\, \nabla w_h) + c_\lambda\,(p^h\,,\, w_h) = \big(y(v) - y_d + \frac{\alpha}{\lambda^2}\,(y(v) - v)\,,\, w_h\big) \quad \forall\, w_h \in W_h \tag{3.14}$$

and similarly, $p_h(v)$ denotes the solution to

$$(\nabla p_h\,,\,\nabla w_h)+c_\lambda\,(p_h\,,\,w_h) = \big(y_h(v)-y_d+\frac{\alpha}{\lambda^2}\,(y_h(v)-v)\,,\,w_h\big) \quad \forall\, w_h\in W_h \tag{3.15}$$

Notice that, with these notations at hand, we have $\bar y = y(\bar v)$, $\bar y_h = y_h(\bar v_h)$, $p = p(\bar v)$, and $p_h = p_h(\bar v_h)$. Before we further exploit (3.11) let us provide some auxiliary results. To begin with we consider

$$(\nabla z\,,\,\nabla w) + c_\lambda\,(z\,,\,w) = (g\,,\,w) \quad \forall\, w\in H^1(\Omega) \tag{3.16}$$

with some $g\in L^2(\Omega)$. Similarly to above, we introduce the discrete version of (3.16) by

$$(\nabla z_h\,,\,\nabla w_h) + c_\lambda\,(z_h\,,\,w_h) = (g\,,\,w_h) \quad \forall\, w\in W_h$$

and denote the associated solution by $z_h(g)$. Now, we derive an estimate analogous to (3.2) which takes into account the dependency on λ.

LEMMA 3.3. *Under Assumption 3.1, there exists a constant $C(\Omega)$ independent of λ such that*

$$\|z_h(g)-z(g)\|_{L^2(\Omega)} \le C(\Omega)\,\big(h^2+\frac{1}{\lambda}\,h^3+\frac{1}{\lambda^2}\,h^4\big)\,\|z(g)\|_{H^2(\Omega)}$$

holds true.

Proof. The proof follows standard arguments. Using the Galerkin orthogonality and standard interpolation error estimates, one obtains

$$\begin{aligned}\|z_h(g)-z(g)\|_{H^1(\Omega)} &\le \|z(g)-I_hz(g)\|_{H^1(\Omega)} + \frac{1}{\lambda}\,\|z(g)-I_hz(g)\|\\ &\le C(\Omega)\,\big(h+\frac{1}{\lambda}\,h^2\big)\,\|z(g)\|_{H^2(\Omega)}\end{aligned}$$

where I_h denotes the linear interpolation operator. Applying the well known argument according to Nitsche then gives the assertion. □

LEMMA 3.4. *Suppose that Assumption 3.1 is fulfilled. Then there exists a constant $C(\Omega)$ independent of λ such that the following estimate is valid*

$$\|y_h(\bar v)-\bar y\| \le C(\Omega)\,\big(h^2+\frac{1}{\lambda}\,h^3+\frac{1}{\lambda^2}\,h^4\big). \tag{3.17}$$

In addition

$$\lambda\,\|p^h(\bar v)-p\| \le C(\alpha,\Omega)\,\big(h^2+\frac{1}{\lambda}\,h^3+\frac{1}{\lambda^2}\,h^4\big) \tag{3.18}$$

holds true with a constant $C(\alpha,\Omega)$ independent of λ.

Proof. By construction, $\bar y = y(\bar v)$ is also the solution of the state equation in (P_λ) with $\bar u = 1/\lambda\,(\bar v-\bar y)$ on the right hand side, i.e. it solves (3.1) with $\bar u$ as inhomogeneity. Therefore, Lemma 3.2 together with (2.8) yields

$$\|\bar y\|_{H^2(\Omega)} \le c\,\|\bar u\| \le c,$$

where the optimality of $\bar{u}$ guarantees its uniform boundedness w.r.t. λ in $L^2(\Omega)$. Together with Lemma 3.3, this implies (3.17).

Moreover, again due to (1.3), i.e. $\bar{u} = 1/\lambda\,(\bar{v} - \bar{y})$, the adjoint state solves

$$\begin{aligned} -\Delta\, p + p &= \bar{y} - y_d - \frac{1}{\lambda}\, p + \frac{\alpha}{\lambda}\,\bar{u} && \text{in } \Omega \\ \partial_n\, p &= 0 && \text{on } \Gamma, \end{aligned}$$

and hence, again by Lemma 3.2,

$$\lambda\,\|p\|_{H^2(\Omega)} \le c\,\Big(\lambda\,\|\bar{y}\| + \lambda\,\|y_d\| + \alpha\,\|\bar{u}\| + \|p\|\Big)$$

follows with a constant c independent of λ. Thanks to their optimality, $\bar{u}$ and $\bar{y}$ are uniformly bounded in $L^2(\Omega)$ independent of λ. Moreover, consider again an arbitrary sequence $\{\lambda_n\}$ tending to zero for $n \to \infty$. Then, from Lemma 2.4, we know that the associated sequence of adjoint states converges weakly in $L^2(\Omega)$, giving in turn its uniform boundedness such that $\|p\| \le c$ independent of λ. Thus, we obtain $\lambda\,\|p\|_{H^2(\Omega)} \le c$ and consequently, Lemma 3.3 gives the assertion. ∎

THEOREM 3.5. *Suppose that Assumption 3.1 is fulfilled. Then, there is a constant $C(\alpha, \Omega, \lambda_{\max})$ independent of λ such that*

$$\|\bar{u} - \bar{u}_h\| + \|\bar{y} - \bar{y}_h\|_{H^1(\Omega)} \le C(\alpha, \Omega, \lambda_{\max})\,\frac{1}{\lambda^2}\,\Big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\Big) \tag{3.19}$$

is satisfied.

Proof. The result will be obtained by estimating the addends on the right hand side of (3.11) by means of the above lemmata. We start with (3.15) with $\bar{v}$ as inhomogeneity and subtract the analogous equation for $\bar{v}_h$ on the right hand side. This gives

$$\begin{aligned} \Big(\nabla\big[p_h(\bar{v}) - p_h(\bar{v}_h)\big]\,,\,\nabla w_h\Big) + c_\lambda\,\Big(p_h(\bar{v}) - p_h(\bar{v}_h)\,,\,w_h\Big) = \\ \Big(y_h(\bar{v}) - y_h(\bar{v}_h) + \frac{\alpha}{\lambda^2}\,\big(y_h(\bar{v}) - y_h(\bar{v}_h) - \bar{v} + \bar{v}_h\big)\,,\,w_h\Big) \quad \forall\, w_h \in W_h. \end{aligned} \tag{3.20}$$

We note that by definition $\bar{y}_h = y_h(\bar{v}_h)$ and $p_h(\bar{v}_h) = p_h$. Now we consider (3.13) with $\bar{v}_h - \bar{v}$ as right hand side, use $p_h(\bar{v}) - p_h$ there as test function, and choose $\bar{y}_h - y_h(\bar{v})$ as test function in (3.20). Next we form the difference of the arising equations and obtain

$$\begin{aligned} &\frac{1}{\lambda}\,\Big(p_h(\bar{v}) - p_h\,,\,\bar{v}_h - \bar{v}\Big) \\ &\qquad = \Big(y_h(\bar{v}) - \bar{y}_h + \frac{\alpha}{\lambda^2}\,\big(y_h(\bar{v}) - \bar{y}_h - \bar{v} + \bar{v}_h\big)\,,\,\bar{y}_h - y_h(\bar{v})\Big), \end{aligned} \tag{3.21}$$

so that B of (3.11) admits the form

$$B = -\Big(1 + \frac{\lambda^2}{\alpha}\Big)\,\|y_h(\bar{v}) - \bar{y}_h\|^2 + 2\,\big(\bar{y}_h - y_h(\bar{v})\,,\,\bar{v}_h - \bar{v}\big) \tag{3.22}$$

Similarly we obtain for A of (3.11)

$$A = \Big(1 + \frac{\lambda^2}{\alpha}\Big)\,\big(\bar{y} - y_h(\bar{v})\,,\,y_h(\bar{v}_h) - y_h(\bar{v})\big). \tag{3.23}$$

Inserting (3.22) and (3.23) into (3.11), straight-forward estimation yields

$$\begin{aligned}0 \le &- \Big[\|\bar v - \bar v_h\|^2 - 2(\bar y - \bar y_h\,,\,\bar v - \bar v_h) + \|\bar y - \bar y_h\|^2\Big] - \frac{\lambda^2}{\alpha}\,\|\bar y - \bar y_h\|^2\\ &+ \big(\bar y - y_h(\bar v)\,,\,\bar v_h - \bar v\big) + \frac{\lambda}{\alpha}\,\big(p - p^h(\bar v)\,,\,\bar v_h - \bar v\big)\\ &- \Big(1 + \frac{\lambda^2}{\alpha}\Big)\big(y_h(\bar v) - \bar y\,,\,\bar y - \bar y_h\big).\end{aligned}$$

Notice that, in view of the transformation formulas (1.3) and (3.4), the term in the squared brackets is equivalent to $\lambda^2\|\bar u - \bar u_h\|^2$. Hence, we replace $\bar v$ and $\bar v_h$ by $\bar u$ and $\bar u_h$, respectively, and obtain

$$\begin{aligned}\alpha\,\|\bar u - \bar u_h\|^2 + \|\bar y - \bar y_h\|^2 \le &\frac{\alpha}{\lambda}\,\big(\bar y - y_h(\bar v)\,,\,\bar u_h - \bar u\big) + \big(p - p^h(\bar v)\,,\,\bar u_h - \bar u\big)\\ &+ \frac{1}{\lambda}\,\big(p - p^h(\bar v)\,,\,\bar y_h - \bar y\big) - \big(y_h(\bar v) - \bar y\,,\,\bar y_h - \bar y\big).\end{aligned}$$

Using Young's inequality we arrive at

$$\begin{aligned}&(\alpha - 2\kappa)\,\|\bar u - \bar u_h\|^2 + (1 - 2\kappa)\|\bar y - \bar y_h\|^2\\ &\qquad\le \Big(\frac{\alpha^2}{\kappa\lambda^2} + \frac{1}{\kappa}\Big)\,\|\bar y - y_h(\bar v)\|^2 + \Big(\frac{1}{\kappa\lambda^2} + \frac{1}{\kappa\lambda^4}\Big)\,\lambda^2\,\|p - p^h(\bar v)\|^2\end{aligned}$$

with $\kappa > 0$ arbitrary. Lemma 3.4 now yields

$$\begin{aligned}&(\alpha - 2\kappa)\,\|\bar u - \bar u_h\|^2 + (1 - 2\kappa)\|\bar y - \bar y_h\|^2\\ &\qquad\le C(\alpha, \Omega, \lambda_{\max})\,\frac{1}{\kappa\lambda^4}\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)^2,\end{aligned}$$

so that choosing κ small enough delivers the result for $\|\bar u - \bar u_h\|$. The estimate for $\|\bar y - \bar y_h\|_{H^1(\Omega)}$ follows from the continuity of the control-to-state operator together with (3.2) and the optimality of $\bar u$ which implies $\|\bar u\|_{L^2(\Omega)} \le c$ independent of λ. ☐

In view of Theorem 3.5, we thus obtain quadratic convergence of the control for a fixed λ as in case of the purely control-constrained case discussed in [14]. On the other hand, the approximation behavior of the solution of $(\mathrm{P}_{\lambda,h})$ strongly depends on the value of λ. For the overall approximation error, we find

$$\|u^* - \bar u_{\lambda,h}\| \le \|u^* - \bar u_\lambda\| + \|\bar u_\lambda - \bar u_{\lambda,h}\|,$$

where, as before, u^* denotes the solution of the original purely state-constrained problem (P). Moreover, $\bar u_\lambda$ is the exact solution of (P_λ) for a given $\lambda > 0$ and $\bar u_{\lambda,h}$ denotes the associated discrete solution. Assuming that the sequence $\{\bar u_\lambda\}_{\lambda\downarrow 0}$ is uniformly bounded in $L^\infty(\Omega)$, it is shown in [19] that

$$\|u^* - \bar u_\lambda\| \le c\,\sqrt{\lambda} \tag{3.24}$$

holds true with a constant c independent of λ. This together with (3.19) prove

Theorem 3.6. *Let Assumption 3.1 be fulfilled and assume that the sequence of optimal solutions to* (P_λ) *for* $\lambda \downarrow 0$*, denoted by* $\{\bar u_\lambda\}$*, is uniformly bounded in* $L^\infty(\Omega)$*. Then, with the notations introduced above there holds*

$$\|u^* - \bar u_{\lambda,h}\| \le C(\alpha, \Omega)\,\Big(\sqrt{\lambda} + \frac{1}{\lambda^2}\big(h^2 + \frac{1}{\lambda}\,h^3 + \frac{1}{\lambda^2}\,h^4\big)\Big), \tag{3.25}$$

with a generic positive constant $C(\alpha,\Omega)$ independent of λ and h.

From (3.25), we deduce the following theoretical prediction concerning the qualitative impact of the Lavrentiev regularization on the numerical approximation of (P).

REMARK 3.7. *We observe that, for the minimization of $\|u^* - \bar u_\lambda\|$ a small value of λ seems to be favorable, while the discretization error $\|\bar u_\lambda - \bar u_{\lambda,h}\|$ may be increased by a reduction of λ. Hence, the two different contributions to the overall error seem to behave contrarily.*

3.2. A error estimate uniform in λ. We now derive an error estimate for $\|\bar u - \bar u_h\|$ which does not depend on λ. In contrast to the theory presented in section 3.1, we do not utilize the auxiliary problem (PV) for the underlying analysis. As before $\bar u$ denotes the solution of (P_λ) for a given λ while $\bar u_h$ is the solution of $(\mathrm{P}_{\lambda,h})$.

THEOREM 3.8. *There exists some $0 < h_0 \le 1$ such that*

$$\|\bar u - \bar u_h\| + \|\bar y - \bar y_h\|_{H^1(\Omega)} \le C h^{1-n/4} \quad \forall\, 0 < h \le h_0 \tag{3.26}$$

holds with a positive constant $C > 0$ which is independent of λ.

Proof. We switch back to $(\mathrm{P}_{\lambda,h})$, i.e. the variational discrete version of (P_λ). The associated optimality system is given by

$$(\nabla \bar y\,,\, \nabla w_h) + (\bar y\,,\, w_h) = (\bar u_h\,,\, w_h) \quad \forall\, w_h \in W_h \tag{3.27}$$

$$(\nabla p_h\,,\, \nabla w_h) + (p_h\,,\, w_h) = \big(\bar y_h - y_d + \mu_h\,,\, w_h\big) \quad \forall\, w_h \in W_h \tag{3.28}$$

$$\alpha \bar u_h(x) + p_h(x) + \lambda \mu_h(x) = 0 \quad \text{a.e. in } \Omega, \tag{3.29}$$

$$\mu_h(x) \ge 0 \text{ a.e. in } \Omega, \quad \lambda \bar u_h(x) + \bar y_h(x) \le y_c(x) \text{ a.e. in } \Omega \tag{3.30}$$

$$\int_\Omega (\lambda \bar u_h + \bar y_h - y_c)\mu_h dx = 0. \tag{3.31}$$

Here, μ_h denotes the discrete Lagrange multiplier associated to the mixed constraints in $(\mathrm{P}_{\lambda,h})$. Notice that, due to (3.29), we have in general $\mu_h \notin W_h$ since $\bar u_h \notin W_h$. Note moreover, that with (3.4), i.e. $\bar v_h = \lambda\, \bar u_h + y_h$, and (3.29), p_h is equivalent to the solution of (3.9), i.e. the solution of the adjoint equation of (PV_h). Now we multiply the difference of (3.29) and its counterpart in (2.7) by $\bar u - \bar u_h$ and integrate over Ω. We obtain

$$\alpha\|\bar u - \bar u_h\|^2 = (-\lambda(\mu - \mu_h), \bar u - \bar u_h) + (p_h - p^h, \bar u - \bar u_h) + (p^h - p, \bar u - \bar u_h) =: (1)+(2)+(3),$$

where p^h denotes the finite element solution to

$$(\nabla p^h\,,\, \nabla w_h) + (p^h\,,\, w_h) = \big(\bar y - y_d + \mu\,,\, w_h\big) \quad \forall\, w_h \in W_h,$$

which coincides with $p^h(\bar v)$, $\bar v = \lambda \bar u + \bar y$, as defined in (3.14). Using the finite element solution y^h of

$$(\nabla y^h\,,\, \nabla w_h) + (y^h\,,\, w_h) = (\bar u, w_h) \quad \forall\, w_h \in W_h,$$

we obtain

$$\begin{aligned}(2) &= (\nabla(y^h - \bar y_h)\,,\, \nabla(p_h - p^h)) + (y^h - \bar y_h\,,\, p_h - p^h)\\ &= (\bar y_h - \bar y, y^h - \bar y_h) + (\mu_h, y^h - \bar y_h) - (\mu, y^h - \bar y_h).\end{aligned}$$

Now, since $\bar{y}_h \le y_c - \lambda\bar{u}_h$ and $\mu \ge 0$ we have, using the complementarity condition for $y_c - \lambda\bar{u} - \bar{y}$ in (2.7),

$$(\mu, \bar{y}_h - y^h) \le (\mu, y_c - \lambda\bar{u}_h - y^h) = (\mu, y_c - \lambda\bar{u}_h - y^h - y_c + \lambda\bar{u} + \bar{y}) = \\ = (\mu, \lambda(\bar{u} - \bar{u}_h)) + (\mu, \bar{y} - y^h).$$

Analogously, we find

$$(\mu_h, y^h - \bar{y}_h) \le (\mu_h, \lambda(\bar{u}_h - \bar{u})) + (\mu_h, y^h - \bar{y}).$$

Thus,

$$\begin{aligned}(1) + (2) + (3) \le\ & (-\lambda(\mu - \mu_h), \bar{u} - \bar{u}_h) + (\mu - \mu_h, \lambda(\bar{u} - \bar{u}_h)) + \\ & + (\bar{y}_h - \bar{y}, y^h - \bar{y}_h) + (\mu, \bar{y} - y^h) + (\mu_h, y^h - \bar{y}) + (p^h - p, \bar{u} - \bar{u}_h) = \\ =\ & (\bar{y}_h - \bar{y}, y^h - \bar{y}) - \|\bar{y}_h - \bar{y}\|^2 + (\mu, \bar{y} - y^h) + (\mu_h, y^h - \bar{y}) + (p^h - p, \bar{u} - \bar{u}_h).\end{aligned}$$

Using the Cauchy–Schwarz inequality, we obtain

$$\alpha\|\bar{u} - \bar{u}_h\|^2 + \|\bar{y}_h - \bar{y}\|^2 \le C\Big\{\|\bar{y} - y^h\|^2 + \big(\|\mu\|_{L^1(\Omega)} + \|\mu_h\|_{L^1(\Omega)}\big)\|\bar{y} - y^h\|_\infty \\ + \frac{1}{\alpha}\|p - p^h\|^2\Big\}.$$

From Lemma 2.2 we infer $\|\mu\|_{L^1(\Omega)} \le C$ uniformly in λ. An inspection of its proof also delivers $\|\mu_h\|_{L^1(\Omega)} \le C$ uniformly in λ, since $\|\bar{y}_h\|, \|\bar{u}_h\|$ are uniformly bounded in h and λ due to their optimality. The uniform boundedness of $\|\mu_h\|_{L^1(\Omega)}$ w.r.t. h, λ also holds true in the case of Dirichlet boundary conditions. This follows immediately, if one replaces S by S_h and G by G_h in the proof of Lemma 2.8.

In [5], it is proven for the case of homogeneous Dirichlet boundary conditions that

$$\|p - p^h\|^2 \le h^{4-n}\left(\|\bar{y} - y_d\|^2 + \|\mu\|^2_{L^1(\Omega)}\right). \tag{3.32}$$

It is easy to see that the same duality argument also applies in case of homogeneous Neumann boundary conditions such that (3.32) holds in both cases (cf. [5, Theorem 3] and the corresponding proof). Furthermore, we have $\|\bar{y} - y^h\|^2 \le Ch^4$, and $\|\bar{y} - y^h\|_{L^\infty(\Omega)} \le Ch^{2-n/2}$ by (3.2) and (3.3) and the optimality of $\bar{u}$ which implies $\|\bar{u}\| \le c$ independent of λ. Hence the estimation of $\|\bar{u} - \bar{u}_h\|$ follows. Then $\|\bar{y} - \bar{y}_h\|_{H^1(\Omega)}$ can be estimated as in the proof of Theorem 3.5 such that claim follows. ☐

Next let us consider (3.1) for $f \in L^\infty(\Omega)$, and let us assume that the corresponding unique solution satisfies $z \in W^{2,q}(\Omega)$ for all $1 \le q < \infty$. Then, due to [11, Lemma 1]

$$\|z - z_h\|_\infty \le Ch^2\,|\log(h)|^2\|v\|_{L^\infty(\Omega)}. \tag{3.33}$$

Now let us assume that $\bar{u}$ is uniformly bounded in $L^\infty(\Omega)$ which is the same assumption that was already needed for the estimation of the regularization error in (3.24). Then, we deduce from the proof of the previous Theorem

Corollary 3.9. *Assume that the sequence of optimal solutions to* (P_λ) *for* $\lambda \downarrow 0$, *denoted by* $\{\bar{u}_\lambda\}$, *is uniformly bounded in* $L^\infty(\Omega)$, *and assume further that the solution of* (3.1) *satisfies* $z \in W^{2,q}(\Omega)$ *for all* $1 \le q < \infty$ *if* $f \in L^\infty(\Omega)$. *Then the sequence of solutions of* $(\mathrm{P}_{\lambda,h})$, *denoted by* $\{\bar{u}_{\lambda,h}\}$ *satisfies*

$$\|\bar{u}_\lambda - \bar{u}_{\lambda,h}\| \le C\max\{h\,|\log(h)|, h^{2-n/2}\} \text{ for all } 0 < h \le h_0. \tag{3.34}$$

with a constant C independent of λ and h. Hence (3.24) immediately implies

$$\|u^* - \bar{u}_{\lambda,h}\| \leq \|u^* - \bar{u}_\lambda\| + \|\bar{u} - \bar{u}_{\lambda,h}\| \leq C\Big(\sqrt{\lambda} + \max\{h\,|\log(h)|, h^{2-n/2}\}\Big) \quad (3.35)$$

with a constant $C > 0$ that does not depend on λ and h.

REMARK 3.10. *Let the assumptions of Corollary 3.9 be satisfied. Then,* (3.35) *implies*

$$\|u^* - \bar{u}_{\lambda,h}\| \sim \sqrt{\lambda} + \begin{cases} h\,|\log(h)| & \text{if } n = 2, \\ h^{1/2} & \text{if } n = 3, \end{cases}$$

which suggests the coupling

$$\sqrt{\lambda} \sim \begin{cases} h & \text{if } d = 2, \\ h^{1/2} & \text{if } d = 3, \end{cases} \quad (3.36)$$

of λ and the finite element grid size h.

4. Numerical investigation. Finally we present a numerical experiment which supports the findings in the previous section. Specifically it turns out that the coupling of λ and the finite element grid size h proposed in (3.36) seems to be optimal.

The test case used for the following numerical investigation is taken from [15], and its numerical implementation using the variational discrete concept is performed along the lines of [20]. It is constructed such that the Lagrange multipliers associated to the pure state constraints are continuous. The considered control problem coincides with (P) unless that there is an additional bound from below in the state constraint, i.e. $y_a(x) \leq y(x) \leq y_b(x)$ a.e. in Ω. It is easy to verify that this additional bound does not influence the theory presented above. We choose $\Omega = (0,1)^2$ as test domain. Moreover, the desired state y_d and the bounds y_a and y_b are defined by

$$y_a(x) = \begin{cases} g(x) & \text{, if } g(x) \leq -0.7 \\ -0.7 & \text{, if } g(x) > -0.7 \end{cases} , \quad y_b(x) = \begin{cases} g(x) & \text{, if } g(x) \geq 0.7 \\ 0.7 & \text{, if } g(x) < 0.7 \end{cases}$$

$$y_d(x) = \begin{cases} \big((2\pi^2\alpha - 1)(2\pi^2 + 1) + 11\big)g(x) - 7 \text{ , if } g(x) \geq 0.7 \\ \big((2\pi^2\alpha - 1)(2\pi^2 + 1) + 11\big)g(x) + 7 \text{ , if } g(x) < 0.7 \end{cases}$$

with $x = (x_1, x_2)$ and $g(x) := \cos(\pi\, x_1)\cos(\pi\, x_2)$. It is straightforward to verify that the exact solution for this problem is given by

$$y^*(x) = g(x)\ ,\ u^*(x) = (2\,\pi^2 + 1)\,g(x)\ ,\ p^*(x) = -\alpha\,(2\,\pi^2 + 1)\,g(x)$$

$$\mu_a(x) = \begin{cases} -10\,g(x) - 7 & \text{, if } g(x) \leq -0.7 \\ 0 & \text{, if } g(x) > -0.7 \end{cases} , \quad \mu_b(x) = \begin{cases} 10\,g(x) - 7 & \text{, if } g(x) \geq 0.7 \\ 0 & \text{, if } g(x) < 0.7. \end{cases}$$

For the numerical solution of the Lavrentiev regularized problems a semi-smooth Newton method is applied to the numerical solution of the variational discretization (PV_h) of (PV). We note that this algorithm is kept on the infinite-dimensional level since the controls v are not discretized. The numerical implementation then is non–standard. For details we refer to [14, 20].

In Tab. 4.1 the dependence of the L^2-error $\|u^* - \bar{u}_h\|$ on λ and h is presented for the case $\alpha = 10^{-2}$. It turns out that the overall error is increased if λ is chosen too small.

TABLE 4.1
Dependence of the L^2 error $\|u^* - \bar{u}_h\|$ on λ and h for $\alpha = 0.01$.

	$\lambda =$							
$h/\sqrt{2}$	$10^{-2.5}$	10^{-3}	$10^{-3.5}$	10^{-4}	$10^{-4.5}$	10^{-5}	$10^{-5.5}$	10^{-6}
0.1	0.6429	1.0828	3.3982	10.715	34.307	107.48	337.80	1065.2
0.05	0.6221	0.3125	0.7956	2.515	7.950	25.09	79.26	250.2
0.025	0.6326	0.2148	0.2111	0.637	2.009	6.34	20.05	63.5
0.0125	0.6358	0.2097	0.0840	0.165	0.516	1.62	5.12	16.2
0.00625	0.6365	0.2101	0.0682	0.045	0.125	0.39	1.23	3.9

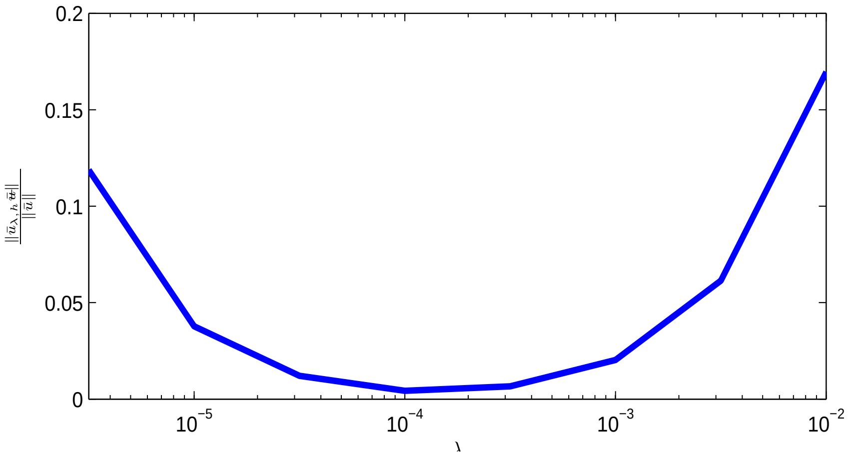

FIG. 4.1. *Relative error $\|u^* - \bar{u}_h\|/\|u^*\|$ over λ for $h = 6.25 \times 10^{-3}$.*

This is also illustrated by Fig. 4.1 which depicts the relative error $\|u^* - \bar{u}_h\|/\|u^*\|$ over λ for $h = 6.25 \times 10^{-3}$. We observe that the theoretical prediction of Remark 3.7, is confirmed by the numerical findings, i.e. the discretization error and the regularization error behave contrarily such that the optimal value of λ is larger than zero and depends on the mesh size h.

Now let us denote by $\lambda(h)$ the Lavrentiev parameter which delivers the smallest L^2-error for a given grid size h. In Fig. 4.2 we present a log plot of $\|u^* - \bar{u}_{\lambda(h),h}\|/\|u^*\|$ for varying values of α. It shows that for the present numerical example we could expect an error behaviour of the form

$$\|u^* - \bar{u}_{\lambda(h),h}\| \sim h,$$

which also would be delivered by the coupling $\lambda \sim h^2$ suggested in (3.36) for the case $n = 2$.

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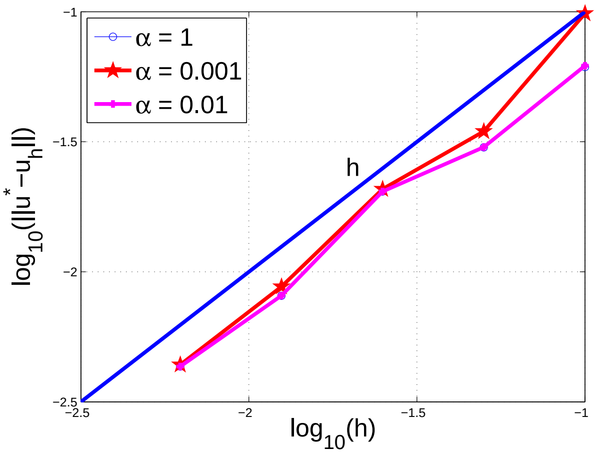

FIG. 4.2. log *plot of* $\|u^* - \bar{u}_{\lambda(h),h}\|/\|u^*\|$ *for varying values of* α.

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