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Searching for critical angles in a convex cone

Dedicated to Boris Polyak on his 70th Birthday

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Abstract. The concept of antipodality relative to a closed convex cone $K \subset \mathbb{R}^d$ has been explored in detail in a recent work of ours. The antipodality problem consists of finding a pair of unit vectors in K achieving the maximal angle of the cone. Our attention now is focused not just in the maximal angle, but in the angular spectrum of the cone. By definition, the angular spectrum of a cone is the set of angles satisfying the stationarity (or criticality) condition associated to the maximization problem involved in the determination of the maximal angle. In the case of a polyhedral cone, the angular spectrum turns out to be a finite set. Among other results, we obtain an upper bound for the cardinality of this set. We also discuss the link between the critical angles of a cone K and the critical angles of its dual cone.

1. Introduction

The present paper is self-contained except for a few results taken from our previous work [4]. The concept of antipodality relative to a closed convex cone is at the core of many interesting questions concerning the geometry of a cone, specially in relation with its angular structure.

For the sake of convenience, we start by introducing some basic terminology and notation. First of all, we write

$$K \in \Xi(\mathbb{R}^d) \Leftrightarrow \begin{cases} K \subset \mathbb{R}^d \text{ is a closed convex cone} \\ \text{different from } \{0\} \text{ and different from } \mathbb{R}^d. \end{cases}$$

The underlying space $\mathbb{R}^d$ is equipped with the usual inner product $\langle u, v\rangle = u^t v$ and the associated norm $\|\cdot\|$. The symbol S_d refers to the unit sphere in $\mathbb{R}^d$. The dimension d is assumed to be greater than or equal to 2.

Recall that the maximal angle of a cone K is given by

$$\theta_{\max}(K) = \sup_{u,v \in K \cap S_d} \arccos\langle u, v\rangle, \tag{1}$$

a number which lies obviously between 0 and π.

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The geometric meaning of $\theta_{\max}(K)$ is important and justifies by itself the study of the variational problem (1). There are also application-oriented motivations. For instance, Peña and Renegar [6] show that the number $\theta_{\max}(K)$ plays a role in estimating the efficiency of certain interior point methods for solving feasibility systems with inequalities described by K. On the other hand, as we explain in [5], the number $\theta_{\max}(K)$ can be used as tool for measuring the degree of pointedness of the cone K. We will come back to this point later in Section 5.

Definition 1. *Let $K \in \Xi(\mathbb{R}^d)$. One says that $(u,v) \in \mathbb{R}^d \times \mathbb{R}^d$ is an antipodal pair of K if $u, v \in K \cap S_d$ and $\arccos\langle u,v\rangle = \theta_{max}(K)$.*

So, antipodality is a matter of achieving the maximal angle of the cone. Observe that (1) is a nonconvex optimization problem, and so is the equivalent problem

$$\cos[\theta_{\max}(K)] = \inf_{u,v\in K\cap S_d} \langle u,v\rangle. \tag{2}$$

A first-order necessary condition for antipodality is recalled in the next theorem. The notation

$$K^+ = \{y \in \mathbb{R}^d : \ \langle y,x\rangle \geq 0 \ \ \forall x \in K\}$$

stands for the dual cone of K.

Theorem 1. *If (u,v) is an antipodal pair of $K \in \Xi(\mathbb{R}^d)$, then*

$$u, v \in K \cap S_d, \ \ v - \langle u,v\rangle u \in K^+, \ \ u - \langle u,v\rangle v \in K^+. \tag{3}$$

Proof. This result is obtained by writing the Karush-Kuhn-Tucker optimality conditions for the problem (2). Only the constraints $\|u\| = 1, \|v\| = 1$ are dualized and therefore one considers the Lagrangian function $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$L(u,v,\lambda_1,\lambda_2) = \langle u,v\rangle - \frac{\lambda_1}{2}\big(\|u\|^2 - 1\big) - \frac{\lambda_2}{2}\big(\|u\|^2 - 1\big).$$

After working out the variational inequality

$$\big\langle \nabla_u L(u,v,\lambda_1,\lambda_2), u' - u\big\rangle + \big\langle \nabla_v L(u,v,\lambda_1,\lambda_2), v' - v\big\rangle \geq 0 \quad \forall u', v' \in K,$$

and getting rid of the Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$, one ends up with the system (3). The details can be consulted in [4]. $\square$

Due to the lack of convexity in (2), the stationarity condition (3) is necessary for antipodality, but not sufficient. The following definition captures the idea that a cone may admit critical pairs that are not antipodal.

Definition 2. *By a critical pair of $K \in \Xi(\mathbb{R}^d)$ we understand any pair (u,v) of vectors satisfying the stationarity condition (3). The angle*

$$\theta(u,v) = \arccos\langle u,v\rangle$$

formed by a critical pair is called a critical angle. The adjective proper is added when u and v are not collinear, that is to say, $|\langle u,v\rangle| \neq 1$. The set of all proper critical angles of K, denoted by $\Omega(K)$, is called the angular spectrum of K.

The purpose of our work is exploring in detail the concept of criticality (or stationarity) as defined above. The reason for doing so is that the angular spectrum of a cone provides a very useful information on the geometric structure of the cone itself. To the best of our knowledge, this kind of angular analysis has not been undertaken before.

The paper is organized as follows. In Section 2 we state the basic ingredients of the theory of critical angles. Except for Propositions 2 and 3, this long section is entirely new. Special attention is paid to the relationship existing between the critical angles of a cone and those of its dual cone. Section 3 is devoted to the angular analysis of elliptic cones. This is a nontrivial class of convex cones for which it is possible to compute explicitly the full set of critical angles. A different class of convex cones is considered in Section 4. In this section the objects under discussion are polyhedral cones. From an angular point of view, polyhedral cones don't behave as nicely as elliptic cones, but they do enjoy a series of noteworthy properties.

2. Critical angles and duality

2.1. Preliminary results

The next proposition can be proven in a straightforward manner, but it deserves to be recorded properly. The notation

$$M_{u,v} = \{(\alpha,\beta) \in \mathbb{R}^2 : \alpha + \beta\langle u,v\rangle \geq 0,\ \alpha\langle u,v\rangle + \beta \geq 0\}$$

is introduced for convenience.

Proposition 1. *Suppose that $u, v \in K \cap S_H$ are not collinear. Then, the following three conditions are equivalent:*

(a) (u, v) *is a critical pair of* K*,*
(b) $(\gamma - \delta\langle u,v\rangle)u + (\delta - \gamma\langle u,v\rangle)v \in K^+ \quad \forall \gamma, \delta \geq 0$,
(c) $\alpha u + \beta v \in K^+ \quad \forall (\alpha,\beta) \in M_{u,v}$.

In particular, if (u, v) is critical pair of K, then the sum $u + v$ is in K^+.

Proof. This is essentially a matter of solving the linear system

$$\begin{aligned} \gamma - \delta\langle u,v\rangle &= \alpha \\ \delta - \gamma\langle u,v\rangle &= \beta \end{aligned}$$

and recalling the definition of a critical pair. The invertibility of such system is guaranteed by the non-collinearity assumption. For the second part of the proposition, it is enough to observe that $(1,1) \in M_{u,v}$. □

The relative interior of K, denoted by $\mathrm{ri}(K)$, is not the right place for finding the components of a proper critical pair. They are to be found in the relative boundary of K, which is defined as the set difference $K \backslash \mathrm{ri}(K)$ (cf. [7]).

Proposition 2. *If (u, v) is a proper critical pair of $K \in \Xi(\mathbb{R}^d)$, then u and v belong to the relative boundary of K.*

Proof. This result has been established already in [4], but we give here a shorter proof. We show that if either u or v belong to the relative interior of K, then the critical pair (u, v) fails to be proper. Assume, for instance, that $u \in \mathrm{ri}(K)$. Hence, there exists an $\varepsilon > 0$ such that

$$u + \varepsilon z \in K \quad \forall z \in B_d \cap \mathrm{span}(K),$$

where B_d denotes the closed unit ball in $\mathbb{R}^d$. If one sets $\lambda = \langle u, v\rangle$, then

$$0 \leq \langle u + \varepsilon z, v - \lambda u\rangle = \langle u, v\rangle - \lambda + \varepsilon\langle z, v - \lambda u\rangle = \varepsilon\langle z, v - \lambda u\rangle. \tag{4}$$

If $\lambda u = v$, then (u, v) isn't proper and we are done. Assume then that $\lambda u \neq v$. Since $\lambda u - v$ belongs to $\mathrm{span}(K)$, we can take $z = \|\lambda u - v\|^{-1}(\lambda u - v)$ in (4), getting $0 \leq -\varepsilon\|v - \lambda u\|$, which implies that $v = \lambda u$, a contradiction with our last assumption. A similar argument applies if we assume that $v \in \mathrm{ri}(K)$. □

Proposition 3. *If (u, v) is a proper critical pair of $K \in \Xi(\mathbb{R}^d)$, then the vectors $v - \langle u, v\rangle u$ and $u - \langle u, v\rangle v$ are in the relative boundary of K^+.*

This result was stated in [4]. It can be proven in a straightforward manner by combining Proposition 2 and the next duality theorem.

2.2. Duality

If K belongs to $\Xi(\mathbb{R}^d)$, then so does the dual cone K^+. The following result is at the core of the forthcoming discussion.

Theorem 2. *If (u, v) is a proper critical pair of $K \in \Xi(\mathbb{R}^d)$, then the vectors*

$$y = \frac{u - \langle u, v\rangle v}{\sqrt{1 - \langle u, v\rangle^2}}\,, \quad z = \frac{v - \langle u, v\rangle u}{\sqrt{1 - \langle u, v\rangle^2}} \tag{5}$$

form a proper critical pair of K^+.

Proof. One can check that $y, z \in K^+$ are unit vectors satisfying $\langle y, z\rangle = -\langle u, v\rangle$. Hence y and z are not collinear. Observe that the vectors

$$y - \langle y, z\rangle z = \sqrt{1 - \langle u, v\rangle^2}\; u\,, \quad z - \langle y, z\rangle y = \sqrt{1 - \langle u, v\rangle^2}\; v$$

belong to K. It follows that (y, z) is a proper critical pair of K^+. □

The proof of Theorem 2 is quite simple, but some thinking was needed before recognizing the expressions (5) as being of particular interest. A striking feature of the relations (5) is that they can be inverted in order to produce

$$u = \frac{y - \langle y, z\rangle z}{\sqrt{1 - \langle y, z\rangle^2}}\,, \quad v = \frac{z - \langle y, z\rangle y}{\sqrt{1 - \langle y, z\rangle^2}}\,.$$

This means that we have found a bijection

$$(u,v) \mapsto \Phi(u,v) = \left(\frac{u - \langle u,v\rangle v}{\sqrt{1-\langle u,v\rangle^2}} \, , \frac{v - \langle u,v\rangle u}{\sqrt{1-\langle u,v\rangle^2}} \right)$$

between the proper critical pairs of K and the proper critical pairs of K^+. In fact, Φ is not just a bijection, but also an involution.

Theorem 3. *Let $K \in \Xi(\mathbb{R}^d)$. For $\theta \in]0,\pi[$, the following two statements are equivalent:*

(a) θ is a critical angle of K,
(b) $\pi - \theta$ is a critical angle of K^+.

Proof. Let $\theta \in]0,\pi[$ be a critical angle of K formed with the pair (u,v) and let $\lambda = \langle u,v\rangle$. By Theorem 2, we know that the vectors

$$\frac{u-\lambda v}{\sqrt{1-\lambda^2}} \, , \quad \frac{v-\lambda u}{\sqrt{1-\lambda^2}}$$

form a proper critical pair of K^+. These vectors produce the angle

$$\arccos \langle \frac{u-\lambda v}{\sqrt{1-\lambda^2}} \, , \frac{v-\lambda u}{\sqrt{1-\lambda^2}} \rangle \ = \arccos \left[\frac{\lambda^3-\lambda}{1-\lambda^2} \right]$$

$$= \arccos(-\lambda) = \arccos(-\cos\theta) = \pi - \theta.$$

We have shown in this way that (a) implies (b). The reverse implication follows by applying the same argument starting from K^+. ☐

As a by-product of the Theorem 3, we see that the angular spectra $\Omega(K)$ and $\Omega(K^+)$ have the same cardinality. In fact,

$$\Omega(K^+) = \{\pi - \theta : \ \theta \in \Omega(K)\}.$$

For a better understanding of the next corollary, it is helpful to keep in mind the general relationship

$$\theta_{max}(K) + \theta_{max}(K^+) \geq \pi \qquad \forall K \in \Xi(\mathbb{R}^d). \tag{6}$$

This inequality was established in [5] by using a rather cumbersome argument, but, in fact, it is implicit in the proof of Theorem 3.

Corollary 1. *For any $K \in \Xi(\mathbb{R}^d)$, one has*

$$\Omega(K) \subset [\pi - \theta_{max}(K^+), \theta_{max}(K)]. \tag{7}$$

Furthermore,

(a) $\theta_{max}(K)$ is in $\Omega(K)$ iff K is pointed and not a ray,
(b) $\pi - \theta_{max}(K^+)$ is in $\Omega(K)$ iff K is solid and not a half-space.

Proof. It is a matter of exploiting Theorem 3 and the general inequality (6). Everything is more or less straightforward, so the details are omitted. ☐

2.3. Purity

Besides the angle 0 (which is an improper critical angle) and the maximal angle $\theta_{max}(K)$ (which may be proper or not), the cone K may posses other critical angles. As a general rule, cones with few critical angles have a simpler geometric structure.

Definition 3. *A cone $K \in \Xi(\mathbb{R}^d)$ is declared pure if $\Omega(K)$ contains at most one element, namely, the maximal angle $\theta_{max}(K)$.*

The question of characterizing the purity of a cone can be answered in an elegant manner with the help of the following theorem.

Theorem 4. *Let $K \in \Xi(\mathbb{R}^d)$ be solid and pointed. Satisfying the equality*

$$\theta_{max}(K) + \theta_{max}(K^+) = \pi \tag{8}$$

is necessary and sufficient for K to be pure.

Proof. If the reflection law (8) holds, then the interval $[\pi - \theta_{max}(K^+), \theta_{max}(K)]$ reduces to a singleton, and therefore K is pure. Conversely, suppose that K is pure. Since K is assumed to be solid and pointed, $\pi - \theta_{max}(K^+)$ and $\theta_{max}(K)$ are both in $\Omega(K)$. Since $\Omega(K)$ contains exactly one element, one necessarily has has $\pi - \theta_{max}(K^+) = \theta_{max}(K)$. □

If the cone fails to be pointed or solid, then the reflection law (8) is sufficient for purity but it is no longer necessary. To see this, just consider a two-dimensional subspace in $\mathbb{R}^4$.

Corollary 2. *Examples of cones that are pure include:*

(a) every linear subspace,
(b) every half-space,
(c) every revolution cone,
(d) every self-dual cone.

Proof. The first case can be handled directly and offers no difficulty. The last three cases correspond to cones satisfying the reflection law (8). □

We mention that both the nonnegative orthant and the cone of symmetric positive semidefinite matrices are self-dual, hence pure.

Remark 1. Denote by $\mathcal{C}_d$ the cone of symmetric copositive matrices of size $d \times d$. As pointed out by one of the referees, this cone deserves some attention because of its extensive use in mathematical programming (see [1],[2], and the references therein). Contrarily to the case of the cone of positive semidefinite matrices, $\mathcal{C}_d$ is not self-dual, and even worst it does not satisfy the reflection law (8). In addition to its maximal angle, the cone $\mathcal{C}_d$ exhibits then other critical angles that could be interesting to identify. By applying Corollary 1(b) one can show that $\pi/2$ is a proper critical angle of $\mathcal{C}_d$, and also that it is the smallest one. Indeed, the dual cone $\mathcal{C}_d^+$, which is formed by the so-called completely nonnegative matrices, has a maximal angle equal to $\pi/2$. By-the-way, for seeing that the reflection law is being violated, note that $\mathcal{C}_d$ has a maximal angle which is strictly greater than $\pi/2$. Indeed, the copositive matrices

$$X = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad Y = \frac{\sqrt{2}}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

have unit length and $\arccos\langle X, Y\rangle = \arccos(-\sqrt{2}/2) = 3\pi/4 > \pi/2$. For notational simplicity we are working with 2×2 matrices but the same example can be extended to matrices of higher size.

3. Angular spectra of elliptic cones

In this section we compute the angular spectrum of a nondegenerate elliptic cone. By this term we understand a closed convex cone of the form

$$\mathcal{E}(A) = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : \sqrt{x^t A x} \leq r\}, \tag{9}$$

with $A \in \mathbb{R}^{n\times n}$ being a symmetric positive definite matrix. The inner product in $\mathbb{R}^n \times \mathbb{R}$ is the usual one, to wit

$$\langle (x, r), (y, s)\rangle = x^t y + rs.$$

It is not difficult to check that $\mathcal{E}(A)$ is pointed and solid. It has been proved elsewhere [3] that

$$[\mathcal{E}(A)]^+ = \mathcal{E}(A^{-1}). \tag{10}$$

We will also use the following lemma, surely well known, whose proof we omit since the result follows easily by applying a standard spectral decomposition technique.

Lemma 1. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. If $x \in \mathbb{R}^n$ is a nonzero vector such that*

$$(x^t A x)(x^t A^{-1} x) \leq \|x\|^4, \tag{11}$$

then x is an eigenvector of A.

We now are ready to present our result on elliptic cones. As shown in the next theorem, computing a critical angle of the elliptic cone $\mathcal{E}(A)$ is essentially the same job as computing an eigenvalue of the matrix A. Critical pairs of $\mathcal{E}(A)$ are constructed from the eigenvectors of A.

Theorem 5. *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. The vectors (x, r) and (y, s) form a proper critical pair of $\mathcal{E}(A)$ if and only if the following three conditions hold:*

(a) $y = -x$,
(b) $s = r = \sqrt{1 - \|x\|^2} = \sqrt{x^t A x}$,

(c) x is an eigenvector of A.

In this case, the corresponding critical angle takes the value

$$\theta = \arccos\left(\frac{\mu - 1}{\mu + 1}\right), \tag{12}$$

where μ is the eigenvalue of A associated to x.

Proof. Suppose that (a)–(c) are in force. Clearly $u = (x, r)$ and $v = (y, s)$ are unit vectors belonging to $\mathcal{E}(A)$. Let $\lambda = \langle u, v\rangle$. We shall prove that

$$u - \lambda v \in \mathrm{bd}[\mathcal{E}(A)]^+ \quad \text{and} \quad v - \lambda u \in \mathrm{bd}[\mathcal{E}(A)]^+ \,. \tag{13}$$

To avoid repetitions, we work out only the first condition in (13). Due to the general identity (10), such condition takes the equivalent form

$$(1 + \lambda)\sqrt{x^t A^{-1} x} = (1 - \lambda)r. \tag{14}$$

Recall that x is assumed to be an eigenvector of A. Denote by μ the associated eigenvalue. It follows that

$$\sqrt{x^t A^{-1} x} = \mu^{-1/2}\|x\| \,, \quad r = \sqrt{x^t A x} = \sqrt{\mu}\|x\|.$$

After a short simplification, the equality (14) becomes

$$(1 + \lambda) = \mu(1 - \lambda). \tag{15}$$

Since $\lambda = -\|x\|^2 + r^2 = 1 - 2\|x\|^2$, one has $1 + \lambda = 2(1 - \|x\|^2)$ and $1 - \lambda = 2\|x\|^2$. But

$$1 = \|x\|^2 + r^2 = \|x\|^2 + x^t A x = \|x\|^2 + \mu\|x\|^2 = (1 + \mu)\|x\|^2,$$

so that $\|x\|^2 = (1+\mu)^{-1}$. One gets finally $1+\lambda = 2\mu/(1+\mu)$ and $1-\lambda = 2/(1+\mu)$, confirming in this way the validity of (15). The conclusion is that (u, v) is a critical pair of $\mathcal{E}(A)$. Observe, incidentally, that (15) gives the formula (12) as by-product.

We now prove the "only if" part. We assume that (x, r) and (y, s) form a critical pair of $\mathcal{E}(A)$, and we must check that (a)–(c) hold. Let $\lambda = \langle (x, r), (y, s)\rangle$. By criticality, we have

$$(x, r) - \lambda(y, s) \in [\mathcal{E}(A)]^+, \tag{16}$$

$$(y, s) - \lambda(x, r) \in [\mathcal{E}(A)]^+. \tag{17}$$

Note that, by definition of $\mathcal{E}(A)$, both $(-x, r)$ and $(-y, s)$ also belong to $\mathcal{E}(A)$. Multiplying the left hand side of (16) by $(-x, r)$, one gets

$$0 \le \big\langle (-x, r), (x, r) - \lambda(y, s)\big\rangle = -\|x\|^2 + r^2 - \lambda(-x^t y + rs)$$

Plugging $\lambda = x^t y + rs$ and rearranging, one obtains

$$0 \le r^2(1 - s^2) - \|x\|^2 + (x^t y)^2. \tag{18}$$

Recall that (x, r) and (y, s) are vectors of unit length, that is to say,

$$\|x\|^2 + r^2 = 1, \quad \|y\|^2 + s^2 = 1. \tag{19}$$

The combination of (18) and (19) produces

$$0 \leq (1 - \|x\|^2)\|y\|^2 - \|x\|^2 + (x^t y)^2,$$

that is to say

$$\|x\|^2 - \|y\|^2 \leq (x^t y)^2 - \|x\|^2 \|y\|^2. \tag{20}$$

By the same token, multiplying the left hand side of (17) by $(-y, s)$, one arrives at

$$\|y\|^2 - \|x\|^2 \leq (x^t y)^2 - \|x\|^2 \|y\|^2. \tag{21}$$

By combining (20), (21) and Cauchy- Schwarz inequality, one gets

$$0 = \|x\|^2 - \|y\|^2 = (x^t y)^2 - \|x\|^2 \|y\|^2.$$

Hence, $y = \pm x$ and $r = s$. The case $y = x$ must be ruled out because we are assuming properness of the critical pair $\{(x, r), (y, s)\}$. We conclude that $y = -x$, establishing (a). The first equality in (b) has also been proved. The second one is contained in (19), and the third one follows from the fact that (x, r) lies in the boundary of $\mathcal{E}(A)$. Next we prove (c). From (16), and the fact that $y = -x$, $r = s$, and $\|x\|^2 + s^2 = 1$, we get, as in the proof of the "if" part,

$$(1 + \lambda)\sqrt{x^t A^{-1} x} \leq (1 - \lambda) r = (1 - \lambda)\sqrt{1 - \|x\|^2}. \tag{22}$$

As before, we get $1 + \lambda = 2(1 - \|x\|^2)$, $1 - \lambda = 2\|x\|^2$, so that (22) becomes

$$4(1 - \|x\|^2)^2 (x^t A^{-1} x) \leq 4\|x\|^4 (1 - \|x\|^2).$$

Since $1 - \|x\|^2 = r^2 = x^t A x > 0$, one arrives at $(x^t A x)(x^t A^{-1} x) \leq \|x\|^4$. We invoke now Lemma 1 to conclude that x is an eigenvector of A. $\square$

We have learned from Theorem 5 how to compute explicitly the angular spectrum of an elliptic cone. It is interesting to observe that the angular spectrum of an elliptic cones in $\mathbb{R}^n \times \mathbb{R}$ has at most n elements. This upper bound is attained, of course, by choosing a matrix A whose eigenvalues are all different.

4. Angular spectra of polyhedral cones

4.1. Preliminary results

We now look at the particular case of a polyhedral cone, that is to say, a set which can be expressed as the intersection of a finite collection of half-spaces. Recall that a polyhedral cone can always be represented in the form

$$K = \text{cone}\{g^1, \cdots, g^p\} = \Big\{ \sum_{i=1}^{p} x_i g^i : \; x \in \mathbb{R}^p_+ \Big\}. \tag{23}$$

Following a standard practice, we say that $\{g^1, \cdots, g^p\} \subset \mathbb{R}^d$ is a set of generators for K. There is no loss of generality in assuming that all generators have unit length and that no generator is a positive linear combination of the remaining ones.

Is the angular spectrum of a polyhedral cone necessarily discrete, or, on the contrary, is it possible to find a whole interval of critical angles? The answer to this question agrees with our geometric intuition:

Proposition 4. *The angular spectrum $\Omega(K)$ of a polyhedral cone $K \subset \mathbb{R}^d$ is finite.*

Proposition 4 has been established in our previous work [4], so we don't need to write here a proof. What we shall do instead is going far beyond this rough description of the angular spectrum of a polyhedral cone. In fact, saying that the angular spectrum is finite doesn't provide any relevant indication on its actual size. How does the cardinality of $\Omega(K)$ depend on the number p of generators of the cone K, or on the dimension d of the underlying Euclidean space? Answering to these questions is not a trivial matter. In order to obtain sharp bounds for the cardinality of the angular spectrum, we need first to prepare the ground by establishing a few preliminary results, some of them having an interest by their own.

To start with, we will rewrite the stationarity condition (3) in a more explicit way for the case of polyhedral cones. We will use the Gramian matrix $M \in \mathbb{R}^{p\times p}$ associated to K, whose entries are defined as $M_{ij} = \langle g^i, g^j\rangle$ $(1 \le i, j \le p)$.

Proposition 5. *Let $K \subset \mathbb{R}^d$ be a polyhedral cone with generators $g^1, \ldots, g^p$ and M the associated Gramian matrix. A pair $(u, v) \in \mathbb{R}^d \times \mathbb{R}^d$ is critical if and only if there exist $x, y \in \mathbb{R}^p$ such that*

$$u = \sum_{i=1}^{p} x_i g^i, \quad v = \sum_{i=1}^{p} y_i g^i, \tag{24}$$

and the following relations hold:

$$Mx - (x^t My)My \ge 0, \tag{25}$$

$$My - (x^t My)Mx \ge 0, \tag{26}$$

$$x^t Mx = 1, \tag{27}$$

$$y^t My = 1, \tag{28}$$

$$x \ge 0, \;\; y \ge 0, \tag{29}$$

$$y_i \left[Mx - (x^t My)My\right]_i = 0 \quad (1 \le i \le p), \tag{30}$$

$$x_i \left[My - (x^t My)Mx\right]_i = 0 \quad (1 \le i \le p). \tag{31}$$

Proof. Observe that the dual cone of (23) is given by

$$K^+ = \{z \in \mathbb{R}^d : \langle g^1, z\rangle \geq 0, \ldots, \langle g^p, z\rangle \geq 0\},$$

so the criticality conditions $u - \langle u,v\rangle v \in K^+$ and $v - \langle u,v\rangle u \in K^+$ can be rewritten as

$$\langle g^i, u\rangle - \langle u,v\rangle\langle g^i, v\rangle \geq 0 \qquad (1 \leq i \leq p), \tag{32}$$

$$\langle g^i, v\rangle - \langle u,v\rangle\langle g^i, u\rangle \geq 0 \qquad (1 \leq i \leq p). \tag{33}$$

If we replace now (24) in (32) and (33), taking into account the definition of the Gramian matrix M, we obtain precisely (25) and (26). Now we look at the remaining condition in the definition of a critical pair, i.e. $u, v \in K \cap S_d$. In view of (24), the fact that u, v belong to K is equivalent to (29), while the fact that u, v belong to S_d is equivalent to (27)-(28). Regarding (30) and (31), which are indeed the complementarity conditions associated with the problem of minimizing $\langle u, v\rangle$ subject to $u, v \in K \cap S_d$ when we write u, v in terms of x, y using (24), for this particular optimization problem they are a consequence of the remaining conditions (25)–(29), as we show next. Note that

$$\sum_{i=1}^{p} y_i \left[Mx - (x^tMy)My\right]_i = y^tMx - (x^tMy)y^tMy = y^tMx - x^tMy = 0, \tag{34}$$

using (28) in the second equality and symmetry of M in the third one. In view of (25), (29), each term in the leftmost expression of (34) is nonnegative. Since their sum is 0, all of them vanish, i.e. (30) holds. For (31) we use a similar argument, with (27) and (26) instead of (28) and (25). □

Remark 2. Observe that in the notation of Proposition 5, the critical angle associated to the critical pair (u, v) is equal to $\arccos(x^tMy)$.

As it is customary in complementarity theory, we can use the complementarity conditions (30), (31) to transform some of the inequalities in (25), (26) into equalities, by identifying the appropriate sets of indices. We proceed to do so. For a vector $x \in \mathbb{R}^p$, let $I(x) = \{i : x_i \neq 0\}$.

Corollary 3. *Under the assumptions of Proposition 5 the following relations hold*

$$(Mx)_i - (x^tMy)(My)_i = 0 \quad (i \in I(y)), \tag{35}$$

$$(My)_i - (x^tMy)(Mx)_i = 0 \quad (i \in I(x)). \tag{36}$$

Proof. These equations follow directly from (30), (31). □

Now we state one of the main results of this section. Theorem 6 will be commented in detail after its proof has been completed.

Theorem 6. *Given a polyhedral cone K with generators $g^1, \ldots, g^p$, for each pair of nonempty sets $J, L \subset \{1, \ldots, p\}$, there exists at most one proper critical angle formed by a pair (u, v), written in terms of vectors $x, y \in \mathbb{R}^p_+$ through (24), such that $I(x) = J$, $I(y) = L$.*

Proof. Let M be the Gramian matrix of K. Take nonempty subsets J, L of $\{1, \ldots, p\}$. Assume that q, r are the cardinality of J, L respectively. Define the matrices $A \in \mathbb{R}^{q\times q}$, $B \in \mathbb{R}^{r\times r}$ and $C \in \mathbb{R}^{q\times r}$ as follows: the entries of A are the entries M_{ik} of M with $i, k \in J$, the entries of B are the entries M_{ik} of M with $i, k \in L$, and the entries of C are the entries M_{ik} of M with $i \in J$ and $k \in L$. Now consider two proper critical pairs $(u, v), (u', v')$ of K with the following property:

$$u = \sum_{i=1}^{p} x_i g^i, \quad v = \sum_{i=1}^{p} y_i g^i, \quad u' = \sum_{i=1}^{p} x'_i g^i, \quad v' = \sum_{i=1}^{p} y'_i g^i,$$

where both (x, y) and (x', y') satisfy (25)–(29), (35), (36), and

$$I(x) = I(x') = J, \quad I(y) = I(y') = L. \tag{37}$$

We must prove that $\langle u, v\rangle = \langle u', v'\rangle$, or equivalently that $x^t My = (x')^t My'$. Let $\xi, \xi' \in \mathbb{R}^q$ be the vectors containing the nonnull components of x, x' respectively, and $\eta, \eta' \in \mathbb{R}^r$ be the vectors containing the nonnull components of y, y' respectively. Since x, y, x', y' are nonnegative by (29), the vectors ξ, η, ξ', η' are strictly positive, i.e.

$$\xi' \in \operatorname{int}(\mathbb{R}^q_+), \quad \eta' \in \operatorname{int}(\mathbb{R}^r_+). \tag{38}$$

Let $\sigma = x^t My$, $\sigma' = (x')^t My'$. We will rewrite (35), (36), (25) and (26) in terms of $A, B, C, \xi, \eta, \xi', \eta', \sigma$ and σ'. In the case of (35) and (36), this rewriting leads to:

$$C^t\xi = \sigma B\eta, \tag{39}$$

$$C\eta = \sigma A\xi, \tag{40}$$

$$C^t\xi' = \sigma' B\eta', \tag{41}$$

$$C\eta' = \sigma' A\xi'. \tag{42}$$

These equations result from (37) and the facts that $x_i = \xi_i$, $x'_i = \xi'_i$ for $i \in J$, $x_i = x'_i = 0$ for $i \notin J$, $y_i = \eta_i$, $y'_i = \eta'_i$ for $i \in L$ and $y_i = y'_i = 0$ for $i \notin L$. Now we look at (25) and (26). We consider them component-wise, i.e. as $(Mx)_i \geq \sigma(My)_i$, $(My)_i \geq \sigma(Mx)_i$. They hold for all $i \in \{1, \ldots, p\}$, but we will be concerned only with indices $i \in J$ for the first system of inequalities and $i \in L$ for the second one. The resulting inequalities are:

$$A\xi \geq \sigma C\eta, \tag{43}$$

$$B\eta \geq \sigma C^t\xi, \tag{44}$$

Now we premultiply (39) by η', (40) by ξ', (41) by η and (42) by ξ, obtaining

$$(\eta')^t C^t\xi = \sigma(\eta')^t B\eta, \tag{45}$$

$$(\xi')^t C\eta = \sigma(\xi')^t A\xi, \tag{46}$$

$$\eta^t C^t\xi' = \sigma'\eta^t B\eta', \tag{47}$$

$$\xi^t C\eta' = \sigma'\xi^t A\xi'. \tag{48}$$

Adding (45) to (46), and then subtracting (47) and (48), we get, in view of the symmetry of A, B,

$$0 = (\sigma - \sigma')(\xi^t A \xi' + \eta^t B \eta'). \tag{49}$$

Assume now that $\sigma \neq \sigma'$. We show next that this assumption leads to a contradiction. By (49), we have

$$0 = \xi^t A \xi' + \eta^t B \eta'. \tag{50}$$

Replace (40) in (43), and (39) in (44), and get $A\xi \geq \sigma^2 A\xi$, $B\eta \geq \sigma^2 B\eta$, or equivalently $(1-\sigma^2)A\xi \geq 0$, $(1-\sigma^2)B\eta \geq 0$. Remember now that $\sigma = \langle u, v\rangle$. Since $\|u\| = \|v\| = 1$, we have $\sigma \in [-1, 1]$. Properness of the pair (u, v) excludes the case $\sigma = \pm 1$. Thus, $1 - \sigma^2 > 0$, and we conclude that $A\xi \geq 0, B\eta \geq 0$. Due to (38), we get from (50) that $A\xi = 0$, $B\eta = 0$. Premultiplication of the first of these systems by ξ produces the obvious contradiction

$$0 = \xi^t A \xi = x^t M x = 1,$$

using (27) in the rightmost equality. Thus, $\sigma = \sigma'$, i.e. $\langle u, v\rangle = \langle u', v'\rangle$. $\square$

We make now a few remarks on Theorem 6. We need some additional notation. Given a polyhedral cone K with generators $\{g^1, \ldots, g^p\}$ and a nonempty set $I \subset \{1, \ldots, p\}$, we define

$$K_I^\circ = \Big\{ \sum_{i \in I} x_i g^i : x_i > 0, \forall\, i \in I \Big\}.$$

For convenience, we will use the term *configuration* for referring to a pair (J, L) of nonempty subsets of the set of indices of the generators of the given polyhedral cone K.

We emphasize first that the expressions of u, v in terms of the generators, as given in (24) are not unique, so that different configurations (J, L) can be associated to the same critical pair (u, v). In other words, a vector $z \in K$ can belong to K_I° for several choices of the set of indices I. In such a case, it is clear that the result of Proposition 5 for a given critical pair (u, v) holds for all possible configurations (J, L) such that $(u, v) \in K_J^\circ \times K_L^\circ$.

We observe now that the result of Theorem 6 does not mean that for each configuration (J, L) there exists at most one critical pair (u, v) with $u \in K_J^\circ$ and $v \in K_L^\circ$; the uniqueness refers to the value of $\langle u, v\rangle$, not the vectors which realize the critical pair.

In this respect, it is useful to consider the cone $K = \mathbb{R}^d_+$, whose generators can be taken as the canonical vectors $e^i \in \mathbb{R}^d$ $(1 \leq i \leq d)$. Since obviously $\langle u, v\rangle \geq 0$ for all $u, v \in K$, the maximal angle is $\pi/2$, and hence any orthogonal pair of unit vectors in K is critical, because they achieve the maximal angle. Take now any configuration (J, L) with $J \cap L = \emptyset$. Clearly, any pair $(u, v) \in (K_J^\circ \cap S_d) \times (K_L^\circ \cap S_d)$ is orthogonal, hence critical, with angle $\pi/2$. It follows from Theorem 6 that the angle associated to any disjoint pair (J, L) is $\pi/2$, but whenever either J or L has cardinality at least 2 such angle is achieved by an infinity of pairs (u, v) associated to the configuration (J, L) (parenthetically, we mention that $\mathbb{R}^d_+$, being self-dual, is pure, in view of Corollary 2, so that $\pi/2$ is indeed its unique critical angle). We conjecture, nonetheless, that multiplicity of pairs associated to a given configuration with the same critical angle only can occur when the critical angle is a right one, or, in other words, that given a configuration (J, L) with an associated critical angle different from $\pi/2$, there exists only one critical pair $(u, v) \in K_J^\circ \times K_L^\circ$.

4.2. Cardinality estimates for angular spectra

Theorem 6 gives a first upper bound for the cardinality of the angular spectrum of a polyhedral cone with p generators, namely $2^{p-1}(2^p-1)$, which is the number of non-ordered pairs of nonempty subsets of $\{1, \ldots, p\}$. Perhaps a more elaborate explanation on the number $2^{p-1}(2^p-1)$ is welcome. Notice that a nonempty set of $\{1, \ldots, p\}$ can be formed in 2^p-1 different ways. Hence, there are $(2^p-1)\times(2^p-1)$ ways of forming a configuration (J, L). However, once a configuration (J, L) has been counted, it is unnecessary to count the symmetric configuration (L, J) because it produces the same critical angle. Geometrically speaking, exchanging the order of J and L corresponds to exchanging the order of u and v. By dropping the superfluous configurations, one passes from $(2^p-1)\times(2^p-1)$ to the sharper upper bound $2^{p-1}(2^p-1)$. We will improve upon the later upper bound in the sequel.

A configuration (J, L) will be said to be *successful* if there exists a proper critical pair (u, v) in $K_J^\circ \times K_L^\circ$. One could think that for a successful configuration (J, L) it holds that $J \cap L = \emptyset$. Albeit intuitive, this turns out to be false, as the following example show.

Example 1. Fix $\alpha \in]0, 1[$ and take a cone $K \subset \mathbb{R}^3$ with generators $g^1 = (0, 0, 1)$, $g^2 = (\sqrt{1-\alpha^2}, 0, -\alpha)$ and $g^3 = (0, \sqrt{1-\alpha^2}, -\alpha)$. The Gramian matrix is given by

$$M = \begin{bmatrix} 1 & -\alpha & -\alpha \\ -\alpha & 1 & \alpha^2 \\ -\alpha & \alpha^2 & 1 \end{bmatrix}$$

Consider the vectors $u = (1, 0, 0)$, $v = (0, 1, 0)$, which can be written in terms of g^1, g^2 and g^3, according to (24), with coefficients $x = (1-\alpha^2)^{-1/2}\,(\alpha, 1, 0)$ and $y = (1-\alpha^2)^{-1/2}\,(\alpha, 0, 1)$ respectively. Note that $u \in K_J^\circ$, $v \in K_L^\circ$ with $J = \{1, 2\}$, $L = \{1, 3\}$. It is easy to check that u, v are unit vectors, $x^t M y = \langle u, v \rangle = 0$, and

$$Mx = \sqrt{1-\alpha^2}\,(0, 1, 0) \geq 0, \quad My = \sqrt{1-\alpha^2}\,(0, 0, 1) \geq 0,$$

in which case (25)–(29) hold. Since (30) and (31) are a consequence of (25)–(29), it follows from Proposition 5 that (u, v) is critical, despite the fact that $J \cap L = \{1\} \neq \emptyset$.

Nevertheless the case of a successful configuration with $J \cap L \neq \emptyset$ is somewhat special. Note that in the previous example we have $\langle u, v \rangle = 0$, and also $\langle g^1, u \rangle = \langle g^1, v \rangle = 0$. The first equality is specific of this example (one could get $\langle u, v \rangle \neq 0$

by a suitable perturbation of u, v), but the remaining two are general for the situation $J \cap L \neq \emptyset$, as the following proposition shows.

Proposition 6. *Consider a configuration* (J, L) *and let* $(u, v) \in K_J^\circ \times K_L^\circ$ *be a proper critical pair of a polyhedral cone* K *with generators* $g^1, \ldots, g^p$. *Then,*

i) $\langle g^i, u\rangle \geq 0$ *for all* $i \in J$ *and* $\langle g^i, v\rangle \geq 0$ *for all* $i \in L$,
ii) $\langle g^i, u\rangle = \langle g^i, v\rangle = 0$ *for all* $i \in J \cap L$.

Proof. Item (i) has been almost established along the proof of Theorem 6. We repeat the argument: write u, v as in (24), so that $I(x) = J$, $I(y) = L$, and apply Proposition 5 and Corollary 3. From (25) and (36), we get

$$(Mx)_i \geq (x^t My)^2 (Mx)_i \quad \forall\, i \in J.$$

Since $x^t My = \langle u, v\rangle$ belongs to $]-1, 1[$ because (u, v) is proper, we get

$$\langle g^i, u\rangle = (Mx)_i \geq 0 \quad \forall\, i \in J,$$

using (24) and the definition of the Gramian matrix M in the equality. A similar argument starting from (26) and (35) establishes that

$$\langle g^i, v\rangle = (My)_i \geq 0 \quad \forall\, i \in L.$$

For (ii), we use in an analogous way (35) and (36): take $i \in J \cap L = I(x) \cap I(y)$, so that both (35) and (36) hold for this index i. We get from these equations

$$\begin{aligned} (Mx)_i &= (x^t My)^2 (Mx)_i, \\ (My)_i &= (x^t My)^2 (My)_i, \end{aligned}$$

and hence, since $(x^t My)^2 \neq 1$, we conclude that $\langle g^i, u\rangle = (Mx)_i = 0 = (My)_i = \langle g^i, v\rangle$. $\square$

Proposition 6 has four interesting corollaries.

Corollary 4. *Let* K *be a polyhedral cone with* p *generators, and* (J, L) *a configuration. If* $(u, v) \in K_J^\circ \times K_L^\circ$ *is a proper critical pair, then neither* $J \subset L$ *nor* $L \subset J$.

Proof. Assume that $J \subset L$, so that $J = J \cap L$. Write u, v as in (24), with $I(x) = J$, $I(y) = L$. By Proposition 6(ii), we have $(Mx)_i = 0$ for all $i \in J$. Since $x_i = 0$ for $i \notin J$ we get, using (27), $0 = x^t Mx = 1$, a contradiction. A similar contradiction is arrived at if one assumes that $L \subset J$. $\square$

Corollary 5. *The cardinality of the angular spectrum* $\Omega(K)$ *of a polyhedral cone* K *with* p *generators is less than or equal to the integer*

$$s_p = 2^{p-1}(2^p + 1) - 3^p.$$

Proof. The result follows from Theorem 6 and Corollary 4: the number of proper critical angles cannot exceed the number of non-ordered pairs of subsets J, L of $\{1, \dots, p\}$ such that neither J contains L nor L contains J. Such number is the difference between the number of all non-ordered pairs of nonempty subsets, namely $2^{p-1}(2^p - 1)$, and the number of pairs where one of the subsets contains the remaining one, namely $3^p - 2^p$. $\square$

A closed and convex cone $K \subset \mathbb{R}^d$ is said to be *acute* if $\langle u, v\rangle \geq 0$ for all $u, v \in K$. A polyhedral cone K with generators $g^1, \dots, g^p$ is acute if and only if $\langle g^i, g^j\rangle \geq 0$ $(1 \leq i, j \leq p)$, i.e. if all entries of the Gramian matrix M are nonnegative. The following result shows that the presence of negative entries in the Gramian matrix of the cone in Example 1 above was unavoidable.

Corollary 6. *Let K be an acute polyhedral cone with p generators, and (J, L) a configuration. If $(u, v) \in K_J^\circ \times K_L^\circ$ is a proper critical pair, then $J \cap L = \emptyset$.*

Proof. Write u, v as in (24), with $I(x) = J$, $I(y) = L$. Assume that $J \cap L \neq \emptyset$. Take any $i \in J \cap L$. By Proposition 6(ii), one has

$$0 = \langle g^i, u\rangle = (Mx)_i = \sum_{j \in J} M_{ij} x_j.$$

Since $x_j > 0$ for $j \in J$ and the entries of M are nonnegative by acuteness of K, we get that $0 = M_{ij} = \langle g^i, g^j\rangle$ for all $i \in J \cap L$ and all $j \in J$. In particular, we can take $j = i$, in which case we have $0 = M_{ii} = \|g^i\|^2$, contradicting the fact that g^i is a unit vector. $\square$

Corollary 7. *The cardinality of the angular spectrum $\Omega(K)$ of an acute polyhedral cone $K \subset \mathbb{R}^d$ with p generators is bounded above by the integer*

$$r_p = \frac{1}{2}(3^p + 1) - 2^p.$$

Proof. The result follows from Theorem 6 and Corollary 6, because r_p is the cardinality of the family of non-ordered pairs of nonempty and pairwise disjoint subsets of $\{1, \dots, p\}$. $\square$

4.3. Cardinality versus dimensionality

The upper bounds obtained up to now for the cardinality of $\Omega(K)$ can be improved through the use of dimensional arguments, which provide bounds on the cardinality of the sets J, L in a successful configuration. The first observation arises from the known result that any point in a closed convex cone contained in $\mathbb{R}^n$ can be written as a positive combination of points belonging to up to n linearly independent extreme rays. Without loss of generality, one may take as n the dimension of the vector subspace spanned by K, which becomes thus an upper bound on the cardinality of J, L. Furthermore, such upper bound can be reduced by one unit, as a consequence of Proposition 2. The next proposition provides the details for this estimate.

Proposition 7. *Let $K \subset \mathbb{R}^d$ be a polyhedral cone with p generators, and n the dimension of the linear subspace spanned by K. If (u, v) is a proper critical pair of K, then there exist subsets J, L of $\{1, \ldots, p\}$ of cardinality less than or equal to $n-1$ such that $(u, v) \in K_J^\circ \times K_L^\circ$.*

Proof. Recall that we are assuming that no generator of K is a positive combination of the remaining ones. The cone version of Caratheodory's Theorem [8, Section 1.6] implies that each point in K can be written as a combination of up to n linearly independent generators of K. Thus, both u and v belong to sets K_J°, K_L°, where the sets $\{g^i : i \in J\}$, $\{g^i : i \in L\}$ are linearly independent and

$$\max\{\mathrm{card}(J), \mathrm{card}(L)\} \leq n.$$

We claim now that this bound can be reduced indeed to $n-1$ rather than n. Assume that u belongs to K_J° with $\mathrm{card}(J) = n$. By using the very definition of K_J° and the linear independence of $\{g^i : i \in J\}$, one can show that K_J° is the relative interior of the closed cone with generators $\{g^i : i \in J\}$ (consider e.g. the linear map which takes the generators into the canonical basis of $\mathbb{R}^n$. It is an homeomorphism and maps K_J° into the strictly positive orthant of $\mathbb{R}^n$, obviously open relative to $\mathbb{R}^n$). Thus K_J° is open relative to the linear subspace V spanned by the cone generated by $\{g^i : i \in J\}$. Since V has dimension n, the same as the linear span of K, in which it is contained, both subspaces coincide, so that $K_J^\circ \subset K$ is open relative to the linear span of K, and hence it is contained in the relative interior of K. Since $u \in K_J^\circ$, we have that u belongs to the relative interior of K, which contradicts Proposition 2. It follows that $\mathrm{card}(J) \leq n-1$. The same reasoning applies to L. $\square$

Corollary 8. *The cardinality of the angular spectrum $\Omega(K)$ of an acute polyhedral cone K with p generators is less than or equal to the integer*

$$\Phi(p, m) = \frac{1}{2} \sum_{\substack{1 \leq j, \ell \leq m \\ j+\ell \leq p}} \frac{p!}{j!\ell!(p-j-\ell)!}\ , \tag{51}$$

where $m = dim(span(K)) - 1$.

Proof. The result follows from Proposition 7, because the announced upper bound is equal the number of non-ordered pairs of nonempty and pairwise disjoint subsets of $\{1, \ldots, p\}$ with cardinality not exceeding m. Indeed,

$$\mathrm{card}[\Omega(K)] \ \leq \ \frac{1}{2} \sum_{\substack{1 \leq j, \ell \leq m \\ j+\ell \leq p}} C_{j,\ell}^p\ , \tag{52}$$

with

$$C_{j,\ell}^p = \begin{cases} \text{number of pairs } (J, L) \text{ such that } J, L \subset \{1, \ldots, p\}, \\ \mathrm{card}(J) = j,\ \mathrm{card}(L) = \ell, \text{ and } J \cap L = \emptyset. \end{cases}$$

In terms of the usual binomial coefficients

$$\binom{p}{k} = \frac{p!}{k!(p-k)!},$$

one can write, of course,

$$C^p_{j,\ell} = \binom{p}{j}\binom{p-j}{\ell} = \frac{p!}{j!\ell!(p-j-\ell)!}\,.$$

Notice, incidentally, that the coefficient 1/2 in (52) takes care of the fact that the configurations (J, L) and (L, J) produce the same proper critical pair, and therefore they should be counted only once. □

A similar refinement is possible for the upper bound given in Corollary 5. Notice that $\Phi(p, m)$ can be decomposed in the form

$$\Phi(p,m) = \sum_{\substack{1\le j<\ell\le m\\ j+\ell\le p}} \frac{p!}{j!\ell!(p-j-\ell)!} \;+\; \frac{1}{2}\sum_{\substack{1\le j\le m\\ 2j\le p}} \frac{p!}{(j!)^2(p-2j)!}\,.$$

For the case of a cone that is not necessarily acute, one has:

Corollary 9. *Let K be a polyhedral cone generated by p vectors, and let $m = dim(span(K)) - 1$. Then, the cardinality of $\Omega(K)$ is less than or equal to*

$$\Gamma(p,m) = \sum_{\substack{1\le j<\ell\le m\\ j+\ell\le p}} \binom{p}{j}\left\{\binom{p}{\ell} - \binom{p-j}{\ell-j}\right\} + \frac{1}{2}\sum_{\substack{1\le j\le m\\ 2j\le p}} \binom{p}{j}\left\{\binom{p}{j} - 1\right\}. \tag{53}$$

Proof. We use again (52), but now

$$C^p_{j,\ell} = \begin{cases} \text{number of pairs } (J, L) \text{ such that } J, L \subset \{1, \ldots, p\}, \\ \text{card}(J) = j,\ \text{card}(L) = \ell,\ J \not\subset L \text{ and } L \not\subset J. \end{cases}$$

A simple combinatorial argument does the rest of the job. □

4.4. Polynomial growth principle

Although the expression $\Gamma(p, m)$ looks quite involved, it is easy to be evaluated in practice. From a theoretical point of view, Corollary 9 allows us to derive the following Polynomial Growth Principle:

Proposition 8. *Consider a polyhedral cone K lying in a space of prescribed dimension, say d. Then, the cardinality of $\Omega(K)$ grows at most polynomially with respect to the number p of generators of K.*

Proof. Let us examine more carefully the behavior of $\Gamma(p, m)$ as function of p. We consider m as an integer that has been fixed once and for all. In the worst case, one can take $m = d - 1$ (because span(K) is at most d-dimensional). For p large enough, namely $p \ge 2m$, the constraints $j + k \le p$ and $2j \le p$ in (53) are superfluous. It is not difficult to see that the function $p \mapsto \Gamma(p,m)$ is then a polynomial of degree $2m$. The terms of highest degree shows up in

$$\frac{1}{2} \sum_{1 \le j \le m} \binom{p}{j} \left\{ \binom{p}{j} - 1 \right\}$$

when $j = m$. After a due simplification, one ends up, for $p \geq 2m$, with a polynomial expansion of the form

$$\Gamma(p,m) = \frac{1}{2(m!)^2}\, p^{2m} \; + \; \text{terms of lower degree.}$$

This expansion and Corollary 9 not only complete the proof of the proposition, but also provides an additional insight on the growth of the polynomial that serve to bound the cardinality of $\Omega(K)$. ☐

Recall that Corollary 7 provides an upper bound that grows exponentially with respect to p. This deficiency is due to the fact that Corollary 7 neglects the dimension of the underlying space. We conjecture, however, that the upper bound given in Corollary 7 is tight, in the sense that

> *for any integer $p \geq 2$, there exist a dimension d (depending on p) and an acute polyhedral cone $K \subset \mathbb{R}^d$ with p generators and r_p critical angles.*

Due to the Polynomial Growth Principle, the dimension d cannot be kept fixed, but, on the contrary, it is forced to grow if p increases. We do have some partial evidence supporting the above mentioned conjecture. In fact, we have been able to construct a polyhedral cone with one critical pair for each pair of nonempty and pairwise disjoint subsets of $\{1, \ldots, p\}$. The corresponding critical angles, however, are not pairwise different (in fact the value of the angle depends only on the cardinality of the subsets), but it seems likely that a careful perturbation of the generators of this cone produces an acute cone with precisely r_p critical angles. Discussing all the details would lead us too far away from the context of this journal. These more elaborate combinatorial aspects of our work are still under investigation and will be reported elsewhere.

Remark 3. We have also some additional results on the upper bounds for the cardinality of $\Omega(K)$ as a function of the number p of generators of K, when the dimension d is fixed. This cardinality might be unbounded as a function of p. This is a consequence of the fact that it is possible to construct non-polyhedral cones with infinite (and even noncountable) angular spectra in any space with dimension $d \geq 3$. The construction procedure, which is quite elaborate and space consuming, will be presented in a forthcoming technical note.

5. By way of application

As mentioned in Section 1, the term $\theta_{\max}(K)$ can be used as tool for measuring the degree of pointedness of the cone K. Recall that an *index of pointedness* is a continuous function $f : \Xi(\mathbb{R}^d) \to \mathbb{R}$ satisfying the following set of axioms:

1. $f(K) = 0$ iff K is not pointed,
2. $f(K) = 1$ iff K is a ray,
3. $f(U(K)) = f(K)$ for any orthonormal matrix U,
4. $K_1 \subset K_2$ implies $f(K_1) \geq f(K_2)$

Continuity of f refers to the usual bounded Pompeiu-Hausdorff metric on $\varXi(\mathbb{R}^d)$, to wit

$$\delta(K_1, K_2) = \sup_{\|x\| \leq 1} \big|\mathrm{dist}[x, K_1] - \mathrm{dist}[x, K_2]\big|.$$

The theory of pointedness indices has been introduced and developed in the paper [5], although the set $\varXi(\mathbb{R}^d)$ is there slightly larger (it includes the trivial cone $\{0\}$ and the whole space $\mathbb{R}^d$).

As proven in our work [5], the function $\hat{f} : \varXi(\mathbb{R}^d) \to \mathbb{R}$ defined by

$$\hat{f}(K) = \cos\Big(\frac{\theta_{\max}(K)}{2}\Big)$$

fulfills all the requirements to qualify as an index of pointedness.

We stress the fact that $\theta_{\max}(K)$ is not the only critical angle of interest. As explained next, the smallest nonzero proper critical angle plays also a relevant role in the description of the cone, namely, it can be used as tool for measuring its degree of solidity.

By an *index of solidity* we understand a continuous function $g : \varXi(\mathbb{R}^d) \to \mathbb{R}$ satisfying the axioms:

1. $g(K) = 0$ iff K is not solid,
2. $g(K) = 1$ iff K is a half-space,
3. $g(U(K)) = g(K)$ for any orthonormal matrix U,
4. $K_1 \subset K_2$ implies $g(K_1) \leq g(K_2)$

In the last axiom, monotonicity occurs now in the upward sense: the bigger the cone, the more solid it should be.

A nice example of index of solidity that one may consider is the function $\hat{g} : \varXi(\mathbb{R}^d) \to \mathbb{R}$ given by

$$\hat{g}(K) = \cos\Big(\frac{\theta_{\max}(K^+)}{2}\Big). \tag{54}$$

What is bothering about the expression (54) is that it involves the dual cone K^+ and not the original cone K itself. However, this problem can be remediated since it is possible to write $\hat{g}$ in the equivalent form

$$\hat{g}(K) = \begin{cases} \sin\big(\frac{\theta_{\min}(K)}{2}\big) & \text{if } K \text{ is solid,} \\ 0 & \text{if } K \text{ is not solid,} \end{cases} \tag{55}$$

where the term

$$\theta_{\min}(K) = \text{smallest nonzero critical angle of } K \tag{56}$$

is well defined for any solid cone K (the definition of $\theta_{\min}(\cdot)$ can be extended to nonsolid cones as well, but the pathological case of a ray must be settled in a sui generis way; the natural convention is that $\theta_{\min}(\cdot)$ vanishes at any ray).

The formulation (55) of $\hat{g}$ is presented here for the first time. The passage from the old definition (54) to the new characterization (55) is not straightforward since it relies on a clever use of Corollary 1.

Remark 4. The concept of angular width involves the functions $\theta_{\max}(\cdot)$ and $\theta_{\min}(\cdot)$ at the same time. By definition, the angular width of a cone $K \in \Xi(\mathbb{R}^d)$ is the nonnegative number

$$\mathrm{aw}(K) = \theta_{\max}(K) - \theta_{\min}(K).$$

For instance, the cone $\mathcal{C}_d$ has an angular width greater or equal than $\pi/4$, i.e, between the largest and the smallest critical angle of $\mathcal{C}_d$ there are at least 45^0 . By constrast, the cone of symmetric positive semidefinite matrices has no angular width. Although both cones yield the same value for $\theta_{\min}(\cdot)$ (and therefore they have the same degree of solidity with respect to the index $\hat{g}$), they do not have the same maximal angle (and therefore they have a different measure of pointedness with respect to $\hat{f}$).

In classical spectral analysis of symmetric matrices, the largest and smallest eigenvalues play the leading role. The intermediate eigenvalues are certainly less relevant but they provide also a piece of information concerning the structure of the matrix. The same remark applies to the intermediate critical angles, except that now the computation and analysis of these objects are far more complicated.

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