Dynamics of two dimensional Blaschke products.

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Abstract. In this paper we study the dynamics on $\mathbb{T}^2$ and $\mathbb{C}^2$ of a two dimensional Blaschke product. We prove that in the case that the Blaschke product is a diffeomorphisms of $\mathbb{T}^2$ with all periodic points hyperbolic then the dynamics is hyperbolic. If a two dimensional Blaschke product diffeomorphism of $\mathbb{T}^2$ is embedded in a two dimensional family given by composition with translations of $\mathbb{T}^2$, then we show that there is an open set of parameter values for which the dynamics is Anosov or has an expanding attractor with a unique SRB measure.

1. Introduction

A (finite) Blaschke product is a map of the form

$$B(z) = \theta_0 \prod_{i=1}^{n} \frac{z - a_i}{1 - z\overline{a_i}}$$

where $n \geq 2$, $a_i \in \mathbb{C}, |a_i| < 1$, $i = 1 \dots n$ and $\theta_0 \in \mathbb{C}$ with $|\theta_0| = 1$. B is a rational mapping of $\mathbb{C}$, it is an analytic function in a neighborhood of the unit disc $\mathbb{D}$, and B maps the unit circle $\mathbb{T}$ to itself. Blaschke products are interesting in their own right but also from the point of view of more general complex dynamics. For example, any meromorphic map such that its Julia set bounds an invariant simply connected neighborhood in $\mathbb{C}$ is conjugate to a Blaschke product which is expanding on the unit circle.

In [**PRS**] we studied the family of Blaschke products

$$\{B_\theta\}_{\{\theta \in \mathbb{T}\}} = \{\theta B\}_{\{\theta \in \mathbb{T}\}}$$

as dynamical systems on $\mathbb{T}$ from the point of view of finding lower bounds for the average entropy of members of the family. In [**PRS**] we first show that either $\{B_\theta\}$ has a fixed sink or indifferent point or the dynamics is expanding. We show that for an open set U of θ, $\{B_\theta\}$ is expanding. Then we give a lower bound for the integral of the entropy $h(B_\theta)$ over U, where the entropy is the measure theoretic entropy with respect to the absolutely continuous invariant measure for $\{B_\theta\}$. This measure is also the SRB measure for $\{B_\theta\}$.

In the present brief paper we make a start on the study of the dynamics of a two dimensional Blaschke product of the form

$$F(z, w) = (A(z)B(w), C(z)D(w)), \tag{1.1}$$

and of the form

$$F(z, w) = (\frac{A(z)}{B(w)}, \frac{D(w)}{C(z)}), \tag{1.2}$$

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considered as dynamical systems on $\mathbb{T} \times \mathbb{T}$ where A, B, C, D are one dimensional Blaschke products and we allow the possibility that some of the degrees of A, B, C, D may also be 1. We concentrate our study in the case of Blaschke products that are diffeomorphisms of $\mathbb{T}^2$. We will call the Blaschke product of type (1.1) that are diffeomorphism of $\mathbb{T}^2$, *Blaschke product diffeomorphisms*; the Blaschke products of type (1.2) that are diffeomorphism of $\mathbb{T}^2$ will be called *quotient Blaschke product diffeomorphisms.* If the periodic points of a Blaschke product diffeomorphism are hyperbolic the diffeomorphism is hyperbolic (see Theorem 3.10). Moreover, some are Anosov diffeomorphisms. In some cases we show that the Julia set of a Blaschke product considered as a map of $\mathbb{C}^2$ is contained in $\mathbb{T}^2$. Observe that the result staed in theorem 3.10 is similar to the one obtained for one dimensional Blaschke product endomorphisms.

These results could help to get some insight into the dynamics of certain meromorphic maps on $\mathbb{C}^2$.

In [**PRS**], we raise the question of whether similar results as the one stated for one dimensional Blaschke families might hold for two dimensional Blaschke products, The family now depends on two parameters $(\theta, \phi) \in \mathbb{T} \times \mathbb{T}$,

$$F_{(\theta,\phi)}(z, w) = (\theta A(z)B(w), \phi C(z)D(w)).$$

We show in Theorem 4.1 that if F is a diffeomorphism of $\mathbb{T} \times \mathbb{T}$ then there is an open set of parameter values for which there is an SRB measure with positive entropy that is either supported on the whole torus in which case the diffeomorphism is Anosov or on a hyperbolic attractor. This is analogous to our result in one dimension. As opposed to [**PRS**] which employs techniques from complex analysis our proofs rely on results from real dynamics, especially [**PS1**], [**PS2**]. We wonder if our results on Blaschke products have simpler proofs more along the lines of complex dynamics. The Blaschke product diffeomorphisms we consider as above are precisely the analytic maps on a neighborhood of $\mathbb{D} \times \mathbb{D}$ mapping $\mathbb{D} \times \mathbb{D}$ to itself and $\mathbb{T} \times \mathbb{T}$ diffeomorphically to itself (see [**R**]).

2. Examples.

The Blaschke product

$$B(z) = \theta_0 \prod_{i=1}^{n} \frac{z - a_i}{1 - z\overline{a_i}}$$

is a degree n, orientation preserving immersion of the circle. It follows that the two dimensional Blaschke product diffeomorphisms F we are considering induce isomorphisms of the fundamental group of the torus, $\mathbb{Z} \times \mathbb{Z}$, which given as a matrix are of the form

$$N_F = \begin{bmatrix} n & m \\ k & j \end{bmatrix}$$

with n, m, k, j positive integers, the degrees of A, B, C, D respectively. Moreover, $det(N) = 1$ since F is an orientation preserving diffeomorphism of $\mathbb{T} \times \mathbb{T}$. From the form of N_F we see that N_F maps the positive orthant into itself and N_F is a hyperbolic linear map. It is sometimes convenient to consider the torus $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$ and to write F additively, as follows: Given a Blaschke product $B : \mathbb{C} \to \mathbb{C}$, there exists $b : \mathbb{R} \to \mathbb{R}$ with

$$b' > 0.$$

such that

$$B(e^{2\pi i x}) = e^{2\pi i b(x)}.$$

Now, given $F(z, w) = (A(z)B(w), C(z)D(w))$, let a, b, c, d be the corresponding transformations acting on $\mathbb{R}$. Therefore

$$F(e^{2\pi i x}, e^{2\pi i y}) = (e^{2\pi i(a(x)+b(y))}, e^{2\pi i(c(x)+d(y))}).$$

So, given F we can take the map

$$\hat{F} : \mathbb{T}^2 \to \mathbb{T}^2,$$

$$\hat{F}(x,y) = (a(x) + b(y), c(x) + d(y)).$$

Observe that for $(z,w) = (e^{2\pi ix}, e^{2\pi iy}) \in \mathbb{T}^2$ we have

$$F^n(z,w) = e^{2\pi i \hat{F}^n(x,y)}.$$

The linear part of $\hat{F}$ is the matrix N above, ie $\hat{F} = N_F + \Theta$ where Θ is doubly periodic. An invariant set Λ for $f : M \to M$ is called hyperbolic if it is compact,the tangent bundle $T_\Lambda M$ can be decomposed as $T_\Lambda M = E^s \oplus E^u$ invariant under Df and there exist $C > 0$ and $0 < \lambda < 1$ such that

$$|Df^n_{/E^s(x)}| \leq C\lambda^n, \ |Df^{-n}_{/E^u(x)}| \leq C\lambda^n$$

for all $x \in \Lambda$ and for every positive integer n. Moreover, a diffeomorphism f,is called Anosov, if $L(f) = M$ is a hyperbolic set, where $L(f)$ is the limit set of f (the closure of the accumulation points of any backward and forward trajectory).

2.1. Blaschke products induced by a linear Anosov diffeomorphisms. Now let us start with a matrix $N \in SL(\mathbb{Z}, 2)$,

$$N = \begin{bmatrix} n & m \\ k & j \end{bmatrix}$$

with n, m, k, j positive integers. Let

$$F_N(z,w) = (z^n w^m, z^k w^j)$$

and observe that F_N is a Blaschke product diffeomorphism. It follows that

$$\hat{F}_N(x,y) = N(x,y) = (nx + my, kx + jy),$$

so $\hat{F}_N$ is a linear Anosov diffeomorphism induced by N on $\mathbb{T}^2$ and $\mathbb{T}^2$ is a hyperbolic set for F_N.

If we take

$$N^{-1} = \begin{bmatrix} j & -m \\ -k & n \end{bmatrix}$$

it follows that

$$F_{N^{-1}}(z,w) = (z^j w^{-m}, z^{-k} w^n)$$

is a quotient Blaschke product diffeomorphism.

From the discussion above we see that F is in the homotopy class class of the linear Anosov diffeomorphism N_F. So question 4.2 of section 4 makes sense from the homotopy point of view.

2.1.1. *The Dynamics in $\mathbb{C}^2$ of Blaschke products induced by a linear Anosov diffeomorphisms.* Now we discuss the dynamics of F_N in $\mathbb{C}^2$.

LEMMA 2.1. *F_N is an Axiom A diffeomorphisms such that $\Omega(F_N) = \{(0,0)\} \cup \mathbb{T}^2\}$. Moreover, the point at infinity (∞, ∞) is an attractor, while $(\infty, 0)$ and $(0, \infty)$ are repeller.*

PROOF. Writing z and w in polar coordinates, $z = r\exp(2\pi ix), w = s\exp(2\pi iy)$, it follows that

$$F_N(r\exp(2\pi ix), s\exp(2\pi iy)) = (r^n s^m \exp 2\pi i(nx + my), r^k s^j \exp 2\pi i(kx + jy)).$$

Studying the map $(r,s) \to (r^n s^m, r^k s^j)$ for $r, s \geq 0$, it follows that its nonwandering set consists of five points: $(0,0)$, $(1,1)$, $(\infty, 0)$, $(0, \infty)$, (∞, ∞). □

The stable manifold of $\mathbb{T}^2$ is contained in $\{|z| < 1\} \times \{|w| > 1\} \cup \{|z| > 1\} \times \{|w| < 1\}$ and the unstable manifold of $\mathbb{T}^2$ is contained in $\{|z| < 1\} \times \{|w| < 1\} \cup \{|z| > 1\} \times \{|w| > 1\}$. The map F_N fails to be a diffeomorphisms at $\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}$ which are critical points for F_N.

2.1.2. *Perturbation of Blaschke products induced by a linear Anosov diffeomorphisms.* The next theorem follows from classical results on the stability of Axiom A systems :

Theorem 2.2. *Let F_N be a Blaschke product diffeomorphism induced by $N \in Sl(\mathbb{Z}, 2)$. Then, for any rational map G on $\mathbb{C}^2$ close enough to F_N it follows that G is an Axiom A diffeomorphism such that $\Omega(G) = \{\{S_1\} \cup \{S_2\} \cup \{R_1\} \cup \{R_2\} \cup \mathcal{H}\}$, such that S_1, S_2 are fixed attracting points, R_1, R_2 are fixed repelling points and $\mathcal{H}$ is a hyperbolic set homeomorphic to a two dimensional torus.*

3. General dynamics of the Blaschke products.

Definition 3.1. An f-invariant set Λ is said to have a dominated splitting, if the tangent bundle over Λ is decomposed in two invariant subbundles $T_\Lambda M = \mathcal{E} \oplus \mathcal{F}$, such that there exist $C > 0$ and $0 < \lambda < 1$ with the following property:

$$|Df^n_{|\mathcal{E}(x)}||Df^{-n}_{|\mathcal{F}(f^n(x))}| \leq C\lambda^n, \text{ for all } x \in \Lambda, n \geq 0.$$

This concept was introduced independently by Mañé, Liao and Pliss, as a first step toward proving that structurally stable systems satisfy a hyperbolicity condition on the tangent map. A dominated splitting is a natural way to relax hyperbolicity.

Theorem 3.2. *Let a, b, c, d be C^1-smooth immersions from S^1 to S^1 that preserve orientation. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be given by*

$$f(\theta, \phi) = (a(\theta) + b(\phi), c(\theta) + d(\phi))$$

then the positive cone field is preserved by Tf and if f is a diffeomorphism then f has a dominated splitting on all of $\mathbb{T}^2$.[1]

Proof. Observe that

$$Df = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}.$$

Let $\mathcal{C}^+$ be the positive cone bounded by the directions spanned by the vectors $(1, 0)$ and $(0, 1)$. Observe that $DF(\mathcal{C}^+)$ is properly contained in $\mathcal{C}^+$. The existence of a dominated splitting follows. □

Corollary 3.3. *Given a two dimensional Blaschke product diffeomorphism*

$$F(z, w) = (A(z)B(w), C(z)D(w)),$$

then $\hat{F}$ has a dominated splitting on all of $\mathbb{T}^2$ and the complexification of this dominated splitting is a dominated splitting for $F|\mathbb{T}^2$ with respect to the complex derivative of F.

Theorem 3.4. *Let a, b, c, d be C^1-smooth immersions from S^1 to S^1 such that a, d preserve orientation and b, c reverse orientation . Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be given by*

$$f(\theta, \phi) = (a(\theta) + b(\phi), c(\theta) + d(\phi))$$

then the positive cone field bounded by the directions spanned by the vectors $(1, 0)$ and $(0, -1)$ is preserved by Tf and if f is a diffeomorphism then f has a dominated splitting on all of $\mathbb{T}^2$.[2]

Proof. The proof is similar to the one of theorem 3.2. □

[1] If f is not a diffeomorphism there is a dominated splitting on the inverse limit space.

[2] If f is not a diffeomorphism there is a dominated splitting on the inverse limit space.

Corollary 3.5. *Given a two dimensional Blaschke product diffeomorphism*

$$F(z,w) = (\frac{A(z)}{B(w)}, \frac{D(w)}{C(z)}),$$

then $\hat{F}$ has a dominated splitting on all of $\mathbb{T}^2$ and the complexification of this dominated splitting is a dominated splitting for $F|\mathbb{T}^2$ with respect to the complex derivative of F.

Now we can apply results from real dynamics about diffeomorphisms with dominated splittings. Recall that $\Omega(f)$ is the nonwandering set of f.

Theorem 3.6. *([**PS1**]) Let $f \in Diff^2(M^2)$ and assume that $\Lambda \subset \Omega(f)$ is a compact invariant set exhibiting a dominated splitting such that any periodic point is a hyperbolic periodic point. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is hyperbolic and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, \ldots \mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugate to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

In the next two propositions we show that periodic simple closed curves do not occur for Blaschke product diffeomorphisms and that there is at most one sink.

Proposition 3.7. *Let F be a Blaschke product diffeomorphism. Then F does not have any normally hyperbolic simple closed invariant curves.*

Proof. If C is an invariant simple closed curve for F the either C represents a non-trivial element of the fundamental group fixed by N_F or C bounds a disc. Since N_F is a hyperbolic linear map it has no non-trivial fixed elements, so the first eventuality is impossible. In the second case the bundle E which is tangent to C cannot be extended over the disc bounded by C contradicting the fact that there is a dominated splittng defined on all of $\mathbb{T}^2$. □

Proposition 3.8. *A two dimensional Blaschke product, F, has at most one attracting or semiattracting periodic point, which is necessarily a fixed point, in $\mathbb{D}^2$.*

Proof. Since $\mathbb{D}^2$ is invariant for F, F^n is a normal family in $\mathbb{D}^2$. If $z_0 \in \mathbb{D}^2$ is an attracting or a semiattracting periodic point of F then some subsequence F^{n_k} converges to the constant map z_0 on an open set in, and hence all of the interior of , $\mathbb{D} \times \mathbb{D}$. Let j be the period of z_0. It follows that $F^{jn}(z)$ converges to z_0 for all z in the interior of $\mathbb{D}^2$. Thus z_0 is the unique attracting periodic point in $\mathbb{D}^2$. □

Question 3.9. We don't know if there is a version of proposition 3.8 which counts the number of repellors, even for Blaschke products which are birational equivalences.

We can now describe the dynamics of Blaschke product diffeomorphisms.

Theorem 3.10. *Let F be either a Blaschke product diffeomorphism or a quotient Blaschke product diffeomorphism such that any periodic point in $\mathbb{T}^2$ is a hyperbolic periodic point. Then, $F_{|\mathbb{T}^2}$ is an Axiom A diffeomorphism. Moreover, one of the next option holds:*

1. *$F_{|\mathbb{T}^2}$ is Anosov and $L(F_{|\mathbb{T}^2}) = \mathbb{T}^2$,*
2. *$L(F_{|\mathbb{T}^2}) = \mathcal{S} \cup \mathcal{H} \cup \mathcal{S}a \cup \mathcal{R}$,*
3. *$L(F_{|\mathbb{T}^2}) = \mathcal{S} \cup \mathcal{H}$,*
4. *$L(F_{|\mathbb{T}^2}) = \mathcal{H} \cup \mathcal{S}a \cup \mathcal{R}$;*

Where $\mathcal{S}$ is a set formed by a single attracting fixed point, $\mathcal{R}$ is a set formed by a finite number of repelling periodic points, $\mathcal{S}a$ is a finite number of isolated saddles and $\mathcal{H}$ is a non-trivial maximal transitive hyperbolic invariant set in $\mathbb{T}^2$. In the last case it follows that $\mathcal{H}$ is an attractor in $\mathbb{T}^2$. Moreover, the order relation is given by $\mathcal{R} \to \mathcal{S}a \to \mathcal{H} \to \mathcal{S}$ (where $A \to B$ if $W^u(A) \cap W^s(B) \neq \emptyset$). In the case that $\mathcal{S}$ is empty, $F|\mathbb{T}^2$ has a unique SRB measure with positive entropy.

If F is a Blaschke product diffeomorphisms it follows that F always has an attracting or semi-attracting fixed point in $\mathbb{D}^2$, and any forward orbit in the interior of $\mathbb{D}^2$ converges to that fixed point. If this fixed point is in the interior of $\mathbb{D}^2$ it is attracting.

PROOF. Since $\hat{F}$ has a dominated splitting on all of $\mathbb{T}^2$ we can apply theorem 3.6. From proposition 3.7, it follows that there are no normally hyperbolic curves. Therefore, $L(F_{|\mathbb{T}^2})$ is a hyperbolic set. This implies that the number of periodic repellers and attractors is finite, the periodic points are dense in $L(F_{|\mathbb{T}^2})$ and $L(F_{|\mathbb{T}^2})$ has a spectral decomposition. From proposition 3.8 F has at most one attracting periodic point which is a fixed point. From the fact that N_F is a hyperbolic linear map with positive entries it follows that $Trace(F_*^n) \to +\infty$ as $n \to \infty$ and this implies by the Lefschetz formula and [**SS**] or Nielsen theory that the number of periodic points goes to infinity. So $Closure(Per(F))$ contains a non-trivial homoclinic class. To prove that $\mathcal{H}$ is a non-trivial hyperbolic set, first we prove that that the closure of all the periodic points such that at least one branch of its unstable manifold and at least one branch of its stable manifold is not bounded, is a unique homoclinic class denoted with $\mathcal{H}$. Later we prove that the set of periodic points with stable and unstable manifold with bounded length are a finite number of periodic points.

LEMMA 3.11. *There exists $h : \mathbb{T}^2 \to \mathbb{T}^2$ continuous, onto and homotopic to the identity such that $h \circ F = A \circ h$, where A is the linear Anosov map induced by N_F. That is, F is semiconjugate to A. Moreover, let p be a periodic point of F such that one of it its unstable branch has unbounded length. Then it follows that $h(\gamma^u(p)) \subset W^u(h(p))$ and $h(\gamma^u(p))$ has unbounded length, where $\gamma^u(p)$ is the branch of $W^u(p) \setminus \{p\}$ such that the corresponding unstable branch has unbounded length. The same holds for the stable branches.*

PROOF. To prove the lemma we use the following lemma:

LEMMA 3.12. *Let $\hat{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the lift of F to $\mathbb{R}^2$. Then $\hat{F} = A + P$ such that A is a linear Anosov map and P is periodic and there exists $H : \mathbb{R}^2 \to \mathbb{R}^2$ continuous and onto such that $H \circ \hat{F} = A \circ H$, that is, $\hat{F}$ is semiconjugate to A. Moreover, there exits a constant K_1 such that $\|H - Id\| < K_1$. Also, for any $m \in \mathbb{Z}^2$ we have $H(x + m) = H(x) + m$.*

PROOF. The proof can be found in theorem 2.2 of [**Fr**]. The main ingredient to construct the semiconjugacy is the global product structure of A. An alternate approach uses the fact that the shadowing lemma holds for any pseudo orbit.

□

Observe that this lemma implies the existence of h. Moreover, since h is a semiconjugacy we also conclude that $h(\gamma^u(p)) \subset W^u(h(p))$. In fact, if $x \in \gamma^u(p)$ then $\exists x_i \to p$ such that $f^i(x_i) = x$. So $h(x_i) \to h(p)$ and $A^i(h(x_i) = h(f^i(x_i) = h(x)$ and $h(x) \in W^u(h(p))$. We have to show now that the length of $h(\gamma^u(p))$ is unbounded. Since γ^u has infinite length we may find periodic saddle points whose stable manifolds cross γ^u in infintely many points. Consider an embedded loop γ consisting of a connected subarc of γ^u and a short piece of stable arc of a periodic saddle r. This loop is not contractable to a point. In fact, if it is not the case, let D be the disc bounded by γ. Observe that there it is a non singular vector field X defined on D induced by the subbundle $\mathcal{F}$ of the dominated splitting with the property that X is inward pointing on D, which is a contradiction. Therefore, it follows that $N_F^n([\gamma]) = [F^n(\gamma)]$ (where N_F is the linear map acting on the fundamental group) goes to infinity . Since $F^n(\gamma) = F^n(\tilde{\gamma^u}) \cup F^n(\tilde{\gamma^s})$ and $F^n(\tilde{\gamma^u}) \subset \gamma^u$ and $F^n(\tilde{\gamma^s})$ is a stable arc the length of which goes to zero, then given $M > 0$ there exists n large enough and $x \in \mathbb{R}^2$ such that $x \in F^n(\tilde{\gamma^u})$ and $d(x, p) > M$. Observe that this implies that the length of $h(\gamma^u)$ is unbounded. In fact, let us assume that this is not the case; so there is K_0 such that for any $x \in \gamma^u$ then $dist(H(x), H(p)) < K_0$. Let M be a positive integer and let $n > 0$ and $x \in \mathbb{R}^2$ such that $x \in F^n(\hat{\gamma_\varepsilon^u})$ and $d(x, p) > M$. Since $dist(H(x), H(p)) < K_0$ and $dist(H(p), p) < K_1$ it follows that

$$dist(H(x), x) > dist(p, x) - dist(p, H(x)) > dist(p, x) - [dist(p, H(p)) + dist(H(p), H(x))] > M - (K_1 + K_0).$$

Since $|H - Id| < K_1$ it follows that for M large enough we get a contradiction.

□

Using the lemma 3.11, we conclude that the closure of all the periodic points such that at least one branch of its unstable manifold and at least one branch of its stable manifold is not bounded, is a unique homoclinic class. To do that, we show that given a pair of periodic points p and q such that p has an unbounded unstable branch and q has an unbounded stable branch, it follows that these branch intersects homoclinically.

Let p be a periodic point with an unbounded unstable branch and let q be a periodic point with an bounded unstable branch . Let $\gamma^u(p)$ be the branch of $W^u(p) \setminus \{p\}$ with unbounded length. Let $\gamma^s(q)$ be the branch of $W^s(q)) \setminus \{q\}$ with unbounded length. From lemma 3.11 it follows that $h(\gamma^u(p))$ is an arc of unbounded length contained in the stable manifold of $h(p)$. Since $W^u(h(p))$ and $W^s(h(q))$ intersects transversally (recall that since A is a linear Anosov, then that for any pair of periodic point of A, any of their stable and unstable branches intersect each other), it follows that $h(\gamma^u(p))$ and $h(\gamma^s(q))$ intersect infinitely often and the long arcs in $h(\gamma^u(p))$ and $h(\gamma^s(q))$ have arbitrarily large intersections. Now if we complete a long arc in $\gamma^u(p)$ to a closed embedded loop $\gamma(p)$ adding a small arc in $W^s(r_1)$ and a long arc in $\gamma^s(q)$ to a closed embedded loop $\gamma(q)$ adding a small arc in $\gamma^u(r_2)$ for appropriately chosen saddles r_1, r_2 we see that $h(\gamma(p))$ and $h(\gamma(q)$ intersect homologically with as large an interesection number as we wish. Since h induces the identity map in homology so do $\gamma(p)$ and $\gamma(q)$ and then by construction so do $\gamma^u(p)$ and $\gamma^s(q)$.

To check the last part of the theorem, that there is always an attracting or semi-attracting fixed point in $\mathbb{D}^2$, we will use the following lemma:

LEMMA 3.13. *There exists N_0 such that for any $z \in \mathbb{D}^2$, there exists a periodic point $z_0 = z_0(z)$ of F with period smaller than N_0 and a subsequences $\{k_i\} = \{k_i(z)\}$ such that $F^{k_i}(z) \to z_0$.*

PROOF. Recall that $\mathbb{D}^2$ is invariant. Therefore, given $z \in \mathbb{D}^2$ it follows that $\{F^n(z)\}$ has a convergent subsequences. Let $\{n_i\}$ be such that $F^{n_i}(z) \to z_0$ for some z_0 and let $g_i = F^{n_{i+1}-n_i}$ and observe that it is a bounded sequences and therefore, there exists a subsequences g_{i_k} converging to some holomorphic map h. It follows that $h(z_0) = z_0$. Then, from the fact that h is a limit of iterates of F follows that F and h commute and therefore it follows that for any k it holds that $F^k(z_0)$ is also a fixed point of h. In other words, F maps the fixed points of h into fixed points of h. Observe that h has a finite number of fixed points; otherwise, h is the identity in $\mathbb{D}^2$ and so it the identity in the closure of $\mathbb{D}^2$ which is not possible because the identity map in $\mathbb{T}^2$ can not be obtained as the limit of iterates of F restricted to $\mathbb{T}^2$ (recall that DF^n diverges in $\mathbb{T}^2$). Therefore, since h has a finite fixed points and F maps them into themselves, it follows that the point z_0 is eventually periodic for F with period smaller and equal to the number of fixed points of h. Without loss of generality, we can assume that it is periodic and the lemma follows. □

COROLLARY 3.14. *For any $z \in \mathbb{D}^2$, there exists a periodic point $z_0 = z_0(z)$ of F with period smaller than N_0 such that $F^k(z)$ converges to the orbit of z_0.*

PROOF. Since F has a finite number of periodic point of period smaller than N_0 we can assume that the accumulation points of any sequence $\{F^k(z)\}$ is a fixed point. Let $\{z_i\}$ the set of fixed points and we can take a pair of finite number of disjoints neighborhoods $\{V_i\}$ and $\{U_i\}$ of each of those fixed points. We assume that $z_i \in V_i \subset U_i$ and if $z \in V_i$ then $f(z) \in U_i$. Now, given z let us subdivide $\mathbb{N}$ into a finite number a set $\{\mathbb{N}_i\}$ such that if n is large and $n \in \mathbb{N}_i$, then $f^n(z) \in W_i$. For each $\mathbb{N}_i$ let us subdivide it in blocks $\mathbb{N}_i = \cup_j B_i^j$ such that it blocks is formed by maximal chains of consecutive positive integers in $\mathbb{N}_i$. We can assume that they are infinitely many of these blocks, otherwise the sequences $f^n(z)$ is convergent. For each block B_i^j let n_i^j the larger integer in it. In particular, $f^{n_i^j+1}(z) \notin W_i$ and so, it belongs to some U_k $(k \neq i)$, but this is a contradiction because $f^{n_i^j}(z) \in W_i$ and so $f^{n_i^j+1}(z) \in U_i$.

□

As a consequences of this corollary, it follows that at least one of the periodic points F with period smaller than N_0, has to attract an open set. Therefore, the function h obtained in the proof of lemma 3.13 is constant in an open set. Since is holomorphic, it follows that it has to be constant in $\mathbb{D}^2$. And this also implies that F has only one semi-attracting periodic point z_0. Replacing F by F^k (where k is the period of z_0) follows that it has to be fixed.

□

Using the spectral decomposition theorem for dominated splitting (see [**PS2**]) similar results as the one stated in theroem 3.10 can be obtained without assuming that the periodic points are hyperbolic.

4. Families of Blaschke products

Now, given a two dimensional Blaschke product diffeomorphism $F(z,w) = (A(z)B(w), C(z)D(w))$, we consider the following two parameter family

$$F_{(\theta,\phi)}(z,w) = (\theta A(z)B(w), \phi C(z)D(w)), \quad (\theta,\phi) \in \mathbb{T} \times \mathbb{T}.$$

THEOREM 4.1. *Let F be a Blaschke product diffeomorphism . Then there exists an open set of $U \subset \mathbb{T}^2$ such that if $(\theta,\phi) \in U$ then it follows that $F_{(\theta,\phi)|\mathbb{T}^2}$ satisfies the first or last item of theorem 3.10; i.e.: $F_{(\theta,\phi)}$ is Anosov or $L(F_{(\theta,\phi)|\mathbb{T}^2})$ is the union of a finite number of repelling periodic points and saddles and a hyperbolic non trivial attractor. In either case $F_{(\theta,\phi)|\mathbb{T}^2}$ has a unique SRB measure of positive entropy.*

QUESTION 4.2. Are there always parameter values (θ,ϕ) such that $F_{(\theta,\phi)}$ is Anosov?

PROOF. First we show that there is an open set of parameter (θ,ϕ) such that $F_{(\theta,\phi)}$ has neither a sink or a semiattracting fixed point in $\mathbb{T}^2$. If it is not the case since there is at most only one sink or a semiattracting fixed point then there is a continuous function from $\mathbb{T}^2$ to $\mathbb{T}^2$, $(\theta,\phi) \to s(\theta,\phi)$ such that $s(\theta,\phi)$ is either a sink or a semiattracting fixed point. However, if (θ,ϕ) is a semiattracting fixed point, from the fact that F has a dominated splitting, it follows that $s(\theta,\phi)$ is a saddle-node and there is an invariant center manifold containing the saddle node; then we can choose a nearby parameter value to bifurcate the point to make it dissapear. Therefore, it follows that $s(\theta,\phi)$ is a sink for every value of (θ,ϕ). As $s(\theta,\phi)$ is a sink $Id - DF_{(\theta,\phi)}(\theta,\phi)$ is an isomorphism and by the implicit function theorem, s is a covering map from the torus to itself. Hence s is surjective and for every $(x,y) \in \mathbb{T}^2$, $det(DF(x,y)) = det(DF_{(\theta,\phi)}(s(\theta,\phi))) < 1$. So F is volume decreasing, which contradicts the fact that F is a diffeomorphism.

To finish, it is enough to show that there is an open and dense set of parameter values such that all the periodic point are hyperbolic. By theorem 3.10 it is enough to show there is a dense set of parameter values such that all the periodic points are hyperbolic since it follows from theorem 3.10 that these systems are stable. This last fact follows from a standard Kupka-Smale argument. There are finitely many periodic points of any period, and if q is a non-hyperbolic periodic point, since F has a dominated splitting, we find an invariant center manifold containing the point q and we can choose a nearby parameter value to bifurcate the point into a hyperbolic one. It follows that the set of parameter values with diffeomorphisms with all periodic points of a given period hyperbolic are open and dense. Hence the set with all periodic points hyperbolic is residual and hence dense. □

Observe that in the case that $F_{(\theta,\phi)|\mathbb{T}^2}$ has a unique SRB measure $\mu_{(\theta,\phi)}$ of positive entropy, it follows that

$$h_{\mu_{(\theta,\phi)}} = \int_{\mathbb{T}^2} log||DF||d\mu_{(\theta,\phi)} = \lambda^+_{(\theta,\phi)} > 0,$$

where $\lambda^+_{(\theta,\phi)}$ is the positive Laypunov exponent. On the other hand, when $F_{(\theta,\phi)|\mathbb{T}^2}$ has a either a sink or a semiattracting fixed point, then the SRB measure is a Dirac measure on the sink or the semiattracting fixed point and therefore we can also define the metrical entropy related to these measures. We wonder if possibe to obtain lower bounds for

$$\int_{\mathbb{T}\times\mathbb{T}} h_{\mu_{(\theta,\phi)}} d\theta d\phi,$$

as the ones we have obtained in [**PRS**].

5. Final remarks, questions and some generalizations.

5.1. Julia sets. Let $F : \mathbb{C}^2 \setminus P \to \mathbb{C}^2$ be an analytic function, where P is a set of poles given by a finite union of codimension one submanifolds. Let us assume that given F analytic on $\mathbb{C}^2 \setminus \{P\}$ (where P are the poles) there exists a finite number of one dimensional complex submanifolds $\{l_i\}$ such that F has an analytic inverse

$$F^{-1} : \mathbb{C}^2 \setminus [\cup_i l_i] \to \mathbb{C}^2.$$

Observe that this is the case for the quotient Blaschke diffeomorphisms. It is possible to define K^+ as the sets

$$K^+ = \{(z, w) \in \mathbb{C}^2 \setminus P : \exists \text{ neighborhood U of (z,w) such that } F^n_{|U} \text{ is a normal family}\}$$

$$K^- = \left\{ \begin{array}{c} (z, w) \in \mathbb{C}^2 \setminus [\cup_i l_i] : \exists \text{ neighborhood U of (z,w) such that} \\ F^{-n}_{|U} \text{ is a normal family and } [\cup_i l_i] \cap F^{-n}(U) = \emptyset \, \forall n > 0 \end{array} \right\}.$$

We define the Julia set J as

$$J = \mathbb{C}^2 \setminus [K^+ \cup K^- \cup_i l_i \cup P].$$

Tt follows immediately that

$$\mathcal{H} \subset J(F).$$

Question 5.1. Is it true that

$$\mathcal{H} = J(F).$$

5.2. Some general version for analytic diffeomorphisms. Some of the result stated here can be stated in a more general setting using classical results of complex analysis (see [**R**]).

Remark 5.2. Any analytic function on $cl(\mathbb{D}^2)$ that keeps $\mathbb{T}^2$ invariant is a Blaschke product.

In fact, this can be easily checked in the following way: fixing $w \in S^1$ it follows that $z \to F_1(z, w)$ (where F_1 is the first coordinate of F) is an analytic function on $cl(\mathbb{D}^1)$ that keeps S^1 invariant and therefore, is a one-dimensional Blaschke product

$$F_1(z, w) = B(w) \prod_{i=1}^{n} \frac{z - a_i(w)}{1 - z\overline{a_i(w)}}$$

where $w \to B(w)$ is an analytic function that preserves S^1 and $\prod_{i=1}^{n} \frac{z-a_i(w)}{1-z\overline{a_i(w)}}$ is an analytic function. In particular, it follows that $B(w)$ is also a Blaschke product. To finish, observe that in the quotient complex conjugation is used so to keep analyticity it follows that the symmetric functions of the $a_i(w)$ are constant functions.

Using the Riemann mapping theorem and theorem 3.10 the next remark follows:

Remark 5.3. Let F be an analytic map in $cl(U) \times cl(V) \to \mathbb{C}^2$ such that U, V are two connected open sets in $\mathbb{C}$. Let us assume that $F(U \times V) \subset U \times V$, $F(\partial U \times \partial V) = \partial U \times \partial V$, and $F_{|\partial U \times \partial V}$ is a diffeomorphisms (where ∂W denotes the boundary of W). Then, the thesis of theorem 3.10 holds on $\partial U \times \partial V$.

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