Ambiguity through Confidence Functions *

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Abstract

We characterize preference relations over bounded below Anscombe and Aumann's acts and give necessary and sufficient conditions that guarantee the existence of a utility function u on consequences, a confidence function φ on the set of all probabilities over states of nature and a positive threshold level of confidence α_0 such that our preference relation has a functional representation J, where given an act f

$$J(f) = \min_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int_S u(f)dp$$

The level set $L_{\alpha_0}\varphi := \{p : \varphi(p) \geq \alpha_0\}$ reflects the priors held by the decision maker and the value $\varphi(p)$ captures the relevance of prior p for his decision. The combination of φ and α_0 may describe the decision-maker's subjective assessment of available information. An important feature of our representation is the characterization of the maximal confidence function which allows us to obtain results on comparative ambiguity aversion and on special cases, namely the subjective expected utility, the Choquet expected utility with convex capacity, and the maxmin expected utility.
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1 Introduction

The presence of vagueness in probability judgements is an important issue in decision making, as Frank Knight (1921, page 227) commented: *The action which follows upon an opinion depends as much upon the amount of confidence in that opinion as it does upon the favorableness of the opinion itself.* Here,

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we may understand an *opinion* as some probability judgement and following Knight's argument a decision maker may have different degrees of confidence on probability assignments which is a crucial factor in the decision making process.

In order to make the preceding discussion more concrete we consider the Ellsberg's seminal article (1961) that presented the following *mind experiments*: there are two urns A and B, each containing one hundred balls. Each ball is either red or black. In urn A there are fifty balls of each color and there is no additional information about urn B. One ball is chosen at random from each urn. There are four states of nature, denoted by $S = \{(r,r),(r,b),(b,r),(b,b)\}$ where (r,r) denotes the event that the ball chosen from urn A is red and the ball chosen from urn B is red, etc. We can construct four bets denoted by A^r, A^b, B^r, B^b, where the bet A^r yields \$100 if the state (r,r) or (r,b) occurs and *zero* if it does not, *i.e.*, A^r is a bet on a red ball in urn A. According to Ellsberg, most decision makers are indifferent between betting on a red ball in urn A and betting on a black ball in urn A, and are similary indifferent between bets on a red ball in urn B or a black ball in urn B. However, there is a nonnegligible proportion of decision makers who prefer every bet from urn A (red or black) to every bet from urn B (red or black).

By a *confidence function* we mean a mapping from the set of priors to the unity interval $[0,1]$ describing the degree of confidence on the alternative probabilistic model governing the relevant phenomenon. For instance, if we assume the existence of a confidence function φ_A over probabilities concerning urn A, it is plausible to take φ_A such that

$$\varphi_A(\beta) := \varphi_A((\beta, 1-\beta)) = 0 \text{ if } \beta \neq 1/2 \text{ and } \varphi_A(1/2) = 1,$$

where $(\beta, 1-\beta)$ denotes the distribution that assign weight β for a red ball and $1-\beta$ for a black ball. On the other hand, in urn B the situation is less simple due to the lack of information about the proportion of balls. Clearly, such an example exhibits symmetries *a la* Laplace´s principle of insufficient reason, however, as it is well known the Ellsberg paradox refuses the possibility of 50%-50% distribution on urn B. But, in some sense we can preserve such symmetrical information by considering a symmetric confidence function about the likelihoods on urn B, for example, consider a confidence function φ_B such that $\varphi_B(\beta) = 4(\beta - \beta^2)$ is the degree of confidence in the distribution $(\beta, 1-\beta)$. Such confidence function illustrates a situation where a decision maker has a subjective judgement that reflects a better amount of confidence in distributions closer to the symmetrical case $(\frac{1}{2}, \frac{1}{2})$[1]. So, the notion of confidence function can capture the essence of the Knigthian argument about the vagueness on the process of probabilistic judgment.

Contrary to the notion of vague probabilities, as nicely discussed by Gilboa et. al. (2007), the first tenet of widely adopted Bayesian approach says that

[1] This problem can be reformulated using a single confidence function φ on the set of all probability measures over S. For instance, consider a probability p, the marginals p_A and p_B such that $p(s_A, s_B) = p_A(s_A)p_B(s_B)$ and suppose that φ satisfies $\varphi(p) = \varphi_A(p_A)\varphi_B(p_B)$, hence $\varphi(p) = 4(\beta - \beta^2)$ if $p_A(r) = 1/2$ and $p_B(r) = \beta$ (i.e., if $p(r,r) = p(b,r) = \beta/2$ and $p(r,b) = p(b,b) = (1-\beta)/2)$ for some $\beta \in [0,1]$ and $\varphi(p) = 0$ otherwise.

whenever a fact is not known, the decision maker should have probabilistic beliefs about it and such beliefs should be given by a single probability measure defined over a state space in which every state resolves all relevant uncertainty. Inspired by Ramsey and de Finetti´s works, Savage (1954) proposed a theory for choice under uncertainty that relies solely upon behavioral data and gave a set of axioms upon preferences amongst *acts* (*i.e.*, maps from states to consequences) under which choice under uncertainty reduces to choice under risk, *i.e.*, the decision maker's preference can be represented by a pair u and p, where u is a utility function over the consequences and p is a probability measure over the states of nature as in the Bayesian theory[2]. So, if we assume the axiomatizations of subjective expected utility (SEU) as basis for the Bayesian behavior pattern and consider the Ellsberg's preceding experiment, the decision maker's confidence function in urn B would have assigned full confidence in a unique probability p_B while differents believes are dismissed[3]. However, a decision maker consistent with the observations from Ellsberg's experiment is not consistent with the SEU characterization: in situations where some events come with probabilistic information and some events do not, subjective probabilities do not always suffice to fully encode all aspects of an individual's uncertain beliefs.

Ellsberg Paradox and some normative failures of SEU[4] have inspired the development of non-probabilistic models of preferences over subjectively uncertain acts. One important line of research replaces the subjective expected utility function with a more general functional, such as the Choquet expected utility (CEU) of Schmeidler (1989) or the maxmin expected utility (MEU) of Gilboa and Schmeidler (1989). Decision makers with MEU preferences evaluate an act using the minimun expected utility over a given nonempty, convex and (weakly*) compact subset C of the set Δ of all probabilities on states, while decision makers with CEU preferences evaluate an act using its expected utility computed according to a capacity (a non additive probability). Although these models are not the same in general, they coincide in the case of ambiguity aversion, that is, CEU with a convex capacity. In this case, the Choquet expected utility with respect to a capacity v reduces to the minimum expected value over the set of probability distribution given by the core of the capacity v (defini-

[2]In a setting where objective probabilities are embedded in the consequence space, Anscombe and Aumann (1963) gave an alternative and simpler axiomatic treatment. This treatment is especially apparent in Fishburn's (1970) well-known reformulation and extension of Anscombe and Aumann's approach which employs usual linear-space arguments and derives the same representation. Ghirardato *et al.* (2003) provide a simple definition of *subjective mixture* of acts that makes it possible to exploit all the advantages of the set-up pioneered by Anscombe and Aumann and Fishburn relying solely on behavioral data, and hence retaining the conceptual appeal of Savage's approach.

[3]One stronger example of such extreme binary assignment of confidence degree over probability assigments in economics is the *rational expectation hypothesis*: under this assumption all agents share the same probability on some relevant economic phenomenon. But it is important to highlight that the axioms from Savage or Anscombe and Aumann imply no restrictions on the form of the probabilistic expectations, in particular, they do not imply that expectations are rational.

[4]For a stimulant discussion see Gilboa *et al.* (2007, section 3).

tions can be found in the section 6.2). Formally, a decision maker with MEU preferences ranks acts according to the following criterion

$$J(f) = \min_{p \in C} \int u(f)dp.$$

An ambiguity aversion decision maker *a la* CEU or MEU exibits a behavior compatible with the Ellsberg Paradox. In this case, we may think that the decision-maker's confidence function satisfies the following rule: if the prior p belongs to C the (normalized) confidence is one, otherwise the confidence is null. Hence, the MEU criterion can be written as follows

$$J(f) = \min_{\{p:\mathbf{1}_C(p) \geq \alpha\}} \frac{1}{\mathbf{1}_C(p)} \int u(f)dp$$

for any level $\alpha \in (0, 1]$, where $\mathbf{1}_C : \Delta \to [0, 1]$ is the characteristic function of C given by

$$\mathbf{1}_C(p) = \begin{cases} 1, \; p \in C, \\ 0, \; p \notin C. \end{cases}$$

However, it seems unreasonable that the decision-maker presents a uniform degree of confidence on the priors relevant to his decision. Also, is necessary, for instance, that a multiple prior decision maker puts null weights on priors that do not belong to the set of multiple priors?

In order to capture this intuition, we want to characterize the preferences of an agent that ranks (bounded below Anscombe-Aumann) acts f according to the following criterion

$$J(f) = \min_{p \in \{q \in \Delta : \varphi(q) \geq \alpha_0\}} \frac{1}{\varphi(p)} \int u(f)dp.$$

where $\varphi : \Delta \to [0, 1]$ is a function representing the agent´s degree of confidence on the possible models p in Δ, α_0 is the threshold level of confidence below which a model is discarded, u is a utility index. So, in general, vagueness about the true probability law in our model are captured by a fully subjective *fuzzy* set of priors φ (a confidence function), which appeared to us as a meaningful way for modeling a decision maker who has a relative assesment of probability measures over states of natures. Also, our preference is ambiguity averse (see, for instance, Proposition 7) and it turns out that a decision maker with a confidence function φ and confidence level α_0 is less ambiguity averse than a MEU decision maker with set of multiple priors given by $C := \{q \in \Delta : \varphi(q) \geq \alpha_0\}$ (see, for details, Proposition 8).

An interesting feature of our representation is the characterization of the *maximal confidence function*, which specifies the upper bound on the confidence level held by the decision maker in order to be consistent with our main representation result. In particular, we obtain that a multiple prior decision maker with maximal confidence function assign positive confidence levels to almost *irrelevant* priors, but the corresponding behavior is very sensible to arbitrary pertubations in his confidence, as we obtain in Example 15. Otherwise, a decision maker with confidence function $\mathbf{1}_C$ still behaves as a MEU agent even through small changes in his confidence about irrelevant priors.

We axiomatize the representation above by showing how it rests on a simple set of axioms that generalizes the MEU axiomatization of Gilboa and Schmeidler (1989).

The rest of the paper is organized as follows. After introducing the setup in Section 2 and the set of axioms in Section 3, we present the main representation result in Section 4. In Section 5, we discuss the ambiguity attitudes, in the sense of Ghirardato and Marinacci (2002), featured by the class of preferences characterized in the main result. In Section 6, we study some special cases, namely the multiple priors preferences of Gilboa and Schmeidler (1989) and its special case given by the Choquet expected utility with convex capacity proposed by Schmeidler (1989). Proofs and related material are collected in the Appendix.

2 Notation and Framework

Consider a set S of *states of nature (world)*, endowed with a σ-algebra Σ of subsets called *events*, and a non-empty set X of *consequences*. We denote by $\mathcal{F}$ the set of all (simple) *acts*: finite-valued functions $f : S \to X$ which are Σ-measurable[5]. Moreover, we denote by $B_0(S, \Sigma)$ the set of all real-valued Σ-measurable simple functions $a : S \to \mathbb{R}$. The norm in $B_0(S, \Sigma)$ is given by $\|a\|_\infty = sup_{s\in S} |a(s)|$ (called *sup norm*) and we can define the space of all bounded and Σ-measurable functions by $B(S, \Sigma) := \overline{B_0(S, \Sigma)}^{\|\cdot\|_\infty}$, i.e., $B(S, \Sigma)$ consists of all uniform limits of finite linear combinations of characteristic functions of sets in Σ (see Dunford and Schwartz, 1988, page 240).

Clearly, note that $u(f) \in B_0(S, \Sigma)$ whenever $u : X \to \mathbb{R}$ and f belongs to $\mathcal{F}$, where the function $u(f) : S \to \mathbb{R}$ is the mapping defined by $u(f)(s) = u(f(s))$, for all $s \in S$.

Let x belong to X, define $x \in \mathcal{F}$ to be the constant act such that $x(s) = x$ for all $s \in S$. Hence, we can identify X with the set $\mathcal{F}_c$ of constant acts in $\mathcal{F}$. Given $f, g \in \mathcal{F}$ and $E \in \Sigma$, we denote by $fEg \in \mathcal{F}$ the act that yields the consequence $f(s)$ if $s \in E$ and the consequence $g(s)$, otherwise.

Additionally, we assume that X is a convex subset of a vector space. For instance, this is the case if X is the set of all finite-support lotteries on a set of prizes Z, as it happens in the classic setting of Anscombe and Aumann (1963).

Using the linear structure of X we can define as usual for every $f, g \in \mathcal{F}$ and $\alpha \in [0, 1]$ the act:

$$\begin{aligned} \alpha f + (1-\alpha)g &\quad : \quad S \to X \\ (\alpha f + (1-\alpha)g)(s) &\quad = \quad \alpha f(s) + (1-\alpha)g(s) \end{aligned}$$

[5]Let $\succsim_0$ be a binary relation on X, we say that a function $f : S \to X$ is Σ-measurable if, for all $x \in X$, the sets $\{s \in S : f(s) \succsim_0 x\}$ and $\{s \in S : f(s) \succ_0 x\}$ belong to Σ.

The decision maker's preferences are given by a binary relation $\succsim$ on $\mathcal{F}$, whose usual symmetric and asymmetric components are denoted by $\sim$ and $\succ$. Finally, for any $f \in \mathcal{F}$, an element $c_f \in X$ is a *certainty equivalent* of f if $c_f \in \{x \in X : x \sim f\}$.

3 Axioms

We assume there exists $x_* \in X$ such that $f \succsim x_*$ for every f belonging to $\mathcal{F}$, x_* is called the *worst consequence.*

(**Axiom 1**) Weak order non-degenerate. If $f, g, h \in \mathcal{F}$:

(completeness) either $f \succsim g$ or $g \succsim f$

(transitivity) $f \succsim g$ and $g \succsim h$ imply $f \succsim h$

there exists $(f, g) \in \mathcal{F}^2$ such that $(f, g) \in \succ$

(**Axiom 2**) Continuity. For all $f, g, h \in \mathcal{F}$ the sets:

$$\{\alpha \in [0,1] : \alpha f + (1-\alpha) g \succsim h\},\ \{\alpha \in [0,1] : h \succsim \alpha f + (1-\alpha) g\} \text{ are closed.}$$

(**Axiom 3**) Monotonicity. For all $f, g \in \mathcal{F}$:

$$\text{if } f(s) \succsim g(s) \text{ for all } s \in S \text{ then } f \succsim g.$$

(**Axiom 4**) Uncertainty aversion. If $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha) g \succsim f$$

(**Axiom 5**) Worst independence. For all $f, g \in \mathcal{F}$ and $\alpha \in (0,1)$:

$$f \sim g \Rightarrow \alpha f + (1-\alpha) x_* \sim \alpha g + (1-\alpha) x_*.$$

(**Axiom 6**) Independence on X. For all $x, y, z \in X$:

$$x \sim y \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}z \sim \tfrac{1}{2}y + \tfrac{1}{2}z.$$

(**Axiom 7**) Bounded attraction for certainty. There exists $\delta \geq 1$ such that for all $f \in \mathcal{F}$ and $x, y \in X$:

$$x \sim f \Rightarrow \tfrac{1}{2}x + \tfrac{1}{2}y \succsim \tfrac{1}{2}f + \tfrac{1}{2}(\tfrac{1}{\delta}y + (1 - \tfrac{1}{\delta})x_*).$$

Axioms 1, 2, 3 and 6 are standard and well understood[6]. We note that these axioms imply that the restriction of $\succsim \subset \mathcal{F} \times \mathcal{F}$ to $X \times X$, denoted by $\succsim|_{X \times X}$, has a von Neumann-Morgenstern representation (Lemma 19). Moreover, it is well known that if a preference relation $\succsim$ satisfies axioms $1, 2$ and 3 then each act $f \in \mathcal{F}$ admits a certainty equivalent $c_f \in X$.

[6] Axiom 3 says that the preference is monotone and is essentially a state-independent condition saying that the decision maker always weakly prefers acts delivering statewise weakly better payoffs, regardless of the state where better payoffs occur.

Axiom 4 is due to Schmeidler (1989) and it says that the decision maker will, in general, prefer the mixture to its components.

The classical independence axiom among acts used in the Anscombe and Aumann´s derivation of subjective probabilities says that if $f, g, h \in \mathcal{F}$ and $\beta \in (0, 1]$ then

$$f \sim g \Rightarrow \beta f + (1 - \beta)h \sim \beta g + (1 - \beta)h,$$

and it says that the preference among mixtures $\beta f + (1-\beta)h$ and $\beta g + (1-\beta)h$ is completely determined by the preference between f and g. An important weakening of this axiom, called certainty independence axiom, was introduced by Gilboa and Schmeidler (1989) in their characterization of MEU preferences: it imposes that h must belong to the set of constant acts X. Our Axiom 5 requires that independence holds whenever acts are mixed with the worst consequence x_*.

Our motivation for relaxing the certainty independence axiom to Axiom 5 of worst independence, while maintaining Axiom 6 of independence on X, and simultaneously introducing Axiom 7 of bounded attraction for certainty can be illustrated in the simple case where consequences are "monetary payoffs" (or degenerated lotteries).

We aim at taking into account the fact that if $f \sim x$ where f is uncertain and x is a constant act, then mixing f with a positive constant act y could be preferred to mixing both constant acts x and y, *i.e.*, $\frac{1}{2}f + \frac{1}{2}y \succ \frac{1}{2}x + \frac{1}{2}y$, indifference being retained if y is null, thus retaining the certainty independence in case of mixing with the worst consequence. So, our Axiom 7 allows to model such an "attraction for smoothing an uncertain act with the help of a positive constant act", a pattern which would be inconsistent with the MEU model. Moreover our axioms will impose some boundedness for this kind of attraction for mixing with a constant act, namely $\frac{1}{2}f + \frac{1}{2}y \precsim \frac{1}{2}x + \frac{1}{2}\delta y$; indeed consistency will impose here that $\delta > 1$.

Let us illustrate this flexibility of our model when compared to the MEU model in the simple following situation where there are two states of nature s_1 and s_2, $f := (1, 4) \sim (2, 2) =: x$ and y is merely chosen to be x. Note that the preference

$$\frac{1}{2}(2, 2) + \frac{1}{2}(1, 4) = (3/2, 3) \succ (2, 2) = x,$$

that we would like to model is inconsistent with the certainty independence axiom of Gilboa and Schmeidler which requires that $(3/2, 3) \sim (2, 2)$. Assume now that Axiom 7 is satisfied with $\delta > 1$, and consider the constant act $z := (2\delta, 2\delta)$, since here $x_* = (0, 0)$ it comes that

$$\begin{aligned}
& (1+\delta, 1+\delta) \\
= \; & \frac{1}{2}(2, 2) + \frac{1}{2}(2\delta, 2\delta) \overbrace{\succsim}^{\text{Axiom 7}} \frac{1}{2}(1, 4) + \frac{1}{2}\left(\frac{1}{\delta}(2\delta, 2\delta) + \left(1 - \frac{1}{\delta}\right)(0, 0)\right) \\
= \; & (3/2, 3)\,.
\end{aligned}$$

This jointly with Axiom 4 of uncertainty aversion gives

$$(1+\delta,1+\delta)\ \overbrace{\succsim}^{\text{Axiom 7}}\ (3/2,3)=\frac{1}{2}(2,2)+\frac{1}{2}(1,4)\ \overbrace{\succsim}^{\text{Axiom 4}}\ (2,2)\,,$$

and allows $(3/2,3)$ to be strictly prefered to $(2,2)$ but in a "bounded way". Note also that clearly such a strict preference would not be possible if $\delta=1$. In fact we will prove later in the main theorem 3 that if $\delta=1$, then Axiom 1 to Axiom 7 imply certainty independence, and thus that we recover in this particular case the MEU model, in our framework.

4 Main Theorem

We can now state our main theorem, which characterizes preferences satisfying axioms A.1-A7.

Let $\mathfrak{X}$ be an arbitrary set, a fuzzy set in $\mathfrak{X}$ is any function $\varphi:\mathfrak{X}\to[0,1]$. This notion due to Zadeh (1965) extends that of characteristic function $\mathbf{1}_A:\mathfrak{X}\to[0,1]$ where $A\subset\mathfrak{X}$, $\mathbf{1}_A(x)=1$ if $x\in A$ and $\mathbf{1}_A(x)=0$ if $x\notin A$. Here we take $\mathfrak{X}=ba^1_+(S,\Sigma)$, the set of all finitely additive probabilities on Σ endowed with the natural restriction of the weak* topology on $ba(S,\Sigma)$[7].

Let $\mathcal{C}_c(\Delta)$ denote the collection of all nonempty convex weak* closed subsets of $\Delta:=ba^1_+(S,\Sigma)$. As an extension of $\mathcal{C}_c(\Delta)$ we define:

Definition 1 *The set* $\mathrm{F}_{\mathcal{R}^*}(\Delta)$ *of regular* fuzzy sets consists of all mappings* $\varphi:\Delta\to[0,1]$ *with the properties*[8]*:*

(a) φ *is normal;*

$$\{p\in\Delta:\varphi(p)=1\}\neq\emptyset$$

(b) φ *is weakly* upper semicontinuous;*

$$\{p\in\Delta:\varphi(p)\geq\alpha\}\ \textit{is weakly}^*\ \textit{closed for any}\ \alpha\in[0,1]$$

(c) φ *is quasi-concave;*

$$\varphi(\beta p_1+(1-\beta)p_2)\geq min\{\varphi(p_1),\varphi(p_2)\}\ \textit{for any}\ \beta\in[0,1].$$

[7] The Banach space $ba(S,\Sigma)$ is the family of all bounded finitely additive set functions with domain Σ and range $\mathbb{R}$ endowed with the norm $\|\lambda\|_{ba}:=\sup_{E\in\Sigma}|\lambda(E)|$. A well-known result says that $(ba(S,\Sigma),\|\cdot\|_{ba})$ is isometrically isomorphic to the norm dual of the Banach space $(B(S,\Sigma),\|\cdot\|_\infty)$ (see Dunford and Schwartz (1958)). Hence, given a subset M of $ba(S,\Sigma)$, the weak* topology is the weakest topology for which all functionals $\lambda\mapsto\langle\lambda,a\rangle=\int_S a(s)\lambda(ds)$ are continuous, where $a\in B(S,\Sigma)$ and $\lambda\in M$.

[8] For an exposition of the concept of regular fuzzy sets over $\mathbb{R}^n$ see Puri and Ralescu (1985, page 1374). We note that the weak* support of φ, denoted by $supp^*\varphi:=\overline{\{p\in\Delta:\varphi(p)>0\}}^*$, is weak* compact by Banach-Alaoglu-Bourbaki's theorem [see Dunford and Schwartz (1958, page 424)].

Remark 2 *We can embed* $\mathcal{C}_c(\Delta)$ *into* $\mathrm{F}_{\mathcal{R}^*}(\Delta)$ *by the natural mapping* $P \mapsto \mathbf{1}_P$. *We will use the notation for level sets*

$$L_\alpha \varphi = \{p \in \Delta : \varphi(p) \geq \alpha\} \text{ for any } \alpha \in (0,1].$$

Moreover, we note that $\varphi \in \mathrm{F}_{\mathcal{R}^*}(\Delta)$ *if and only if the correspondence*

$$\alpha \mapsto L_\alpha \varphi$$

takes values only on $\mathcal{C}_c(\Delta)$. *Because of this previous result, a quasi-concave fuzzy set* φ *is called* fuzzy convex *(all level sets* $L_\alpha \varphi$ *are convex)*[9]. *Hence,* $\mathrm{F}_{\mathcal{R}^*}(\Delta)$ *denotes the set of all weak* closed and convex fuzzy sets while* $\mathcal{C}_c(\Delta)$ *denotes its sub-family of all classical weak* closed and convex sets.*

In the setting of all bounded below Anscombe and Aumann's acts, our representation has as its main component a mapping φ from the set Δ into the unit interval $[0,1]$, a positive threshold level of confidence $\alpha_0 \in (0,1]$, and a real-valued affine function u on X. Following our initial discussion on this paper, a mapping $\varphi \in F_{\mathcal{R}^*}(\Delta)$ is called a *confidence function,* and this class of functions models the ambiguity concerning the true probability law on the state space S. Hence, the term *ambiguity* refers purely to the vague perception of the likelihood subjectively associated with an event by a decision maker and it is captured by a set of probabilities with different degrees of confidence. The number $\alpha_0 \in (0,1]$ is a threshold level of confidence below which a model is discarted and if $\varphi(p_1) \geq \varphi(p_2) \geq \alpha_0$ then the decision maker presents a greater confidence on p_1 than p_2. The main theorem follows as:

Theorem 3 *Let* $\succsim$ *be a binary relation on* $\mathcal{F}$*, the following conditions are equivalent:*

(i) The preference relation $\succsim$ *satisfies Axioms A.1-A.7*

(ii) There exist a unique non-constant affine function $u : X \to \mathbb{R}_+$*, such that* $u(x_*) = 0$*, defined up to a positive multiplication, a minimal confidence level* $\alpha_0 \in (0,1]$*, and a regular* fuzzy set* $\varphi : \Delta \to [0,1]$ *such that, for all* $f, g \in \mathcal{F}$

$$f \succsim g \Leftrightarrow \min_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int u(f)dp \geq \min_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int u(g)dp$$

Moreover, δ *in (i) and* α_0 *in (ii) are linked by the relation* $\alpha_0 = \delta^{-1}$. *Also importantly it turns out that our model reduces to the MEU model as soon as* $\delta = 1$[10].

Remark 4 *As soon as* $\delta > 1\,(\alpha_0 < 1)$ *our preference is not invariant, i.e., the utility function* u *can not be dropped by a positive affine transformation* $v := \lambda u + \beta$ *for* $\beta \neq 0$. *In fact, it is not a surprise because Ghirardato et al. (2005) proved that invariance is equivalent to the certainty independence axiom of Gilboa and Schmeidler(1989).*

[9] We recall that $C \subset \Delta$ is weak* compact iff $\mathbf{1}_C$ is weak* upper semicontinous.

[10] For more details, see Remark 23.

Note also that if $\varphi = \mathbf{1}_P$, where $P \in \mathcal{C}_c(\Delta)$, we obtain the representation of Gilboa and Schmeidler (1989) under the existence of a worst consequence. A very useful result follows as:

Corollary 5 *Under the conditions of Theorem 3, there exists a unique maximal confidence function φ^* given by*

$$\varphi^*(p) = \inf_{f \in \mathcal{F}} \left(\frac{\int u(f)dp}{u(c_f)} \right)$$

such that, for all $f, g \in \mathcal{F}$

$$f \succsim g \Leftrightarrow \min_{p \in L_{\alpha_0}\varphi^*} \frac{1}{\varphi^*(p)} \int u(f)dp \geq \min_{p \in L_{\alpha_0}\varphi^*} \frac{1}{\varphi^*(p)} \int u(g)dp.$$

Futhermore, under the maximal confidence function α_0 is not relevant[11]*, i.e., for all $f, g \in \mathcal{F}$*

$$f \succsim g \Leftrightarrow \min_{p \in \Delta} \frac{1}{\varphi^*(p)} \int u(f)dp \geq \min_{p \in \Delta} \frac{1}{\varphi^*(p)} \int u(g)dp.$$

Next we give the immediate uniqueness properties of our representation.

Corollary 6 *Two pairs (u_1, φ_1^*) and (u_2, φ_2^*) represent the same preference $\succsim$ on $\mathcal{F}$ as in Corollary 5 if and only if there exists $\lambda > 0$ such that $u_1 = \lambda u_2$ and $\varphi_1^* = \varphi_2^*$.*

Corollary 5 states that a preference $\succsim$ on $\mathcal{F}$ that satisfies Axioms A.1-A.7 can be represented by a pair (u, φ^*). The function φ^* should be viewed as the *upper confidence function*, specifying maximal confidence among priors that the decision maker may face in order to be consistent with our main representation.

Note that a natural candidate for being a lower (minimal) confidence function is $\varphi_*(\cdot) = \mathbf{1}_{L_1\varphi^*}$. It turns out that φ_* fits our model if only if $\delta = 1$, i.e., if and only if our model reduces to the MEU model. Accordingly we will mainly focus in the sequel on the upper confidence function φ^*, without elaborating more on φ_*.

In contrast with the MEU model, a decision maker that presents a behavior consistent with our set of axioms, in general, does not evaluate the acts by their minimal expected utility on the set of priors $L_{\alpha_0}\varphi^*$ which matters to him. Hence, our representation is coherent with a non-extremely pessimistic behavior (with respect to $L_{\alpha_0}\varphi^*$). For instance, we may observe two decision makers who share the same sets of priors but one is more cautious than the other. This can occur when they have different personal confidence functions[12].

[11] In fact, the role of α_0 is implicit in the determination of the maximal confidence function.

[12] Another model of decision making under ambiguity that has a similar *non-extremely pessimistic behavior interpretation* is the smooth model proposed by Klibanoff et al. (2005). In their representation the doubt about the *right* probability is given by a subjective probability over Δ while in our case this vagueness is given by a subjective fuzzy set of priors.

Another preference representation that presents the multiple prior model as particular case is the variational preferences[13] model proposed by Maccheroni et al. (2006): variational preferences have the following representation

$$V(f) = \min_{p\in\Delta}\left(\int u(f)dp + c^*(p)\right)$$

where $c^* : \Delta \to \mathbb{R}\cup\{+\infty\}$ is a convex, weak* lower semicontinuous function, such that $\{p \in \Delta : c^*(p) = 0\}$ is nonempty.

We note that function c^* generalizes the indicator functions from Convex Analysis[14], and if $c^* = \delta_C$ then we obtain the multiple prior model with set of priors C. Function c^* can be interpreted as the index of ambiguity aversion and has a nice expression for the minimal index of ambiguity aversion:

$$c^*(p) = \sup_{f\in\mathcal{F}}\left(u(c_f) - \int u(f)dp\right).$$

Moreover, if $u(X)$ is unbounded then the index of ambiguity aversion c^* is unique.

Our preference and the variational preference captures different designs of behavior under uncertainty. Note that variational preferences is represented by the *inf* of affine functionals (a *niveloid* which is always Lipschitz-continuous with constant one) whereas our preferences are represented by the *inf* of linear functionals (which is, in general, Lipschitz-continuous with a constant greater than one (see Lemma 22 in the Appendix)).

An alternative interpretation of variational preferences, where the function $c^* : \Delta \to \mathbb{R}\cup\{+\infty\}$ could be viewed as a cost function of a malevolent Nature[15], can be translated for our preference in considering the function $\varphi^* : \Delta \to [0,1]$ as a *plausibility function* of a malevolent nature. Each number $\varphi^*(p)$ captures the decision maker's perception of the relative plausibility of the different models p that Nature can choose in order to make the decision maker the most possible worst off; if $\varphi^*(p_1) \geq \varphi^*(p_2)$ then model p_1 is weakly more plausible than model p_2. Hence, the decision maker's play follows the rule

$$\max_{f\in\mathcal{F}}\min_{p\in L_{\alpha_0}\varphi^*}\left\{\frac{1}{\varphi^*(p)}\int u(f)dp\right\}$$

where the strategies are pairs $(f,p) \in \mathcal{F}\times L_{\alpha_0}\varphi^*$, and $\mathcal{F}$ is the decision maker's set of pure strategies and $L_{\alpha_0}\varphi^*$ is the Nature's set of pure strategies.

[13] The multiplier preferences of Hansen and Sargent (2001) and the mean-variance preference of Markovitz (1952) and Tobin (1958) are also special cases. The worst consequence is not required in the axiomatization of the variational preferences.

[14] The indicator function is the mapping $\delta_P : \Delta \to \mathbb{R}\cup\{+\infty\}$ where $\delta_P(p) = 0$ if $p \in P$ and $\delta_P(p) = +\infty$ if $p \notin P$. In our representation, we saw that φ^* generalizes the characteristic function from Measure Theory.

[15] As briefly discussed by Maccheroni et. al. (2006), in this interpretation an agent must to make choices under limited information and without a full knowledgement of what is going on. In this case the agent envisions a malevolent nature that represents an opponent who might take advantage of his relative ignorance.

5 Ambiguity Attitudes

We now analyse ambiguity attitude featured by the class of preferences characterized in this paper. By Corollary 5, our class of preference relation is represented by a utility functional J on $\mathcal{F}$, such that:

$$J(f) = \min_{p \in \Delta} \frac{1}{\varphi^*(p)} \int u(f) dp$$

where $u : X \to \mathbb{R}_+$ is an affine utility function such that $u(x_*) = 0$ and φ^* is the maximal confidence function:

$$\varphi^*(p) = \inf_{f \in \mathcal{F}} \left(\frac{\int u(f) dp}{u(c_f)} \right)$$

We follow the approach proposed by Ghirardato and Marinacci (2002). They proposed a notion of absolute ambiguity aversion by building on a notion of comparative ambiguity. The comparative ambiguity attitude says that, given two preferences $\succsim_1$ and $\succsim_2$, the preference relation $\succsim_1$ is *more ambiguity averse than* $\succsim_2$, if for all $x \in X$ and $f \in \mathcal{F}$,

$$f \succsim_1 x \Rightarrow f \succsim_2 x$$

Note that two preference relations $\succsim_1$ and $\succsim_2$ satisfying the comparative ambiguity attitude above induces preferences relation on X that can be representated by the same utility index u on consequences[16]. Hence, ambiguity aversion is comparable across two decision makers only if their risk attitudes coincide.

The absolute notion of ambiguity aversion defined by Ghirardato and Marinacci (2002) consider SEU preferences as benchmarks for ambiguity neutrality: We say that a preference relation $\succsim$ is *ambiguity averse* if it is more ambiguity averse than an SEU preference.

Now, by considering the behavioral assumptions stipulated in our main result, which includes the preference for randomization of Schmeidler (1989) described by the uncertainty aversion axiom, we obtain in a precise sense the next result concerning a negative attitude toward ambiguity:

Proposition 7 *The preference $\succsim$ as in our main theorem is ambiguity averse.*

Any preference relations as in our main result can be identified with a pair (u, φ^*) of an affine utility index, such that $u(x_*) = 0$, and maximal confidence function φ^*. The following result shows that comparative ambiguity attitudes for preferences as in our main result are determined by the confidence function φ^*.

[16] In fact, given $y, x \in X$, $y \succsim_1 x \Rightarrow y \succsim_2 x$, hence if u_1 and u_2 are the respectives nonconstant affine functions that represents $\succsim_1$ and $\succsim_2$, we obtain that $u_1(y) \geq u_1(x) \Rightarrow u_2(y) \geq u_2(x)$ for every $y, x \in X$. By Ghirardato et. al. (2004), Corollary B.3, we can assume $u_1 = u_2 = u$.

Proposition 8 *Given two preferences in accordance with our main theorem. The following conditions are equivalent:*

(1) $\succsim_1$ *is more ambiguity averse than* $\succsim_2$*;*

(2) There exist pairs (u, φ_i^*) *that represents* $\succsim_i (i = 1, 2)$*, where* $\varphi_1^* \geq \varphi_2^*$*.*

This proposition says that more ambiguity averse preference relations are characterized, up to index normalization, by greater functions φ^*. In particular, note that if the pair (u_1, φ_1^*) represents a more ambiguity averse preference than the pair (u_2, φ_2^*) , then there exists $\lambda > 0$ where $u_2 = \lambda u_1$ and $\varphi_1^* \geq \varphi_2^*$. Also, we may view a confidence function as the *ambiguity index* because, by the previous result, less confidence among the priors is associated with a greater ambiguity aversion.

Example 9 *The maximal ambiguity aversion behavior is characterized by* $\varphi^*(p) = 1$ *for any* $p \in \Delta$*. In this case*

$$J(f) = \min_{p \in \Delta} E_p(u(f)) = \min_{s \in S} \{u(f(s))\}$$

is an expression that reflects extreme ambiguity aversion.

Example 10 *The minimal ambiguity aversion corresponds here to ambiguity neutrality, as we know that our preferences are ambiguity averse. The least ambiguity averse functions* φ^* *are associated with SEU preferences. In this case we obtain that*

$$\varphi^*(p) = \inf_{E \in \Sigma} \frac{p(E)}{q(E)}$$

where $q \in \Delta$ *is the subjective probability of the decision maker. For details see Corollary 18.*

Example 11 *Consider*

$$J_v(f) = \int u(f) dv,$$

where $v : \Sigma \to [0, 1]$ *is a capacity (see Subsection 6.2) such that there exists* $\lambda \in (0, 1)$ *and* $q \in \Delta$

$$\begin{aligned} v(E) &= \lambda q(E), \text{ if } \Sigma \ni E \neq S, \\ v(S) &= 1. \end{aligned}$$

The functional $J_{\lambda q}$ *is the well known* ε*-contaminated model. Denote by* φ_q^* *the maximal confidence function of an SEU preference with subjective probability* q*, and* $\varphi_{\lambda q}^*$ *the maximal confidence function of* ε*-contamined model for* $\lambda = 1 - \varepsilon$*. We then obtain that,*

$$\varphi_{\lambda q}^*(p) = \inf_{\{E \in \Sigma : q(E) > 0\}} \left\{ \frac{p(E)}{\lambda q(E)} \wedge 1 \right\} = \frac{\varphi_q^*(p)}{\lambda} \wedge 1.$$

So, $\varphi^*_{\lambda q}(p) = 1$ *iff* $\varphi^*_q(p) \geq \lambda$, i.e.,

$$L_1\left(\varphi^*_{\lambda q}\right) = L_\lambda\left(\varphi^*_q\right).$$

Note that the previous equality of sets of priors clarify the interpretation of the ε*-contaminated model: For a* ε*-contaminated agent (w.r.t.* q*) the probabilities that matter are the same as the set of priors with confidence greater than* $1-\varepsilon$ *held by an SEU agent with subjective probability with confidence function* φ^*_q*.*

The next two example present confidence functions for which there exists only one prior with full confidence, and provides a preference relation consistent with the Ellsberg type behavior.

Example 12 *Consider the Ellsberg two color urn and a decision maker who has a confidence function* φ *as described in the Introduction and a minimal confidence level* $\alpha_0 \in (0,1]$*: given a probability* p *and the marginals* p_A *and* p_B *such that* $p(s_A, s_B) = p_A(s_A)p_B(s_B)$ *we have that* $\varphi(p) = \varphi_A(p_A)\varphi_B(p_B)$*, so* $\varphi(p) = 4(\beta - \beta^2)$ *if* $p_A(r) = 1/2$ *and* $p_B(r) = \beta$ *(i.e., if* $p(r,r) = p(b,r) = \beta/2$ *and* $p(r,b) = p(b,b) = (1-\beta)/2$*) for some* $\beta \in [0,1]$ *and* $\varphi(p) = 0$ *otherwise. Without loss of generality, set* $u(0) = 0$ *and* $u(100) = 1$*. Note that*

$$L_{\alpha_0}\varphi = \left\{ p \in \Delta : \begin{array}{c} p_A(r) = 1/2 \text{ and } p_B(r) = \beta \\ \text{with } \beta \in \left[\frac{1-(1-\alpha_0)^{1/2}}{2}, \frac{1+(1-\alpha_0)^{1/2}}{2}\right] \end{array} \right\},$$

hence, denoting $\Lambda_{\alpha_0} := \left[\frac{1-(1-\alpha_0)^{1/2}}{2}, \frac{1+(1-\alpha_0)^{1/2}}{2}\right]$

$$\begin{aligned} J(A^r) &= J\left(A^b\right) = \min_{\beta \in \Lambda_{\alpha_0}} \frac{1/2}{4(\beta - \beta^2)} = 1/2, \\ J(B^r) &= \min_{\beta \in \Lambda_{\alpha_0}} \frac{\beta/2 + \beta/2}{4(\beta - \beta^2)} = \frac{2}{4 + 4(1-\alpha_0)^{1/2}}, \\ J\left(B^b\right) &= \min_{\beta \in \Lambda_{\alpha_0}} \frac{(1-\beta)/2 + (1-\beta)/2}{4(\beta - \beta^2)} = \frac{2}{4 + 4(1-\alpha_0)^{1/2}}, \end{aligned}$$

thus, for any $\alpha_0 < 1$*, delivering the Ellsberg pattern*

$$A^r \sim A^b \succ B^r \sim B^b.$$

Example 13 *Now, consider the Ellsberg three color urn, with 30 red balls and 60 balls either green or blue. The usual Ellsberg bets are given as*

bets\color	*red*	*green*	*blue*
f_r	100	0	0
f_g	0	100	0
f_{rg}	100	100	0
f_{gb}	0	100	100

where f_r pays hundred dolars if a red ball is drawn and nothing otherwise, f_g pays hundred dolars if a green ball is drawn and nothing otherwise, and so on. The well known Ellsberg argument says that most subjects rank these acts as

$$f_r \succ f_g \text{ and } f_{gb} \succ f_{rg}.$$

Consider a decision maker with confidence function φ such that for each probability measure $p = (\alpha, \beta, 1 - \alpha - \beta)$ on $S = \{r, g, b\}$,

$$\varphi(p) = \begin{cases} 9\beta\left(\frac{2}{3} - \beta\right), \ \alpha = 1/3 \\ 0, \ otherwise \end{cases}$$

Also, taking a utility index u as in the previous example and (for simplicity) $\alpha_0 \in (0, 0.9)$, we obtain that

$$L_{\alpha_0}\varphi = \left\{ p \in \Delta : \ \alpha = 1/3 \text{ and } \beta \in \left[\frac{1 - (1 - \alpha_0)^{1/2}}{3}, \frac{1 + (1 - \alpha_0)^{1/2}}{3}\right]\right\},$$

hence, denoting $\Theta_{\alpha_0} := \left[\frac{1-(1-\alpha_0)^{1/2}}{3}, \frac{1+(1-\alpha_0)^{1/2}}{3}\right]$

$$\begin{aligned} J(f_r) &= \min_{\beta \in \Theta_{\alpha_0}} \frac{1/3}{9\beta(2/3 - \beta)} = 1/3, \\ J(f_g) &= \min_{\beta \in \Theta_{\alpha_0}} \frac{\beta}{9\beta(2/3 - \beta)} = 1/\left\{3\left[3 - \left(1 + (1 - \alpha_0)^{1/2}\right)\right]\right\} < 1/3, \\ J(f_{gb}) &= \min_{\beta \in \Theta_{\alpha_0}} \frac{\beta + \left(1 - \frac{1}{3} - \beta\right)}{9\beta(2/3 - \beta)} = 2/3, \\ J(f_{rg}) &= \min_{\beta \in \Theta_{\alpha_0}} \frac{\frac{1}{3} + \beta}{9\beta(2/3 - \beta)} \approx 0.6220 < 2/3, \end{aligned}$$

which is consistent with the Ellsberg argument above.

6 Special Cases

Choosing suitably the confidence function, we can obtain well known cases in the literature:

6.1 Maxmin Expected Utility

Gilboa and Schmeidler (1989) characterized maximin expected utility preferences (also known as *multiple prior model*), which has as numerical representation a functional J on $\mathcal{F}$ that satisfies the formula

$$J(f) = \min_{p \in C} \int u(f)dp$$

where $C \subset \Delta$ is non empty, convex and weak* compact set. As we mentioned in Section 3, the multiple prior model is characterized by the certainty independence axiom. Assuming our bounded below assumption of the preference relation we are able to study the consequences of certainty independence axiom for the maximal confidence function related to the multiple prior model. By our main result, the maximal confidence function $\varphi^* : \Delta \to \mathbb{R}$ related to the multiple prior model is given by

$$\varphi^*(p) = \inf_{f \in \mathcal{F}} \frac{\int u(f)dp}{u(c_f)} = \inf_{a \in B^+} \frac{\int adp}{I(a)},$$

where I is the functional on $B^+(S, \Sigma)$ given by $I(a) = \min_{p \in C} \int adp$.

Proposition 14 *Let $\succsim$ be a bounded below preference that satisfies Axioms A1, A2, A3, A4 and the certainty independence axiom then $\succsim$ is a maxmin expected utility preference and its maximal confidence function φ^* is such that $\varphi^*(p) = 1$ if and only if $p \in C$, and*

$$J(f) = \min_{p \in L_1\varphi^*} \int u(f)dp = \min_{p \in \Delta} \frac{1}{\varphi^*(p)} \int u(f)dp.$$

An interesting fact is that a decision maker *a la* Gilboa and Schmeidler is characterized by a maximal confidence function which assigns positive confidence levels among many priors out of C.

Example 15 *Taking the Gilboa and Schmeidler´s functional I on $\mathbb{R}^2_+$ with $C = \{(\lambda, 1-\lambda) : \lambda \in [0.4, 0.6]\}$, follows that:*

$$\varphi^*(\lambda) = \begin{cases} 1, \text{ if } \lambda \in [0.4, 0.6] \\ \lambda/0.4, \text{ if } \lambda \in [0, 0.4) \\ (1-\lambda)/0.4, \text{ if } \lambda \in (0.6, 1], \end{cases}$$

hence, $\varphi^* \neq \mathbf{1}_C$.

It is worth noting that the confidence function decreases while the probability moves away from the full confidence set of priors C and, in a sufficiently fast way, in order to keep our decision maker a maxmin expected utility agent with respect to C. We know that φ^* is maximal by Lemma 30. Consider, for example, a distortion φ^*_r of φ^* given by

$$\varphi^*_r(\lambda) = \begin{cases} 1, \text{ if } \lambda \in [0.4, 0.6] \\ (1-\lambda)/0.4, \text{ if } \lambda \in (0.6, 1] \\ \left(\frac{0.5(\lambda-0.4)}{0.2+r}\right) + 1, \text{ if } \lambda \in [0.2-r, 0.4) \\ \left(\frac{0.5}{0.2-r}\right)\lambda, \text{ if } \lambda \in [0, 0.2-r) \end{cases}$$

where $r \in (0, 0.2)$ and note that $\lim_{r \searrow 0} \varphi^*_r(\lambda) = \varphi^*(\lambda)$ for any $\lambda \in [0,1]$. Define for any $a \in \mathbb{R}^2_+$

$$I^*(a) = \min_{\lambda \in [0,1]} \left(\frac{\lambda a_1 + (1-\lambda) a_2}{\varphi^*_r(\lambda)} \right),$$

in this case we obtain that $I^*((1,0)) = 0.4 - 2r < 0.4 = I((1,0))$.

6.2 Choquet Expected Utility

Choquet expected utility is a well known model proposed by Schmeidler (1989) using the notion of capacities or non-additive probabilities. The key feature of a preference $\succsim$ in the class of Choquet expected utility preferences is the *comonotonic independence axiom*: We say that two acts f and g are comonotonic if for no states $s_1, s_2 \in S$

$$f(s_1) \succ g(s_1) \text{ and } g(s_2) \succ f(s_2).$$

The comonotonic independence axiom says that: For all pairwise comonotonic acts f, g and h and $\alpha \in [0,1]$,

$$f \sim g \Rightarrow \alpha f + (1-\alpha)\, h \sim \alpha g + (1-\alpha)\, h.$$

The main concept used in the Schmeidler´s representation theorem is the Choquet integral. This kind of integral is related to the notion of a set-function $v : \Sigma \to [0,1]$ which is a capacity, *i.e.,*

(*i*) $v(\emptyset) = 0$, $v(S) = 1$

(*ii*) $E, F \in \Sigma$ such that $E \subset F \Rightarrow v(E) \leq v(F)$.

Now, consider a non-negative, bounded and Σ-measurable function $a : S \to \mathbb{R}$. The Choquet integral of a with respect to v is given by

$$\int a dv = \int_0^{+\infty} v\left(\{a \geq \lambda\}\right) d\lambda$$

Choquet expected utility preferences has as numerical representation a functional J on $\mathcal{F}$ that satisfies

$$J(f) = \int u(f) dv,$$

where u is a utility index and v is a capacity.

A suficient condition for ambiguity aversion attitudes in the Choquet expected utility theory is the convexity of the capacity v :

(*iii*) Capacity v is convex if for all events $E, F \in \Sigma$:

$$v(E \cup F) + v(E \cap F) \geq v(E) + v(F).$$

When v is convex, the well known result of Schmeidler (1986) says that the core of v

$$\mathcal{C}(v) = \{p \in \Delta : p(E) \geq v(E), \forall E \in \Sigma\}$$

is nonempty (convex and weak* compact). Moreover,

$$\int u(f) dv = \min_{p \in \mathcal{C}(v)} \int u(f) dp.$$

This explains why the Choquet expected utility is a subclass of the maxmin expected utility when the capacity is convex.

Proposition 16 *Let $\succsim$ be a bounded below preference that satisfies Axioms A1, A2, A3, A4 and the comonotonic independence axiom then $\succsim$ is a Choquet expected utility preference and its maximal confidence function φ^* satisfies*

$$\varphi^*(p) = \inf_{E \in \Sigma} \frac{p(E)}{v(E)}.$$

Removing the restriction of non-negativity, one obtains in the general case the following result:

Proposition 17 *If we define the confidence function φ for any $p \in \Delta$ by:*

$$\varphi(p) = \inf_{E \in \Sigma} \left\{ \frac{p(E)}{v(E)} \wedge \frac{1 - v(E)}{1 - p(E)} \right\}$$

then, for every function $a \in B(S, \Sigma)$, we have that

$$\int a dv = \min_{p \in L_{\alpha_0} \varphi} \left\{ \frac{\int a^+ dp}{\varphi(p)} + \varphi(p) \int a^- dp \right\}, \tag{2}$$

for any level $\alpha_0 \in (0, 1]$; where $a^+ = a \vee 0$, $a^- = a \wedge 0$, and the Choquet integral of a with respect to v is given by

$$\int a dv = \int_{-\infty}^{0} [v(\{a \geq \lambda\}) - 1] d\lambda + \int_{0}^{+\infty} v(\{a \geq \lambda\}) d\lambda.$$

It is immediately obvious that comonotonic independence axiom is weaker than the classical independence axiom and an immediate consequence of the Proposition 16 is the following corollary:

Corollary 18 *Let $\succsim$ be a bounded below preference that satisfies Axioms A1, A2, A3, A4 and the independence axiom, then $\succsim$ is a expected utility preference and its maximal confidence function $\varphi^* := \varphi_q^*$ satisfies*

$$\varphi_q^*(p) = \inf_{E \in \Sigma} \frac{p(E)}{q(E)}, \forall p \in \Delta$$

for some subjective probability q.

We note that a decision maker *a la* SEU not necessarily presents non-null confidence only in a unique prior q, but the confidence among priors different from q implies that such priors are negligible. However, small pertubations in the decision maker´s confidence level may destroy the subjective expected utility pattern of behavior.

7 Concluding Remarks

Proposition 17 suggests a more general functional that defines a preference without the bounded below assumption: There exists a referential consequence $\overline{x} \in X$ in such way that the functional J on $\mathcal{F}$ is determined by a confidence function $\varphi : \Delta \to [0,1]$, a unique non-constant affine function $u : X \to \mathbb{R}$, such that $u(\overline{x}) = 0$, defined up to a positive multiplication, and a minimal confidence level $\alpha_0 \in (0,1]$, such that for all $f \in \mathcal{F}$,

$$J(f) = \min_{p \in L_{a_0}\varphi} \int_S u(f)\varphi(p)^{-sgn\{u(f)\}} dp.$$

This extension of our model is the subject of future research.

We also intend, following a nice suggestion of Castagnoli (2006), to investigate applications of our functional to insurance pricing; such a functional being intended through an admission fee proportional to the *pure price* to cover expenses and to eliminate or reduce the probability of ruin.

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8 APPENDIX

PROOF OF THEOREM 3:

Part (ii)⇒(i) is straightforward. (i)⇒(ii) will result from Lemma 19 to Lemma 28.

Lemma 19 *There exists an affine $u : X \to \mathbb{R}$ non-constant function such that for all $x, y \in X : x \succsim y$ iff $u(x) \geq u(y)$. Moreover, we can choose u such that $u(x_*) = 0$.*

Proof: By axioms 1,2 and 6 the premises of the von Neumann-Morgenstern theorem are satisfied [see Schmeidler (1989, page 577) or Herstein and Milnor (1953)] and there exists an affine function $u : X \to \mathbb{R}$ such that for all $x, y \in X$:

$x \succsim y$ iff $u(x) \geq u(y)$. Therefore, we can choose $u(x_*) = 0$. By axiom 1, there exist $f, g \in \mathcal{F}$ s.t. $f \succ g$; given $x, y \in X$ such that $x \succsim f(s)$ and $g(s) \succsim y$ for all $s \in S$, then by monotonicity (axiom 3) we have that $x \succ y$, then u cannot be constant. Finally, we can suppose that there exists $x \in X$ s.t. $u(x) = 1$.

Lemma 20 *For any $u : X \to \mathbb{R}$ satisfying Lemma 19 there exists a unique $J : \mathcal{F} \to \mathbb{R}$ such that*

(i) $f \succeq g$ iff $J(f) \geq J(g)$ for all $f, g \in \mathcal{F}$.

(ii) If $f = x\mathbf{1}_S \in \mathcal{F}_c \equiv X$ (the set of constant functions) then $J(f) = u(x)$.

Proof: On $\mathcal{F}_c$ the functional J is uniquely determined by (ii). Since for all $f \in \mathcal{F}$ there exists a $c_f \in \mathcal{F}_c$ such that $f \sim c_f$, we set $J(f) = u(c_f)$ and by construction J satisfies (i), hence it is also unique.

We denote by $B_0(S, \Sigma, K)$ the functions in $B_0(S, \Sigma)$ that assume finitely many values in an interval $K \subset \mathbb{R}$ and by $B_0^+(S, \Sigma) = B_0(S, \Sigma, \mathbb{R}_+)$. For $k \in \mathbb{R}$, let $k\mathbf{1}_S \in B_0(S, \Sigma)$ be the constant function on S such that $k\mathbf{1}_S(S) = \{k\}$.

Lemma 21 *Let u and J be defined as in Lemmas 19 and 20, then there exists a functional*

$$I : B_0^+(S, \Sigma) \to \mathbb{R}$$

where for every $f \in \mathcal{F}$ $J(f) = I(uof)$ such that:

(i) *I is superadditive, i.e., for $a, b \in B_0^+(S, \Sigma) : I(a + b) \geq I(a) + I(b)$;*

(ii) *I is positively homogeneous,i.e., for $a \in B_0^+(S, \Sigma), \lambda \geq 0 : I(\lambda a) = \lambda I(a)$;*

(iii) *I is monotonic, i.e., for $a, b \in B_0^+(S, \Sigma) : a \geq b \Rightarrow I(a) \geq I(b)$;*

(iv) *I is normalized, i.e., $I(\mathbf{1}_S) = 1$;*

(v) *For every $a \in B_0^+(S, \Sigma)$ and $\xi \geq 0$*

$$I(a + \xi\mathbf{1}_S) \leq I(a) + \delta\xi.$$

Proof: We begin with $B_0(S, \Sigma, u(X))$ and then extend I to all $B_0^+(S, \Sigma)$. If $f \in \mathcal{F}$ then $u(f) \in B_0(S, \Sigma, u(X))$. Now, if $a \in B_0(S, \Sigma, u(X))$ we have that there exists $\{E_i\}_{i=1}^n \subset \Sigma$ a partition of S and $\{x_i\}_{i=1}^n \subset X$ such that

$$a := \sum_{i=1}^{n} u(x_i)\mathbf{1}_{E_i},$$

hence, we can choose $f \in \mathcal{F}$ such that $f(s) = x_i$ when $s \in E_i$ and we conclude that $a = u(f)$.

From this, we can write $B_0(S, \Sigma, u(X)) = \{u(f) : f \in \mathcal{F}\}$; therefore, $u(f) = u(g) \Leftrightarrow u(f(s)) = u(g(s))$, $\forall s \in S \Leftrightarrow f(s) \sim g(s)$, $\forall s \in S$; and, by axiom 3 (monotonicity), $f \sim g$, *i.e.*, $u(f) = u(g) \Leftrightarrow J(f) = J(g)$.

Define $I(a) = J(f)$ whenever $a = u(f)$. Hence, we have that I is well defined over $B_0(S, \Sigma, u(X))$.

Now, if $a = u(f)$ and $b = u(g) \in B_0(S, \Sigma, u(X))$ and $a \geq b$, then $u(f(s)) \geq u(g(s))$ for any $s \in S$ and, by axiom 3 (monotonicity), we have that $f \succsim g$, *i.e.*, $J(f) \geq J(g)$ and $I(a) = I(u(f)) = J(f) \geq J(g) = I(u(g)) = I(b)$; which proves that I is monotonic.

Set $k \in u(X)$, then there exists some $x \in X$ such that $k = u(x)$ and $I(k\mathbf{1}_S) = I(u(x)\mathbf{1}_S) = J(x) = u(x) = k$, *i.e.*, I is normalized. In particular, since $1 \in u(X), I(\mathbf{1}_S) = 1$.

We now show that I is positively homogeneous. Assume $a = \alpha b$, where $a, b \in B_0(S, \Sigma, u(X))$ and $0 < \alpha \leq 1$. Let $g \in \mathcal{F}$ satisfy $u(g) = b$ and define $f = \alpha g + (1-\alpha)x_*$. Hence $u(f) = \alpha u(g) + (1-\alpha)u(x_*) = \alpha b = a$, so $I(a) = J(f)$. We have $J(c_g) = J(g) = I(b)$. By axiom 5 (worst independence), $\alpha c_g + (1-\alpha)x_* \sim \alpha g + (1-\alpha)x_* = f$, hence $J(f) = J(\alpha c_g + (1-\alpha)x_*) = \alpha J(c_g) + (1-\alpha)J(x_*) = \alpha J(c_g)$ and we can write

$$I(\alpha b) = I(a) = J(f) = \alpha J(c_g) = \alpha I(b).$$

Furthermore, this implies positive homogeneity for $\alpha > 1 : a = \alpha b \Rightarrow b = \alpha^{-1} a \Rightarrow I(b) = \alpha^{-1} I(a) \Rightarrow I(a) = \alpha I(b)$.

Now, by positive homogeneity we can extend I to all $B_0^+(S, \Sigma)$, since $u(X)$ is a non-empty interval of $\mathbb{R}_+$ containing 0.

Next, we show that *(v)* is satisfied. Let there be given $a \in B_0^+(S, \Sigma)$ and $\xi \geq 0$. By homogeneity we may assume without loss of generality that $2a$ and $2\delta\xi\mathbf{1}_S \in B_0(S, \Sigma, u(X))$. Now we define $\beta = I(2a) = 2I(a)$. Let $f \in \mathcal{F}$ such that $u(f) = 2a$ and $y, z \in X$ satisfy $u(y) = \beta$ and $u(z) = 2\delta\xi$, then $J(f) = I(u(f)) = 2I(a) = \beta = I(\beta\mathbf{1}_S) = I(u(y)) = J(y)$, *i.e.*, $f \sim y$. By axiom 7 (bounded attraction for certainty), there exists $\delta \geq 1$ such that

$$\frac{1}{2}y + \frac{1}{2}z \succeq \frac{1}{2}f + \frac{1}{2}(\frac{1}{\delta}z + (1 - \frac{1}{\delta})x_*)$$

hence

$$\frac{1}{2}J(y) + \frac{1}{2}J(z) \geq J(\frac{1}{2}f + \frac{1}{2}(\frac{1}{\delta}z + (1 - \frac{1}{\delta})x_*))$$

then

$$\frac{1}{2}I(u(y)) + \frac{1}{2}I(u(z)) \geq I(\frac{1}{2}u(f) + \frac{1}{2}u(\frac{1}{\delta}z + (1 - \frac{1}{\delta})x_*))$$

from the facts above

$$\frac{1}{2}I(\beta\mathbf{1}_S) + \frac{1}{2}I(2\delta\xi\mathbf{1}_S) \geq I(\frac{1}{2}2a + \frac{1}{2}(\frac{1}{\delta}u(z) + (1 - \frac{1}{\delta})u(x_*)))$$

we obtain

$$I(a) + \delta\xi \geq I(a + \frac{1}{\delta}\delta\xi\mathbf{1}_S) = I(a + \xi\mathbf{1}_S)$$

It remains to show that I is superadditive. Let there be given $a, b \in B_0^+(S, \Sigma)$ and, once again, by homogeneity we assume that $a, b \in B_0(S, \Sigma, u(X))$. First, we note that axiom 4 (uncertainty aversion) implies that I is quasi-concave, in fact:

Since $a, b \in B_0(S, \Sigma, u(X))$ we can choose $f, g \in \mathcal{F}$ such that $a = u(f)$ and $b = u(g)$, since $\alpha a + (1-\alpha)b = \alpha u(f) + (1-\alpha)u(g) = u(\alpha f + (1-\alpha)g)$, we obtain $I(\alpha a + (1-\alpha)b) = J(\alpha f + (1-\alpha)g)$ and, by axiom 4 (uncertainty aversion), $\alpha f + (1-\alpha)g \succsim g$ if $f \succsim g$, hence $J(\alpha f + (1-\alpha)g) \geq \min\{J(f), J(g)\}$, i.e., $I(\alpha a + (1-\alpha)b) \geq \min\{I(a), I(b)\}$.

Now, since I is positively homogeneous it follows that I is concave [see Berge(1959)], then $\frac{1}{2}I(a+b) = I(\frac{1}{2}a+\frac{1}{2}b) \geq \frac{1}{2}I(a)+\frac{1}{2}I(b)$, that is, $I(a+b) \geq I(a)+I(b)$.

Lemma 22 *There exists a unique continuous extension of I to $B^+(S,\Sigma)$. Clearly, this extension satisfies on $B^+(S,\Sigma)$ properties (i) to (v) defined in Lemma 21.*

Proof: Since $a = b+a-b \leq b+\|a-b\|_\infty$, by monotonicity:

$$I(a) \leq I(b+\|a-b\|_\infty)$$

and by (v):

$$I(a) \leq I(b)+\delta\,\|a-b\|_\infty$$

that is

$$I(a)-I(b) \leq \delta\,\|a-b\|_\infty\,,$$

therefore

$$|I(a)-I(b)| \leq \delta\,\|a-b\|_\infty$$

and by equality $B^+(S,\Sigma) = \overline{B_0^+(S,\Sigma)}^{\|\cdot\|_\infty}$, there exists a unique continuous extension of I.

Remark 23 *According to Lemma 22, note that I defined on $B^+(S,\Sigma)$ is such that for any $a \in B^+(S,\Sigma)$ and any $\xi \in \mathbb{R}_+$:*

$$I(a)+\xi = I(a)+I(\xi\mathbf{1}_S) \leq I(a+\xi\mathbf{1}_S) \leq I(a)+\delta\xi.$$

So if $\delta = 1$, it comes that $I(a+\xi\mathbf{1}_S) = I(a)+\xi$, this clearly implies that if $\delta = 1$, the Certainty Independence Axiom will be satisfied. More precisely if $\delta = 1$, the functional I is clearly a monotone, superlinear and C-independent functional on $B^+(S,\Sigma)$ with $I(\mathbf{1}_S) = 1$, therefore (see for instance Lemma 3.5 in Gilboa and Schmeidler (1989)) in this case one recovers the MEU model.

Building upon Fan's Theorem 24 below, we give in the next Lemma 25, the key result for our representation Theorem 3. This Lemma can be seen as a generalization of the representation Theorem proposed by Chateauneuf (1991) for Gilboa and Schmeidler's model (1989). In fact, as mentioned previously both models coincide if $\delta = 1$.

Consider a real Banach space E and denote by E^* the dual space of E:

Theorem 24 *[Fan, (1956, page 126)] Given an arbitrary set Λ, let the system*

$$\langle f, x_i\rangle \geq \alpha_i, i \in \Lambda \qquad (\pounds)$$

of linear inequalities; where $\{x_i\}_{i\in\Lambda}$ be a family of elements, not all 0, in real normed linear space E, and $\{\alpha_i\}_{i\in\Lambda}$ be a corresponding family of real numbers.

Let $\sigma := sup \sum_{j=1}^{n} r_j \alpha_{i_j}$ *when* $n \in \mathbb{N}$, *and* r_j *vary under conditions:* $r_j \geq 0$, $\forall j \in \{1,...,n\}$*and* $\left\| \sum_{j=1}^{n} r_j x_{i_j} \right\|_E = 1$. *Then the system* $(\pounds)$ *has a solution* $f \in E^*$ *if and only if* σ *is finite. Moreover, if the system* $(\pounds)$ *has a solution* $f \in E^*$, *and if the zero-functional is not a solution of* $(\pounds)$, *then* $\sigma = min \{\|f\|_{E^*} : f$ *is a solution of* $(\pounds)\}$.

Lemma 25 *Let* Σ *be a* σ*-algebra of subsets of a set* S *and let* I *be a functional on the set* $B^+(S, \Sigma)$.*The following two assertions are equivalent:*

Assertion 1*:* I *satisfies the properties:*

1) I *is superadditive: for* $a, b \in B^+(S, \Sigma)$

$$I(a + b) \geq I(a) + I(b);$$

2) I *is positively homogeneous: for* $a, b \in B^+(S, \Sigma)$, $\lambda \geq 0$:

$$I(\lambda a) = \lambda I(a);$$

3) I *is monotonic: for* $a, b \in B^+(S, \Sigma)$:

$$a \geq b \Rightarrow I(a) \geq I(b);$$

4) I *is normalized:*

$$I(\mathbf{1}_S) = 1;$$

5) *There exists* $\delta \geq 1$ *such that for all* $a \in B^+(S, \Sigma)$ *and* $k \geq 0$*:*

$$I(a + k\mathbf{1}_S) \leq I(a) + \delta k.$$

Assertion 2*: there exists* $\alpha_0 \in (0, 1]$ *and a normal fuzzy set* $\varphi : \Delta \to [0, 1]$ *such that for any* $a \in B^+(S, \Sigma)$:

$$I(a) = \inf_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int_S adp$$

Proof: In order to simplify the notation we set $B^+(S, \Sigma) = B^+$, and $\int adp = E_p(a)$ for every $(a, p) \in B^+ \times \Delta$.

Assertion 2 implies Assertion 1 is straighforward.

In order to prove that Assertion 1 implies Assertion 2 we need the following lemma:

Lemma 26 *The mapping*

$$\begin{array}{rcl} \varphi^* & : & \Delta \to \mathbb{R} \\ p & \mapsto & \varphi^*(p) = \inf_{a \in B^+} \dfrac{E_p(a)}{I(a)} \end{array}$$

is a normal fuzzy set[17]*. Moreover, the functional*

$$\begin{array}{rcl} I^* & : & B^+ \to \mathbb{R} \\ a & \mapsto & I^*(a) = \inf\limits_{p \in \Delta} \dfrac{E_p(a)}{\varphi^*(p)} \end{array}$$

satisfies $I^*(a) = I(a)$*, for any* $a \in B^+$*.*

Proof: Since for all $a \in B^+$, $E_p(a) \geq 0$ and $I(a) \geq 0$, clearly $\varphi^*(p) \geq 0$ and $\frac{E_p(\mathbf{1}_S)}{I(\mathbf{1}_S)} = 1$ implies that $\varphi^*(p) \in [0,1]$ for all $p \in \Delta$.

Let us show that φ^* is normal, i.e., that there exists a $p_0 \in \Delta$ such that $\varphi^*(p_0) = 1$, since $\varphi^*(p_0) \leq 1$ it is enough to show that there exists $p_0 \in \Delta$ such that

$$E_{p_0}(a) \geq I(a) \ \forall a \in B^+$$

Setting $E = B$, we need to show that there exists $f \in E^*$ such that $f(\mathbf{1}_S) \geq 1$, $f(-\mathbf{1}_S) \geq -1$ and $f(a) \geq I(a)$ for all $a \in B^+$. Then we have a system of linear inequalities and can now use Fan's theorem:

Let us consider $\lambda_1, \lambda_2, ..., \lambda_n \geq 0$ and $\mathbf{1}_S$, $-\mathbf{1}_S$, $a_j \in B^+$, $3 \leq j \leq n$ such that:

$$\left\| \lambda_1 \mathbf{1}_S + \lambda_2(-\mathbf{1}_S) + \sum_{j=3}^{n} \lambda_j \, a_j \right\|_\infty = 1$$

it follows that

$$\lambda_1 \mathbf{1}_S - \lambda_2 \mathbf{1}_S + \sum_{j=3}^{n} \lambda_j \, a_j \leq \mathbf{1}_S$$

hence

$$\lambda_1 \mathbf{1}_S + \sum_{j=3}^{n} \lambda_j \, a_j \leq (\lambda_2 + 1)\mathbf{1}_S$$

from *(1)*,*(3)*,*(4)* and *(2)* it comes that:

$$\lambda_1 + \sum_{j=3}^{n} \lambda_j I(\, a_j) \leq \lambda_2 + 1$$

therefore

$$\lambda_1 - \lambda_2 + \sum_{j=3}^{n} \lambda_j I(\, a_j) \leq 1$$

i.e., $\sum\limits_{j=1}^{n} \lambda_j \alpha_{_j} \leq 1$; where $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_j = I(a_j), 3 \leq j \leq n$. Hence σ is finite and from Fan's theorem there exists $p_0 \in \Delta$ such that $E_{p_0}(a) \geq I(a)$ for all $a \in B^+$.

[17] Note that we adopt the usual convention $0/0 = 1$ and $r/0^+ = +\infty$ if $r > 0$.

Now, we have that for any $a \in B^{+}$, $I^{*}(a) = \inf_{p\in\Delta} \frac{E_p(a)}{\varphi_+(p)} \in \mathbb{R}_+$. It remains to prove that $I^{*}(a) = I(a)$, for any $a \in B^{+}$.

Let a_0 be chosen in B^{+}, and first prove that $I^{*}(a_0) \geq I(a_0)$: If $I(a_0) = 0$ this is immediate. Assume, now, $I(a_0) > 0$. Note that it is enough to prove $I^{*}(a_0) \geq I(a_0)$ if $1 \geq I(a_0) > 0$. Actually, let a_0 be such that $I(a_0) > 1$ and choose $\lambda > 0$ such that $\lambda I(a_0) \leq 1$, since I^{*} and I are positively homogeneous, one obtains:

$$\lambda I(a_0) = I(\lambda a_0) \leq I^{*}(\lambda a_0) = \lambda I^{*}(a_0)$$

hence $I(a_0) \leq I^{*}(a_0)$. Considering $a_0 \in B^{+}$ such that $1 \geq I(a_0) > 0,$ we have that

$$\varphi^{*}(p) = \inf_{a\in B^{+}} \frac{E_p(a)}{I(a)} \leq \frac{E_p(a_0)}{I(a_0)},\ \forall p \in \Delta,$$

hence,

$$I(a_0) \leq \frac{E_p(a_0)}{\varphi^{*}(p)},\ \forall p \in \Delta$$

and from the definition of $I^{*} : I^{*}(a_0) \geq I(a_0)$.

Let us now prove that $I^{*}(a_0) \leq I(a_0)$ for any chosen $a_0 \in B^{+}$. Clearly, it is enough to prove this inequality when $I^{*}(a_0) > 0$. Since $I^{*}(a_0)$ is the greatest lower bound of the set of real numbers given by $\left\{\frac{Eq(a_0)}{\varphi^{*}(q)} : q \in \Delta\right\}$ if we find $p \in \Delta$ such that $\frac{Ep(a_0)}{\varphi^{*}(p)} \leq I(a_0)$ then the result will be proved:

Let us first show that there exists $f \in E^{*}$ such that $\delta \geq f(\mathbf{1}_S) \geq 1$, $f(a_0) = I(a_0)$ and $f(a) \geq I(a)$ for all $a \in B^{+}$., i.e., $f \in E^{*}$ such that

$$\begin{aligned} f(\mathbf{1}_S) &\geq 1,\ f(-\mathbf{1}_S) \geq -\delta,\ f(a_0) \geq I(a_0) \\ f(-a_0) &\geq -I(a_0) \text{ and } f(a) \geq I(a)\ \forall a \in B^{+} \end{aligned}$$

Again, we use Fan's theorem:

Let us consider $\lambda_1, ..., \lambda_n \geq 0$ and $\mathbf{1}_S$, $-\mathbf{1}_S$, a_0, $-a_0$, $a_j \in B^{+}$, $5 \leq j \leq n$ such that:

$$\left\| \lambda_1 \mathbf{1}_S + \lambda_2(-\mathbf{1}_S) + \lambda_3 a_0 + \lambda_4(-a_0) + \sum_{j=5}^{n} \lambda_j\, a_j \right\|_{\infty} = 1$$

it follows that

$$\lambda_1 \mathbf{1}_S - \lambda_2 \mathbf{1}_S + \lambda_3 a_0 - \lambda_4 a_0 + \sum_{j=5}^{n} \lambda_j\, a_j \leq \mathbf{1}_S$$

hence

$$\lambda_1 \mathbf{1}_S + \lambda_3 a_0 + \sum_{j=5}^{n} \lambda_j\, a_j \leq \lambda_4 a_0 + (\lambda_2 + 1)\mathbf{1}_S$$

By properties of I in assertion 1 it comes that:

$$\lambda_1 + \lambda_3 I(a_0) + \sum_{j=5}^{n} \lambda_j I(\,a_j) \leq \lambda_4 I(a_0) + (\lambda_2 + 1)\delta$$

therefore

$$\lambda_1 - \lambda_2\delta + \lambda_3 I(a_0) - \lambda_4 I(a_0) + \sum_{j=5}^{n} \lambda_j I(\,a_j) \leq \delta$$

By Fan's theorem, it comes that there exists $\eta \in [1,\delta]$, $p \in \Delta$ such that:

$$\begin{array}{rcl} (1)\,\eta E_p(a_0) & = & I(a_0), \text{ and} \\ (2)\,\eta E_p(a) & \geq & I(a) \text{ for all } a \in B^+ \end{array}$$

From (2) it comes that $E_p(a)/I(a) \geq \eta^{-1}$, for all $a \in B^+$. Actually, by the initial convention, $E_p(a) = 0$ implies $I(a) = 0$ and then $E_p(a)/I(a) = 1 \geq \eta^{-1}$. Moreover, if $E_p(a) > 0$ and $I(a) = 0$ then $E_p(a)/I(a) = +\infty \geq \eta^{-1}$.

Consequentely, $\varphi^*(p) \geq \eta^{-1}$, and therefore $\varphi^*(p) > 0$.

Let us show that this entails $E_p(a_0) > 0$. In fact, $0 < I^*(a_0) \leq E_p(a_0)/\varphi^*(p)$, so we get $E_p(a_0) > 0$. Hence, (1) entails $I(a_0) > 0$. Consequently,

$$\frac{E_p(a_0)}{I(a_0)} = \frac{1}{\eta} \leq \varphi^*(p),$$

that is,

$$\frac{E_p(a_0)}{\varphi^*(p)} \leq I(a_0),$$

as desired.

Lemma 27 *The mapping $\varphi^* : \Delta \to \mathbb{R}$ is a regular fuzzy set.*

Proof: We know that φ^* is a normal fuzzy set. Now, let us show that φ^* is fuzzy convex. In fact, we have it that φ^* is concave: taking $p_1, p_2 \in \Delta$ and $r \in [0,1]$, denote by $p^r = rp_1 + (1-r)p_2$. Hence for every $a \in B^+$ $E_{p^r}(a) = rE_{p_1}(a) + (1-r)E_{p_2}(a)$ and

$$\begin{array}{rcl} \varphi^*(p^r) & = & \inf\limits_{a \in B^+} \dfrac{rE_{p_1}(a) + (1-r)E_{p_2}(a)}{I(a)} \\ & \geq & r \inf\limits_{a \in B^+} \dfrac{E_{p_1}(a)}{I(a)} + (1-r) \inf\limits_{a \in B^+} \dfrac{E_{p_2}(a)}{I(a)} \\ & = & r\varphi^*(p_1) + (1-r)\varphi^*(p_2). \end{array}$$

in particular, φ^* is quasiconcave.

Finally, let us show that φ^* is weak* upper semicontinuous. For each $a \in \{b \in B^+ : I(b) > 0\} := \{I > 0\}$, define

$$\begin{array}{rcl} \psi_a & : & \Delta \to \mathbb{R} \\ p & \mapsto & \psi_a(p) = E_p(a)/I(a). \end{array}$$

By the definition of weak* topology we have that ψ_a is weak* upper semicontinuous for any $a \in \{I > 0\}$. Note that

$$\varphi^*(p) = \inf_{a \in B^+} \frac{E_p(a)}{I(a)} = \inf_{\{I>0\}} \frac{E_p(a)}{I(a)}$$

Since the sets $\{p \in \Delta : \psi_a(p) \geq \alpha\}$ are weak* closed for any $a \in \{I > 0\}$ and for any $\alpha \in [0,1]$, we obtain that

$$\{p \in \Delta : \varphi^*(p) \geq \alpha\} = \bigcap_{\{I>0\}} \{p \in \Delta : \psi_a(p) \geq \alpha\}$$

is weak* closed as desired (in fact, we have an infimun over continuous functions and it is well known that it is upper semicontinuous).

Finally the proof of Theorem 3 is completed through Lemma 28 below.

Lemma 28 *Set $\alpha_0 = 1/\delta$ and $L_{\alpha_0}\varphi^* = \{p \in \Delta : \varphi^*(p) \geq \alpha_0\}$, then for every $a \in B^+$*

$$I(a) = \min_{p \in L_{\alpha_0}\varphi^*} \frac{E_p(a)}{\varphi^*(p)}$$

Proof: Denoting by

$$I'(a) := \inf_{p \in L_{\alpha_0}\varphi^*} \frac{E_p(a)}{\varphi^*(p)},$$

we claim that $I(a) = I'(a)$, $\forall a \in B^+$: First, it is immediate that $I'(a) \geq I^*(a) = I(a)$ for any $a \in B^+$. In order to show that $I'(a) = I(a)$ for every $a \in B^+$, it is enough to show that for a given a_0 belonging to B^+ such that $I'(a_0) > 0$, there exists $p_0 \in L_{\alpha_0}\varphi^*$ such that $E_{p_0}(a_0)/\varphi^*(p_0) \leq I(a_0)$. In fact, by Lemma 26 we know that there exists $p_0 \in \Delta$ such that $E_{p_0}(a)/I(a) \geq 1/\delta$ for every $a \in B^+$, i.e., $p_0 \in L_{\alpha_0}\varphi^*$. Since $I'(a) > 0$, it follows that $E_{p_0}(a_0) > 0$ and, again by Lemma 26, $E_{p_0}(a_0)/I(a_0) = \varphi^*(p_0)$, and then $I(a_0) = E_{p_0}(a_0)/\varphi^*(p_0)$.

Now it is enough to show that the inf given in the definition of I' is in fact a "min" for any $a \in B^+$. The result is immediate for any

$$a \in \left\{b \in B^+ : \exists p_1 \in L_{\alpha_0}\varphi^* \text{ s.t. } E_{p_1}(b) = 0 \right\}$$

because $p \to E_p(a)/\varphi^*(p)$ is non-negative on $L_{\alpha_0}\varphi^*$, so

$$I(a) = E_{p_1}(a)/\varphi^*(p_1) = 0.$$

Now suppose that $E_p(a) > 0$ for any $p \in L_{\alpha_0}\varphi^*$. Note that the mapping

$$L_{\alpha_0}\varphi^* \ni p \to E_p(a)/\varphi^*(p),$$

is weak* lower semicontinuous: In fact, we need to show that for any real number λ the set $\Gamma_\lambda := \{p \in L_{\alpha_0}\varphi^* : E_p(a)/\varphi^*(p) \leq \lambda\}$ is weak* closed. For $\lambda \leq 0$ the result is trivial because $\{p \in L_{\alpha_0}\varphi^* : E_p(a)/\varphi^*(p) \leq \lambda\} = \emptyset$. Otherwise,

$$\Gamma_\lambda = \{p \in L_{\alpha_0}\varphi^* : E_p(a) \leq \lambda\varphi^*(p)\},$$

since $\varphi^*(p) = \inf_{b \in B^+} \{E_p(b)/I(b)\}$, it comes that

$$\begin{aligned}
\Gamma_\lambda &= \left\{p \in L_{\alpha_0}\varphi^* : \lambda^{-1}E_p(a) \leq E_p(b)/I(b),\ \forall b \in B^+\right\} \\
&= \left\{p \in L_{\alpha_0}\varphi^* : \lambda^{-1}I(b) \leq E_p(b)/E_p(a),\ \forall b \in B^+\right\} \\
&= \bigcap_{b \in B^+} \left\{p \in L_{\alpha_0}\varphi^* : \lambda^{-1}I(b) \leq E_p(b)/E_p(a)\right\} \\
&= \bigcap_{b \in B^+} \left\{\tau_{a,b}^{-1}\left([\lambda^{-1}I(b), \infty)\right)\right\},
\end{aligned}$$

where $\tau_{a,b} : L_{\alpha_0}\varphi^* \ni p \to \tau_{a,b}(p) = E_p(b)/E_p(a)$. Since $\tau_{a,b}$ is weak* continuous (for each $c \in \{a, b\}$, $p \to E_p(c)$ is weak* continuous and $p \to E_p(a)$ is stricly positive) we obtain that each set $\{p \in L_{\alpha_0}\varphi^* : \lambda^{-1} I(b) \leq E_p(b)/E_p(a)\}$ is weak* closed, so Γ_λ is an intersection of weak* closed sets which is weak* closed too.

By Banach-Alaoglu-Bourbaki's theorem (see Dunford and Schwartz (1958, page 424)) $L_{\alpha_0}\varphi^*$ is weak* compact which together with the generalized Weierstrass´s theorem (see, for instance Aliprantis and Border (1999), Theorem 2.40) entails that the mapping $L_{\alpha_0}\varphi^* \ni p \to E_p(a)/\varphi^*(p)$ attains a minimum value, i.e., there exists $q \in L_{\alpha_0}\varphi^*$ such that $I(a) = E_q(a)/\varphi^*(q) \leq E_p(a)/\varphi^*(q)$ for any $p \in L_{\alpha_0}\varphi^*$.

Lemma 29 *Let φ be a regular fuzzy set satisfying the model, i.e.*

$$I(a) = \min_{p \in L_{\alpha_0}\varphi} \frac{1}{\varphi(p)} \int adp \, \textit{for all } a \in B^+,$$

and let φ^* be defined as previously by

$$\varphi^*(p) = \inf_{a \in B^+} \frac{E_p(a)}{I(a)} \textit{ or equally } \varphi^*(p) = \inf_{f \in \mathcal{F}} \frac{E_p(u(f))}{u(c_f)},$$

then for any $p \in L_{\alpha_0}\varphi$ one obtains $\varphi^*(p) \geq \varphi(p)$.

Proof: Let $p \in L_{\alpha_0}\varphi$ then for all $a \in B^+$, $I(a) \leq E_p(a)/\varphi(p)$. Hence, $\varphi(p)I(a) \leq E_p(a)$ for all $a \in B^+$. Since $\varphi(p) > 0$, if $E_p(a) = 0$ then $I(a) = 0$ and in this case $E_p(a)/I(a) = 1 \geq \varphi(p)$. If $E_p(a) > 0$ in any case, due to the convention $r/0 = +\infty$ if $r > 0$, one obtains that $E_p(a)/I(a) \geq \varphi(p)$. Hence, $\varphi(p) \leq E_p(a)/I(a)$ for all $a \in B^+$ and, therefore, $\varphi^*(p) \geq \varphi(p)$.

Lemma 30 *From Lemmas 26 and 27, there exists a regular fuzzy set φ such that*

$$I(a) = \inf_{p \in \Delta} \frac{E_p(a)}{\varphi(p)} \textit{ for all } a \in B^+.$$

In fact, for any such φ, one obtains that $\varphi^*(p) \geq \varphi(p)$ for all $p \in \Delta$.

Proof: Take $p \in \Delta$, then for all $a \in B^+$, $I(a) \leq E_p(a)/\varphi(p)$; if $\varphi(p) = 0$, clearly $E_p(a)/I(a) \geq \varphi(p)$ for all $a \in B^+$ and $\varphi^*(p) \geq \varphi(p)$. If $\varphi(p) > 0$, the same proof as for Lemma 29 applies.

PROOF OF COROLLARY 5:

The proof results from Lemmas 29, 30, and note that $\forall a \in B^+$

$$I(a) = \inf_{p \in \Delta} \frac{E_p(a)}{\varphi^*(p)} = \min_{p \in L_{\alpha_0}\varphi^*} \frac{E_p(a)}{\varphi^*(p)},$$

so $\forall a \in B^+$, $\exists p_a \in L_{\alpha_0}\varphi^*$ such that

$$\frac{E_p(a)}{\varphi^*(p)} \geq \frac{E_{p_a}(a)}{\varphi^*(p_a)}, \ \forall p \in \Delta,$$

hence

$$I\left(a\right)=\min_{p\in\Delta}\frac{E_p(a)}{\varphi^*(p)}.$$

PROOF OF COROLLARY 6:

Proof: Let (u_1,φ_1^*) represents $\succsim$ as in Corollary 5. If (u_2,φ_2^*) is another representation of $\succsim$, by the fundamental equivalence obtained in Corollary 5, u_1 and u_2 are nonconstant affine representations $\succsim|_{X\times X}$, and by standard uniqueness results there exists $\lambda>0$ and $\delta\in\mathbb{R}$ such that $u_1=\lambda u_2+\delta$. But, our main result imposes that $u_1\left(x_*\right)=u_2\left(x_*\right)=0$, so $\delta=0$. Building on the characterization of the maximal confidence function and that $u_1=\lambda u_2$, for any $p\in\Delta$

$$\varphi_1^*(p)=\inf_{f\in\mathcal{F}}\left(\frac{\int u_1(f)dp}{u_1\left(c_f\right)}\right)=\inf_{f\in\mathcal{F}}\left(\frac{\int\lambda u_2(f)dp}{\lambda u_2\left(c_f\right)}\right)=\varphi_2^*(p),$$

as desired. The converse is obvious.

PROOF OF PROPOSITION 7:

Proof: We have that $J(f)=\inf_{p\in\Delta}\left\{\frac{1}{\varphi^*(p)}\int u(f)dp\right\}$ with $\varphi^*\in F_{\mathcal{R}^*}(\Delta)$, in particular the normality of φ^* says that we can take some $p'\in\Delta$ such that $\varphi^*(p')=1$. Now, we define $V(f)=\int(u(f))dp'$ which induces the SEU preference $\succsim_V$. Furthermore, the inequality $V(f)\geq J(f)$ implies that $f\succsim_J x\Rightarrow f\succsim_V x$.

PROOF OF PROPOSITION 8:

Proof: (1) $\Rightarrow$ (2): We saw that we can take $u_1=u_2=u$. Now, for any $f\in\mathcal{F}$, if $f\sim_1 x$ then $f\succsim_2 x$, moreover:

$$J_1(u(f))=u(x)\leq J_2(u(f))$$

i.e., $J_1\leq J_2$. Hence,

$$\varphi_1^*(p)=\inf_{f\in\mathcal{F}}\frac{\int u(f)dp}{J_1(u(f))}\geq\inf_{f\in\mathcal{F}}\frac{\int u(f)dp}{J_2(u(f))}=\varphi_2^*(p)$$

as desired.

(2) $\Rightarrow$ (1) : For any $f\in\mathcal{F}$ and $x\in X$, if $f\succsim_1 x$ then

$$\inf_{p\in\Delta}\left(\frac{\int u(f)dp}{\varphi_1^*(p)}\right)\geq u(x),$$

since $\varphi_1^*\geq\varphi_2^*$ implies that

$$\inf_{p\in\Delta}\left(\frac{\int u(f)dp}{\varphi_2^*(p)}\right)\geq\inf_{p\in\Delta}\left(\frac{\int u(f)dp}{\varphi_1^*(p)}\right)$$

we conclude that $f\succsim_2 x$.

PROOF OF PROPOSITION 14:

Proof: By Axioms A1, A2, A3, A4 and the certainty independence axiom it is simple to see that the functional $I:B^+\to\mathbb{R}$ obtained in the Lemmas 21 and 22 satisfies properties (i) to (iv) of Lemma 21 and for any $a\in B^+$ and $\xi\geq 0$ it is true that $I\left(a+\xi\right)=I\left(a\right)+\xi$. So, we obtain the particular version of our Lemma 25 as proved in Chateauneuf (1991). Hence, there exists a weak* compact and convex set $C \subset \Delta$ such that for any $a \in B^+$

$$I(a) = \min_{p \in C} \int a dp.$$

Recall that for any $p \in \Delta$,

$$\varphi^*(p) = \inf_{a \in B^+} \frac{\int a dp}{I(a)}.$$

Now, we note that for all $p \in C$, $I(a) \leq \int a dp$ for any $a \in B^+$, since $I(\mathbf{1}_S) = p(S) = 1$ we obtain that $\varphi^*(p) = 1$, $\forall p \in C$. If $p \notin C$ by a separation theorem for locally convex linear topological space [Dunford and Schwartz, (1988, page 418)] there exists $a_0 \in B^+$ such that

$$\int a_0 dp < \min\{\int a_0 dp : p \in C\} = I(a_0)$$

therefore

$$\varphi^*(p) = \inf_{a \in B^+} \frac{\int a dp}{I(a)} \leq \frac{\int a_0 dp}{I(a_0)} < 1$$

and we conclude that $\varphi^*(p) = 1$ if and only if $p \in C$. In particular, for any $a \in B^+(S, \Sigma, u(X))$ (i.e., $\exists f \in \mathcal{F}$ s.t. $a = u(f)$),

$$\begin{aligned} J(f) &= I(a) = \min_{p \in C} \int a dp = \min_{p \in L_1 \varphi^*} \int a dp \\ &= \min_{p \in L_1 \varphi^*} \int u(f) dp = \min_{p \in \Delta} \frac{1}{\varphi^*(p)} \int u(f) dp. \end{aligned}$$

PROOF OF PROPOSITION 16:

Proof: By Axioms A1, A2, A3, A4 and the comonotonic independence axiom it is simple to see that the functional $I : B^+ \to \mathbb{R}$ obtained in the Lemmas 21 and 22 satisfies properties (i) to (iv) of Lemma 21 and for any $a, b \in B^+$ such that a and b are comonotonics it is true that $I(a+b) = I(a) + I(b)$. So, by Schmeidler (1986, 1989) there exists a convex capacity $v : \Sigma \to [0,1]$ such that for any $a \in B^+$

$$I(a) = \int a dv.$$

Now, we define the following mapping

$$\begin{aligned} \widetilde{\varphi} &: \Delta \to \mathbb{R} \\ p &\mapsto \widetilde{\varphi}(p) = \inf_{E \in \Sigma} \frac{p(E)}{v(E)}. \end{aligned}$$

We obtain the proof by the following next two lemmas.

Lemma 31 *The mapping $\widetilde{\varphi}$ is a normal fuzzy set and for every $a \in B(S, \Sigma)$,*

$$(1) \qquad I(a) = \underset{p \in L_{\alpha_0}\varphi}{inf} \frac{\int adp}{\widetilde{\varphi}(p)},$$

for any level of minimal confidence $\alpha_0 \in (0, 1]$.

Proof: Let us first prove that $\widetilde{\varphi}$ is a normal fuzzy set. Take $p \in \Delta$, clearly $\widetilde{\varphi}(p) \in \overline{\mathbb{R}}_+$, and since $p(S) = v(S) = 1$ it turns out that $\widetilde{\varphi}(p) \in [0, 1]$.

Finally $\widetilde{\varphi}$ is normal: since v is convex we know that $\mathcal{C}(v)$ is nonempty. Note that $\widetilde{\varphi}(p) = 1$ if and only if $p \in \mathcal{C}(v)$.

Let us first prove equality (1), when a belongs to $B_0^+(S, \Sigma)$, the set of real-valued Σ-measurable, non-negative simple functions. Then

$$a = \sum_{i=1}^{m} x_i \mathbf{1}_{E_i}$$

where $\{E_i\}_{i=1}^m \subset \Sigma$ is a partition of S where $x_1 > x_2 > ... > x_m \geq 0 = x_{m+1}$.

First, let us prove that

$$I'(a) := \underset{p \in L_{\alpha_0}\varphi}{inf} \frac{\int adp}{\widetilde{\varphi}(p)} \geq I(a).$$

It is enough to show that for any given $p \in L_{\alpha_0}\widetilde{\varphi}$ we have:

$$(2) \quad \frac{\int adp}{\widetilde{\varphi}(p)} \geq \int adv.$$

Set $d_1(p) = \int adp - \widetilde{\varphi}(p) \int adv$; hence (2) is equivalent to $d_1(p) \geq 0$. We note that

$$d_1(p) = \sum_{i=1}^{m} (x_i - x_{i+1}) \left[p\left(\bigcup_{j=1}^{i} E_i \right) - \widetilde{\varphi}(p) v \left(\bigcup_{j=1}^{i} E_i \right) \right],$$

since for all $i \in \{1, ..., m\}$

$$\widetilde{\varphi}(p) \leq \frac{p\left(\bigcup_{j=1}^{i} E_i \right)}{v\left(\bigcup_{j=1}^{i} E_i \right)}$$

and $(x_i - x_{i+1}) \geq 0$, we obtain $d_1(p) \geq 0$.

It remains to show that $I'(a) \leq I(a)$: taking $p_0 \in \mathcal{C}(v)$ such that

$$\int adv = \underset{p \in \mathcal{C}(v)}{min} \int adp = \int adp_0.$$

Since $\widetilde{\varphi}(p_0) = 1$, we obtain

$$I'(a) \leq \int adp_0 = I(a).$$

Therefore, $I'(a) = I(a)$ for all $a \in B_0^+$.

Let us take now a belonging to B^+: We know that there exists $a_n \in B_0^+$, $a_n \to a$ uniformly. From the previous case, $I'(a_n) = I(a_n)$ for all $n \geq 1$. From Lemma 22, $I'(a_n) \to I'(a)$, but $I(a_n) \to I(a)$. Hence, $I'(a) = I(a)$.

In fact, it is true that

Lemma 32 *For any probability $p \in \Delta$ we have that $\widetilde{\varphi}(p) = \varphi^*(p)$.*

Proof: Note first that $E \in \Sigma$ implies that $\mathbf{1}_E \in B$, then $0 \leq \varphi^*(p) \leq \widetilde{\varphi}(p)$, so the proof has only to be done if $\widetilde{\varphi}(p) > 0$.

In the previous proposition, when restricting to B_0^+, we obtain that

$$\frac{\int adp}{\widetilde{\varphi}(p)} \geq \int adv, \text{ for every } a \in B_0^+$$

so, by continuity,

$$\frac{\int adp}{\widetilde{\varphi}(p)} \geq \int adv, \text{ for every } a \in B^+.$$

Hence, $\int adp \geq \widetilde{\varphi}(p)I(a)$ for any $a \in B^+$. If $I(a) = 0$ either $\int adp = 0$ and $\int adp/I(a) = 1 \geq \widetilde{\varphi}(p)$, or $\int adp > 0$ and $\int adp/I(a) = +\infty \geq \widetilde{\varphi}(p)$. Finally, if $I(a) > 0$, clearly $\int adp/I(a) \geq \widetilde{\varphi}(p)$ and therefore $\varphi^*(p) \geq \widetilde{\varphi}(p)$.

PROOF OF PROPOSITION 17

Proof: It is similar to the proof of Lemma 31: Take $p \in \Delta$, clearly $\varphi(p) \in [0, 1]$.

In order to prove that φ is normal, note that since v is convex then $\mathcal{C}(v)$ is non empty. Moreover, $p \in \mathcal{C}(v)$ if and only if $p(E) \geq v(E)$, or equivalently, $1 - v(E) \geq 1 - p(E)$ for every $E \in \Sigma$, and then $\varphi(p) = 1$.

Let us prove now equality (2), where a belongs to $B_0(S, \Sigma)$ the set of real-valued Σ-measurable simple functions. Then

$$a = \sum_{i=1}^{m} x_i \mathbf{1}_{E_i} + \sum_{k=m+1}^{n} y_i \mathbf{1}_{E_k} = a^- + a^+$$

where $\{E_i\}_{i=1}^n, \subset \Sigma$ is a partition of S and $x_1 < x_2 < ... < x_m < 0 \leq x_{m+1} < x_{m+2} < ... < x_n$.

First let us prove that

$$I'(a) := \inf_{p \in L_{\alpha_0}\varphi} \left(\frac{\int a^+ dp}{\varphi(p)} + \varphi(p) \int a^- dp \right) \geq I(a) = I(a^+) + I(a^-)$$

It is enough to show that for a given $p \in L_{\alpha_0}\widetilde{\varphi}$ we have:

$$(2) \quad \frac{\int a^+ dp}{\widetilde{\varphi}(p)} \geq \int a^+ dv$$

$$(3) \varphi(p) \int a^- dp \geq \int a^- dv$$

Set $d_1(p) = \int a^+ dp - \varphi(p) \int a^+ dv$; hence (2) is equivalent to $d_1(p) \geq 0$, which we proved in Lemma 31.

Setting now $d_2(p) = \varphi(p) \int a^- dp - \int a^- dv$, (3) is equivalent to $d_2(p) \geq 0$.

Note that

$$\begin{aligned} d_2(p) &= \varphi(p) \sum_{i=1}^{m} x_i \left(p\left(\bigcup_{j=i}^{n} E_j \right) - p\left(\bigcup_{j=i+1}^{n} E_j \right) \right) \\ &\quad - \sum_{i=1}^{m} x_i \left(v\left(\bigcup_{j=i}^{n} E_j \right) - v\left(\bigcup_{j=i+1}^{n} E_j \right) \right). \end{aligned}$$

Since,

$$\varphi(p) \leq \frac{1 - v(A)}{1 - p(A)}, \ \forall A \in \Sigma,$$

it follows that

$$\varphi(p) \left(1 - p\left(\bigcup_{j=2}^{n} E_j \right) \right) - \left(1 - v\left(\bigcup_{j=2}^{n} E_j \right) \right) \leq 0,$$

therefore,

$$\begin{aligned} & x_1 \left[\varphi(p) \left(1 - p\left(\bigcup_{j=2}^{n} E_j \right) \right) - \left(1 - v\left(\bigcup_{j=2}^{n} E_j \right) \right) \right] \\ \geq \ & x_2 \left[\varphi(p) \left(1 - p\left(\bigcup_{j=2}^{n} E_j \right) \right) - \left(1 - v\left(\bigcup_{j=2}^{n} E_j \right) \right) \right], \end{aligned}$$

this entails that

$$\begin{aligned} & \varphi(p) \sum_{i=1}^{2} x_i \left(p\left(\bigcup_{j=i}^{n} E_j \right) - p\left(\bigcup_{j=i+1}^{n} E_j \right) \right) \\ & - \sum_{i=1}^{2} x_i \left(v\left(\bigcup_{j=i}^{n} E_j \right) - v\left(\bigcup_{j=i+1}^{n} E_j \right) \right) \\ = \ & x_1 \left[\varphi(p) \left(1 - p\left(\bigcup_{j=2}^{n} E_j \right) \right) - \left(1 - v\left(\bigcup_{j=2}^{n} E_j \right) \right) \right] \\ & + x_2 \left[\varphi(p) \left(p\left(\bigcup_{j=2}^{n} E_j \right) - p\left(\bigcup_{j=3}^{n} E_j \right) \right) - \left(v\left(\bigcup_{j=2}^{n} E_j \right) \right) \right] \\ \geq \ & x_2 \left[\varphi(p) \left(1 - p\left(\bigcup_{j=3}^{n} E_j \right) \right) - \left(1 - v\left(\bigcup_{j=3}^{n} E_j \right) \right) \right]. \end{aligned}$$

Since $x_2 < x_3$ and

$$\varphi(p)\left(1 - p\left(\bigcup_{j=3}^{n} E_j\right)\right) - \left(1 - v\left(\bigcup_{j=3}^{n} E_j\right)\right) \leq 0,$$

we obtain also that,

$$\begin{aligned}
& \varphi(p)\sum_{i=1}^{3} x_i \left(p\left(\bigcup_{j=i}^{n} E_j\right) - p\left(\bigcup_{j=i+1}^{n} E_j\right)\right) \\
& -\sum_{i=1}^{3} x_i \left(v\left(\bigcup_{j=i}^{n} E_j\right) - v\left(\bigcup_{j=i+1}^{n} E_j\right)\right) \\
\geq \quad & x_3\left[\varphi(p)\left(1 - p\left(\bigcup_{j=4}^{n} E_j\right)\right) - \left(1 - v\left(\bigcup_{j=4}^{n} E_j\right)\right)\right],
\end{aligned}$$

in fact, we can iterate and obtain that

$$d_2\left(p\right) \geq x_m\left[\varphi(p)\left(1 - p\left(\bigcup_{j=m+1}^{n} E_j\right)\right) - \left(1 - v\left(\bigcup_{j=m+1}^{n} E_j\right)\right)\right] \geq 0.$$

It remains to show that $I'(a) \leq I(a)$: taking $p_0 \in \mathcal{C}(v)$ such that

$$\int a dv = \underset{p \in \mathcal{C}(v)}{min} \int a dp = \int a dp_0.$$

Since $\varphi(p_0) = 1$, we obtain

$$I'(a) \leq \int a^+ dp_0 + \int a^- dp_0 = I(a).$$

Therefore, $I'(a) = I(a)$ for all $a \in B_0(S, \Sigma)$.

Let us take now a belonging to $B(S, \Sigma)$: We know that there exists $a_n \in B_0(S, \Sigma)$, $a_n \to a$ uniformly. From the previous case $I'(a_n) = I(a_n)$ for all $n \geq 1$. From Lemma 22, $I'(a_n) \to I'(a)$, but $I(a_n) \to I(a)$. Hence, $I'(a) = I(a)$.

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