Frobenius theorem for foliations on singular varieties

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Abstract. *We generalize Frobenius singular theorem due to Malgrange, for a large class of codimension one holomorphic foliations on singular analytic subsets of $\mathbb{C}^N$. As a consequence we obtain the following : let M be a smooth complete intersection sub-variety of $\mathbb{P}^N$, where $dim(M) \geq 3$. Then the singular set of any codimension one foliation on M has at least one component of codimension two.*

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1. Introduction and Statement of Results

In 1976 B. Malgrange proved the following result (cf. [M]) :
Malgrange's Theorem. *Let ω be a germ at $0 \in \mathbb{C}^N$ of a holomorphic integrable 1-form. Suppose that the singular set of ω has complex codimension greater than or equal to three. Then there exist germs of holomorphic functions f and g, where $g(0) \neq 0$, such that $\omega = g.df$.*

In other words, if we take representatives of the germs ω and f in a neighborhood U of $0 \in \mathbb{C}^N$, then f is a first integral of the codimension one foliation on U defined by the differential equation $\omega = 0$. In this paper we generalize this result, in certain cases, for germs of foliations in a germ of an analytic subset of $\mathbb{C}^N$. Before stating our main result, we need a definition.

Let X be a germ at $0 \in \mathbb{C}^N$ of an irreducible analytic set of complex dimension $n \geq 2$, with singular set $sing(X)$. Let $X^* = X \setminus sing(X)$. Consider an open neighborhood B of $0 \in \mathbb{C}^N$ such that X, $sing(X)$ and X^* have representatives, wich will be denoted by X_B, $sing(X_B)$ and $X_B^* := X_B \setminus sing(X_B)$, respectively. If B is small enough then X_B^* is a smooth connected manifold of complex dimension n. In this case, we define a singular complex codimension one foliation on X_B^* as usual (cf. [LN-BS]). The singular set of a foliation $\mathcal{F}$ on a complex manifold M will be denoted by $sing(\mathcal{F})$. We observe that it is always possible to suppose that

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$cod_M(sing(\mathcal{F})) \geq 2$, in the sense that there exists a foliation $\mathcal{G}$ on M such that $cod_M(sing(\mathcal{G})) \geq 2$ and $\mathcal{G} \equiv \mathcal{F}$ on $M \setminus sing(\mathcal{F})$ (cf. [LN-BS]).

Definition 1.0.1. *A germ $\mathcal{F}$, of codimension one holomorphic foliation on X^*, is defined by the following data :*

(a). *There exists an open neighborhood B of $0 \in \mathbb{C}^N$ as above and a codimension one foliation $\mathcal{F}_B$ on X^*_B.*

(b). *If $0 \subset U \subset B$ is an open set such that X^*_U is connected then there exists a codimension one foliation $\mathcal{F}_U$ on X^*_U such that $\mathcal{F}_B|_{X^*_U} = \mathcal{F}_U$.*

The germ $\mathcal{F}$ is the collection $\{\mathcal{F}_U\}_{0 \in U \subset B}$. The singular set of $\mathcal{F}$ is the germ of analytic subset of X^* defined by the collection $\{sing(\mathcal{F}_U)\}_{0 \in U \subset B}$.

For example, let ω be a germ at $0 \in \mathbb{C}^N$ of holomorphic 1-form such that $\omega|_{X^*} \not\equiv 0$ and $\omega \wedge d\omega|_{X^*} \equiv 0$. If $cod_{X^*}(sing(\omega|_{X^*})) \geq 2$ then it defines a germ of codimension one foliation on X^*. This germ will be denoted by $\mathcal{F}_\omega$.

We would like to observe that when $cod_{X^*}(sing(\omega|_{X^*})) \geq 1$ then there exists a germ of foliation $\mathcal{F}$, with $cod_{X^*}(sing(\mathcal{F})) \geq 2$, such that $\mathcal{F}|_{X^* \setminus sing(\omega)}$ coincides with the foliation induced by ω on $X^* \setminus sing(\omega)$ (cf. [LN-BS]). This is the case of example 1.1 after the statement of the main theorem.

Definition 1.0.2. *Let X be an irreducible germ of analytic set at $0 \in \mathbb{C}^N$ with dimension $n < N$. We say that X is k-regular, $0 \leq k \leq n$, if there exists a neighborhood U of $0 \in \mathbb{C}^N$ and representatives X_U, $sing(X_U)$ and X^*_U of X, $sing(X)$ and X^*, respectively, such that : For any germ of holomorphic k-form η on X^*_U there exists a holomorphic k-form θ on U such that $\theta|_{X^*_U} \equiv \eta$.*

Main Theorem. *Let X be a germ of irreducible analytic set at $0 \in \mathbb{C}^N$, of dimension n, $3 \leq n \leq N$, and $\mathcal{F}$ be a germ of holomorphic codimension one foliation on X^*. Suppose that :*

(a) $H^1(X^*, \mathcal{O}) = 0$.
(b) *X is k-regular for $k = 0, 1$.*
(c) *$\mathcal{F}$ is defined by a holomorphic (germ of) 1-form ω on X^* such that $cod_{X^*}(sing(\omega)) \geq 3$.*
(d) $dim(sing(X)) \leq dim(X) - 3$.

Then there exist germs of analytic functions f and g at $0 \in \mathbb{C}^N$ such that $g(0) \neq 0$ and $\omega = g.df|_{X^*}$. In other words, $f|_{X^*}$ is a first integral of $\mathcal{F}$.

In the appendix, we will see that hypothesis (a) and (b) of the main theorem are fulfilled when X is a complete intersection and $dim_{\mathbb{C}}(sing(X)) \leq dim_{\mathbb{C}}(X) - 3$. This implies the following :

Corollary 1. *Let X be a germ of irreducible analytic set at $0 \in \mathbb{C}^N$, of dimension $3 \leq n \leq N$, and $\mathcal{F}$ be a germ of holomorphic codimension one foliation on X^*. Suppose that :*

(a) *X is a complete intersection.*
(b) $dim_{\mathbb{C}}(sing(X)) \leq dim(X) - 3$.
(c) *$\mathcal{F}$ is defined by a holomorphic (germ of) 1-form ω on X^* such that $cod_{X^*}(sing(\omega)) \geq 3$.*

Then $\mathcal{F}$ has a holomorphic first integral.

Another fact that will proved in the appendix is that when X is a complete intersection with an isolated singularity at $0 \in \mathbb{C}^N$ and $dim(X) \geq 4$ then $H^1(X^*, \mathcal{O}^*) = 1$. This implies hypothesis (c) of the main theorem and we get the following consequence :

Corollary 2. *Let X be a germ of irreducible analytic set at $0 \in \mathbb{C}^N$, of dimension $4 \leq n \leq N$, and $\mathcal{F}$ be a germ of holomorphic codimension one foliation on X^*. Suppose that X is a complete intersection with an isolated singularity at $0 \in \mathbb{C}^N$ and that $cod_{X^*}(sing(\mathcal{F})) \geq 3$. Then $\mathcal{F}$ has a holomorphic first integral.*

As an application, we obtain a generalization of a result due to F. Touzet (private communication) : if $n \geq 3$ and M^n is a smooth hypersurface of $\mathbb{P}^{n+1}$ then there is no non-singular holomorphic codimension one foliation on M.

Corollary 3. *Let M^n be a smooth algebraic submanifold of $\mathbb{P}^N$ with dimension $n \geq 3$ and $\mathcal{G}$ be a codimension one holomorphic foliation on M. If M is a complete intersection then $sing(\mathcal{G})$ has at least one component of codimension two in M.*

The proof can be done as follows : let $X \subset \mathbb{C}^{N+1}$ be the cone over M and $\pi\colon \mathbb{C}^{N+1} \to \mathbb{P}^N$ be the natural projection. Note that X is a complete intersection of dimension ≥ 4. Suppose by contradiction that M admits a foliation $\mathcal{F}$ such that $cod(sing(\mathcal{F})) \geq 3$. Consider the foliation $\mathcal{G} = \pi^*(\mathcal{F})$ on X^*. Its singular set has codimension ≥ 3 and $dim(X) \geq 4$, and so by corollary 2 it has a non-constant holomorphic first integral. In particular, it has a finite number of leaves accumulating at the origin. On the other hand, all leaves of $\mathcal{G}$ must accumulate at the origin, because $\mathcal{G} = \pi^*(\mathcal{F})$, a contradicition.

We observe that corollary 3 was already known for $M = \mathbb{P}^n$, $n \geq 3$ (cf. [LN]). It was used in [LN] to prove that codimension one foliations on $\mathbb{P}^n$, $n \geq 3$, have no non trivial minimal sets.

Example 1.1. *An example without holomorphic first integral.* Let X be the quadric in $\mathbb{C}^4$ given as

$$X = \{(x, y, z, t)\,;\, x\,y = z\,t\}\ .$$

In this case, $sing(X) = \{0\}$ and $X^* = X \setminus \{0\}$. Let $\Pi\colon \mathbb{C}^4 \setminus \{0\} \to \mathbb{P}^3$ be the natural projection. It is known that $\Pi(X^*) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, $\pi := \Pi|_X^*\colon X^* \to \mathbb{P}^1 \times \mathbb{P}^1$ is a submersion. Let $\mathcal{G}$ be the non-singular foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ whose leaves are the rules $\mathbb{P}^1 \times \{pt\}$. Then $\mathcal{F} := \pi^*(\mathcal{G})$ is a non-singular codimension one foliation on X^*. Note that any leaf of $\mathcal{F}$ is a 2-plane passing through 0. This implies that the germ of $\mathcal{F}$ at $0 \in X$ has no holomorphic first integral, because a foliation with a holomorphic first integral has only a finite number of leaves through $0 \in X$. We would like to remark that X satisfies hypothesis (a), (b) and (d) of the main theorem (see the apendix), but $\mathcal{F}$ do not satisfy (c). In fact, $\mathcal{F}$ has the merophorphic first integral $z/x = y/t$ on X^*. In this way, $\mathcal{F}$ can be defined in the set $U_1 = \{(x, y, z, t) \in X^*\,;\, z \neq 0\,\text{or}\, x \neq 0\}$ by the 1-form $\omega_1 = z\,dx - x\,dz$ and in the set $U_2 = \{(x, y, z, t)\,;\, y \neq 0\,\text{or}\, t \neq 0\}$ by the 1-form $\omega_1 = t\,dy - y\,dt$. In the intersection $U_1 \cap U_2$ we have $\omega_1 = g_{12}.\omega_2$, where $g_{12} = x^2/t^2 = z^2/y^2$.

Example 1.2. *An example in which the conclusion of the main theorem is true, but which do not satisfy hypothesis (b).* Let $\phi\colon \mathbb{C}^3 \to \mathbb{C}^9$ be defined by

$$\phi(x, y, z) = (x^2, y^2, z^2, x\,y, x\,z, y\,z, x^3, y^3, z^3)\ .$$

As the reader can check, $\phi|_{\mathbb{C}^3\setminus\{0\}}\colon \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}^9 \setminus \{0\}$ is an immersion. Therefore, $X := \phi(\mathbb{C}^3)$ has an isolated singularity at $0 \in \mathbb{C}^9$ and $X^* = X \setminus \{0\}$. Since X^* is biholomorphic to $\mathbb{C}^3 \setminus \{0\}$, we have $H^1(X^*, \mathcal{O}) = 0$ and $H^1(X^*, \mathcal{O}^*) = 1$. Hence, X satisfies hypothesis (a) and (d) of the main theorem. If $\mathcal{F}$ is a foliation on X^* then it is defined by a holomorphic 1-form on X^* and the conclusion of the main theorem is true : if $\mathcal{F}$ has no singularities on X^* then it has a holomorphic first integral, by Malgrange's theorem. However, X do not satisfy hypothesis (b) of the main theorem : the function $f \in \mathcal{O}(X^*)$ defined by $f = x \circ \phi^{-1}\colon X^* \to \mathbb{C}$ has no holomorphic extension to a neighborhood of $0 \in \mathbb{C}^9$.

Example 1.3. *An example of singular variety which is not a complete intersection and which admits foliations without meromorphic first integral.* Let $T \subset \mathbb{P}^n$ be a complex tori of dimension ≥ 2 and $\mathcal{G}$ be a codimension one foliation on T without singularities and with dense leaves. Let $\pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the natural projection. Set $X^* = \pi^{-1}(T)$ and $\mathcal{F} = \pi^*(\mathcal{G})$. In this case, $X = X^* \cup \{0\}$ has an isolated singularity at $0 \in \mathbb{C}^{n+1}$. Each leaf of $\mathcal{F}$ is dense in X^*, and so it has no meromorphic first integral.

Let us state some problems which arise naturally from the above results and examples. The first one concerns the quadric of example 1.1.

Problem 1. Let X be the quadric $(x\,y - z\,t = 0) \subset \mathbb{C}^4$ and $\mathcal{G}$ be a germ at $0 \in \mathbb{C}^4$ of non-singular codimension one foliation on X^*. Suppose that $\mathcal{G}$ is not defined by a holomorphic 1-form as in (c) of the main theorem. Does there exists a germ of automorphism $\varphi\colon (X,0) \to (X,0)$ such that $\mathcal{G} = \varphi^*(\mathcal{F})$, where $\mathcal{F}$ is the foliation of example 1.1 ?

Example 1.2 motivates the following :

Problem 2. Can we substitute hypothesis (b) of the main theorem by another more general, in such a way that the result applies in the case of example 1.2 ?

As mentioned before the fact that the singular set of a codimension one foliation $\mathcal{F}$ on $\mathbb{P}^n$, $n \geq 3$, has at least one codimension two irreducible component was used in [LN] to prove that $\mathcal{F}$ has no non-trivial minimal set. Corollary 3 motivates the following :

Problem 3. Let $M \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n \geq 3$. Is it possible that M admits a codimension one foliation $\mathcal{F}$ with a non-trivial minimal set ?

This work will be organized as follows. In §2 we will state some basic results that will be used in the proof of the main theorem, specially the construction of the Godbillon-Vey sequence associated to an integrable 1-form ω such that $cod(sing(\omega)) \geq 3$. The main theorem will be proved in §3.

We would like to mention that the problem of extending Malgrange's theorem for singular germs was posed to us by R. Moussu. He told us that the problem was posed to him by H. Hauser. We would like to acknowledge them and also A. Dimca for some helpfull suggestions.

2. Basic results

2.1. **Godbillon-Vey sequences.** One of the tools that will be used in the proof of the main theorem is the so called "Godbillon-Vey sequence" associated to a foliation (cf. [Go]). Let M be a holomorphic manifold of dimension $n \geq 2$ and ω be a holomorphic integrable 1-form on M.

Definition 2.1.1. *A holomorphic Godbillon-Vey sequence (briefly h.g.v.s.) for ω, is a sequence $(\omega_k)_{k\geq 0}$ of holomorphic 1-forms on M such that $\omega_0 = \omega$ and the formal 1-form Ω on $(\mathbb{C},0)\times M$ defined by the power series*

$$\Omega := dt + \sum_{j=0}^{\infty} \frac{t^j}{j!}\,\omega_j$$

is formally integrable, that is

$$\Omega \wedge d\Omega = 0$$

It is not dificult to prove that the above relation is equivalent to

$$d\omega_k = \omega_0 \wedge \omega_{k+1} + \sum_{j=1}^{k} \binom{k}{j} \omega_j \wedge \omega_{k+1-j}\ ,\ \forall\, k \geq 0\ . \tag{1}$$

By using (1), it can be proved by induction on $k \geq 0$, that a sufficient condition for the existence of a h.g.v.s. for ω is that it satisfies the *2-division property*, which is defined below :

ℓ-division property (briefly ℓ-d.p.). *We say that ω satisfies the ℓ-d.p., if for any $\Theta \in \Omega^\ell(M)$ such that $\omega \wedge \Theta = 0$ then there exists a $\eta \in \Omega^{\ell-1}(M)$ such that $\Theta = \omega \wedge \eta$ (cf. [M] and [Mo]).*

For instance, if ω satisfies the 2-d.p., the first three steps of the h.g.v.s. can be obtained as follows

$$\omega_0 \wedge d\omega_0 = 0 \implies d\omega_0 = \omega_0 \wedge \omega_1 \implies d\omega_0 \wedge \omega_1 - \omega_0 \wedge d\omega_1 = 0 \implies \omega_0 \wedge d\omega_1 = 0 \implies$$

$$\implies d\omega_1 = \omega_0 \wedge \omega_2 \implies d\omega_0 \wedge \omega_2 - \omega_0 \wedge d\omega_2 = 0 \implies \omega_0 \wedge (d\omega_2 - \omega_1 \wedge \omega_2) \implies$$

$$\implies d\omega_2 = \omega_0 \wedge \omega_3 + \omega_1 \wedge \omega_2 = \omega_0 \wedge \omega_3 + \binom{2}{1} \omega_1 \wedge \omega_2 + \binom{2}{2} \omega_2 \wedge \omega_1 \implies \ldots$$

Remark 2.1.1. *If $cod_M(sing(\omega)) \geq 2$ then ω satisifies the 1-d.p., that is, if $\Theta \in \Omega^1(M)$ is such that $\omega \wedge \Theta = 0$ then there exists $g \in \mathcal{O}(M)$ such that $\Theta = g.\omega$.*

In the next result we give a sufficient condition for ω to satisfy the 2-d.p..

Lemma 2.1.1. *Let M be a complex manifold of dimension $n \geq 3$ and ω be a holomorphic 1-form on M. Assume that $cod_M(sing(\omega)) \geq 3$ and $H^1(M,\mathcal{O}) = 0$. Then ω satisfies the 2-division property.*

Proof. Let $\Theta \in \Omega^2(M)$ be such that $\Theta \wedge \omega = 0$. Since $cod_M(sing(\omega)) \geq 3$, the 2-d.p. is true locally on M (cf. [M]). It follows that there exists a Leray covering $\mathcal{U} = (U_j)_{j\in J}$ of M and a collection $(\eta_j)_{j\in J}$, $\eta_j \in \Omega^1(U_j)$, such that $\Theta|_{U_j} = \eta_j \wedge \omega|_{U_j}$, for all $j \in J$. If $U_{ij} := U_i \cap U_j \neq \emptyset$, then

$$(\eta_j - \eta_i) \wedge \omega|_{U_{ij}} = 0 \implies \eta_j - \eta_i = g_{ij}\,.\,\omega|_{U_{ij}}\ ,$$

where $g_{ij} \in \mathcal{O}(U_{ij})$. Note that the collection $(g_{ij})_{U_{ij}\neq\emptyset}$ can be considered as an aditive cocycle in $C^1(\mathcal{U},\mathcal{O})$. Since $H^1(M,\mathcal{O}) = 0$, there exists $(f_j)_{j\in J} \in C^0(\mathcal{U},\mathcal{O})$ such that $g_{ij} = f_j - f_i$ on $U_{ij} \neq \emptyset$. Hence there exists $\eta \in \Omega^1(M)$ such that $\eta|_{U_j} := \eta_j - f_j.\omega|_{U_j}$. This form satisfies $\Theta = \eta \wedge \omega$. $\square$

Now, let X be a germ of irreducible analytic set at $0 \in \mathbb{C}^N$, of dimension n, $3 \leq n \leq N$, such that $H^1(X^*, \mathcal{O}) = 0$. Let ω be a germ of holomorphic integrable 1-form on X^* with $cod_{X^*}(sing(\omega)) \geq 3$. In this case, if we take a ball $B \subset \mathbb{C}^N$ with small radius then we can assume that :
(I). X, $sing(X)$ and X^* have representatives on B, say X_B, $sing(X_B)$ and $X_B^* = X_B \setminus sing(X_B)$, respectively, where X_B^* is a connected complex manifold with dimension $n \geq 3$.
(II). ω has a representative $\omega_B \in \Omega^1(X_B^*)$ such that $cod_{X_B^*}(sing(\omega_B)) \geq 3$.
(III). $H^1(X_B^*, \mathcal{O}) = 0$.

Since B will be fixed from now on, for simplicity we will use the old notations : $X_B = X$, $X_B^* = X^*$, $sing(X_B) = sing(X)$, $\omega_B = \omega$.

As a consequence of lemma 2.1.1, we have the following :

Corollary 2.1.1. *In the above situation there exists a h.g.v.s. for ω, say $(\omega_k)_{k\geq 0}$, where $\omega_0 = \omega$.*

2.2. **Resolution of X and h.g.v.s.** Let $B \subset \mathbb{C}^N$, X, $sing(X)$, X^* and the h.g.v.s. $(\omega_j)_{j\geq 0}$ of $\omega_0 = \omega$ be as in section 2.1. In this section we will suppose that X is 0 and 1-regular. In particular, we can take the ball B in such a way that, for any $j \geq 0$ there exists a holomorphic 1-form η_j on B such that $\eta_j|_{X^*} = \omega_j$.

Consider a resolution of (B, X) by blowing-ups $\Pi\colon \tilde{B} \to B$ (cf. [ ]). The complex manifold $\tilde{B}$ and the holomorphic map Π are obtained in such a way that :
(A). The strict transform $\tilde{X}$ of X by Π is a connected smooth complex submanifold of $\tilde{B}$ of complex dimension $n = dim(X)$. Set $\pi := \Pi|_{\tilde{X}}\colon \tilde{X} \to X$.
(B). $E := \Pi^{-1}(sing(X)) \cap \tilde{X}$ is a connected codimension one analytic subset of $\tilde{X}$. Moreover, E is a normal crossing sub-variety of $\tilde{X}$, which means that for any $p \in E$ there exists a neighborhood V of p in $\tilde{X}$ such that $V \cap E$ is bimeromorphic to an union of at most n pieces of $(n-1)$-planes in general position.
(C). The maps $\Pi|_{\tilde{B}\setminus E}\colon \tilde{B} \setminus E \to B \setminus sing(X)$ and $\pi|_{\tilde{X}\setminus E}\colon \tilde{X} \setminus E \to X^*$ are bimeromorphisms.

Let $\tilde{X}^* := \Pi^{-1}(X^*) = \tilde{X} \setminus E$. If we set $\tilde{\eta}_j := \Pi^*(\eta_j)$, $j \geq 0$, then $\tilde{\eta}_j \in \Omega^1(\tilde{B})$ and $\tilde{\eta}_j|_{\tilde{X}^*} = \pi^*(\omega_j)$, so that $\pi^*(\omega_j)$ can be extended to a holomorphic 1-form $\tilde{\omega}_j := \tilde{\eta}_j|_{\tilde{X}}$ on $\tilde{X}$, for all $j \geq 0$. Set $\tilde{\omega} = \tilde{\omega}_0$.

We can assume that the blowing-up process begins by a blowing-up at $0 \in \mathbb{C}^N$. In this case, $\Pi^{-1}(0)$ has codimension one in $\tilde{B}$. This implies that :
(D). The analytic set $D := \Pi^{-1}(0) \cap \tilde{X} \subset E$ has codimension one in $\tilde{X}$, is a normal crossing codimension one sub-variety of $\tilde{X}$ and is connected (because X is irreducible).

Remark 2.2.1. *The sequence $(\tilde{\omega}_j)_{j\geq 0}$ is a h.g.v.s. for $\tilde{\omega} = \tilde{\omega}_0$.*

Lemma 2.2.1. *For any $k \geq 0$ we have $\tilde{\omega}_k|_D = 0$.*

Proof. Let p be a smooth point of D. Since $dim(D) = n - 1 = dim(\tilde{X}) - 1$, we can find a local coordinate system $[U, (u, z, v) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}^{N-n}]$ such that $U \cap \tilde{X} = (v = 0)$ and $U \cap D = (z = 0) \cap (v = 0)$. In this coordinate system we can write $\Pi|_U = (X_1, ..., X_N)$, where $X_j\colon U \to \mathbb{C}$ and $X_j(u, 0, 0) = 0$. This implies that

$$X_j(u, z, v) = z.A_j(u, z, v) + \sum_{i=1}^{N-n} v_i.B_{ij}(u, z, v) \ .$$

It follows that

$$\Pi^*(dx_j) = A_j.dz + z.dA_j + \sum_{i=1}^{N-n} B_{ij} dv_i + v_i.dB_{ij} \implies \Pi^*(dx_j)|_{D\cap U} = 0 \ .$$

Hence, $\tilde{\omega}_k|_{D\cap U} = \Pi^*(\eta_k)|_{D\cap U} = 0$. $\square$

Remark 2.2.2. *Note that the foliation $\pi^*(\mathcal{F}_\omega)$, which in principle is defined only on $\tilde{X}^*$, can be extended to the foliation $\mathcal{F}_{\tilde{\omega}}$ on $\tilde{X}$.*

On the other hand, in general the singular set of $\tilde{\omega}$ has components of codimension one. After dividing $\tilde{\omega}$ by the local equations of the codimension one irreducible components of $sing(\tilde{\omega})$, we obtain a foliation $\tilde{\mathcal{F}}$, which will be called the *strict transform* of $\pi^*(\mathcal{F}_\omega)$.

Lemma 2.2.2. *Any irreducible component of D is invariant for the strict transform $\tilde{\mathcal{F}}$.*

Proof. Let p be a smooth point of D and $[U, (u, z, v) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}^{N-n}]$ be a coordinate system around p as in the proof of lemma 2.2.1. We can write $\tilde{\omega}|_{\tilde{X}\cap U} = z^\ell.\omega_U$, where $\omega_U \in \Omega^1(\tilde{X} \cap U)$ is integrable, $\ell \geq 0$ and $cod(sing(\omega_U)) \geq 2$. The foliation $\tilde{\mathcal{F}}$ is defined on $\tilde{X} \cap U$ by the form ω_U. If $\ell = 0$ then the result follows from lemma 2.2.1. If $\ell \geq 1$, then it follows from $d\tilde{\omega} = \tilde{\omega} \wedge \tilde{\omega}_1$ that

$$\ell\, z^{\ell-1}\, dz \wedge \omega_U + z^\ell\, d\omega_U = z^\ell\, \omega_U \wedge \tilde{\omega}_1 \implies dz \wedge \omega_U = z.\theta \ ,$$

where $\theta = \ell^{-1}(\omega_U \wedge \tilde{\omega}_1 - d\omega_U) \in \Omega^2(U)$. This implies that $(z = 0) = D \cap U$ is invariant for $\tilde{\mathcal{F}}$. $\square$

3. Proof of the main theorem

3.1. **Formal first integrals in the resolution.** Let $\Pi \colon (\tilde{B}, \tilde{X}) \to (B, X)$ be a resolution of X, satisfying properties **(A)**, **(B)**, **(C)** and **(D)** of the last section. Consider also the h.g.v.s. $(\tilde{\omega}_k)_{k\geq 0}$ of $\tilde{\omega}$ and the formal integrable 1-form

$$\tilde{\Omega} = dt + \tilde{\omega} + \sum_{j=1}^{\infty} \frac{t^j}{j!}\, \tilde{\omega}_j \tag{2}$$

Recall that $\tilde{\omega}_j|_D = 0$, where $D = \tilde{X} \cap \Pi^{-1}(0)$. By doing more blowing-ups along the normal crossings of $E = \Pi^{-1}(sing(X)) \cap \tilde{X}$, we can assume that
(E). All irreducible components of E are smooth. In particular, all irreducible components of D are smooth.

The aim of this section is to prove that $\tilde{\mathcal{F}}$ has a "formal" first integral. This formal first integral will be a global section of the formal (or m-adic) completion of $\tilde{X}$ along D (cf. [B-S] and [Mi]).

Definition 3.1.1. *Let M be a complex manifold and $Y \subset M$ be an analytic subset of M. Let $\mathcal{I} \subset \mathcal{O}_M$ be the sheaf of ideals defining Y. The formal completion of M in Y, denoted by $\mathcal{O}_{\hat{Y}}$ (see [B-S]), is the sheaf of ideals defined by*

$$\mathcal{O}_{\hat{Y}} = (\underset{\underset{n}{\leftarrow}}{\ell im}\, \mathcal{O}_M/\mathcal{I}^n)|_Y$$

Similarly, when $\mathcal{M}$ is a sheaf of $\mathcal{O}_M$-modules, we define

$$\mathcal{M}_{\hat{Y}} = (\lim_{\stackrel{\leftarrow}{n}} \mathcal{M}/\mathcal{I}^n.\mathcal{M})|_Y$$

Note that $\Omega^k_{\hat{Y}}$ is a sheaf of modules over $\hat{O}_Y$. A global section of $\hat{O}_Y$ (resp. $\Omega^k_{\hat{Y}}$) will be called a formal function (resp. k-form) along Y.

Remark 3.1.1. *Since $\mathcal{O}_Y \simeq (\mathcal{O}_M/\mathcal{I})|_Y$, we have a natural projection $\mathcal{O}_{\hat{Y}} \xrightarrow{r} \mathcal{O}_Y$, called the restriction to Y. Given a formal function $\hat{f}$ along Y we will use the notation $r(\hat{f}) := \hat{f}|_Y$. Note that, if Y is compact then $\hat{f}|_Y$ is a constant.*

Notation. Let A be an integral domain and $k \geq 1$. We use the notation $A[[z]] := A[[z_1, ..., z_k]]$ for the set of complex formal power series F in k variables with coefficients in A of the form :

$$F = \sum_\sigma a_\sigma \, z^\sigma \ , \ a_\sigma \in A \ ,$$

where $z = (z_1, ..., z_k)$, $\sigma = (\sigma_1, ..., \sigma_k)$, $\sigma_j \geq 0$, $1 \leq j \leq k$, and $z^\sigma = z_1^{\sigma_1} ... z_k^{\sigma_k}$. Remark that $A[[z]]$ is also an integral domain, with the operations of sum and multiplication of formal power series.

Suppose that Y is a codimension k smooth submanifold of M and $dim_{\mathbb{C}}(M) = n \geq k + 1$. Let $[W, (u, z) \in \mathbb{C}^{n-k} \times \mathbb{C}^k]$ be a holomorphic coordinate system such that $U := Y \cap W = (z = 0)$ is non-empty and connected. We have the following interpretation for a formal function along $U \subset Y$, say $\hat{f}$: $\hat{f}|_U$ can be thought as a formal power series in $\mathcal{O}(U)[[z]]$ of the form $S = \hat{f}(u, z) = \sum_\sigma f_{U,\sigma}(u) \, z^\sigma$, where $f_{U,\sigma} \in \mathcal{O}(U)$ for all σ.

Notation. Given a coordinate system $[W, (u, z)]$, $U = Y \cap W = (z = 0)$ and the series S, as above, we will call S a *representative of* $\hat{f}$ over U.

Note that, $\hat{f}|_U = f(u, 0) = f_{U,\overline{0}}(u) \in \mathcal{O}(U)$, where $\overline{0} = (0, ..., 0) \in \mathbb{Z}^k$. If $\sum_\sigma f_{U,\sigma}(u) \, z^\sigma \in \mathcal{O}(U)\{z\}$, that is the series converges, then it represents a holomorphic function in a neighborhood of U in M. In this case, we will say that $\hat{f}$ converges over U.

Similarly, if $\hat{\eta}$ is a formal 1-form along Y, then

$$\hat{\eta}|_U = \sum_{j=1}^{n-k} \hat{g}_j . du_j + \sum_{i=1}^{k} \hat{h}_i . dz_i \ , \ \hat{g}_j, \hat{h}_i \in \mathcal{O}(U)[[z]] \ , \ \forall \ i, j \ .$$

Observe that $\tilde{\Omega}$ can be thought as a formal 1-form (on $\tilde{X} \times \mathbb{C}$) along $\tilde{X} \times \{0\}$.

The aim of this section is to prove the following :

Theorem 3.1. *There exist formal functions $\hat{f}$ and $\hat{g}$ along $D \subset \tilde{X}$ such that $\tilde{\omega} = \hat{g}.d\hat{f}$, $\hat{g}|_D = 1$ and $\hat{f}|_D = 0$. In particular, $\hat{f}$ is a formal first integral of $\tilde{\mathcal{F}}$.*

Proof. Let $D = \cup_{j=1}^r D_j$ be the decomposition of D into smooth irreducible components. Fix an irreducible component D_ℓ and a coordinate system of $\tilde{X}$, say $[W, (u, z) \in \mathbb{C}^{n-1} \times \mathbb{C}]$ such that $U := D_\ell \cap W = (z = 0)$, is connected and non-empty.

Lemma 3.1.1. *Let $h(t) = \sum_{j \geq 1} a_j \, t^j \in \mathbb{C}[[t]] \setminus \{0\}$. Then there exists a unique formal power series $F^h \in \mathcal{O}(U)[[z, t]]$,*

$$F^h(u, z, t) = \sum_{i,j \geq 0} f_{ij}(u) \, z^i . t^j$$

such that $F^h_t.\tilde{\Omega} = dF^h$ and $F^h(u,0,t) = h(t)$. In particular, F^h is a formal first integral of $\tilde{\Omega}$.

Proof. Recall that $\tilde{\omega}_k|_D = 0$ for all $j \geq 0$. This implies that $\tilde{\omega}_k$ can be written in the coordinate system $[W,(u,z)]$ as

$$\tilde{\omega}_k = A_k(u,z)\,dz + \sum_{i=1}^{n-1} z.B_{ki}(u,z)\,du_i \ ,$$

where $A_k, B_{ki} \in \mathcal{O}(W)$. In a neighborhood $W_1 = U \times (|z| < \epsilon)$ of U in W, we can represent the $A_{k's}$ and $B_{ki's}$ by power series in $\mathcal{O}(U)\{z\}$. By doing that and adding the forms $\frac{t^k}{k!}\tilde{\omega}_k$ to obtain $\tilde{\Omega}$, it is not difficult to see that we can write :

$$\tilde{\Omega} = dt + G(u,z,t)\,dz + \sum_{i=1}^{n-1} z.H_i(u,z,t)\,du_i \ ,$$

where $G, H_i \in \mathcal{O}(U)[[z,t]]$. Note that $F_t.\tilde{\Omega} = dF$ is equivalent to

$$F_z = G.F_t \text{ and } F_{u_i} = z.H_i.F_t \ , \ i = 1,..,n-1 \ . \tag{3}$$

Uniqueness. Suppose that $F(u,z,t) = \sum_{i,j\geq 0} f_{ij}(u)\,z^i.t^j$ is a solution of the problem. If we set $f_i(u,t) = \sum_j f_{ij}(u)\,t^j \in \mathcal{O}(U)[[t]]$, then we can write F as an element of $(\mathcal{O}(U)[[t]])[[z]]$: $F(u,z,t) = \sum_i f_i(u,t)\,z^i$. Note that $f_0(u,t) = F(u,0,t) = h(t)$. Similarly, we can write $G(u,z,t) = \sum_i g_i(u,t)\,z^i$. Therefore, the first relation in (3), $F_z = G.F_t$, gives

$$(k+1).f_{k+1}(u,t) = \sum_{i+j=k} g_j(u,t).\frac{\partial f_i}{\partial t}(u,t) \ , \ k \geq 0 \ , \tag{4}$$

where $\frac{\partial f_i}{\partial t}(u,t) = \sum_j (j+1)f_{ij+1}(u)\,t^j \in \mathcal{O}(U)[[t]]$. This, of course, implies that F is unique. Note that (4) implies that, if $K \in \mathcal{O}(U)[[z,t]]$ satisfies $K_z = G.K_t$ and $K(u,0,t) = 0$ then $K \equiv 0$.

Existence. Relation (4) allows us to find, by induction on $k \geq 0$, the coefficients $f_i \in \mathcal{O}(U)[[t]]$ of $F \in (\mathcal{O}(U)[[t]])[[z]] = \mathcal{O}(U)[[z,t]]$ in such a way that $F_z = G.F_t$. It remains to prove that F satisfies the others relations in (3). Let $\Theta := F_t.\tilde{\Omega}$. Remark that Θ is integrable. On the other hand, we can write

$$\Theta = F_t.dt + F_t.G.dz + \sum_i z.F_t.H_i\,du_i = dF + \sum_i K_i\,du_i \ ,$$

where $K_i = z.H_i.F_t - F_{u_i}$. We want to prove that $K_i \equiv 0$ for all $i = 1,...,n-1$. The reader can check that the coefficient of $dz \wedge dt \wedge du_i$ in $\Theta \wedge d\Theta$ is $F_z.K_{it} - F_t.K_{iz}$. Since $\Theta \wedge d\Theta = 0$, we get

$$0 = F_z.K_{it} - F_t.K_{iz} = F_t\,(G.K_{it} - K_{iz}) = 0 \implies K_{iz} = G.K_{it} \ ,$$

because $F_t \not\equiv 0$. On the other hand, we have

$$K_i(u,0,t) = -F_{u_i}(u,0,t) = -\frac{\partial}{\partial u_i}(F(u,0,t)) = -\frac{\partial h(t)}{\partial u_i} = 0 \ .$$

This implies that $K_i \equiv 0$. $\square$

Notations. Set $Y_\ell = D_\ell \times \{0\} \subset \tilde{X} \times \mathbb{C}$, $1 \le \ell \le r$. We will denote by F_U the solution given by lemma 3.1.1, for which $F_U(u,0,t) = t$. Note that $F_U \in \Gamma(U, \mathcal{O}_{\hat{Y}_\ell})$.

Let $\{[W_j,(u_j,z_j)]\}_{j\in J}$ be a collection of coordinate systems on $\tilde{X}$ such that for all $U_j := W_j \cap D_\ell = (z_j = 0) \neq \emptyset$ is connected and $\cup_{j\in J} U_j = D_\ell$. We will set $U_{ij} = U_i \cap U_j$.

Corollary 3.1.1. *If $U_{ij} \neq \emptyset$ then the sections F_{U_i} and F_{U_j} coincide over U_{ij}. In particular, there exist formal functions $\hat{F}_\ell$ and $\hat{G}_\ell$ along Y_ℓ such that $\tilde{\Omega} = \hat{G}_\ell.d\hat{F}_\ell$, $\hat{G}_\ell|_{Y_\ell} = 1$ and $\hat{F}_\ell|_{Y_\ell} = 0$, $1 \le \ell \le r$.*

Proof. The fact that F_{U_i} and F_{U_j} coincide over U_{ij} follows from the uniqueness in lemma 3.1.1. We leave the details of its proof for the reader. It implies that there exists $\hat{F}_\ell \in \mathcal{O}_{\hat{Y}_\ell}$ such that $\hat{F}_\ell|_{U_j} = F_{U_j}$ for all $j \in J$. Recall that the formal power series F_{U_j} satisfies

$$\frac{\partial F_{U_j}}{\partial t}.\tilde{\Omega} = dF_{U_j} \ .$$

Since $F_{U_j}(u,0,t) = t$ we get $\frac{\partial F_{U_j}}{\partial t}(u,0,t) = 1$, and so $\frac{\partial F_{U_j}}{\partial t}(u,0,0) = 1$. It follows that $\frac{\partial F_{U_j}}{\partial t}$ is an unit of the ring $\mathcal{O}(U_j)[[z,t]]$. Therefore we can define $G_{U_j} := (\frac{\partial F_{U_j}}{\partial t})^{-1} \in \mathcal{O}(U_j)[[z,t]]$, so that $\tilde{\Omega} = G_{U_j}.dF_{U_j}$ for all $j \in J$. Of course, the first part of the lemma implies that the sections G_{U_i} and G_{U_j} coincide over $U_{ij} \neq \emptyset$. Hence, there exists $\hat{G}_\ell \in \Gamma(Y_\ell, O_{\hat{Y}_\ell})$ such that $\tilde{\Omega} = \hat{G}_\ell.d\hat{F}_\ell$. □

Recall that $\tilde{\omega} = \tilde{\Omega}|_{(t=0)}$. If we set $\hat{f}_\ell := \hat{F}_\ell|_{(t=0)}$ and $\hat{g}_\ell := \hat{G}_\ell|_{(t=0)}$, then Corollary 3.1.1 implies the following :

Remark 3.1.2. *For all $\ell \in \{1,...,r\}$ there exist $\hat{f}_\ell, \hat{g}_\ell \in \mathcal{O}_{\hat{D}_\ell}$ such that $\tilde{\omega} = \hat{g}_\ell.d\hat{f}_\ell$, $\hat{f}_\ell|_{D_\ell} \equiv 0$ and $\hat{g}_\ell|_{D_\ell} \equiv 1$. In particular, $\hat{f}_\ell$ is a formal first integral of $\tilde{\mathcal{F}}$ along D_ℓ.*

Now we consider a point $p \in sing(D)$ which is a normal crossing of two irreducible components of D, say D_m and D_n, $m \neq n$. In this case, we can find a local coordinate system around p, $[W,(u,z_m,z_n) \in \mathbb{C}^{n-2} \times \mathbb{C}^2]$, such that $u(p) = 0 \in \mathbb{C}^{n-2}$, $z_m(p) = z_n(p) = 0 \in \mathbb{C}$, $U_{mn} := (z_m = z_n = 0)$ and $U_j := D_j \cap W = (z_j = 0)$ are connected, for $j = m,n$.

With the above conventions, we can consider, in a natural way, $\mathcal{O}(U_m)[[z_n,t]]$ and $\mathcal{O}(U_n)[[z_m,t]]$ as sub-rings of $\mathcal{O}(U_{mn})[[z_m,z_n,t]]$. Let $F_m(u,z_m,z_n,t) := F_{U_m}(u,z_m,z_n,t) \in \mathcal{O}(U_m)[[z_n,t]]$ and $F_n(u,z_m,z_n,t) := F_{U_n}(u,z_m,z_n,t) \in \mathcal{O}(U_n)[[z_m,t]]$ be as in corollary 3.1.1. As the reader can check, the uniqueness in lemma 3.1.1 implies the following :

Remark 3.1.3. *The formal power series F_m and F_n coincide, when we consider them as elements of $\mathcal{O}(U_{mn})[[z_m,z_n,t]]$. In particular, there exists a formal function along $Y_m \cup Y_n$, say $\hat{F}_{mn}$, such that $\hat{F}_{mn}$ coincides with $\hat{F}_m$ over Y_m and with $\hat{F}_n$ over Y_n.*

Let us finish the proof of theorem 3.1. Remark 3.1.3 implies that there exist a formal function along $D \times \{0\} \subset \tilde{X} \times \mathbb{C}$, say $\hat{F}$, such that $\hat{F}$ coincides with $\hat{F}_\ell$ over Y_ℓ, for all $\ell \in \{1,...,r\}$. On the other hand, we have seen in corollary 3.1.1 that $\tilde{\Omega} = \hat{G}_\ell.d\hat{F}_\ell$ over Y_ℓ, where $\hat{G}_\ell = (\partial\hat{F}_\ell/\partial t)^{-1}$, for all ℓ. This implies that $\tilde{\Omega} = \hat{G}.d\hat{F}$, where $\hat{G} = (\partial\hat{F}/\partial t)^{-1}$. Note that, by construction, we have

$\hat{G}|_{D\times\{0\}} = 1$ and $\hat{F}|_{D\times\{0\}} = 0$. If we set $\hat{f} := \hat{F}|_{(t=0)}$ and $\hat{g} := \hat{G}|_{(t=0)}$, then we get $\tilde{\omega} = \hat{g}.d\hat{f}$, as in remark 3.1.2. This finishes the proof of theorem 3.1. □

3.2. **Convergence of formal first integrals.** Let $\hat{f}$ and $\hat{g}$ be as in theorem 3.1, so that $\tilde{\omega} = \hat{g}\,d\hat{f}$, $\hat{f}|_D = 0$ and $\hat{g}|_D = 1$. The aim of this section is to give conditions for the convergence of $\hat{f}$ and $\hat{g}$.

Let $\hat{h}$ be a formal function along $D \subset \tilde{X}$. Given $p \in D_\ell$, $1 \leq \ell \leq r$, consider a representative $\hat{h}(u,z) = \sum_{j\geq 0} h_j(u)\, z^j \in \mathcal{O}(U)[[z]]$ of $\hat{h}$ over U, where $p \in U \subset D_\ell$. We say that $\hat{h}$ converges over U, if for every $u \in U$ the series $\sum_{j\geq 0} h_j(u)\, z^j \in \mathbb{C}[[z]]$ converges. In this case, the power series defines a holomorphic function on a neighborhood of U in $\tilde{X}$. Conversely, a holomorphic function in a neighborhood of U in $\tilde{X}$ can be expanded as a power series in $\mathcal{O}(U)[[z]]$ and defines a section of $\mathcal{O}_{\hat{D}}$ over U. This implies that the definition is independent of the coordinate system used to express the power series.

We say that $\hat{h}$ converges, if for any $p \in D$ and any irreducible component D_ℓ of D such that $p \in D_\ell$, there exist a neighborhood U of p in D_ℓ and a representative of $\hat{h}$ over U that converges. After this discussion, we have the following :

Remark 3.2.1. *If $\hat{h}$ converges then there exists a holomorphic function h on a neighborhood of D in $\tilde{X}$ such that the section defined by h on $\Gamma(D, \mathcal{O}_{\hat{D}})$ coincides with $\hat{h}$.*

Given $p \in D$, we will denote by $\hat{O}_p$ (resp. $\mathcal{O}_p$) the ring of formal functions along $\{p\} \subset \tilde{X}$ (resp. germs at p of holomorphic functions on $\tilde{X}$). Recall that $\hat{O}_p$ and $\mathcal{O}_p$ are Noetherian rings. Note that, given a formal function $\hat{h}$ along D, $p \in D$ and a formal power series that represents $\hat{h}$ over some neighborhood of p in D, say $\hat{h}(u,z) = \sum_j h_j(u)\, z^j$, then it can expanded as a formal power series in $(u - u(p), z)$, so defining an element $\hat{h}_p \in \hat{O}_p$. We will call $\hat{h}_p$ the germ of $\hat{h}$ at p.

Lemma 3.2.1. *Let $\hat{f}$ be the formal first integral of $\tilde{\mathcal{F}}$ given by theorem 3.1. Suppose that there is $p \in D$ such that the germ $\hat{f}_p \in \hat{O}_p$ converges. Then $\hat{f}$ converges.*

Proof. Let $A = \{q \in D \,|\,$ the germ $\hat{f}_q$ converges $\} \neq \emptyset$. We will prove that A is open and closed in D. Since D is connected, this will imply that $A = D$ and the lemma.

I. A is open in D. Let $q \in A$. Suppose that $q \in D_\ell$, $1 \leq \ell \leq r$. Since $\hat{f}_q$ converges, we can find a coordinate system $[W,(u,z)]$ such that $u(q) = 0 \in \mathbb{C}^{n-1}$, $z(q) = 0 \in \mathbb{C}$, $q \in W \cap D_\ell = (z = 0)$ is connected and $\hat{f}_q$ can be represented by a convergent series $\hat{f}(u,z) = \sum_{\sigma,j}^{\infty} a_{\sigma,j}\, u^\sigma.z^j$. Suppose that the series converges in the set $V := \{(u,z)\,|\, max(||u||,|z|) < \rho\} \subset W$. In this case, for all $j \geq 1$, the series $f_j(u) = \sum_\sigma a_{\sigma,j}\, u^\sigma$ converges in the set $U := \{(u,0)\,|\,||u|| < \rho\} \subset D_\ell$. Hence, the series $\hat{f}(u,z) = \sum_j f_j(u)\, z^j \in \mathcal{O}(U)\{z\}$, so that $\hat{f}$ converges over U and $\hat{f}_x$ converges for every $x \in U$. This implies that A is open in D_ℓ. Since the argument is true for every ℓ such that $q \in D_\ell$, it follows that A is open in D.

II. If $A \cap D_\ell \neq \emptyset$ then $A \supset D_\ell$. Since $cod_{\tilde{X}}(sing(\tilde{\mathcal{F}})) \geq 2$, we get $cod_{D_\ell}(sing(\tilde{\mathcal{F}})) \geq 1$. It follows that the set $B_\ell = D_\ell \setminus sing(\tilde{\mathcal{F}})$ is open, connected and dense in D_ℓ.

Claim 3.2.1. *If B_ℓ is as above then $A \supset B_\ell$.*

Proof. First of all, $A \cap B_\ell$ is a non-empty open subset of D_ℓ, because B_ℓ is open and dense in D_ℓ. Fix $q \in B_\ell$. Since $q \notin sing(\tilde{\mathcal{F}})$ and D_ℓ is invariant for $\tilde{\mathcal{F}}$ (lemma 2.2.2), we can find a coordinate system $[W,(u,z)]$ such that $q \in U := W \cap D_\ell = (z=0)$ is connected and $\tilde{\mathcal{F}}|_W$ is defined by $dz = 0$. It follows that $d\hat{f} \wedge dz = 0$, and so $\hat{f}$ can be represented over U by a power series of the form $\sum_{j=0}^{\infty} a_j.z^j \in \mathbb{C}[[z]]$. This implies that : $A \cap U \neq \emptyset \iff A \supset U$. Hence, $A \cap B_\ell$ is closed in B_ℓ and $A \supset B_\ell$. $\square$

Now, fix $q \in sing(\tilde{\mathcal{F}}) \cap D_\ell$ and let us prove that $q \in A$. At this point, we will use the following result (cf. [M-M]) :

Theorem 3.2.1. *Let η be a germ of holomorphic integrable 1-form at $0 \in \mathbb{C}^n$, with $\eta(0) = 0$ and $cod_{\mathbb{C}^n}(sing(\eta)) \geq 2$. If η has a non-constant formal first integral then η has a non-constant holomorphic first integral. Moreover, the holomorphic first integral, say $g \in \mathcal{O}_n$, can be choosen in such a way that $g(0) = 0$ and it is not a power in $\mathcal{O}_n$, that is $g \neq g_1^\ell$, $\ell \geq 2$. In this case, any formal first integral f of η is of the form $f = \zeta \circ g$, where $\zeta \in \mathbb{C}[[t]]$ (power series in one variable).*

Theorem 3.2.1 is consequence of Theorem A, page 472 in [M-M]. Given $q \in D_\ell \cap sing(\tilde{\mathcal{F}})$, write the germ of $\tilde{\omega}$ at q as : $\tilde{\omega}_q = k.\eta$, where $k \in \mathcal{O}_q$, η is integrable and $cod(sing(\eta)) \geq 2$. The germ $\hat{f}_q$ is a non-constant formal first integral of η. Hence, by theorem 3.2.1, $\tilde{\mathcal{F}}$ has a non-constant holomorphic first integral, say $g_q \in \mathcal{O}_q$, with $g_q(0) = 0$, and such that $\hat{f}_q = \zeta \circ g_q$, where $\zeta \in \mathbb{C}[[t]]$. Note that $\zeta(0) = 0$, because $g_q(q) = \hat{f}_q(q) = 0$. Since D_ℓ is invariant for $\tilde{\mathcal{F}}$ we must have $g_q|_{D_{\ell,q}} = 0$, where $D_{\ell,q}$ denotes the germ of D_ℓ at q.

Consider a representative g of g_q in some polydisk Δ around q. Note that $g|_{\Delta \cap D_\ell} \equiv 0$. The polydisk Δ is given in some coordinate $[\Delta,(u,z)]$ as $(||u|| < \epsilon, |z| < \epsilon)$ and $U := \Delta \cap D_\ell = (z = 0)$. Let $\hat{f}(u,z) = \sum_{j \geq 1} f_j(u)\, z^j \in \mathcal{O}(U)[[z]]$ be a representative of $\hat{f}$ over U. We can also consider $g \in \mathcal{O}(\Delta)$ as an element of $\mathcal{O}(U)[[z]]$. Since $g|_U \equiv 0$, we can compose the series $\zeta \in \mathbb{C}[[t]]$ and $g \in \mathcal{O}(U)[[z]]$, so that we can consider $\zeta \circ g \in \mathcal{O}(U)[[z]]$. Note that $\zeta \circ g \in \mathcal{O}(U)[[z]]$ and $\hat{f}$ coincide as elements of $\mathcal{O}(U)[[z]]$, because $\hat{f}_q = \zeta \circ g_q$. Since $B_\ell \cap U \neq \emptyset$ and $A \supset B_\ell$, there exists $(u_o, 0) \in U$ such that the power series $\hat{f}(u_o, z)$ is convergent. It follows that the series $\zeta \in \mathbb{C}[[t]]$ is convergent, because $g(u_o,z), \hat{f}(u_o,z) \in \mathbb{C}\{z\}$ and $\zeta \circ g(u_o,z) = \hat{f}(u_o,z)$. Hence, $\hat{f} \in \mathcal{O}(U)\{z\}$, which implies that $q \in A$.

Note, that **II** implies that A is the union irreducible components of D, and so it is closed in D. This finishes the proof of lemma 3.2.1. $\square$

Corollary 3.2.1. *Under the hypothesis of lemma 3.2.1, $\hat{g}$ converges. Moreover, there exist a ball $B_1 \subset B$ around $0 \in \mathbb{C}^N$ and $f, g \in \mathcal{O}(B_1)$ such that $f(0) = 0$, $g(0) = 1$ and $\omega = g.df$ on $X^* \cap B_1$.*

Proof. Fix $q \in D$. Since $d\hat{f}$ converges and $\tilde{\omega}_q = \hat{g}_q.d\hat{f}_q \in \Omega^1_q$, it follows that $\hat{g}_q \in \mathcal{O}_q$. This implies that $\hat{g}$ converges. Therefore, we can consider $\hat{f}$ and $\hat{g}$ as holomorphic functions defined in a neighborhood $\tilde{V}$ of D in $\tilde{X}$. We can suppose that $\tilde{V} = \pi(B_1 \cap X)$, where $B_1 \subset B$ is a ball around $0 \in \mathbb{C}^N$. Since $\pi\colon X^* \to \tilde{X} \setminus E$ is a biholomorphism, these functions induce holomorphic functions $f_1, g_1 \in \mathcal{O}(V^*)$, satisfying $\omega|_{V^*} = g_1.df_1$, where $V^* = \pi^{-1}(V \setminus E)$. Now, f_1 and g_1 can be extended to holomorphic functions $f, g \in \mathcal{O}(B_1)$, because X is 0-regular, and this proves the corollary. $\square$

3.3. **End of the proof of the main theorem.** The idea is to prove that there exists $\zeta \in \mathbb{C}[[t]]$ such that $\zeta \circ \hat{f}$ converges. The composition $\zeta \circ \hat{f}$ is defined in such a way that, if $\zeta(t) = \sum_{i \geq 0} a_i\, t^i$ and $\hat{f}(u,z) = \sum_{j \geq 1} f_j(u)\, z^j$ is a representative of $\hat{f}$ over some open set $U \subset D_\ell$, $1 \leq \ell \leq r$, then $\zeta \circ \hat{f}$ is represented over U by the formal power series in z, $S(u,z) = \zeta \circ \sum_{j \geq 1} f_j(u)\, z^j$. This series is well defined because $\hat{f}|_U \equiv 0$. The next result imples the main theorem :

Lemma 3.3.1. *There exists $\zeta \in \mathbb{C}[[t]]$ such that $\zeta(0) = 0$, $\zeta'(0) = 1$ and $\zeta \circ \hat{f}$ converges. In particular, there exist holomorphic functions $\tilde{f} := \zeta \circ \hat{f}$ and $\tilde{g}$ defined in a neighborhood of D in $\tilde{X}$ such that $\tilde{\omega} = \tilde{g}\, d\tilde{f}$.*

Proof. Let us suppose for a moment that there exists $\zeta \in \mathbb{C}[[t]]$ as in the conclusion of the lemma. Since $\tilde{\omega} = \hat{g}.d\hat{f}$, we have

$$\tilde{f} = \zeta \circ \hat{f} \implies d\tilde{f} = \zeta' \circ \hat{f}.d\hat{f} \implies d\hat{f} = \hat{h}.d\tilde{f}\ ,$$

where $\hat{h} = (\zeta' \circ \hat{f})^{-1}$ and $\hat{h}|_D = 1$. This implies $\tilde{\omega} = \tilde{g}.d\tilde{f}$, where $\tilde{g} = \hat{g}.\hat{h}$. Since $\tilde{\omega}$ and $d\tilde{f}$ are convergent, so is $\tilde{g}$. Moreover, $\tilde{g}|_D = 1$. Let us prove the existence of ζ.

Let D_i be an irreducible component of D and $p \in D_i$ be fixed. Let $[W,(u,z)]$ be a coordinate system around p such that $p \in U := W \cap D_i = (z=0)$ and $\hat{f}$ has a representative $\hat{f}(u,z) \in \mathcal{O}(U)[[z]]$ over U. Since $\hat{f}(u,0) \equiv 0$, we get $\hat{f}(u,z) = z^{k(U)}.f_U(u,z)$, $k(U) \geq 1$, $f_U \in \mathcal{O}(U)[[z]]$ and $f_U(u,0) \not\equiv 0$.

Remark 3.3.1. *The integer $k(U)$ depends only of the irreducible component D_i. It will be called the multiplicity of $\hat{f}$ at D_i and will be denoted by k_i.*

We leave the proof of the above remark for the reader. Since $\hat{O}_p$ is a noetherian ring, the germ $\hat{f}_p \in \hat{O}_p$ of $\hat{f}$ at p can be decomposed as

$$\hat{f}_p = z^{k_i}.\hat{h}_1^{m_1}...\hat{h}_s^{m_s}\ ,$$

where $m_j \geq 1$, $\hat{h}_j(p) = 0$ and $\hat{h}_j$ is irreducible in $\hat{O}_p$ for all $j = 1,...,s$.

Claim 3.3.1. *For each $j \in \{1,...,s\}$ there exist $h_j \in \mathcal{O}_p$ and $\hat{v}_j \in \hat{O}_p$ such that $\hat{v}_j(p) \neq 0$ and $\hat{h}_j = \hat{v}_j.h_j$. In particular, each h_j is invariant for $\tilde{\mathcal{F}}$. Moreover, we can write $\hat{f}_p = \hat{\alpha}.z^{k_i}.h_1^{m_1}...h_s^{m_s}$, where $\hat{\alpha} \in \hat{O}_p$ and $\hat{\alpha}(0) \neq 0$.*

Notation. The germs $(z=0), (h_1 = 0), ..., (h_s = 0)$ will be called the *separatrices* of $\tilde{\mathcal{F}}$ through p. The integer $m_j \geq 1$ will be called the *multiplicity* of the separatrix $(h_j = 0)$, $1 \leq j \leq s$.

Proof. It follows from theorem 3.2.1 that the germ of $\tilde{\mathcal{F}}$ at p has a first integral $g \in \mathcal{O}_p$ such that $\hat{f}_p = \mu \circ g$, where $\mu \in \mathbb{C}[[t]]$ and $\mu(0) = 0$. We can set $\mu(t) = t^m.\beta(t)$, where $m \geq 1$, $\beta \in \mathbb{C}[[t]]$ and $\beta(0) \neq 0$. It follows that $\hat{f}_p = \hat{\gamma}.g^m$, where $\hat{\gamma} = \beta \circ g \in \hat{O}_p$ and $\hat{\gamma}(0) \neq 0$. If we write the decomposition of g into irreducible factors in $\mathcal{O}_p$ as $g = z^\ell.h_1^{\ell_1}...h_r^{\ell_r}$ then we get

$$\hat{f}_p = z^{k_i}.\hat{h}_1^{m_1}...\hat{h}_s^{m_s} = \hat{\gamma}^m.z^{m.\ell}.h_1^{m.\ell_1}...h_r^{m.\ell_r} \implies r = s$$

and we can suppose that $\hat{h}_j = \hat{v}_j.h_j$, where $\hat{v}_j \in \hat{O}_p$ and $\hat{v}_j(0) = 1$, for all $j = 1,...,s$. In this case we get, $k_i = m.\ell$ and $m_j = m.\ell_j$, for all $j = 1,...,s$, and

$\hat{f}_p = \hat{\alpha}.z^{k_i}.h_1^{m_1}...h_s^{m_s}$, where $\hat{\alpha} = \hat{v}_1^{m_1}...\hat{v}_s^{m_s}$. The fact that $\hat{\alpha}.z^{k_i}.h_1^{m_1}...h_s^{m_s}$ is a formal first integral for $\tilde{\mathcal{F}}$ implies that each h_j is invariant for $\tilde{\mathcal{F}}$. We leave the proof of this last assertion for the reader. □

Let $S = (h = 0)$ be a germ of separatrix of $\tilde{\mathcal{F}}$ at p, where $h \in \mathcal{O}_p$ is irreducible. We have two possibilities : either S is contained in some irreducible component of E, or not.

Claim 3.3.2. *Let m be the multiplicity of h in $\hat{f}_p$. If $m \geq 2$ then S is contained in some irreducible component C of E. In this case, C is invariant for $\tilde{\mathcal{F}}$.*

Proof. It follows from claim 3.3.1 that we can write $\hat{f}_p = h^m.\hat{\phi}$, where $\hat{\phi} \in \hat{O}_p$. We have

$$\tilde{\omega}_p = \hat{g}_p.d\hat{f}_p = \hat{g}_p.d(h^m.\hat{\phi}) = h^{m-1}\,\hat{g}_p.(m.\hat{\phi}.dh + h.d\hat{\phi}) \ .$$

Hence, h divides all coefficients of $\tilde{\omega}_p$ and $(h = 0) \subset sing(\tilde{\omega})$. Since $cod_{X^*}(sing(\mathcal{F})) \geq 3$ and $\tilde{\omega}$ represents $\tilde{\mathcal{F}}$ on $\tilde{X} \setminus E$, any irreducible component of $sing(\tilde{\omega})$ which cuts $\tilde{X} \setminus E$ has codimension ≥ 3. This implies that $(h = 0) \subset E$, because $(h = 0)$ has codimension one. Since h is irreducible in $\mathcal{O}_p$, it follows that $(h = 0)$ must be contained in some irreducible component of E, say C. This component C contains a non-empty open subset (in C) which is invariant for $\tilde{\mathcal{F}}$ and this implies that it is invariant for $\tilde{\mathcal{F}}$. □

Lemma 3.3.2. *Suppose that there exists $p \in D$ and a separatrix of $\tilde{\mathcal{F}}$ through p, say $(h_1 = 0)$, with multiplicity one. Then the conclusions of lemma 3.3.1 are true.*

Proof. In this case, the germ $\hat{f}_p \in \hat{O}_p$ can be written as $\hat{f}_p = \hat{\alpha}.h_1.h_2^{m_2}...h_s^{m_s}$, where $\hat{\alpha}$ is an unit in $\hat{O}_p$. It follows from theorem 3.2.1 that $\tilde{\mathcal{F}}_p$ has a first integral $g \in \mathcal{O}_p$ such that $g(p) = 0$ and $\hat{f}_p = \beta \circ g$, where $\beta \in \mathbb{C}[[t]]$. Clearly $\beta(0) = 0$. Let us prove that $\beta'(0) \neq 0$. Set $\beta(t) = t^{\ell}.\mu(t)$, where $\ell \geq 1$, $\mu \in \mathbb{C}[[t]]$ and $\mu(0) \neq 0$. Let $g = g_1^{n_1}...g_k^{n_k}$ be the decomposition of g into irreducible factors. Then

$$\hat{\alpha}.h_1.h_2^{m_2}...h_s^{m_s} = \beta \circ g = g^{\ell}.\,\mu \circ g = \mu \circ g.\,g_1^{\ell.n_1}...g_k^{\ell.n_k} \ .$$

Since $\hat{\alpha}$ and $\mu \circ g$ are units in $\hat{O}_p$, it follows that there is $j \in \{1, ..., k\}$ such that $g_j^{\ell.n_j}$ and h_1 differ by an unit in $\mathcal{O}_p$, so that $\ell.n_j = 1$. Therefore, $\ell = n_j = 1$, which implies that $\beta'(0) \neq 0$.

Since $\beta'(0) \neq 0$ there exists $\zeta \in \mathbb{C}[[t]]$ such that $\zeta \circ \beta(t) = t$. It follows that $g = \zeta \circ \hat{f}_p = (\zeta \circ \hat{f}\,)_p$. This implies that there exists $p \in D$ such that $(\zeta \circ \hat{f}\,)_p$ converges. Hence, $\zeta \circ \hat{f}$ satisfies the hypothesis of lemma 3.2.1, and so it converges. □

The next result will imply lemma 3.3.1 and the main theorem.

Lemma 3.3.3. *There exists $p \in D$ and a separatrix of $\tilde{\mathcal{F}}$ through p with multiplicity one.*

Proof. The proof will be by contradiction. Suppose by contradiction that $\tilde{\mathcal{F}}$ has no separatrix of multiplicity one. Let H be a ℓ-plane of $\mathbb{C}^N$, where $\ell = N - n + 2$, such that $0 \in H$. Denote by $\tilde{H}$ be the strict transform of $H \cap X$ by Π.

We can assume that $\tilde{H} \cap E = \tilde{H} \cap D$. If $sing(X) = \{0\}$ then $D = E$ and the assertion is trivially true. Suppose that $sing(X) \neq \{0\}$. In this case, the closure of $E \setminus D$ in $\tilde{B}$ is the strict transform of $sing(X)$ by Π. Recall that the first blowing-up in the process was a blowing-up at $0 \in B \subset \mathbb{C}^N$. Denote it by

$\Pi_1 \colon (B_1, \mathbb{P}^{N-1}) \to (B, 0)$. Let C and H_1 be the strict transforms of $sing(X)$ and H respectively by Π_1. Since $dim(H_1) = N-n+2$, $dim(C) = dim(sing(X)) \le n-3$ and $(N-n+2)+(n-3) = N-1 < dim(B_1)$, if we choose H in such a way that H_1 is transverse to all strata of C then $C \cap H_1 = \emptyset$. This, of course, implies the assertion.

In the above situation, we have $X \cap H = (X^* \cap H) \cup \{0\}$. It follows from the theory of transversality that we can choose H in such a way that it cuts X^* transversely, so that any irreducible component of $X \cap H$ has an isolated singularity at $0 \in \mathbb{C}^N$ and has dimension $2 = (N-n+2)+n-N$.

Let $S \subset \tilde{X}$ be an irreducible component of the strict transform of $H \cap X$ by Π ($dim(S) = 2$). Since $\tilde{H} \cap E = \tilde{H} \cap D$ and all separatrices $\tilde{\mathcal{F}}$ have multiplicity ≥ 2, it follows from claim 3.3.2 that, if $p \in S \cap D$ and h is a separatrix of $\tilde{\mathcal{F}}$ through p then $(h=0) \subset D$. Note that $S^* := S \setminus D$ is smooth of dimension 2, so that $sing(S) \subset D$. After new blowing-ups involving only points or curves contained in $S \cap D$, we can assume that :

(F). S is smooth and cuts transversely all the irreducible components D_j, $1 \le j \le r$. We can assume also that for each $j \in \{1, ..., r\}$ the curve $S \cap D_j$ is smooth and cuts tranversely $D_j \cap D_i$, for all $i \ne j$.

Let $D_\ell \cap S = \cup_{j=1}^{s_\ell} C_{\ell j}$ be the decomposition of $D_\ell \cap S$ into irreducible components. Denote by $[C_{\ell j}]$ the class in $H^2_{DR}(S)$ of the divisor $C_{\ell j}$. Let $L \in H^2_{DR}(S)$ be defined by

$$L = \sum_{\ell=1}^{r} \sum_{j=1}^{s_\ell} k_\ell . [C_{\ell j}] := \sum_{\sigma} k_\sigma \, [C_\sigma] \; . \tag{5}$$

In (5) we set $k_\sigma = k_\ell$ if $\sigma = (\ell \, j)$. Since $S \cap D$ is contracted to a point by Π, it follows that $L^2 < 0$, because the intersection matrix $([C_\sigma].[C_\mu])_{\sigma\mu}$ is negative defined (cf. [La]).

Let $i \colon S \to \tilde{X}$ be the inclusion map and $\mathcal{G} = i^*(\tilde{\mathcal{F}})$ be the induced foliation. It follows from **(F)** that the singularities of $\mathcal{G}$ are the corners $C_{\ell\, j} \cap C_{m\, i} \ne \emptyset$, where $\ell \ne m$. Moreover, the Camacho-Sad index (cf. [C-S] or [Su]) of $\mathcal{G}$ at a point $p \in C_{\ell\, j} \cap C_{m\, i}$ with respect to $C_{\ell\, j}$, denoted by $CS(\mathcal{G}, C_{\ell j}, p)$, is $-k_m/k_\ell$. This follows from the fact that $\tilde{\mathcal{F}}$ has a first integral of the form $z_m^{k_m}.z_\ell^{k_\ell}$ in a neighborhood of the point, where $(z_\ell = 0)$ and $(z_m = 0)$ are local equations of D_ℓ and D_m, respectively (see theorem 3.2.1). It follows from Camacho-Sad theorem (cf. [C-S] or [Su]) that

$$[C_\sigma]^2 = \sum_{p \in C_\sigma} CS(\mathcal{G}, C_\sigma, p) = - \sum_{\substack{p \in C_\sigma \cap C_\mu \\ \mu \ne \sigma}} k_\mu / k_\sigma = -\frac{1}{k_\sigma} \sum_{\mu \ne \sigma} k_\mu . [C_\sigma].[C_\mu] \tag{6}$$

On the other hand, (5) and (6) imply that

$$L^2 = \sum_{\sigma} k_\sigma^2 \, [C_\sigma]^2 + \sum_{\mu \ne \sigma} k_\mu . k_\sigma \, [C_\mu].[C_\sigma] = 0 \; ,$$

a contradiction. This contradiction implies lemma 3.3.3 and the main theorem. □

4. Appendix.

In this appendix X will be an irreducible complete intersection germ at $0 \in \mathbb{C}^N$ of analytic set. In this case, the generating ideal of X has generators $f_1, ..., f_k \in \mathcal{O}_N$ such that $dim_{\mathbb{C}}(X) + k = N$. From now on we will fix these generators. Let B be a ball around $0 \in \mathbb{C}^N$ such that $f_1, ..., f_k$ have representatives, which by simplicity we will denote by the same letters. The ball B will be taken small in such a way that $(f_1 = ... = f_k = 0)$ is irreducible in B. For simplicity, we will denote $X = (f_1 = ... = f_k = 0)$. We will set $sing(X) = \{p \in B \,|\, df_1(p) \wedge ... \wedge df_k(p) = 0\}$ and $X^* = X \setminus sing(X)$. We will suppose that $sing(X) \neq \emptyset$. Note that X^* is a holomorphic sub-manifold of complex dimension $n = N - k$ of $B \setminus sing(X)$.

With these conventions in mind, we will prove the following results :

Proposition 4.0.1. *Suppose that $dim(sing(X)) \leq dim(X) - 2$. Then any holomorphic function $g \in \mathcal{O}(X^*)$ can be extented to a holomorphic function $\tilde{g} \in \mathcal{O}(B)$. In particular, the germ of X at $0 \in \mathbb{C}^N$ is 0-regular.*

Proposition 4.0.2. *Suppose that $dim(sing(X)) \leq dim(X) - 3$. Then any holomorphic 1-form $\omega \in \Omega^1(X^*)$ can be extended to a holomorphic 1-form $\tilde{\omega} \in \Omega^1(B)$. In particular, the germ of X at $0 \in \mathbb{C}^N$ is 1-regular.*

Proposition 4.0.3. *If $dim(sing(X)) \leq dim(X) - 3$ then $H^1(X^*, \mathcal{O}) = 0$.*

In the next result, we will consider the case of a complete intersection $X = (f_1 = ... = f_k = 0) \subset B$, with an isolated singularity at $0 \in \mathbb{C}^N$. In this case, $X^* = X \setminus \{0\}$.

Proposition 4.0.4. *Suppose that $sing(X) = \{0\}$ and $dim_{\mathbb{C}}(X) \geq 4$. If the ball B is small enough then $H^1(X^*, \mathcal{O}^*) = 1$.*

Next we state some facts that will be used in the proof of the above results. The first one is the following (cf. [G-R] page 133) :

Theorem 4.0.1. *Let Z be an analytic subset of a Stein manifold M with $dim(M) = N$. If $dim(Z) \leq N - \ell - 2$ then $H^j(M \setminus Z, \mathcal{O}) = 0$ for $1 \leq j \leq \ell$.*

The second one, is a consequence of De Rham-Saito division theorem (cf. [D-R] and [S]) and the fact that $X = (f_1 = ... = f_k = 0) \subset B$ is a complete intersection. Let U be a Stein open subset of $B \setminus sing(X)$ and $V = X^* \cap U \neq \emptyset$. Let $e_j \in \mathcal{O}(B)^k$ be defined as $e_j = (0, ..., 0, 1, 0, ..., 0)$, where the 1 appears in the j^{th} position. Set

$$F = \sum_{j=1}^{k} f_j \, . \, e_j \in \Lambda^1(\mathcal{O}^k(B)) \ . \tag{7}$$

Theorem 4.0.2. *If $G^j \in \Lambda^j(\mathcal{O}(U)^k)$ is such that $G^j \wedge F = 0$ and $1 \leq j \leq k-1$ then there exists $H^{j-1} \in \Lambda^{j-1}(\mathcal{O}(U)^k)$ such that $G^j = H^{j-1} \wedge F$.*

The third is also a consequence of the fact that X is a complete intersection and that $sing(X) = \{q \in X \,|\, df_1(q) \wedge ... \wedge df_k(q) = 0\}$.

Remark 4.0.2. *Let U be a Stein open subset of $B \setminus sing(X)$ and $V = X \cap U \subset X^*$. If $h \in \mathcal{O}(U)$ is such that $h|_V \equiv 0$ then there exist $h_1, ..., h_k \in \mathcal{O}(U)$ such that*

$$h = \sum_{j=1}^{k} h_j \, . \, f_j|_U \ .$$

Fix a Leray covering $\mathcal{U} = (U_j)_{j\in J}$ of $B \setminus sing(X)$.

Definition 4.0.1. *Let $\Sigma^\ell = \{E_\sigma := e_{\sigma_1} \wedge ... \wedge e_{\sigma_\ell} \,|\, 1 \le \sigma_1 < ... < \sigma_\ell \le k\}$. An ℓ-vector of s-cochains in $\mathcal{U}$ is an element $G_s^\ell = \sum_{\sigma\in\Sigma^\ell} g_\sigma E_\sigma$, where $g_\sigma \in C^s(\mathcal{U},\mathcal{O})$ for all $\sigma \in \Sigma^\ell$. Its coboundary, defined by $\delta G_s^\ell = \sum_\sigma \delta g_\sigma E_\sigma$, is an ℓ-vector of $(s+1)$-cochains. The set of ℓ-vectors of s-cochains in $\mathcal{U}$ will be denoted by $\Lambda_s^\ell(\mathcal{U})$. In the case $\ell = 0$ we set $\Lambda_s^0(\mathcal{U}) = C^s(\mathcal{U},\mathcal{O})$.*

The following consequence of theorem 4.0.2 will be usefull :

Lemma 4.0.4. *Fix s, ℓ integers with $s \ge 1$ and $1 \le \ell \le k-1$. Assume that $H^j(B \setminus sing(X),\mathcal{O}) = 0$ for $1 \le j \le s+\ell$. Let $G_s^\ell \in \Lambda_s^\ell(\mathcal{U})$ be be such that $\delta G_s^\ell \wedge F = 0$. Then there exist $H_s^{\ell-1} \in \Lambda_s^{\ell-1}(\mathcal{U})$ and $H_{s-1}^\ell \in \Lambda_{s-1}^\ell(\mathcal{U})$ such that $G_s^\ell = H_s^{\ell-1} \wedge F + \delta H_{s-1}^\ell$.*

Proof. Note that $\delta G_s^\ell \wedge F = 0$ and theorem 4.0.2 imply that there exists $G_{s+1}^{\ell-1} \in \Lambda_{s+1}^{\ell_1}(\mathcal{U})$ such that

$$\delta G_s^\ell = G_{s+1}^{\ell-1} \wedge F \implies \delta G_{s+1}^{\ell-1} \wedge F = 0 \ .$$

When $\ell = 1$, the last relation implies that $\delta G_{s+1}^0 = 0$, and so $G_{s+1}^0 = \delta H_s^0$ for some $H_s^0 \in C^s(\mathcal{U},\mathcal{O})$, because $H^{s+1}(\mathcal{U},\mathcal{O}) = 0$. In this case, we get

$$\delta(G_s^1 - H_s^0.F) = 0 \implies G_s^1 = H_s^0.F + \delta H_{s-1}^1 \ ,$$

where $H_{s-1}^1 \in \Lambda_{s-1}^1(\mathcal{U})$, because $H^s(\mathcal{U},\mathcal{O}) = 0$.

When $\ell > 1$, we get by induction that for all $j \in \{0,...,\ell-1\}$ there exists $G_{s+\ell-j}^j \in \Lambda_{s+\ell-j}^j(\mathcal{U})$ such that

$$\delta G_{s+\ell-j-1}^{j+1} = G_{s+\ell-j}^j \wedge F \implies \delta G_{s+\ell-j}^j \wedge F = 0 \ . \tag{8}$$

If we do $j = 0$ in the second relation in (8) we get

$$\delta G_{s+\ell}^0 = 0 \implies G_{s+\ell}^0 = \delta H_{s+\ell-1}^0 \ ,$$

because $H^{s+\ell}(\mathcal{U},\mathcal{O}) = 0$. Hence,

$$\delta(G_{s+\ell-1}^1 - H_{s+\ell-1}^0 \wedge F) = 0 \implies G_{s+\ell-1}^1 = H_{s+\ell-1}^0 \wedge F + \delta H_{s+\ell-2}^1 \ ,$$

because $H^{s+\ell-1}(\mathcal{U},\mathcal{O}) = 0$. It follows that

$$\delta G_{s+\ell-2}^2 = (H_{s+\ell-1}^0 \wedge F + \delta H_{s+\ell-2}^1) \wedge F = \delta H_{s+\ell-2}^1 \wedge F \implies$$

$$\delta(G_{s+\ell-2}^2 - H_{s+\ell-2}^1 \wedge F) = 0 \implies G_{s+\ell-2}^2 = H_{s+\ell-2}^1 \wedge F + \delta H_{s+\ell-3}^2$$

and by induction that there exists $H_s^{\ell-1} \in \Lambda_s^{\ell-1}(\mathcal{U})$ such that

$$... \ \delta(G_s^\ell - H_s^{\ell-1} \wedge F) = 0 \implies G_s^\ell = H_s^{\ell-1} \wedge F + \delta H_{s-1}^\ell$$

where $H_{s-1}^\ell \in \Lambda_{s-1}^\ell(\mathcal{U})$. ☐

Proof of proposition 4.0.1 Observe first that $dim(X) = N-k$, and so $dim(sing(X)) \le N-k-2$. It follows from theorem 4.0.1 that $H^j(B\setminus sing(X),\mathcal{O}) = 0$ for $1 \le j \le k$.

Fix a holomorphic function $g \in \mathcal{O}(X^*)$ and let us prove that it can be extended to a holomorphic function $\tilde{g} \in \mathcal{O}(B)$.

Let $\mathcal{U} = (U_j)$ be a Leray covering of $B \setminus sing(X)$. Define $V_j = U_j \cap X^*$ and $\mathcal{V} = (V_j)$. We will use the notations $U_{ij} = U_i \cap U_j$ and $V_{ij} = V_i \cap V_j$.

For each j let $g_j \in \mathcal{O}(U_j)$ be an extension of $g|_{V_j}$ to U_j. If $V_j = \emptyset$ we define $g_j = 0$. Since $(g_j - g_i)|_{V_{ij}} \equiv 0$, it follows from remark 4.0.2 that there exist 1-cochains $g_1^1, ..., g_1^k$, $g_1^r = (g_{ij}^r)_{U_{ij} \neq \emptyset}$, $r = 1, ..., k$, such that

$$g_j - g_i = \sum_{r=1}^{k} g_{ij}^r \, f_r \ . \tag{9}$$

Let F be as in (7). Consider the $(k-1)$-vector of 1-cochains G_1^{k-1} defined by

$$(G_1^{k-1})_{ij} = \sum_{r=1}^{k} (-1)^{r-1} \, g_{ij}^r \, e_1 \wedge ... \wedge \hat{e}_r \wedge ... \wedge e_k$$

and the k-vector of 0-cochains G_0^k defined by $(G_0^k)_j = g_j \, e_1 \wedge ... \wedge e_k$. Then (9) is equivalent to

$$\delta \, G_0^k = G_1^{k-1} \wedge F \implies \delta G_1^{k-1} \wedge F = 0 \ .$$

Since $H^j(B \setminus sing(X), \mathcal{O}) = 0$ for $1 \leq j \leq k$, we get from lemma 4.0.4 that there exist $H_1^{k-2} \in \Lambda_1^{k-2}(\mathcal{U})$ and $H_0^{k-1} \in \Lambda_0^{k-1}(\mathcal{U})$ such that $G_1^{k-1} = H_1^{k-2} \wedge F + \delta H_0^{k-1}$, which implies

$$\delta(G_0^k - H_0^{k-1} \wedge F) = 0 \tag{10}$$

If we set

$$(H_0^{k-1})_j = \sum_{r=1}^{k} (-1)^{r-1} \, h_j^r \, e_1 \wedge ... \wedge \hat{e}_r \wedge ... \wedge e_k \ , \ h_j^r \in \mathcal{O}(U_j) \ .$$

then (10) is equivalent to

$$g_j - g_i = \sum_{r=1}^{k} (h_j^r - h_i^r).f_r \implies \exists \, \tilde{g} \in \mathcal{O}(B \setminus sing(X)) \text{ s.t. } \tilde{g}|_{U_j} = g_j - \sum_{r=1}^{k} h_j^r.f_r \ .$$

The function $\tilde{g}$ extends g to $X \setminus sing(X)$. Since $cod(sing(X)) \geq 2$, it follows from Hartogs' theorem that $\tilde{g}$ can be extended to B. $\square$

Proof of proposition 4.0.3 It follows from theorem 4.0.1 that $H^j(B \setminus sing(X), \mathcal{O}) = 0$ for $1 \leq j \leq k+1$, because $dim(sing(X)) \leq N - k - 3 = N - (k+1) - 2$.

Let $\mathcal{U} = (U_j)_{j \in J}$ be a Leray covering of $B \setminus sing(X)$ and $\mathcal{V} = (V_j)_{j \in J}$ be defined by $V_j = U_j \cap X^*$. Since $\mathcal{V}$ is a Leray covering of X^* it is sufficient to prove that $H^1(\mathcal{V}, \mathcal{O}) = 0$.

Fix $g_1 = (g_{ij})_{V_{ij} \neq \emptyset} \in Z^1(\mathcal{V}, \mathcal{O})$. We want to prove that $g_1 = \delta h_0$, where $h_0 = (h_j)_j \in C^0(\mathcal{V}, \mathcal{O})$. Extend each g_{ij} to $\tilde{g}_{ij} \in \mathcal{O}(U_{ij})$, thus obtaining $\tilde{g}_1 = (\tilde{g}_{ij})_{U_{ij} \neq \emptyset} \in C^1(\mathcal{U}, \mathcal{O})$. Set $\delta \, \tilde{g}_1 := (\tilde{g}_{ij\ell})_{U_{ij\ell} \neq \emptyset}$, where $\tilde{g}_{ij\ell} = \tilde{g}_{ij} + \tilde{g}_{j\ell} + \tilde{g}_{\ell i}$. If $V_{ij\ell} \neq \emptyset$ then $\tilde{g}_{ij\ell}|_{V_{ij\ell}} = g_{ij} + g_{j\ell} + g_{\ell i} = 0$. It follows from remark 4.0.2 that

$$\tilde{g}_{ijk} = \sum_{r=1}^{\ell} g_{ijk}^r \, .f_r \ , \tag{11}$$

where $g_{ijk}^r \in \mathcal{O}(U_{ijk})$. Let $G_2^{k-1} \in \Lambda_2^{k-1}(\mathcal{U})$ be defined by

$$(G_2^{k-1})_{ij\ell} = \sum_{r=1}^{k} (-1)^{r-1} \, g_{ij\ell}^r \, e_1 \wedge ... \wedge \hat{e}_r \wedge ... \wedge e_k \ .$$

Then (11) is equivalent to

$$\delta\tilde{g}_1\, e_1 \wedge ... \wedge e_k = G_2^{k-1} \wedge F \implies \delta G_2^{k-1} \wedge F = 0 \ , \tag{12}$$

where F is as in (7). Therefore, lemma 4.0.4 implies that there exist $H_2^{k-2} \in \Lambda_2^{k-2}(\mathcal{U})$ and $H_1^{k-1} \in \Lambda_1^{k-1}(\mathcal{U})$ such that

$$G_2^{k-1} = H_2^{k-1} \wedge F + \delta H_1^{k-1} \implies \delta\tilde{g}_1\, e_1 \wedge ... \wedge e_k = \delta H_1^{k-1} \wedge F \implies$$

$$\delta(\tilde{g}_1\, e_1 \wedge ... \wedge e_k - H_1^{k-1} \wedge F) = 0 \implies \tilde{g}_1\, e_1 \wedge ... \wedge e_k = H_1^{k-1} \wedge F + \delta H_0^k \ ,$$

where $H_0^k = h_0 e_1 \wedge ... \wedge e_k$, $h_0 = (h_j)_j \in C^0(\mathcal{U}, \mathcal{O})$. Since $F|_{X^*} = 0$, it follows that

$$g_{ij} - (h_j - h_i)|_{V_{ij}} = [\tilde{g}_{ij} - (h_j - h_i)]\,|_{V_{ij}} = 0 \ ,$$

if $V_{ij} \neq \emptyset$. Hence $H^1(X^*, \mathcal{O}) = 0$. □

Proof of proposition 4.0.2 Recall that $X = (f_1 = ... = f_k = 0) \subset B$. For $1 \leq \ell \leq k$, set $X_\ell = (f_1 = ... = f_\ell = 0)$ and $X_\ell^* = X_\ell \setminus sing(X_\ell)$. Note that $sing(X_\ell) = \{p \in X_\ell \mid df_1(p) \wedge ... \wedge df_\ell(p) = 0\}$ and that X_ℓ is a complete intersection of dimension $N - \ell$. We set also, $X_0^* = B \setminus sing(X)$. In this way, we have $B \setminus sing(X) = X_0^* \supset X_1^* \supset ... \supset X_k^* = X^*$. Note that $H^1(X_0^*, \mathcal{O}) = 0$ (see theorem 4.0.1). We need a lemma.

Lemma 4.0.5. *For all $1 \leq \ell \leq k$ we have $dim(sing(X_\ell)) \leq dim(X_\ell) - 3$. In particular, $H^1(X_\ell^*, \mathcal{O}) = 0$ for all $0 \leq j \leq k$.*

Proof. For $\ell = k$ this is the hypothesis. Let $1 \leq \ell < k$. If we set $W = (f_{\ell+1} = ... = f_k = 0)$ then $dim(W) = N - (k - \ell)$. On the other hand,

$$W \cap sing(X_\ell) = (f_1 = ... = f_k = 0) \cap (df_1 \wedge ... \wedge df_\ell = 0) \subset sing(X) \ .$$

This implies that

$$dim(W \cap sing(X_\ell)) \leq dim(sing(X)) \leq dim(X) - 3 = N - k - 3 \ .$$

On the other hand, we have

$$dim(W \cap sing(X_\ell)) \geq dim(W) + dim(sing(X_\ell)) - N = dim(sing(X_\ell)) - k + \ell \implies$$

$$dim(sing(X_\ell)) \leq N - \ell - 3 = dim(X_\ell) - 3 \ .$$

□

Fix $\omega^* \in \Omega^1(X^*)$. Let $\mathcal{U} = (U_j)$ be a Leray covering of $B \setminus sing(X)$. Set $V_j = U_j \cap X^*$ and $\mathcal{V} = (V_j)$. Since U_j is Stein, we can extend $\omega^*|_{V_j}$ to $\omega_j \in \Omega^1(U_j)$.

Assertion 4.0.1. *We can find the extensions ω_j of $\omega^*|_{V_j}$ in such a way that, if $U_{ij} \neq \emptyset$ then*

$$\omega_j - \omega_i = \sum_{r=1}^{k} f_r.\alpha_{ij}^r \ \textit{where}\ \alpha_{ij}^r \in \Omega^1(U_{ij}) \ . \tag{13}$$

Proof. Since $\omega_j - \omega_i\,|_{V_{ij}} = 0$, we can write

$$\omega_j - \omega_i = \sum_{r=1}^{k} g_{ij}^r.df_r + \sum_{r=1}^{k} f_r.\alpha_{ij}^r \ \text{where}\ g_{ij}^r \in \mathcal{O}(U_{ij}) \ \text{and}\ \alpha_{ij}^r \in \Omega^1(U_{ij}) \ .$$

Let $g^r \in C^1(\mathcal{V}, \mathcal{O})$ be defined by $g^r = (g_{ij}^r|_{V_{ij}})_{V_{ij} \neq \emptyset}$. We assert that $g^r \in Z^1(\mathcal{V}, \mathcal{O})$, for all $r \in \{1, ..., k\}$.

Let us prove the assertion for $r = 1$. Fix $p \in V_{ij\ell} \neq \emptyset$. Then $\omega_j(p) - \omega_i(p) = \sum_{r=1}^{k} g_{ij}^r(p).df_r(p)$. Since $df_1(p) \wedge ... \wedge df_k(p) \neq 0$, we get

$$(\omega_j(p) - \omega_i(p)) \wedge df_2(p) \wedge ... \wedge df_k(p) = g_{ij}^1(p).df_1(p) \wedge ... \wedge df_k(p) \implies$$
$$\implies g_{ij}^1(p) + g_{j\ell}^1(p) + g_{\ell i}^1(p) = 0 \text{ , if } \implies \delta\, g^1 = 0 \text{ .}$$

In a similar way, we get $\delta g^r = 0$ for all $r \geq 2$. Since $H^1(X^*, \mathcal{O}) = 0$, for all $r = 1, ..., k$, there exists $h^r = (h_j^r)_{V_j \neq \emptyset} \in C^0(\mathcal{V}, \mathcal{O})$ such that $g^r = \delta h^r$. Extend $h_j^r \in \mathcal{O}(V_j)$ to $\tilde{h}_j^r \in \mathcal{O}(U_j)$ (if $V_j = \emptyset$ set $\tilde{h}_j^r = 0$). Define $\tilde{\omega}_j = \omega_j - \sum_{r=1}^{k} \tilde{h}_j^r.df_r$. For $p \in V_{ij} \neq \emptyset$ we have

$$\tilde{\omega}_j(p) - \tilde{\omega}_i(p) = \sum_{r=1}^{k} (g_{ij}^r(p) - h_j^r(p) + h_i^r(p)).df_r(p) = 0 \text{ .}$$

This implies that all coefficients of $\tilde{\omega}_j - \tilde{\omega}_i$ vanish on V_{ij}. Hence, there exist 1-forms $\tilde{\alpha}_{ij}^r \in \Omega^1(U_{ij})$ such that

$$\tilde{\omega}_j - \tilde{\omega}_i = \sum_{r=1}^{k} f_r.\tilde{\alpha}_{ij}^r \qquad \square$$

Assertion 4.0.2. *Let $1 \leq \ell \leq k$. Suppose that there exists $\omega_\ell^* \in \Omega^1(X_\ell^*)$ such that $\omega_\ell^*|_{X^*} = \omega^*$. Then there exists $\omega_{\ell-1}^* \in \Omega^1(X_{\ell-1}^*)$ such that $\omega_{\ell-1}^*|_{X_\ell^*} = \omega_\ell^*$.*

Proof. Let ω_ℓ^* be as in the hypothesis. It follows from assertion 4.0.1 that there exist $\omega_j \in \Omega^1(U_j)$ and $\alpha_{ij}^r \in \Omega^1(U_{ij})$ ($U_{ij} \neq \emptyset$) such that

$$\omega_j - \omega_i = \sum_{j=1}^{\ell} f_r.\alpha_{ij}^r$$

Write $\alpha_{ij}^\ell = \sum_{s=1}^{N} a_{ij}^s\, dx_s$, where $a_{ij}^s \in \mathcal{O}(U_{ij})$. If $p \in U_{ij} \cap X_{\ell-1}^*$, then

$$\omega_j(p) - \omega_i(p) = f_\ell(p).\alpha_{ij}^\ell(p) = f_\ell(p).\sum_{s=1}^{N} a_{ij}^s(p)\, dx_s \text{ .} \tag{14}$$

It follows from (14) that, if $U_{ijr} \cap X_\ell^* \neq \emptyset$ then $a_{ij}^s(p) + a_{jr}^s(p) + a_{ri}^s(p) = 0$, which implies that $(a_{ij}^s|_{U_{ij} \cap X_\ell^*})_{U_{ij} \cap X_\ell^* \neq \emptyset} \in Z^1(\mathcal{U} \cap X_\ell^*, \mathcal{O})$.

Since $H^1(X_{\ell-1}^*, \mathcal{O}) = 0$, for all $s = 1, ..., N$, there exists $c^s := (c_j^s)_{U_j \cap X_{\ell-1}^* \neq \emptyset} \in C^0(\mathcal{U} \cap X_\ell^*, \mathcal{O})$ such that $a_{ij}^s|_{U_{ij} \cap X_\ell^*} = c_j^s - c_i^s$.

Extend c_j^s to $h_j^s \in \mathcal{O}(U_j)$ and define $\eta_j \in \Omega^1(U_j)$ by $\eta_j = \sum_{s=1}^{N} h_j^s\, dx_s$. Set

$$\tilde{\omega}_j = \omega_j - f_\ell.\eta_j \text{ .}$$

The reader can check that, if $p \in X_{\ell-1}^*$ then

$$\tilde{\omega}_j(p) - \tilde{\omega}_i(p) = 0 \implies \exists\, \omega_{\ell-1}^* \in \Omega^1(X_{\ell-1}^*)$$

such that $\omega_{\ell-1}^*|_{U_j \cap X_{\ell-1}^*} = \tilde{\omega}_j|_{U_j \cap X_{\ell-1}^*}$. $\square$

The last assertion implies that there exists $\omega_0^* \in \Omega^1(B \setminus sing(X))$ such that $\omega_0^*|_{X^*} = \omega^*$. Finally, Hartogs theorem implies that ω_0^* can be extended to a 1-form $\omega \in \Omega^1(B)$, whose restriction to X^* coincides with ω^*. $\square$

Proof of proposition 4.0.4 Let $X = (f_1 = ... = f_k = 0) \subset B$ be a complete intersection with an isolated singularity at $0 \in B \subset \mathbb{C}^N$ and $dim(X) \geq 4$. We take the ball B with small radius, in such a way that :

(i). For any smaller ball $\overline{B}_r := \{z \in \mathbb{C}^N \,;\, ||z|| \leq r\} \subset B$ then the sphere $S_r = \partial\overline{B}_r$ is transversal to X. This implies that $N_r := S_r \cap X$ is a real smooth compact submanifold of $\mathbb{C}^N$ of $dim_{\mathbb{R}} N_r = 2\, dim_{\mathbb{C}}(X) - 1$.
(ii). X^* has a conical structure, that is, it is homeomorphic to $N_r \times \mathbb{R}$.

We want to prove that $H^1(X^*, \mathcal{O}^*) = 1$. As we have seen, $H^1(X^*, \mathcal{O}) = 0$. It follows from the exact sequence

$$0 = H^1(X^*, \mathcal{O}) \to H^1(X^*, \mathcal{O}^*) \xrightarrow{\delta^*} H^2(M, \mathbb{Z}) \to ...$$

that it is sufficient to prove that $\delta^* = 0$. In fact, we will prove that X^* is simply connected and that $H^2(X^*, \mathbb{Z})$ is finite. Let us prove that this implies that $H^1(X^*, \mathcal{O}^*) = 1$.

Since δ^* is injective, we get that $H^1(X^*, \mathcal{O}^*)$ is finite. Let $r = \#(H^1(X^*, \mathcal{O}^*))$. Fix a Leray covering ,$\mathcal{V} = (V_j)_j$, of X^* and let $g = (g_{ij})_{V_{ij} \neq \emptyset}$ be a multiplicative cocycle. We can assume that $g_{ij}^r = 1$. This implies that if $V_{ij} \neq \emptyset$ then g_{ij} is a constant, a r^{th}-root of the unity. Therefore, g is a cocycle in $H^1(\mathcal{V}, S^1)$, where $S^1 \subset \mathbb{C}$ is the unit circle, considered as a multiplicative group. But, $\Pi_1(X^*) = 1$ implies that $H^1(X^*, S^1) = 1$. Hence, $g \simeq 1$.

It follows from **(ii)** that X^* has the same homotopy type of N_r. Therefore, it is sufficient to prove that $\Pi_1(N_r) = 1$ and $H^2(N_r, \mathbb{Z})$ is finite. For the proof that $H^2(N_r, \mathbb{Z})$ is finite, it is sufficient to prove that $\beta_2(N_r) = 0$, so that we will prove that $H_2(N_r, \mathbb{Z}) = 0$, which implies $\beta_2(N_r) = 0$.

Given $\epsilon = (\epsilon_1, ..., \epsilon_k)$, define

$$F_\epsilon := (f_1 = \epsilon_1, ..., f_k = \epsilon_k) \cap \overline{B}_r \ , \ \dot{F}_\epsilon := F_\epsilon \setminus S_r \text{ and } N_\epsilon := F_\epsilon \cap S_r \ .$$

Since X cuts S_r transversely at N_r, it follows that, if $||\epsilon|| := |\epsilon_1| + ... + |\epsilon_k|$ is small then N_ϵ is homeomorphic to N_r. On the other hand, the following facts are known :
(iii). If $min\{|\epsilon_1|, ..., |\epsilon_k|\} > 0$ and $||\epsilon||$ is small then $\dot{F}_\epsilon$ is smooth and so a Stein manifold. This fact implies that :
(iv). The inclusion $N_\epsilon \xrightarrow{i} F_\epsilon$ induces isomorphisms

$$\Pi_1(N_\epsilon) \xrightarrow{i_*} \Pi_1(F_\epsilon) \text{ ,if } dim_{\mathbb{C}}(F_\epsilon) \geq 3$$

and

$$H_2(N_\epsilon, \mathbb{Z}) \xrightarrow{i_*} H_2(F_\epsilon, \mathbb{Z}) \text{ ,if } dim_{\mathbb{C}}(F_\epsilon) \geq 4 \ .$$

(v). (Milnor-Hamm). F_ϵ has the homotopy type of a finite cell complex of real dimension $dim_{\mathbb{C}}(X)$ and is $dim_{\mathbb{C}}(X) - 1$ connected (cf. [L] pg. 72-73).

Since $dim_{\mathbb{C}}(F_\epsilon) = dim_{\mathbb{C}}(X) \geq 4$, we get from **(iv)** that

$$\Pi_1(N_\epsilon) \simeq \Pi_1(F_\epsilon) \text{ and } H_2(N_\epsilon, \mathbb{Z}) \simeq H_2(F_\epsilon, \mathbb{Z})$$

and from **(v)** that $\Pi_1(F_\epsilon) = 1$ and $H_2(F_\epsilon, \mathbb{Z}) = 0$, which finishes the proof of the proposition. □

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