# A GENERALIZATION OF THE GIULIETTI-KORCHMÁROS MAXIMAL CURVE

Arnaldo Garcia[a,*] Cem Güneri[b] Henning Stichtenoth[b]

[a] IMPA, Estrada Dona Castorina 110, Rio de Janeiro, Brazil
[b] Sabancı University, FENS, 34956 İstanbul, Turkey

Abstract. We introduce a family of algebraic curves over $\mathbb{F}_{q^{2n}}$ (for an odd $n$) and show that they are maximal. When $n = 3$, our curve coincides with the $\mathbb{F}_{q^6}$-maximal curve that has been found by Giulietti and Korchmáros recently. Their curve (i.e., the case $n = 3$) is the first example of a maximal curve proven not to be covered by the Hermitian curve.

## 1. Introduction

Consider a nonsingular, geometrically irreducible projective curve (a curve for short) $\mathcal{X}$ over the finite field $\mathbb{F}_\ell$ of cardinality $\ell$. We say that $\mathcal{X}$ is $\mathbb{F}_\ell$-*maximal* if its number of rational points over $\mathbb{F}_\ell$ reaches the Hasse-Weil upper bound

$$|\mathcal{X}(\mathbb{F}_\ell)| = \ell + 1 + 2g(\mathcal{X})\sqrt{\ell}\ ,$$

where $g(\mathcal{X})$ is the genus of $\mathcal{X}$. Here the cardinality $\ell$ of the finite field will always be a square.

A curve $\mathcal{Y}$ is said to be a *cover* of $\mathcal{X}$ over $\mathbb{F}_\ell$ if there exists a surjective map $\varphi : \mathcal{Y} \to \mathcal{X}$, where $\varphi$ and both curves are defined over $\mathbb{F}_\ell$. By Serre's result (see [7]), a curve covered by an $\mathbb{F}_\ell$-maximal curve is also $\mathbb{F}_\ell$-maximal. There have been extensive studies on maximal curves (see for instance [2, 3, 4]) and most of the known examples have been shown to be subcovers of the Hermitian curve $\mathcal{H}$, which is defined over $\mathbb{F}_\ell$ by

$$\mathcal{H}: \quad Y^{\sqrt{\ell}} + Y = X^{1+\sqrt{\ell}}.$$

This led to the question whether every maximal curve is a subcover of the Hermitian curve or not. In [5], the first and third author illustrated a maximal curve over $\mathbb{F}_{27^2}$ which is not a Galois subcover of the Hermitian curve over the same finite field. Very recently, Giulietti and Korchmáros introduced a new example of a maximal curve and they showed that their curve is not covered by the Hermitian curve (see [6]). Their curve is defined over $\mathbb{F}_\ell$, where $\ell = q^6$ for some prime power $q$ (cf. Equation (2.2)).

In the present article we introduce a family of maximal curves over $\mathbb{F}_\ell$ with $\ell = q^{2n}$ for a prime power $q$ and an odd integer $n \geq 3$ (cf. Equation (2.1)). The member with $n = 3$ of

*This article was written while the first author was visiting Sabancı University in Fall 2007. His visit was partially supported by TÜBİTAK, Sabancı University and also by CNPq # 470163/2006-2.

this family coincides with the curve of Giulietti and Korchmáros (see Remark 2.1). Our proof for maximality is quite different from theirs. While they use results from higher dimensional algebraic geometry (curves on Hermitian varieties), we give a more elementary proof. The main ingredients of our proof are some polynomial identities (cf. Lemmas 2.3 and 2.5) and a certain subcover whose maximality was previously shown by Abdón, Bezerra and Quoos (see [1] and also [5]). We remark here that if $n$ is odd and $n > 3$, we do not know whether our curves are covered by the Hermitian curve or not.

Throughout this article, we denote by $q$ a power of a prime number.

## 2. A Family of Maximal Curves

Let $n$ be an odd positive integer. We consider the curve $\mathcal{C}_n$ over $\mathbb{F}_{q^{2n}}$ defined by the following equations:

$$\mathcal{C}_n \quad : \quad \begin{aligned} X^q + X &= Y^{q+1} \\ Y^{q^2} - Y &= Z^{\frac{q^n+1}{q+1}} \end{aligned} \tag{2.1}$$

**Remark 2.1.** In [6], Giulietti and Korchmáros introduced the first example of a maximal curve that is not covered by the Hermitian curve. This curve is defined over $\mathbb{F}_{q^6}$ by the equations

$$\mathcal{C}' \quad : \quad \begin{aligned} X^q + X &= Y^{q+1} \\ Yh(X) &= Z^{\frac{q^3+1}{q+1}}, \end{aligned} \tag{2.2}$$

where

$$h(X) = \frac{X^{q^2} - X}{X^q + X}.$$

Note that the first equation of (2.2) implies that

$$Y^{q^2-1} = (X^q + X)^{q-1} = \frac{X^{q^2} + X^q}{X^q + X} = h(X) + 1.$$

Hence, we have

$$Y^{q^2} - Y = Y(Y^{q^2-1} - 1) = Yh(X).$$

This shows that $\mathcal{C}' = \mathcal{C}_3$, i.e. the Giulietti-Korchmáros curve is a special case of the curves defined by (2.1).

Note that the first equation in (2.1) defines a maximal curve over $\mathbb{F}_{q^{2n}}$ since it is the equation of the Hermitian curve over $\mathbb{F}_{q^2}$ and $n$ is odd. The curve defined by the second equation, on the other hand, was shown to be maximal for $n = 3$ in [5]. Recently, Abdón et al. noticed that this equation defines a maximal curve over $\mathbb{F}_{q^{2n}}$ for any odd $n$ (see [1]). So, $\mathcal{C}_n$ is a fibre product of two maximal curves over $\mathbb{F}_{q^{2n}}$ and our main result in this article will be that $\mathcal{C}_n$ itself is maximal over $\mathbb{F}_{q^{2n}}$. For this purpose, we first compute the genus of $\mathcal{C}_n$.

In the following, $\mathbb{P}^1$ denotes the projective line.

**Proposition 2.2.** *Consider the covering of curves $\mathcal{C}_n \longrightarrow \mathcal{X}_n \longrightarrow \mathbb{P}^1$ over $\mathbb{F}_{q^{2n}}$, where $\mathcal{C}_n$ is defined as in (2.1) and $\mathcal{X}_n$ is defined by the second equation in (2.1). Then we have*

$$\begin{aligned} g(\mathcal{X}_n) &= \frac{(q-1)(q^n-q)}{2}, \\ g(\mathcal{C}_n) &= \frac{(q-1)(q^{n+1}+q^n-q^2)}{2}. \end{aligned}$$

*Proof.* Set the coordinates on the three curves as $(X,Y,Z)$, $(Y,Z)$ and $Y$ respectively so that $\mathcal{X}_n$ is viewed as a Kummer cover of $\mathbb{P}^1$ and $\mathcal{C}_n$ is viewed as an Artin-Schreier cover of $\mathcal{X}_n$. The only ramified points in $\mathcal{X}_n \longrightarrow \mathbb{P}^1$ are the points $Y \in \mathbb{F}_{q^2}$ and the pole of $Y$. All of these points are totally ramified (cf. [8, Proposition III.7.3]). Hence by Riemann-Hurwitz formula, we have

$$2g(\mathcal{X}_n) - 2 = -2\frac{q^n+1}{q+1} + (q^2+1)\left(\frac{q^n+1}{q+1} - 1\right).$$

A simple manipulation yields the genus formula for $\mathcal{X}_n$.

In the covering $\mathcal{C}_n \longrightarrow \mathcal{X}_n$, the only ramification occurs at the common pole $P_\infty$ of $Y$ and $Z$. We have

$$m_{P_\infty} := -(q+1)v_{P_\infty}(Y) = (q+1)\left(\frac{q^n+1}{q+1}\right) = q^n+1, \quad \text{(cf. [8, Proposition III.7.10])}.$$

Hence the different exponent at this point is $(q-1)(q^n+2)$ and we obtain

$$2g(\mathcal{C}_n) - 2 = ((q-1)(q^n-q)-2)q + (q-1)(q^n+2).$$

From the last equation the result follows easily. □

In order to compute the number of rational points on $\mathcal{C}_n$, we will need some polynomial identities.

**Lemma 2.3.** *Let $n = 2k+1 \geq 1$ be an odd positive integer and $S := Y^{q^2} - Y \in \mathbb{F}[Y]$, where* char $\mathbb{F} = p$ *and $q$ is a power of $p$. Then we have the following:*

$$\begin{aligned} \sum_{i=1}^{k} S^{q^n+q^{2i}} &= Y^{q^{n+2}+q^{n+1}} - Y^{q^{n+2}+q^2} - Y^{q^{n+1}+q^n} + Y^{q^n+q^2} \\ \sum_{i=0}^{k} S^{1+q^{2i+1}} &= Y^{q^{n+2}+q^2} - Y^{q^{n+2}+1} - Y^{q^2+q} + Y^{q+1} \end{aligned}$$

*Proof.* We prove both equations by induction. For $n=1$, both identities hold trivially. Assume the validity of the equations for $n-2 = 2(k-1)+1$. For the first equation we have

$$\begin{aligned} \sum_{i=1}^{k} S^{q^n+q^{2i}} &= S^{q^n+q^2} + \left(S^{q^{n-2}+q^2} + \cdots + S^{q^{n-2}+q^{n-3}}\right)^{q^2} \\ &= \left(Y^{q^{n+2}+q^4} - Y^{q^{n+2}+q^2} - Y^{q^n+q^4} + Y^{q^n+q^2}\right) + \\ &\qquad \left(Y^{q^n+q^{n-1}} - Y^{q^n+q^2} - Y^{q^{n-1}+q^{n-2}} + Y^{q^{n-2}+q^2}\right)^{q^2} \quad \text{(by induction)} \\ &= Y^{q^{n+2}+q^{n+1}} - Y^{q^{n+2}+q^2} - Y^{q^{n+1}+q^n} + Y^{q^n+q^2}. \end{aligned}$$

Similarly, we have the following for the second identity:

$$\begin{aligned}\sum_{i=0}^{k} S^{1+q^{2i+1}} &= S^{1+q^n} + \left(S^{1+q^{n-2}} + \cdots + S^{1+q}\right) \\ &= \left(Y^{q^{n+2}+q^2} - Y^{q^{n+2}+1} - Y^{q^n+q^2} + Y^{q^n+1}\right) + \\ &\qquad \left(Y^{q^n+q^2} - Y^{q^n+1} - Y^{q^2+q} + Y^{q+1}\right) \quad \text{(by induction)} \\ &= Y^{q^{n+2}+q^2} - Y^{q^{n+2}+1} - Y^{q^2+q} + Y^{q+1}.\end{aligned}$$

□

We state the following simple observation without a proof.

**Lemma 2.4.** *If $i, j$ are integers with $i \not\equiv j \bmod 2$, then $(q+1)$ divides $(q^i + q^j)$.*

In the following, we obtain another nontrivial polynomial identity which plays a key role in showing that the curve $\mathcal{C}_n$ is maximal.

**Lemma 2.5.** *Let $n > 1$ be an odd integer of the form $n = 3m + r$ for some $r \in \{0, 1, 2\}$ and $m \geq 1$. Let $q$ be a power of a prime $p$ and define the following polynomials in $\mathbb{F}[Y]$, where* $\operatorname{char} \mathbb{F} = p$*:*

$$T := Y^{q+1} \qquad S := Y^{q^2} - Y \qquad T_n := T^{\frac{q^n+q^2}{q+1}} - T^q + T$$

$$B_n := T^{q^{n-1}} - T^{q^{n-2}} + \cdots - T^q + T \qquad Q_n := \sum_{j=0}^{m-1} (-1)^{r+j} \left(S^{q+1}\right)^{q^{r+3j}}$$

*Then, we have*

$$B_n - Q_n - T_n = P_n, \tag{2.3}$$

*where $P_n$ is a polynomial in $\mathbb{F}[S^{q+1}]$ with coefficients in $\{0, 1, -1\}$.*

*Proof.* Let $U := T^{\frac{q^3+1}{q+1}}$ and note that $S^{q+1} = T^{q^2} - T^q + T - U$. The proof will be given by induction on $n$. We start with the initial values $3, 5, 7$ for each value of $n$ modulo 3.
$(n = 3)$ In this case $B_3 - S^{q+1} - T_3 = -S^{q+1}$.
$(n = 5)$ We have

$$\begin{aligned}B_5 - (S^{q+1})^{q^2} - T_5 &= U^{q^2} - T^{\frac{q^5+q^2}{q+1}} \\ &= Y^{q^5+q^2} - Y^{q^5+q^2} \\ &= 0.\end{aligned}$$

$(n = 7)$ We have

$$\begin{aligned}B_7 - (S^{q+1})^{q^4} + (S^{q+1})^q - T_7 &= U^{q^4} - U^q - T^{\frac{q^7+q^2}{q+1}} + T^q \\ &= Y^{q^7+q^4} - Y^{q^7+q^2} - Y^{q^4+q} + Y^{q^2+q}.\end{aligned}$$

We claim that the last polynomial expression, $P_7(Y)$, can be written as a polynomial in $S^{q+1}$. Note that

$$\begin{aligned} P_7(Y) &= \left(Y^{q^2}-Y\right)^{q^2}\left(Y^{q^6}-Y\right)^{q} \\ &= S^{q^2}\left(S^{q^4}+S^{q^2}+S\right)^{q} \\ &= S^{q^2-1}\left(S^{q^5+1}+S^{q^3+1}+S^{q+1}\right), \end{aligned}$$

and the last expression is a polynomial in $S^{q+1}$. Hence, (2.3) also holds in this case.

Now, assume the validity of Equation (2.3) for all odd integers less than $n$ and consider the differences

$$B_n - Q_n - T_n = \begin{cases} U^{q^{n-3}} - \cdots - U^{q^3} + U - T^{\frac{q^n+q^2}{q+1}} + T^q - T & \text{, if } n \equiv 0 \bmod 3 \\ U^{q^{n-3}} - \cdots + U^{q^4} - U^q - T^{\frac{q^n+q^2}{q+1}} + T^q & \text{, if } n \equiv 1 \bmod 3 \\ U^{q^{n-3}} - \cdots - U^{q^5} + U^{q^2} - T^{\frac{q^n+q^2}{q+1}} & \text{, if } n \equiv 2 \bmod 3 \end{cases} \tag{2.4}$$

Note that the preceding odd integer with the same residue modulo 3 as $n$ is $n-6$. By induction hypothesis, we have that

$$B_{n-6} - Q_{n-6} - T_{n-6} = P_{n-6}$$

is a polynomial in $S^{q+1}$. We also have:

$$P_{n-6} = \begin{cases} U^{q^{n-9}} - \cdots - U^{q^3} + U - T^{\frac{q^{n-6}+q^2}{q+1}} + T^q - T & \text{, if } n \equiv 0 \bmod 3 \\ U^{q^{n-9}} - \cdots + U^{q^4} - U^q - T^{\frac{q^{n-6}+q^2}{q+1}} + T^q & \text{, if } n \equiv 1 \bmod 3 \\ U^{q^{n-9}} - \cdots - U^{q^5} + U^{q^2} - T^{\frac{q^{n-6}+q^2}{q+1}} & \text{, if } n \equiv 2 \bmod 3 \end{cases} \tag{2.5}$$

Combining (2.4) and (2.5), it is enough to prove that the expression below

$$\begin{aligned} A_n(Y) &= U^{q^{n-3}} - U^{q^{n-6}} - T^{\frac{q^n+q^2}{q+1}} + T^{\frac{q^{n-6}+q^2}{q+1}} \\ &= Y^{q^n+q^{n-3}} - Y^{q^{n-3}+q^{n-6}} - Y^{q^n+q^2} + Y^{q^{n-6}+q^2} \end{aligned}$$

is a polynomial in $S^{q+1}$ (for any $n$). We have

$$\begin{aligned} A_n(Y) &= Y^{q^{n-3}}\left(Y^{q^n}-Y^{q^{n-6}}\right) - Y^{q^2}\left(Y^{q^n}-Y^{q^{n-6}}\right) \\ &= \left(Y^{q^n}-Y^{q^{n-6}}\right)\left(Y^{q^{n-3}}-Y^{q^2}\right) \\ &= \left(Y^{q^6}-Y\right)^{q^{n-6}}\left(Y^{q^{n-5}}-Y\right)^{q^2}. \end{aligned}$$

Note that if $n \equiv 0, 1, 2 \pmod 3$, then $n - 5 \equiv 4, 2, 0 \pmod 6$, respectively. Hence we have

$$Y^{q^{n-5}} - Y = \begin{cases} \left(Y^{q^6} - Y\right)^{q^{n-11}} + \left(Y^{q^6} - Y\right)^{q^{n-17}} + \cdots + \left(Y^{q^6} - Y\right)^{q^4} + (Y^{q^4} - Y) & \text{, if } n \equiv 0 \bmod 3 \\ \left(Y^{q^6} - Y\right)^{q^{n-11}} + \left(Y^{q^6} - Y\right)^{q^{n-17}} + \cdots + \left(Y^{q^6} - Y\right)^{q^2} + (Y^{q^2} - Y) & \text{, if } n \equiv 1 \bmod 3 \\ \left(Y^{q^6} - Y\right)^{q^{n-11}} + \left(Y^{q^6} - Y\right)^{q^{n-17}} + \cdots + \left(Y^{q^6} - Y\right) & \text{, if } n \equiv 2 \bmod 3 \end{cases}$$

Since $Y^{q^6} - Y = S^{q^4} + S^{q^2} + S$ and $n$ is odd, this means that $\left(Y^{q^{n-5}} - Y\right)^{q^2}$ only involves terms of the form $S^{q^j}$ with $j$ even. On the other hand

$$\left(Y^{q^6} - Y\right)^{q^{n-6}} = S^{q^{n-2}} + S^{q^{n-4}} + S^{q^{n-6}},$$

i.e. it contains terms $S^{q^i}$ with $i$ odd. Therefore, the product $A_n(Y)$ is a combination of terms $S^{q^i+q^j}$, where $i \not\equiv j \bmod 2$. Hence, $A_n(Y)$ is a polynomial in $S^{q+1}$ by Lemma 2.4. □

We are now ready to prove the main result of this article.

**Theorem 2.6.** *For each odd positive integer $n$, the curve $\mathcal{C}_n$ over $\mathbb{F}_{q^{2n}}$ defined in (2.1) is maximal.*

*Proof.* Let us consider the coverings $\mathcal{C}_n \longrightarrow \mathcal{X}_n \longrightarrow \mathbb{P}^1$ as in Proposition 2.2. For $\mathcal{C}_n$ and $\mathcal{X}_n$ to be maximal, their number of $\mathbb{F}_{q^{2n}}$-rational points must satisfy the following equalities (by the genus formulae in Proposition 2.2):

$$|\mathcal{X}_n(\mathbb{F}_{q^{2n}})| = q^{2n} + 1 + (q-1)(q^n - q)q^n = q^{2n+1} - q^{n+2} + q^{n+1} + 1, \tag{2.6}$$

$$|\mathcal{C}_n(\mathbb{F}_{q^{2n}})| = q^{2n} + 1 + (q-1)(q^{n+1} + q^n - q^2)q^n = q^{2n+2} - q^{n+3} + q^{n+2} + 1. \tag{2.7}$$

We know the maximality of $\mathcal{X}_n$ by the work of Abdón et al. ([1, Theorem 1]); i.e., we know that (2.6) holds. Hence, for $\mathcal{C}_n$ to be maximal we must have

$$|\mathcal{C}_n(\mathbb{F}_{q^{2n}})| = 1 + q \cdot \left(|\mathcal{X}_n(\mathbb{F}_{q^{2n}})| - 1\right),$$

where $q$ is the degree of the covering $\mathcal{C}_n \longrightarrow \mathcal{X}_n$. It is clear that the point at infinity on $\mathcal{X}_n$ is totally ramified. So, we will show that every other $\mathbb{F}_{q^{2n}}$-rational point of $\mathcal{X}_n$ splits completely in $\mathcal{C}_n$.

Note that if $(\alpha, \beta, \gamma)$ is an $\mathbb{F}_{q^{2n}}$-rational point of $\mathcal{C}_n$, then $(\alpha + \delta, \beta, \gamma)$ is also an $\mathbb{F}_{q^{2n}}$-rational point for any $\delta \in \mathbb{F}_{q^2} \subset \mathbb{F}_{q^{2n}}$ with $\mathrm{Tr}_{q^2/q}(\delta) = \delta^q + \delta = 0$. So, we will prove the claim if we can show:

$$\text{If } (\alpha, \beta, \gamma) \text{ is on } \mathcal{C}_n \text{ with } \beta, \gamma \in \mathbb{F}_{q^{2n}}, \text{ then } \alpha \text{ also lies in } \mathbb{F}_{q^{2n}}. \tag{2.8}$$

We will think of the polynomials introduced in Lemma 2.5 as functions on $(X, Y, Z) = (\alpha, \beta, \gamma)$. Hence, $T = \beta^{q+1}$, $S = \beta^{q^2} - \beta$, $T_n = T_n(\beta^{q+1})$, $B_n = B_n(\beta^{q+1})$ and $Q_n = Q_n(S^{q+1})$.

By the first defining equation of $\mathcal{C}_n$, we have

$$B_n^{q^n} - B_n = \sum_{i=0}^{2n-1} (-1)^{i+1} T^{q^i} = X^{q^{2n}} - X.$$

Hence

$$X = \alpha \in \mathbb{F}_{q^{2n}} \iff B_n(\beta^{q+1}) \in \mathbb{F}_{q^n}. \tag{2.9}$$

Observe that if $Y = 0$, then $X^q + X = 0$ and the roots of this equation clearly lie in $\mathbb{F}_{q^2} \subset \mathbb{F}_{q^{2n}}$. So, we can assume that $Y \neq 0$. Suppose $S = \beta^{q^2} - \beta = 0$. This means that $Y = \beta \in \mathbb{F}_{q^2}$ and hence $\beta^{q+1} \in \mathbb{F}_q$. In this case, solutions $X = \alpha$ of $X^q + X = \beta^{q+1}$ lie in $\mathbb{F}_{q^2} \subset \mathbb{F}_{q^{2n}}$, i.e. (2.8) holds. So, we may only consider $S \neq 0$ in the rest of the proof.

By Hilbert's Theorem 90 and the second defining equation of $\mathcal{C}_n$, we have that $Y = \beta$ is in $\mathbb{F}_{q^{2n}}$ if and only if

$$\mathrm{Tr}_{q^{2n}/q^2}(S) = S + S^{q^2} + \cdots + S^{q^{2(n-1)}} = 0.$$

Multiplying both sides of the last equation by $S^{q^n}$, we obtain

$$S^{1+q^n} + S^{q^2+q^n} + \cdots + S^{q^{2(n-1)}+q^n} = 0. \tag{2.10}$$

Since $Z = \gamma \in \mathbb{F}_{q^{2n}}$, we have that its norm $\gamma^{q^n+1} = S^{q+1}$ lies in $\mathbb{F}_{q^n}$. Therefore, by Lemma 2.4, $S^{q^j+1} \in \mathbb{F}_{q^n}$ for any odd integer $j$ and we have

$$S^{q^{n+j}+q^n} = \left(S^{q^j+1}\right)^{q^n} = S^{q^j+1}.$$

Hence, (2.10) can be written as

$$\begin{aligned}
0 &= \left(S^{q^2+q^n} + S^{q^4+q^n} + \cdots + S^{q^{n-1}+q^n}\right) + \left(S^{1+q} + S^{1+q^3} + \cdots + S^{1+q^n}\right) \\
&= -\beta^{q^{n+2}+1} + \beta^{q^{n+2}+q^{n+1}} - \beta^{q^{n+1}+q^n} + \beta^{q^n+q^2} - \beta^{q^2+q} + \beta^{q+1} \qquad \text{(by Lemma 2.3)} \\
&= -\left(\beta^{q^n+q^2} - \beta^{q^2+q} + \beta^{q+1}\right)^{q^n} + \left(\beta^{q^n+q^2} - \beta^{q^2+q} + \beta^{q+1}\right) \\
&= -T_n(\beta^{q+1})^{q^n} + T_n(\beta^{q+1}).
\end{aligned}$$

We know by Lemma 2.5 that

$$B_n = Q_n + P_n + T_n,$$

where $Q_n, P_n$ are polynomials in $S^{q+1} \in \mathbb{F}_{q^n}$ with coefficients in $\{0, 1, -1\}$. We showed above that $T_n(\beta^{q+1}) \in \mathbb{F}_{q^n}$. Hence $B_n(\beta^{q+1}) \in \mathbb{F}_{q^n}$, which finishes the proof by (2.9). □

## References


[1] Abdón, M., Bezerra, J., Quoos, L., "Further examples of maximal curves", preprint, 2007.

[2] Abdón, M., Garcia, A., "On a characterization of certain maximal curves", *Finite Fields Appl.*, vol. 10, pp. 133-158, 2004.

[3] Abdón, M., Torres, F., "On maximal curves in characteristic two", *Manuscripta Math.*, vol. 99, pp. 39-53, 1999.

[4] Fuhrmann, R., Garcia, A., Torres, F., "On maximal curves", *J. Number Theory*, vol. 67, pp.29-51, 1997.

[5] Garcia, A., Stichtenoth, H., "A maximal curve which is not a Galois subcover of the Hermitian curve", *Bull. Braz. Math. Soc. (N.S.)*, vol. 37, no. 1, pp. 139-152, 2006.

[6] Giulietti, M., Korchmáros, G., "A new family of maximal curves over a finite field", preprint, 2007.
[7] Lachaud, G., "Sommes d'Eisenstein et nombre de points de certaines courbes algébriques sur les corps finis", *C.R. Acad. Sci. Paris, Serie I*, vol. 305, pp. 729-732, 1987.
[8] Stichtenoth, H., *Algebraic Function Fields and Codes*, Springer-Verlag, Berlin, 1993.