Pinball billiards with dominated splitting

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Abstract

We study the dynamics of a type of nonconservative billiards where the ball is "kicked" by the wall giving a new impulse in the direction of the normal. For different types of billiard tables we study the existence of attractors with dominated splitting.

1 Introduction. Statement of main results.

Billiards are mathematical models for many physical phenomena where one or more hard balls move in a container and have elastic collisions with its walls and/or with each other. In the usual models a point particle moves with constant velocity on a Riemannian manifold with boundaries. When the particle collides with the boundary its velocity vector gets instantaneously reflected across the tangent line following the classical rule: the angle of incidence is equal to the angle of reflection. The dynamical properties of such models are determined by the shape of the boundaries of this manifold. They may vary from completely regular (integrable) to fully chaotic. Examples of the last ones are the dispersing billiard tables due to Ya. Sinai, introduced as a model of hard balls studied by L. Boltzmann in the XIX century and the Lorentz gas introduced to describe electricity in 1905. In his paper [Si70] showed that *billiards with dispersing walls are prototypes of hyperbolic dynamics.* In contrast, billiards induced by polygonal tables are integrable and so they are non-hyperbolic, although they are generically ergodic [KMS].

The dynamics of "classical" billiards are prototypes of conservative dynamics: the Liouville measure is preserved. Therefore, these billiards are not useful to model rich phenomena that could hold in regimes far from the equilibrium. In this direction, moving towards overcoming these restrictions, in [CELS] one obtains several results about nonequilibrium states in the Lorentz gas, studying the dynamics of a system defined by a single particle travelling in a billiard table (bouncing off the scatterer with elastic collisions) and such that the particle is subjected to an electric field and a momentum-dependent frictional force between collisions with the scatterer. This unusual frictional force is chosen so that the total kinetic energy of the system is conserved although the dynamics do not preserve Liouville measure. The deep study in this system depends on the rather detailed knowledge that it has properties of hyperbolic type (e.g. existence of stable and unstable manifolds and rate of decay of correlations) for billiard systems.

Other types of nonconservative billiards are the *pinball billiards*. The particle moves along straight lines inside the billiard table and when it hits one of the walls with angle α with respect to the normal, it is reflected with angle $\lambda\alpha$ with respect to normal line (with $\lambda \leq 1$): the ball is "kicked" by the wall giving a new impulse in the direction of the normal and thereby increasing its kinetic energy. After a number of collisions this system ends up like a "*particle accelerator*". In fact, we introduce two kinds of perturbations **A1a -A1b** (see Subsection 3.3 for details), depending on the non-uniformity or uniformity of the contraction. Two conditions **A2 - A3**, are introduced for obtaining results on billiards with focusing boundaries and non-uniform contraction (see Subsection 4.4 for details).

We study plane billiards whose boundaries are C^3 curves and develop general formulas for a general class of such *pinball billiards*. Then we restrict the analysis to some perturbation on the reflection angle which only depend on the incidence angle (not on the position on the boundary).

The dynamics of the billiard maps induced by those pinball billiards have a weak form of hyperbolicity called dominated splitting. In our two dimensional case this means that the tangent bundle splits into two invariant directions, the contractive behavior on one of them dominates the other one by a uniform factor. Under the assumption of dominated splitting it is possible to obtain a spectral decomposition for the limit set (see theorem B in section 2.1).

Theorem 1. *The pinball billiard map associated to a billiard table satisfying Assumption* **A1a** *with non negative curvature (semidispersing walls) such that for a fixed $m \in \mathbb{N}$ there are not trajectories with more than m successive bounces on flat points and whose smooth components of the boundary intersect with angle greater than zero has a dominated splitting.*

Theorem 2. *The pinball billiard map associated to a billiard table satisfying Assumption* **A1b** *with non negative curvature (semidispersing walls) has a dominated splitting.*

The second theorem include billiards with cusps and polygonal billiards. It clearly results from the proof (in subsection 4.3) that focusing curves with curvature close to zero are also admitted; so C^3-curves with inflection points can make part of the boundaries.

Theorem 3. *Consider the pinball billiard map associated to a billiard table bounded by C^3 curves that are C^2 close to circle, such that $K_0 + K_1(t_0K_0 + 1) \geq 0$. If it satisfies Assumption* **A1b** *it has dominated splitting.*

Theorem 3a. *Consider the pinball billiard map associated to a billiard table bounded by C^3 curves that are C^2 close to circle, such that $K_0 + K_1(t_0K_0 + 1) \geq 0$. If it satisfies Assumption* **A1a** *and* **A2** *it has dominated splitting.*

Theorem 4. *Consider the pinball billiard map associated to a billiard table with focusing components satisfying Wojtkowski conditions for a elastic billiard map being hyperbolic (non-vanishing Lyapunov exponents). If it satisfies Assumption* **A1b** *it has dominated splitting.*

Theorem 4a. *Consider the pinball billiard map associated to a billiard table with focusing components satisfying Wojtkowski conditions for a elastic billiard map being hyperbolic (non-vanishing Lyapunov exponents). If it satisfies Assumptions* **A1a**, **A2** *and* **A3** *it has dominated splitting.*

In view of theorem B (see next section), we conclude:

Theorem 5. *The pinball billiard maps referred in Theorems 1 - 4 admits a spectral decomposition (as in theorem B) on any compact invariant set that neither contains trajectories that finish on a corner of the table nor are tangent to the boundary of the table.*

There is an extreme case of the one that we considered before: the particle moves along straight lines inside the billiard table and it reflects at the boundary along the normal line. We call these billiards, *slap billiard maps* and they induce a one-dimensional map T defined on the union of a finite number of arcs of finite length. The theorems about these special case are enunciated within section 5. Moreover, in subsection 5.3 we consider small "two-dimensional perturbations" of these slap billiards. The example there gives a general class of examples of pinball billiards.

In section 2 we introduce the notion of dominated splitting, the dynamical consequences of this weak form of hyperbolicity, and its relation with the cone fields and quadratic forms. In section 3 we introduce the formal definition of pinball billiards and we list all the assumptions involved in the theorems stated before. In section 4 we give the proof of theorems 1 - 4.

2 Dominated splitting

Consider the diffeomorphism $f : M \to M' \subset M$, where M is a riemannian manifold. An f-invariant set Λ is said to have a *dominated splitting* if we can decompose its tangent bundle in two invariant continuous subbundles $T_\Lambda M = E \oplus F$, such that:

$$\|Df^n_{|E(x)}\| \; \|Df^{-n}_{|F(f^n(x))}\| \leq Ca^n, \text{ for all } x \in \Lambda, n \geq 0. \tag{1}$$

with $C > 0$ and $0 < a < 1$; *a is called a constant of domination.*

Of course, it is assumed that neither of the subbbundless is trivial (otherwise, the other one has a uniform hyperbolic behavior: contracting or expanding). Also observe that any hyperbolic splitting is a dominated one.

Let us explain briefly the meaning of the above definition: it says that, for n large, the "greatest contraction" of Df^n on E is less than the "greatest expansion" of Df^n on F and by a factor that becomes exponentially small with n. In other words, every direction not belonging to E must converge exponentially fast under iteration of Df to the direction F.

For completeness we give some other definitions.

As usual, let the *limit set* be

$$L(f) = \overline{\bigcup_{x \in M} (\omega(x) \cup \alpha(x))}$$

where $\omega(x)$ and $\alpha(x)$ are the ω and α-limit sets of x, respectively. A point $x \in M$ is *nonwandering* with respect to f if for any open set containing x there is a $N > 0$ such that $f^N(U) \cap U \neq \emptyset$. The set of all nonwandering points of f is denoted by $\Omega(f)$. A set $B \subset M$ is called *transitive* if there exists a point $x \in B$ such that its orbit $\{f^n x\}_{n \in \mathbb{Z}}$ is dense in B.

A diffeomorphism $f : M \to M$ is called *Morse-Smale* if $\Omega(f)$ consists of hyperbolic fixed or periodic points, whose stable and unstable manifolds are transversal. The point $x \in M$ is said *homoclinic* to the point $y \in M$ if $\lim_{|n| \to \infty} d(f^n x, f^n y) = 0$. y is a *transversal homoclinic point* if $y \in W^s(x) \cap W^u(x)$ for a fixed point x.

A compact invariant submanifold V is *normally hyperbolic* if the tangent space to the ambient manifold at any point x can be decompose in three invariant continuous subbundles $T_V M = E^s \oplus TV \oplus E^u$, such that:

$$\inf_x m(D_x f_{|E^u(x)}) > \sup_x \|D_x f_{|TV(x)}\|, \quad \sup_x \|D_x f_{|E^s(x)}\| < \inf_x m(D_x f_{|TV(x)}) \tag{2}$$

where the minimum norm $m(A)$ of a linear transformation A is defined by $m(A) = \inf\{\|Au\| : ||u|| = 1\}$.

2.1 Consequences of dominated splitting. Proof of theorem 5

One of the main goals in dynamics is to understand how the dynamics of the tangent map Df controls or determines the underlying dynamics of f. Actually, this paradigm is motivated by the success of the hyperbolic theory.

In fact, assuming that the limit set $L(f)$ splits into two subbundles, $T_{L(f)}M = E^s \oplus E^u$, invariant under Df and vectors in E^s are contracted by positive iteration of the tangent map (the same holding for E^u but under negative iteration), Smale [S] proved that $L(f)$ can be decomposed into the disjoint union of finitely compact maximal invariant and transitive sets. Moreover, the periodic points are dense in $L(f)$ and the asymptotic behavior of any point in the manifold is represented by an orbit in $L(f)$.

A similar spectral decomposition theorem as the one stated for hyperbolic dynamics holds for smooth surface diffeomorphisms exhibiting a dominated splitting.

Theorem A ([PS00]): *Let M be a compact 2-manifold and f a C^2-diffeomorphism defined as before. Assume that $\Lambda \subset \Omega(f)$ is a compact invariant set exhibiting a dominated splitting such that every periodic point in Λ is hyperbolic. Then $\Lambda = \Lambda_1 \cup \Lambda_2$ where Λ_1 is a hyperbolic set and Λ_2 consists of a finite union of periodic simple closed curves $\mathcal{C}_1, ..., \mathcal{C}_n$, normally hyperbolic, and such that $f^{m_i} : \mathcal{C}_i \to \mathcal{C}_i$ is conjugate to an irrational rotation (m_i denotes the period of $\mathcal{C}_i$).*

A similar description can be obtained for C^2 surface diffeomorphisms having dominated splitting over the limit set $L(f)$ without any assumption on the hyperbolicity of the periodic points:

Theorem B ([PS08]): *Let M be a compact 2-manifold and f a C^2-diffeomorphism defined as before. Assume that $L(f)$ has a dominated splitting. Then $L(f)$ can be decomposed into $L(f) = \mathcal{I} \cup \tilde{\mathcal{L}}(f) \cup \mathcal{R}$ such that*

1. *$\mathcal{I}$ is a set of periodic points with bounded periods contained in a disjoint union of finitely many normally hyperbolic periodic arcs or simple closed curves.*
2. *$\mathcal{R}$ is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.*
3. *$\tilde{\mathcal{L}}(f)$ can be decomposed into a disjoint union of finitely many compact invariant and transitive sets (called basic sets). The periodic points are dense in $\tilde{\mathcal{L}}(f)$ and at most finitely many of them are non-hyperbolic periodic points. The (basic) sets above are the union of finitely many (nontrivial) homoclinic classes. Furthermore $f|\tilde{\mathcal{L}}(f)$ is expansive.*

Roughly speaking, the above theorem says that the dynamics of a C^2 surface diffeomorphism having a dominated splitting can be decomposed into two parts: one where the dynamics consists of periodic and almost periodic motions ($\mathcal{I}$, $\mathcal{R}$) with the diffeomorphism acting equicontinuously; and another, where the dynamics are expansive.

Theorem B also can be formulated in a more general version. Let us take a compact invariant set Λ contained in M and define $L(f_{|\Lambda}) = \overline{\bigcup_{x\in\Lambda}(\omega(x) \cup \alpha(x))}$. Then we have a similar version of theorem B in the present context. Next, we can apply this version to the billiards described in theorems 1 - 4 concluding the proof of theorem 5.

2.2 Quadratic forms and dominated splitting

In this Subsection we recall a general method for establishing hyperbolic properties of dynamical systems. (see, for example, [M88, CM06]) Let M be a compact Riemannian manifold (perhaps, with boundary and corners) of dimension d, $N \subset M$ an open and dense subset and $F : N \to M$ a C^r (with $r \geq 1$) diffeomorphism of N onto $F(N)$. N is the union of a finite number of open connected sets M_i^+. Note that all the iterations of F are defined on the set

$$\tilde{M} = \cap_{n=-\infty}^{\infty} F^n(N).$$

Let m be the Lebesgue measure on M. We will assume that $\tilde{M}$ has full measure: $m(M) = m(\tilde{M})$.

We recall that a quadratic form Q in $\mathbb{R}^d$ is a function $Q : \mathbb{R}^d \to \mathbb{R}$ such that $Q(u) = Q_2(u, u)$, where Q_2 is a bilinear symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$. Equivalently, $Q : \mathbb{R}^d \to \mathbb{R}$ is a quadratic form if there is a symmetric matrix A such that $Q(u) = u^T A u$ for $u \in \mathbb{R}^d$ (here u^T means transposition of a column-vector u). The *positive index of inertia of* Q is the number of positive eigenvalues of the matrix that defines the form Q (i.e., the maximal dimension of a subspace of $\mathbb{R}^d$ on which the form is positive).

A *quadratic form* Q on M is a function $Q : \mathcal{T}M \to \mathbb{R}$ such that its restriction Q_x to $\mathcal{T}_xM$ at m-almost every point $x \in M$ is a quadratic form in the usual sense.

We say that a quadratic form Q is *nondegenerate* at x if for every nonzero vector $u \in \mathcal{T}_xM$, there exists a $v \in \mathcal{T}_xM$ such that $Q_2(v, u) \neq 0$ (equivalently, $\det A \neq 0$ for the corresponding symmetric matrix A). We say that Q is *positive* (nonnegative) if at every point x the form Q_x is positive definite (positive semidefinite); i.e. $Q_x(u) > 0$ (respectively, $Q_x(u) \geq 0$) for all $0 \neq u \in \mathcal{T}_xM$.

We denote by $F^\# Q$ (the pullback of Q by F) the function defined by $(F^\# Q)_x u = Q_{F(x)}(D_xFu)$. One can easily verify that $F^\# Q$ is also a quadratic form, and that $F^\# Q$ is nondegenerate at x iff Q is nondegenerate at $F(x)$. We note that $P = F^\# Q - Q$ is a quadratic form, too.

Let be Q a nondegenerate quadratic form defined on $\mathcal{T}M$ with positive index of inertia equal to p and negative index of inertia equal to n, $p + n = d, p \geq 1, n \geq 1$, for every $x \in M$. We assume that Q is continuous on each M_i^+ and denote by

$$C_\pm(x) = \{v \in \mathcal{T}_xM : \pm Q_x(v) > 0\} \cup \{0\}$$

the open cones of, respectively, positive and negative vectors (with the zero vector included), and by $C_0(x)$ their common boundary, $C_0(x) = \{v \in \mathcal{T}_x M : Q_x(v) = 0\}$.

Let P be a linear subspace of dimension p contained in $C_+(x)$ and N the Q-orthogonal complement of P. The dimension of N is n and it is contained in $C_-(x)$. We can introduce an auxiliary scalar product in $\mathcal{T}_x M$ by using Q in P and $-Q$ in N. We choose coordinates in $P = \mathbb{R}^p$ and $N = \mathbb{R}^n$ in such a way that Q and $-Q$ become the arithmetic scalar products in $\mathbb{R}^p$ and $\mathbb{R}^n$, respectively. We obtain the following coordinate representation: $\mathcal{T}M = \mathbb{R}^p \times \mathbb{R}^n$, and for $v = (v_1, v_2), v_1 \in \mathbb{R}^p, v_2 \in \mathbb{R}^n$,

$$Q(v) = \langle Jv, v\rangle = v_1^2 - v_2^2, \text{ where } J = \begin{pmatrix} I_p & 0 \\ 0 & -I_n \end{pmatrix},$$

I_p, I_n are the identity matrices in $\mathbb{R}^p, \mathbb{R}^n$, respectively.

Definition 1. *$D_xT : \mathcal{T}_x M \to \mathcal{T}_{Tx} M$ is*

1. *Q-separated if $D_xTC_+(x) \subset C_+(Tx)$,*
2. *strictly Q-separated, if $D_xT[(C_+(x)] \cup C_0(x)) \subset C_+(Tx)$,*
3. *Q-monotone, if $Q_{Tx}(D_xTu) \geq Q_x(u)$ for every $u \in \mathcal{T}_x M$,*
4. *strictly Q-monotone if $Q_{Tx}(D_xTu) > Q_x(u)$ for every $u \in \mathcal{T}_x M, u \neq 0$,*
5. *Q-isometry, if $Q_{Tx}(D_xTu) = Q_x(u)$ for every $u \in \mathcal{T}_x M$.*

In [Wo01], following some remarkable works by V. P. Potapov, it is proved that

1. If DT is Q-separated then the set of positive numbers r such that $\frac{1}{r}DT$ is Q-monotone is a closed interval possibly degenerating to a point. In fact, from Theorem 1.2 in [Wo01] it results that $r \in [r_-, r_+],\ r_- > 0$, with

$$r_-^2(x) = \sup_{u \in C_-(x)} \frac{Q_{Tx}(D_xTu)}{Q_x u}\ , \quad r_+^2(x) = \inf_{u \in C_+(x)} \frac{Q_{Tx}(D_xTu)}{Q_x u}\ . \tag{3}$$

2. If DT is strictly Q-separated then the set of positive numbers r such that $\frac{1}{r}DT$ is strictly Q-monotone is an open interval: $(r_-, r_+), r_- > 0$.

Definition 2. *$DT : TM \to TM$ is* eventually uniformly strictly Q-separated *(euss) at x if it is Q-separated in every point $T^i(x),\ i \in \mathbb{Z}$ of the orbit of x, and there exist constants $n \geq 1$ and $0 < d < 1$ (not depending on x and n) such that for each $k \geq 0$*

$$\#\left\{i : D_{F^{k+i}x}FC_+(T^{k+i}x) \textit{ is not strictly contained in } C_+(F^{k+i+1}x)\right\} \leq n \text{ and}$$

$$\#\left\{j : 0 \leq j \leq n,\ \frac{r_-(T^{k+j}x)}{r_+(T^{k+j}x)} \leq d\right\} > 0\ . \tag{4}$$

Definition 3. *The diffeomorphism F is euss in an invariant set N if DF is euss at each point $x \in N$.*

In the hyperbolic setting $r_-(x), (r_+(x))$ correspond to the weaker contraction (minimal expansion) of the derivative at the point x

Proposition 6. *If the diffeomorphism F is euss in an invariant set N then N has a dominated splitting.*

Proof. The proof is similar to the proof of Proposition 4.1 in [Wo01] (see also Proof of Theorem 1 in [M94]). Conditions for F being euss are automatically satisfied in the original proof because it is assumed that F acts on a compact manifold.

If F preserves a probability measure, the exponential contraction of the diameter of the manifold of (positive) linear subspaces contained in C_+ is obtained by standard methods (using the Birkhoff Ergodic Theorem). Since in the present context we do not have a natural invariant measures, we introduce the notion of "eventually uniformly strictly separation" given in definition 2. The conditions on $\frac{r_-^2}{r_+^2}$ allows to construct the invariant direction $F(z)$ as intersection of the nested family of cones obtained as the m-th positive iteration of the cones defined by the quadratic form at $T^{-m}(z)$.

We can consider the map T^n, and the numbers

$$r_-^{(n)}(y) = \sup_{u \in C_-(y)} \frac{Q_{T^n y}(D_y T^n u)}{Q_y u} \ , \quad r_+^{(n)}(y) = \inf_{u \in C_+(y)} \frac{Q_{T^n y}(D_y T^n u)}{Q_y u} \ ,$$

for any y in the trajectory of x.

The number $\sup_y \frac{r_-^{(n)}(y)}{r_+^{(n)}(y)}$ computed for this map will be uniformly less than one. In fact

$$r_-^{(n)}(y) = \sup_{u \in C_-(y)} \prod_{j=1}^{n} \frac{Q_{T^j y}(D_y T^j u)}{Q_{T^{j-1} y}(D_y T^{j-1} u}) \ \leq \ \prod_{j=1}^{n} \sup_{v \in C_-(T^{j-1} y)} \ \frac{Q_{T^j y}(D_y T^j v)}{Q_{T^{j-1} y}(D_y T^{j-1} v)} \ .$$

Then

$$\frac{r_-^{(n)}(y)}{r_+^{(n)}(y)} \leq \prod_{j=1}^{n} \frac{r_-(T^j(y))}{r_+(T^j(y))} \leq d < 1$$

since all the r_-/r_+ factors are smaller or equal than one, and one of them is $\leq d$ by (4)

Finally, the number a that appears in formula (1) is proportional to $d^{1/n}$. □

3 Billiards

Let B be an open bounded and connected subset of the plane whose boundary consists of a finite number of closed C^3-curves $\Gamma_i, \ i = 1, \cdots, k$. In order to simplify all the computations we will assume that B is simple connected.

3.1 Elastic billiards

We begin with elastic classical billiards. For details, see [CM06], Ch. 2. The billiard in B is the dynamical system describing the free motion of a point mass inside B with elastic reflections at the boundary $\Gamma = \cup_{i=0}^{k} \Gamma_i$ where each Γ_i is a C^3 compact curve. Let $n(q)$ be the unit normal of the curve Γ at the point $q \in \Gamma$ pointing toward the interior of B. The phase space of such a dynamical system is given by

$$M = \{(q, v) : \quad q \in \Gamma, \quad |v| = 1, \quad \langle v, n(q) \rangle \ \geq \ 0\}.$$

Let π denote the projection of M onto B, i.e., $\pi(q, v) = q$.

We introduce the set of coordinates (r, ϕ) on M where r is the arc length parameter along Γ and ϕ is the angle between v and the inward normal vector $n(q)$ to the boundary at q. Clearly $-\pi/2 \leq \phi \leq \pi/2$ and $\langle n(q), v \rangle \ = \ \cos\phi$. A natural probability measure on M is $d\nu = c \cos\phi \, dr d\phi$ where $c = (2|\Gamma|)^{-1}$ is the normalizing factor and $|\Gamma|$ stands for the total length of Γ.

The elastic billiard map T is defined by $T(q_0, v_0) = (q_1, v_1)$ where q_1 is the point of Γ hit first by the oriented line through (q_0, v_0) and v_1 is the velocity vector after the reflection at q_1. Formally, $v_1 = v_0 - 2\langle n(q_1), v_0 \rangle n(q_1)$. The angle between v_i and the normal vector $n(q_i)$ at q_i is denoted by ϕ_i, and the Euclidean distance between the bouncing points q_i and q_{i+1} is denoted by t_i. Since the speed of the point mass is one, then t_i is also the time between q_i and q_{i+1}. The negative iterates $z_i = (q_i, v_i), i < 0$ of z_0 are defined analogously. The main relations are $Tz_i = z_{i+1}$ and $q_{i+1} = q_i + t_i v_i$ with $i \in \mathbb{Z}$.

The map T is piecewise C^2. It is not defined at z_0 if $n(q_1)$ is not uniquely defined or if the oriented line through z_0 is tangent to some Γ_k ($\phi_1 = \pm\pi/2$). Finally T is continuous but not differentiable at z_0 if Γ is C^1 but not C^2 at q_1.

The map T preserves the measure ν. The set of points $x = (q, v) \in M$ whose forward or backward trajectory is tangent to Γ or ends in $\Gamma_i \cap \Gamma_j$ has ν-measure zero.

If T is well defined and differentiable at $\widetilde{z}_0 = (\widetilde{r}_0, \widetilde{\phi}_0)$, then for all $z_0 = (r_0, \phi_0)$ in a small neighborhood of $\tilde{z}_0$ the derivative matrix of T is given by

$$D_{z_0}T = -\begin{pmatrix} \frac{t_0 K_0 + \cos\phi_0}{\cos\phi_1} & \frac{t_0}{\cos\phi_1} \\ K_1 \frac{\cos\phi_0 + t_0 K_0}{\cos\phi_1} + K_0 & \frac{K_1 t_0}{\cos\phi_1} + 1 \end{pmatrix} \tag{5}$$

where $K_i = K(z_i),\ i = 0, 1$, is the curvature of Γ at q_i ($K(q)$ is positive if the component of the boundary by q is dispersing with non-zero curvature).

If the curvatures at both q_0, q_1 are not zero, then (5) can be rewritten as

$$-\begin{pmatrix} \frac{t_0 + d_0}{r_0 \cos\phi_1} & \frac{t_0}{\cos\phi_1} \\ \frac{t_0 + d_0 + d_1}{r_0 d_1} & \frac{t_0 + d_1}{d_1} \end{pmatrix} \tag{6}$$

where $r_i = 1/K_i, i = 0, 1$, is the radius of curvature of Γ at q_i and $d_i = r_i \cos\phi_i, i = 0, 1$. Note that if $K_i < 0$ (focusing component), then $-d_i$ is the length of the subsegment of $\overline{q_0 q_1}$ contained in the disk $D(q_i)$ tangent to Γ at q_i with radius $r_i/2$ (*half-osculating disk*).

We remark the main differences with other usual conventions related with these formulas: the curvature of dispersing curves is positive, the angle ϕ, is measured counterclockwise from v to $n(q)$, and goes from $-\pi/2$ to $\pi/2$.

3.2 Pinball Billiards

We will consider perturbations on the angle of reflection. The pinball billiard map will be $P(r_0, \phi_0) = (r_1, \phi_1)$ where r_1 is obtained as in the usual billiard (moving along the direction determined by ϕ_0 beginning at the boundary point determined by r_0) and

$$-\pi/2 \le \phi_1 = -\eta_1 + f(r_1, \eta_1) \le \pi/2 \tag{7}$$

where η_1 is the angle from the incidence vector at q_1 to the outward normal $-n(q_1)$ and $f : [0, |\Gamma|] \times [-\pi/2, \pi/2] \to \mathbb{R}$ is a C^2 function.

The derivative $D_{x_0}P$ of this map at $x_0 = (r_0, \phi_0)$ is given by

$$-\begin{pmatrix} A & B \\ (K_1 A + K_0)(1 - f_\eta(r_1, \eta_1)) + A\ f_r(r_1, \eta_1) & (K_1 B + 1)(1 - f_\eta(r_1, \eta_1)) + B f_r(r_1, \eta_1) \end{pmatrix} \tag{8}$$

$$\text{where} \quad A = \frac{t_0 K_0 + \cos\phi_0}{\cos\eta_1} \ ; \quad B = \frac{t_0}{\cos\eta_1}$$

This formula looks quite strange because it includes the angle of reflection and the angle η of incidence in the perturbed billiard. If $f(r, \eta) \equiv 0$, then $\phi = -\eta$ and we have a elastic billiard map whose derivatives are given in (5).

Remark 7. *We observe that if we restrict ourselves to the case $f = f(\eta)$ (it depends only on the angle of incidence) and $f_\eta = f' = 1$, then the reflecting angle is constant, ϕ_0. Its hyperbolicity could be studied using these formulae, but it is not necessary, because the resulting one dimensional dynamical system has derivative $\frac{t_0 K_0 + \cos\phi_0}{-\cos\eta_1}$ which shows that its dynamical behavior depends on relations between the curvature K_0 and the distance between bouncing points. In the last Section we will do a detailed study of the slap billiard, $\phi_0 = 0$.*

3.3 Assumptions on the perturbation of the reflection.

We now introduce the assumptions used in the theorem listed in the introduction. Assumptions **A1a** and **A1b** exclude each other.

A1. We assume that the perturbation depends only on the angle of incidence: $f = f(r, \eta) = f(\eta)$ for $-\pi/2 \le \eta \le \pi/2$, with $\eta \times f(\eta) \ge 0$. Let us call $\lambda(\eta) = 1 - f'(\eta)$; $\lambda_i = 1 - f'(\eta_i)$.

A1a. We also assume that $f(-\pi/2) = f(0) = f(\pi/2) = 0$, and that there exist $D_1 > 0, D_2 > 1$ such that $D_1 < \lambda(\eta) < D_2$. Typical models for these f's are $\mu \ \sin 2\eta$ and (in this case we describe a half of the curve) $\mu \ \eta(\pi/2 - \eta)$ for $0 \le \eta \le \pi/2$, in both cases with $\mu \ll 1/2$.

A1b. We also assume that $f(0) = 0$, and that $0 \le \lambda(\eta) \le \lambda = \max\{\lambda(\eta) : -\pi/2 \le \eta \le \pi/2\}$ where $0 < \lambda < 1$. A typical model for this case is $\lambda(\eta) = \lambda < 1$: there is uniform contraction, $f(\eta) = (1 - \lambda)\eta$ and the angle of reflection is $\phi = -\lambda\eta$ for $-\pi/2 \le \eta \le \pi/2$.

Condition **A1a** on the values of f assure that P is a diffeomorphism, between two open sets of ν-measure one. Condition **A1b** on the values of f assure that P is a injective.

Any alternatives mean that the trajectory moves approaching to the normal line in the reflection point: the absolute value of the angle (with the normal line) of reflection is smaller than or equal to the angle of incidence.

The derivative $D_{x_0}P$ of this map at $x_0 = (r_0, \phi_0)$ is now given by

$$-\begin{pmatrix} A & B \\ (K_1 A + K_0)\lambda_1 & (K_1 B + 1)\,\lambda_1 \end{pmatrix} \quad \text{where } A \text{ and } B \text{ are as in formula (8):} \tag{9}$$

$$A = \frac{t_0 K_0 + \cos\phi_0}{\cos\eta_1} \quad ; \quad B = \frac{t_0}{\cos\eta_1}.$$

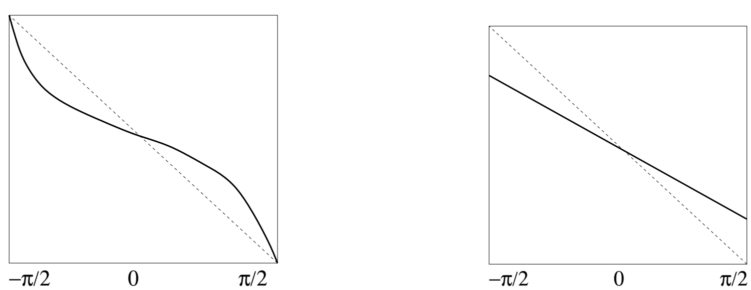

Figure 1: Assumptions A1a and A1b.

3.4 Wave fronts and cones

We want to study the behavior of the derivative map. It is frequently more convenient to work with a bundle of rays in the configuration space B rather than directly with vectors in $\mathcal{T}M$. So, we make a short incursion in wave fronts and cone fields.

Let us take an orthogonal cross-section of that bundle, which passes through the point $x = (q, v)$. We call that cross-section Σ. It is a curve in B that intersects every ray of our bundle perpendicularly. Velocity vectors of the points on that curve are thus normal vectors to it. Hence, Σ is smooth curve equipped with a family of normal vectors pointing in the direction of motion. We call Σ a *wave front*, the term borrowed from physics.

The curvature of the front Σ plays a crucial role in our analysis. The sign of the curvature is chosen according to the following rule. If the front Σ is *divergent*, then its curvature is positive. If the front is *convergent*, its curvature is negative. If the front is made by parallel rays (Σ is then a perpendicular line), then the curvature is zero, and such fronts are said to be *neutral*.

We now turn to exact equations describing the dynamics of wave fronts in elastic billiards (for details, see [CM06]). Then, we will repeat the analysis for pinball billiards.

The evolution of the curvature during time intervals $(0, t)$ with no reflections on the boundary is very simple:

$$\chi_t = \frac{1}{t + 1/\chi_0} \tag{10}$$

where χ_0 the curvature of the front at the point q,

When the wave front bounces off the boundary, its curvature instantaneously jumps. One can say that the curvature of the boundary is then "combined" with the curvature of the front itself. The following is one of the basic laws of geometric optics known as *mirror equation*. Denote by χ_- and χ_+ the curvature of the front before and after reflection, respectively. Also, let K_1 be the curvature of the boundary at the point of reflection (whose sign is set by the above mentioned rules), and ϕ_1 the angle of reflection. Then the mirror equation reads

$$\chi_+ = \chi_- + \frac{2K_1}{\cos\phi_1} \tag{11}$$

The mirror equation changes to

$$\chi_+ = \frac{1}{cos\phi_1}\left[\chi_-(1 - f_1')\cos\eta_1 + K_1(2 - f_1')\right] \tag{12}$$

The proof is quite involved, but it can help to "see" better what is going on[1]. One can compute the curvature of the wave front at any time t. In particular, if χ_0 and χ_1 are the curvatures of the wave front leaving at the points q_0 and q_1, respectively, we have

$$\chi_1 = \chi_+ = \frac{1}{cos\phi_1}\left(\frac{\lambda_1\cos\eta_1}{t_0 + 1/\chi_0} + K_1(1 + \lambda_1)\right) \tag{13}$$

Now, we state a relation between the curvature of a wave front just at the moment after a reflection and vectors in the tangent space of the phase space M. Let $x = (q, v) \in M$ be a point having coordinates (r, ϕ) and $u = (dr, d\phi) \in \mathcal{T}_x M$ a tangent vector at x. Note that dr and $d\phi$ are infinitesimal quantities. Every tangent vector $u \neq 0$ can be represented by a curve in M, for example $(r + h\,dr, \phi + h\,d\phi)$, where $0 < h < \varepsilon$ is a small parameter. This curve gives a bundle of outgoing trajectories, whose cross-section is a wave front Σ passing through the point q. The curvature of this front (χ_+) at the point q can be now computed as

$$\chi_+ = \frac{1}{\cos\phi}\left(\frac{d\phi}{dr} + K(s)\right) \tag{14}$$

This follows from infinitesimal analysis like we used in the computation of the 'perturbed mirror equation' (12).

4 Proofs of theorems 1 - 4.

To prove the theorems we apply the results of Section 2.2. In each case we will define a cone field (and consequently a quadratic form) in such a way that the numbers $r_-(x), r_+(x)$ satisfy the conditions of Proposition 6. This essentially means that $\frac{r_-^2(x)}{r_+^2(x)}$ must be uniformly smaller than one. See Definition 2.

[1]See, for example [CM03], Figure IV.2.; in this book, our ϕ is η. We have $h = dr\cos\phi$; $h_1 = -dr_1\cos\eta_1$; $h_1' = -dr_1\cos\phi_1$; $\alpha = \chi_- h_1$; $\beta = \chi_+ h_1'$ is the angle between the trajectories leaving r_1 and $r_1 + dr_1$; $d\eta_1 = K_1 dr_1 - \alpha$; $\beta = d\phi_1 - K_1 dr_1$.

Let be $x = (r, \phi) \in M$, a point whose image by P is well defined; and $u = u(x), v = v(x) \in \mathcal{T}_x M$ two unit vectors that define the boundaries of $C_+(x)$. Any vector $w \in \mathcal{T}_x M$ can be written $w = au + bv$. Define the quadratic form Q by $Q_x w = ab$.

Let be $Px = (r_1, \phi_1)$, $u_1 = u(Px), v_1 = v(Px)$ two unit vectors that define the boundaries of $C_+(Px)$ and $D_x P w = a_1 u_1 + b_1 v_1$ with $a_1 = \tilde{L}a + \tilde{S}b$ and $b_1 = \tilde{U}a + \tilde{V}b$. Then the cone field is invariant (strictly Q-separated) if $a \times b \geq 0, a^2 + b^2 > 0$ implies $a_1 \times b_1 > 0$. We will precisely compute r_+ and r_-, using formulae (3). We will use the $(dr, d\phi)$ coordinates in $\mathcal{T}M$.

Let us call $h = b/a$; then

$$\frac{Q_{Px}(D_x P u)}{Q_x u} = \frac{a_1 b_1}{ab} = \tilde{S}\tilde{V}h + \left(\tilde{L}\tilde{V} + \tilde{S}\tilde{U}\right) + \tilde{L}\tilde{U}h^{-1}.$$

Finally $r_+^2 (r_-^2)$ are the minimum (maximum) of this function for $h > (<) 0$ (they are obtained at $h = +(-)\sqrt{\tilde{L}\tilde{U}/\tilde{S}\tilde{V}}$):

$$r_\pm^2 = \tilde{L}\tilde{V} + \tilde{S}\tilde{U} \pm \sqrt{\tilde{L}\tilde{V}\tilde{S}\tilde{U}}. \tag{15}$$

So

$$\frac{r_-^2(x)}{r_+^2(x)} = \frac{\tilde{L}\tilde{V} + \tilde{S}\tilde{U} - \sqrt{\tilde{L}\tilde{V}\tilde{S}\tilde{U}}}{\tilde{L}\tilde{V} + \tilde{S}\tilde{U} + \sqrt{\tilde{L}\tilde{V}\tilde{S}\tilde{U}}}$$

which is uniformly smaller than 1 if

$$\sqrt{\tilde{L}\tilde{V}\tilde{S}\tilde{U}} \text{ is uniformly away from zero.} \tag{16}$$

4.1 Dispersing pinball billiards. A1a. Proof of Theorem 1.

For pinball billiards whose boundaries are curves with non-negative curvature (dispersing or flat) we will define the cone $C_+(x)$ as the vectors w such that $\chi_- > 0$. Then, according to the perturbed mirror equation (12), it results that $\chi_+ > (1 + \lambda(\eta))K(r)/\cos\phi$. **Due to our equation (14), this cone is described by**

$$C_+(x) = \{w = (dr, d\phi) \in \mathcal{T}_x M : \lambda(\eta)\ K(r) < d\phi/dr < \infty\}.$$

Let be $u(P^i x) = \frac{1}{M_i}(1, H_i)$ where $M_i = \sqrt{1 + H_i^2}$, $H_i = \lambda_i K_i$, and $v(P^i x) = (0, 1)$.

A simple but tedious computation, using (8), gives

$$D_x P w = \left(\frac{a_1}{M_1}, \frac{a_1}{M_1}H_1 + b_1\right) =$$

$$-\left(Bb + \frac{a}{M_0}(A + BH_0),\ \lambda_1 \left\{b(K_1 B + 1) + \frac{a}{M_0}[K_1 A + K_0 + H_0(K_1 B + 1)]\right\}\right).$$

Then

$$-\frac{a_1}{M_1} = Bb + \frac{a}{M_0}(A + BH_0) =$$

$$Bb + \frac{a}{M_0}\left[K_0 \frac{t_0}{\cos\eta_1}(1 + \lambda_0) + \frac{\cos\phi_0}{\cos\eta_1}\right]; \text{ and}$$

$$-b_1 = \lambda_1 \left\{b(K_1 B + 1) + \frac{a}{M_0}[K_1 A + K_0 + H_0(K_1 B + 1)] - K_1[Bb + \frac{a}{M_0}(A + BH_0)]\right\} =$$

$$\lambda_1 \left\{b + \frac{a}{M_0}K_0(1 + \lambda_0)\right\}.$$

Let be

$$L = K_0 \frac{t_0}{\cos\eta_1}(1+\lambda_0) + \frac{\cos\phi_0}{\cos\eta_1}; \text{ and}$$
$$U = K_0(1+\lambda_0)$$

Since the coefficients $B, L, 1, U$ are greater than zero, the cone field is invariant.

$$\frac{r_-^2(x)}{r_+^2(x)} = \frac{BU + L - 2\sqrt{BLU}}{BU + L + 2\sqrt{BLU}}$$

is uniformly smaller than 1 if and only if

$$\sqrt{BLU} \quad \text{is uniformly away from zero} \quad . \tag{17}$$

So $\sup_x \frac{r_-(x)}{r_+(x)}$ will not be strictly smaller than one if the free path t is not bounded away from zero.

4.1.1 Corners

If the billiard table has corners there can be series of consecutive reflections (near them) that have short free paths ($t(x) \approx 0$). In this case, let us fix a sufficiently small $\delta > 0$ and call a series of consecutive reflections a *corner series* if they all occur in the δ-neighborhood of one corner point. If the angle γ between the components that define the corner is greater than zero[2], the proof of Lemma 2.10 of [CM06] (see also Lemma A.1.5 of [BSC91] and [CM03] IV.2) can be immediately adapted to obtain the following result.

Lemma 8. *The number of reflections in any corner series (corner with angle $\gamma > 0$) is uniformly bounded above (by some number $\hat{m} > 0$ depending only on the angle γ); so there is a constant $t_{\min} > 0$ such that for each $x \in \mathcal{M}$ there is an $i = i_x \in \{0, \cdots \hat{m} - 1\}$ such that $t(T^i x) \geq t_{\min}$.*

Proof. For completeness we give the proof of the existence of a bound for the number of reflections in a corner series if the sides of the corner are flat (the general result follows by a simple approximation argument): let be $\tilde{\phi}_n, \tilde{\eta}_n$ the angles of the incidence and reflection as indicated in Fig 2 (they are measured from 0 to π with respect to the tangent vectors directed to the vertex). Then $\tilde{\eta}_{n+1} = \tilde{\phi}_n + \gamma$. Assumption A implies that after N bounces close to the vertex, $\tilde{\phi}_N \geq N\gamma$. Then, this angle is greater than π after at most $\left[\frac{\pi}{\gamma}\right] + 1$ bounces: the trajectory has left the angle. $\square$

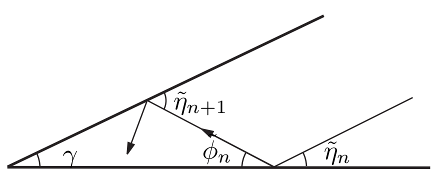

Figure 2: Proof of Lemma 8.

[2]We conjecture that if the perbilliard map is dissipative (the reflected angle moves towards the normal) this results are also true if both components are tangent. In this case, in a classical dispersing billiard, there are trajectories that stay in a corner as much as you want. We explain a little bit the reasons for our conjecture: the first and last reflection of a corner series are the "most tangent" ones. After entering in the corner the trajectory moves approaching to the normal in the reflection point; then if the perturbed map acts in this direction it will accelerate the exit from the corner.

If we use the same cone field in points with $K=0$ it results that D_xT is only Q-separated and we can apply Proposition 6 only if there are not more than $\tilde{m}$ successive bounces on flat parts of the boundary.[3] Then Proposition 6 can be applied taking $m=\hat{m}+\tilde{m}$.

Therefore, we have proved that if the billiard boundary has curvature $K\geq 0$, angles between smooth components of the boundary are greater than zero, and there are not trajectories with more than $\hat{m}$ successive bounces on flat points, then the conditions of Proposition 6 are satisfied and therefore the pinball billiard map exhibits a dominated splitting. This finishes the proof of Theorem 1. $\square$

4.2 Trajectories that only bounce at flat points. A1a.

If all the bounces are on flat points it is necessary to introduce some remarks.

In this case the derivative $D_{x_0}T$ at $x_0=(r_0,\phi_0)$ is given by

$$\frac{-1}{\cos\eta_1}\begin{pmatrix}\cos\phi_0 & t_0\\ 0 & \lambda_1\cos\eta_1\end{pmatrix}\ . \tag{18}$$

In all the cases we have in mind, $\lambda\cos\eta<a<1$. Then for n successive bounces on points with zero curvature, we obtain, for the derivative of T^n a matrix with eigenvalues equal to

$$\prod_{i=0}^{n-1}\cos\phi_i>\left[\min_{0\leq i<n}\cos\phi_i\right]^n\ ,\quad \prod_{j=1}^{n}\lambda_j\cos\eta_j<a^n.$$

Both eigenvalues are smaller than one but, if there are not trajectories close to the tangencies ($\phi=\pm\pi/2$) in an invariant set, it has dominated splitting if a is small enough. This is obviously true for periodic trajectories.

4.3 Semispersing billiards. A1b. Proof of Theorem 2.

In this Subsection we consider semidespersing ($K\geq 0$) billiards satisfying Assumption **A1b**: $0\leq 1-f'(\eta)\leq\lambda$, with $0<\lambda<1$, i.e. we have a kind of uniform contraction of the angle of reflection with respect to the angle of incidence.

As in Subsection 4.1, we define the cone field and make some computations

We will use the $(dr',d\phi)$ coordinates in $\mathcal{P}M$ where $dr'=\cos\phi dr$. Now, the derivative $D_{x_0}T$ of the pinball billiard map at $x_0=(r_0,\phi_0)$ is given by

$$-\begin{pmatrix}\frac{\cos\phi_1}{\cos\phi_0}A & \cos\phi_1 B\\ \frac{1}{\cos\phi_0}(K_1A+K_0)\lambda_1 & (K_1B+1)\,\lambda_1\end{pmatrix}\quad \text{where } A \text{ and } B \text{ are as in formula (8):} \tag{19}$$

$$A=\frac{t_0K_0+\cos\phi_0}{\cos\eta_1}\ \ ;\quad B=\frac{t_0}{\cos\eta_1}$$

Let be $u(P^ix)=\frac{1}{M_i}(1,K_i-\varepsilon)$ where $M_i=\sqrt{1+(K_i-\varepsilon)^2}$, $v(T^ix)=\frac{1}{N_i}(\xi_i,1)$ where where $N_i=\sqrt{1+\xi_i^2}$, and $w=au(x)+bv(x)$.

[3] We will explain another simple way to avoid this problem on trajectories such that $K_{-1}>0, K_0=0$. Let be $\chi_{-1}=\frac{(2-\lambda)K_{-1}}{\cos\phi_{-1}}>0$ the curvature of the "boundary" wave front -that corresponding to $u(T^{-1}x)$. After bouncing on q_0 ($K_0=0$) this wave front will have curvature $\chi_0=\frac{(1-\lambda)\cos\eta_0}{\cos\phi_0\ (t_{-1}+\chi_{-1}^{-1})}>0$. Then,we can take as boundaries of the new cone in x_0 the vectors with slopes $\overline{H}_0=\frac{1}{2}\ \frac{(1-\lambda)\cos\eta_0}{t_{-1}+\chi_{-1}^{-1}}>0$ if $\eta_0\neq\pm\frac{\pi}{2}$ and infinity. With these adapted field of cones, we can define the quadratic form in the same way and obtain that D_xT is strictly Q-separated on trajectories that eventually bounces on points with positive curvature, and do not arrive tangently to all its points with zero curvature. Moreover, using (15), with $\overline{H}_0$ instead of H_0, we obtain that $\frac{r_-(x)}{r_+(x)}<1$.

A simple but tedious computation, using (19), gives

$$D_xTw = \left(\frac{a_1}{M_1} + \frac{b_1\xi_1}{N_1},\ \frac{a_1}{M_1}(K_1-\varepsilon) + \frac{b_1}{N_1}\right) =$$

$$-\left(\begin{array}{c} \frac{\cos\phi_1}{\cos\phi_0}A\left[\frac{a}{M_0}+\frac{b}{N_0}\xi_0\right] + B\cos\phi_1\left[\frac{a}{M_0}(K_0-\varepsilon)+\frac{b}{N_0}\right], \\ \lambda_1\left\{\frac{K_1A+K_0}{\cos\phi_0}\left[\frac{a}{M_0}+\frac{b}{N_0}\xi_0\right] + (K_1B+1)\left[\frac{a}{M_0}(K_0-\varepsilon)+\frac{b}{N_0}\right]\right\} \end{array}\right).$$

Then,

$$\begin{aligned} &-\frac{a_1}{M_1}\left(1-\xi_1(K_1-\varepsilon)\right) = \\ &\frac{a}{M_0}\left(\frac{\cos\phi_1}{\cos\phi_0}A + B\cos\phi_1(K_0-\varepsilon) - \xi_1\lambda_1\left[\frac{K_1A+K_0}{\cos\phi_0} + (K_1B+1)(K_0-\varepsilon)\right]\right) \\ &+\frac{b}{N_0}\left(\frac{\cos\phi_1}{\cos\phi_0}A\xi_0 + B\cos\phi_1 - \xi_1\lambda_1\left[\frac{K_1A+K_0}{\cos\phi_0}\xi_0 + (K_1B+1)\right]\right) \end{aligned}$$

and

$$\begin{aligned} &-\frac{b_1}{N_1}\left(1-\xi_1(K_1-\varepsilon)\right) = \\ &\frac{a}{M_0}\left(\lambda_1\left[\frac{K_1A+K_0}{\cos\phi_0} + (K_1B+1)(K_0-\varepsilon)\right]\right) - \ (K_1-\varepsilon)\left[\frac{\cos\phi_1}{\cos\phi_0}A + (K_0-\varepsilon)B\cos\phi_1\right] \\ &+\frac{b}{N_0}\left(\lambda_1\left[\frac{K_1A+K_0}{\cos\phi_0}\xi_0 + (K_1B+1)\right] - (K_1-\varepsilon)\left[A\frac{\cos\phi_1}{\cos\phi_0}\xi_0 + B\cos\phi_1\right]\right). \end{aligned}$$

Set

$$L = \frac{\cos\phi_1}{\cos\phi_0}A + B\cos\phi_1(K_0-\varepsilon) - \xi_1\lambda_1\left[\frac{K_1A+K_0}{\cos\phi_0} + (K_1B+1)(K_0-\varepsilon)\right],$$

$$S = \frac{\cos\phi_1}{\cos\phi_0}A\xi_0 + B\cos\phi_1 - \xi_1\lambda_1\left[\frac{K_1A+K_0}{\cos\phi_0}\xi_0 + (K_1B+1)\right],$$

$$U = \lambda_1\left[\frac{K_1A+K_0}{\cos\phi_0} + (K_1B+1)(K_0-\varepsilon)\right] - (K_1-\varepsilon)\left[\frac{\cos\phi_1}{\cos\phi_0}A + (K_0-\varepsilon)B\cos\phi_1\right],$$

$$V = \lambda_1\left[\frac{K_1A+K_0}{\cos\phi_0}\xi_0 + (K_1B+1)\right] - (K_1-\varepsilon)\left[\frac{\cos\phi_1}{\cos\phi_0}A\xi_0 + B\cos\phi_1\right].$$

We will study different situation for the coefficients L, S, U, V being greater than zero.

Non-negative curvature, without cusps. If $K_i = 0$, set $\xi_i > 0$; if $K_j > 0$, set $\xi = 0$. Since $t_0 < D$ there exists ε such that $1 - \varepsilon t_0 > \max\lambda$. After making these elections it results that L, S, U, V are uniformly greater than zero. We show these computations in the interesting case of successive reflections on flat points: $K_1 = K_0 = 0$:

$$L_0 = \frac{\cos\phi_1}{\cos\eta_1}\left(1-\varepsilon t_0\right) + \varepsilon\lambda_1\xi_1, \quad S_0 = \frac{\cos\phi_1}{\cos\eta_1}\left(\xi_0 + t_0\right) - \lambda_1\xi_1,$$

$$U_0 = \varepsilon\left[\frac{\cos\phi_1}{\cos\eta_1}\left(1-\varepsilon t_0\right) - \lambda_1\right], \quad V_0 = \lambda_1 + \varepsilon\frac{\cos\phi_1}{\cos\eta_1}\left(\xi_0 + t_0\right).$$

Since $\cos\phi_1/\cos\eta_1 \geq b > 1$ the cone field is invariant and $\sup_x \frac{r_-(x)}{r_+(x)}$ is strictly smaller than one, independently of the values of t_0.

Cusps with positive curvature. In order to simplify the computations and make the results clearer, we will consider a billiard formed by three tangent arcs of circle with radius one. Then $K_0 = K_1 = 1$, and

$$L_1 = \frac{A}{\cos\phi_0}(\cos\phi_1 - \xi_1\lambda_1) + B(1-\varepsilon)(\cos\phi_1 - \xi_1\lambda_1) - \xi_1\lambda_1\left(\frac{1}{\cos\phi_0} + 1 - \varepsilon\right),$$

$$S_1 = \frac{A\xi_0}{\cos\phi_0}(\cos\phi_1 - \xi_1\lambda_1) + B(\cos\phi_1 - \xi_1\lambda_1) - \xi_1\lambda_1(\frac{\xi_0}{\cos\phi_0} + 1),$$

$$U_1 = \frac{A}{\cos\phi_0}(\lambda_1 + (\varepsilon - 1)\cos\phi_1) + B(1-\varepsilon)(\lambda_1 + (\varepsilon-1)\cos\phi_1) + (1 - \varepsilon + 1/\cos\phi_0)\lambda_1,$$

$$V_1 = \frac{A\xi_0}{\cos\phi_0}(\lambda_1 + (\varepsilon-1)\cos\phi_1)) + B(\lambda_1 + (\varepsilon - 1)\cos\phi_1) + \lambda_1\left(1 + \frac{\xi_0}{\cos\phi_0}\right).$$

We take $\varepsilon = 1$ and $\xi_i = \xi\cos\phi_i$, and recall that $\cos\phi_i > a > 0$. It immediately results that $U_1, V_1 \geq \lambda_1$, and

$$L_1 \geq \cos\phi_1\left[\frac{1}{\cos\eta_1}(1 - \lambda_1\xi) - \frac{\lambda_1\xi}{\cos\phi_0}\right]$$

which is bounded away from zero if ξ is small enough. Moreover, if the bounces are away of the vertices, t_0 is bounded away from zero and so is S_1. If we have two successive bounces close to the vertices

$$S_1 \geq \cos\phi_1\xi\left[\frac{\cos\phi_0}{\cos\eta_1}(1 - \lambda_1\xi) - \lambda_1(1+\xi)\right] \geq a\xi\left[\frac{\cos\phi_0}{\cos\eta_1} - \lambda_1\left(1 + \xi(\cos\phi_0/\cos\eta_1 + 1)\right)\right].$$

So, if $\cos\phi_0/\cos\eta_1$ is close to one and ξ is small enough, once again S_1 will be bounded away from zero. In fact, for entering trajectories, $\eta_1 = \phi_0 + \alpha_0 + \alpha_1$ (see, for instance, formula (3.6) in [CM07]), where $\alpha_i = |r_i - \bar{r}|$, $\bar{r}$ stands for the arc length coordinate of the vertex of the cusp (hence α_i is the length of the arc of circle between the vertex and the corresponding collision point). Then $\frac{\cos\phi_0}{\cos\eta_1} = \cos(\alpha_0 + \alpha_1) + \tan\eta_1\sin(\alpha_0 + \alpha_1)$. But now, η_1 is bounded away from $\pi/2$ (it corresponds to the second bounce close to the vertex) and $\alpha_0 + \alpha_1$ is very small; then $\frac{\cos\phi_0}{\cos\eta_1}$ is as close to one as we need, an we are done. If the trajectory is leaving the cusp, we can do a similar analysis.

4.4 Pinball billiards with focusing components.

We introduce a cone field that will be used in two different cases. Other cones fields can be studied but we concentrate the analysis in these ones in order to avoid more technical problems.

The analysis will follow the method applied for semidispersing pinball billiards. We take

$$u(P^i x) = \frac{1}{N_i}(1, -K_i),\ v(P^i x) = \frac{1}{N}(-1, -\epsilon)\ \text{ where }\ N_i = \sqrt{1 + K_i^2},\ N = \sqrt{1+\epsilon^2}.$$

Note that

$$D_x P u = \frac{-\cos\phi_0}{N_0\cos\eta_1}(1,\ \lambda_1 K_1),$$

$$D_x P v = \frac{1}{N\cos\eta_1}\left(t_0(K_0+\epsilon) + \cos\phi_0,\ \lambda_1[K_1 t_0(K_0 + \epsilon) + K_1\cos\phi_0 + (K_0+\epsilon)\cos\eta_1]\right). \tag{20}$$

It results:

$$\frac{b_1}{N}(-K_1 - \epsilon) = \frac{-a\cos\phi_0}{N_0\cos\eta_1}K_1(1+\lambda_1) + \frac{b}{N}\left\{K_1(A + \epsilon B) + \lambda_1[K_1(A+\epsilon B) + K_0 + \varepsilon]\right\},$$

$$\frac{a_1}{N_1}(-K_1 - \epsilon) = \frac{a\cos\phi_0}{N_0\cos\eta_1}(\epsilon - K_1\lambda_1) + \frac{b}{N}\left\{-\epsilon(A+\epsilon B) + \lambda_1[K_1(A + \epsilon B) + K_0 + \epsilon]\right\}.$$

Then, the same computations as before, allows to deduce a condition that corresponds to formula (17) for $\frac{r_-^2(x)}{r_+^2(x)}$ being uniformly smaller than 1. This strict inequality is satisfied if and only if

$$K_1(1+\lambda_1)(\lambda_1 K_1-\epsilon)\left[(A+\epsilon B)K_1(1+\lambda_1)+\lambda_1(K_0+\epsilon)\right]\left[(A+\epsilon B)(-\epsilon+\lambda_1 K_1)+\lambda_1(K_0+\epsilon)\right] \quad (21)$$

is uniformly away from zero, for certain elections of ε that are indicated below.

4.4.1 Proof of Theorems 3 and 3a.

First we study the circle and other focusing C^3- curves close to the circle. The result in the circle derives from the last computations taking $K_0 = K_1 = -1, \quad A = -1, \quad B = 2$. Then, $\frac{r_-^2(x)}{r_+^2(x)}$ is uniformly smaller than 1 if and only if (we recall that in the circle $\phi_0 = \eta_1$)

$$(1+\lambda_1)(\lambda_1+\epsilon)(1-2\epsilon-\lambda_1\epsilon)(\epsilon-\epsilon\lambda_1-2\epsilon^2) \quad \text{is uniformly away from zero} \quad .$$

This condition is not immediately satisfied because
for $\lambda_1 = 1, \ \epsilon = 0$, the last factor is zero; and it is greater than zero, but no uniformly,
for $\lambda_1 < 1, \ 0 < \epsilon < \ (1-\lambda_1)/2$, and
for $\lambda_1 > 1, \ (1-\lambda_1)/2 < \epsilon < \ 0$.

If the pinball billiard satisfies Assumption **A1b** it is sufficient to take a small ε. To avoid new difficulties and simplify computations in the analysis of case **A1a**, we use assumption **A2**.

A2. We assume that f ia an odd function $[f(-\eta) = -f(\eta)]$ and that there is only one point $\hat{\eta} \in [0, \pi/2]$ such that $f'(\hat{\eta}) = 0$; $M = f(\hat{\eta}) < \hat{\eta}$.

Then, if $\lambda_1 = 1$, we have that $\eta_1 = \hat{\eta}$ and $\phi_1 \ = \ \eta_2 = -\hat{\eta} + M$. As a consequence of this remark and Assumption **A2** we deduce that we can select a small $\delta > 0$ such that $|1-\lambda_i| < \delta$ only for at most a finite number m of bounces (that will depend on M). Then we obtain that the first term in (1) is smaller or equal than $C_1 a^{n-m}$ for every point of the trajectory: taking $C = C_1 a^{-m}$, we are done.

If the values of the curvature are close to -1 –that is, if the boundary curve is C^2 close to the unitary circle–, $\frac{r_-^2(x)}{r_+^2(x)}$ can also be made uniformly smaller than 1 if some additional conditions are introduced.

We must take care of the last two factors in (21).

$$(A+\epsilon B)(-\epsilon+\lambda_1 K_1)+\lambda_1(K_0+\epsilon) =$$

$$\frac{t_0 K_0+\cos\phi_0}{\cos\eta_1}\lambda_1 K_1+\lambda_1 K_0+\varepsilon\left[-\frac{t_0 K_0+\cos\phi_0}{\cos\eta_1}+\lambda_1+\frac{t_0}{\cos\eta_1}\left(\lambda_1 K_1-\varepsilon\right)\right].$$

The first terms are equal to

$$\lambda_1\left(\frac{t_0 K_0+\cos\phi_0}{\cos\eta_1}K_1+K_0\right) \quad (*)$$

that can be positive only if

$$\frac{t_0 K_0+\cos\phi_0}{\cos\eta_1} \leq -1 \ \text{ for any trajectory}$$

(if it is equal to -1 everything goes as in the studied case of circle). This condition is implied by the relation established in the statement of Theorem 3: $K_0 + K_1(t_0 K_0 + 1) \geq 0$. Then

$$(*) \geq (-t_0 K_0-\cos\phi_0)(-K_1)+K_0 \geq (-t_0 K_0-1)(-K_1)+K_0 = (t_0 K_0+1)(K_1)+K_0.$$

Since the coefficient of ε in the initial expression is greater than zero, the total expression is bounded away from zero if the condition in Theorem 3 is satisfied. The last factor in (21) is

$$(A+\epsilon B)K_1(1+\lambda_1)+\lambda_1(K_0+\epsilon)=\frac{t_0K_0+\cos\phi_0}{\cos\eta_1}(1+\lambda_1)K_1+\lambda_1K_0+\varepsilon\left(\frac{t_0K_1}{\cos\eta_1}(1+\lambda_1)+\lambda_1\right).$$

Since the coefficient of ε is negative for curves close to the circle, the terms without ε must be strictly greater than zero. As a consequence of $\frac{t_0K_0+\cos\phi_0}{\cos\eta_1}\leq -1$, they a greater than

$$-K_1(1+\lambda_1)+\lambda_1K_0\geq(1+\lambda_1)\min(-K_1)-\lambda_1\max(-K_0).$$

So the following condition must be satisfied for some positive a close to zero

$$\lambda<\frac{\min(-K)-a}{\max(-K)-\min(-K)}$$

Maximum and minimum are taken over all the values of the curvature. This condition states the closeness to the circle. For example, if $\max(-K)=1+1/5)$ and $\min(-K)=1-1/5$, then λ must be smaller than $2-b$ for some small b. These values are satisfied by all our perturbations.

Billiards maps in this billiard tables are C^2 on a compact manifold -there are not singularities- and the results in Theorems A and B can be apply directly.

It is also very interesting to recall that having a finite number of periodic orbits, for each period N (at least one of them being hyperbolic with transversal invariant manifolds), is a generic property for the billiard dynamics on tables bounded by a C^2 strictly focusing curve [DOP07].

4.4.2 Proof of Theorems 4 and 4a.

Next we study pinball billiards bounded by focusing (convex) curves satisfying Wojtkowski condition $t_0>d_0+d_1$ where $d_i=-\cos\phi_i/K_i,\ i=0,1$[4]. If this condition is satisfied along all the points of a regular focusing component of the boundary, then it is equivalent to $\frac{d^2R}{dr^2}<0$, where $R(r)$ is the curvature of the curve. It is well known that such arcs can be the boundary of hyperbolic elastic billiards. See [Wo86], [CM03].

Additional conditions on the other components of the boundary are:
- dispersing components not adjacent to any focusing curve must be outside the union of the disks of semi-curvature of all the focusing components;
- the union of all disks of semi-curvature of different focusing curves do not intersect;
- if two smooth pieces of the boundary meet at a vertex the angle must be greater than π when both curves are focusing, not less than π when one is focusing and the other dispersing, and bigger than $\pi/2$ when one piece is dispersing and the other flat.

The cardioid satisfies the condition on the curvature at all its points and $\frac{d^2R}{dr^2}<c<0$. Then the cardioid admits C^4 perturbations, maintaining the hyperbolicity of the resulting billiard map.

In order to prove Theorem 4 we use Assumption **A3**:

A3. We assume that the negative curvature of the focusing components is bounded away from zero: $-K>c>0$.

Let be $\widetilde{W_0}=t_0-d_0+\cos\eta_1/K_1\geq t_0-d_0-d_1>0$. We first study the invariance of the cone field. The slope of D_xPu is always negative. The second coordinate in (20) is equal to

$$\frac{\lambda_1K_0K_1}{N\cos\eta_1}\left[\widetilde{W_0}+\frac{\epsilon}{K_0}(t_0+\cos\eta_1/K_1)\right].$$

[4]Note that d_i is the length of the subsegment of $\overline{q_0q_1}$ contained in the disk $D(q_i)$ tangent to Γ at q_i with radius $R_i/2=-1/(2K_i)$ (*half-osculating disk, disks of semi-curvature*)

So, the slope of D_xPv is equal to

$$\lambda K_1 \left[\frac{\widetilde{W}_0 + \frac{\epsilon}{K_0}(t_0 + \cos\eta_1/K_1)}{\widetilde{W}_0 - \frac{\cos\eta_1}{K_1} + \frac{\epsilon t_0}{K_0}}\right]$$

which is always negative and greater than λK_1 if $\epsilon > 0$ is small enough.

Next we study the conditions for $\frac{r_-^2(x)}{r_+^2(x)}$ being uniformly smaller than 1. We analyze each of the relevant factors in formula (21).

With respect to the other factors we take into account that $\cos\eta_1 A = K_0(t_0 + \frac{\cos\phi_0}{K_0}) < K_0\widetilde{W}_0 < 0$, and obtain

$$(A+\epsilon B)K_1(1+\lambda_1) + \lambda_1(K_0+\epsilon) = \epsilon[BK_1(1+\lambda_1)+\lambda_1] + AK_1(1+\lambda_1) + \lambda_1 K_0 =$$

$$\epsilon[BK_1(1+\lambda_1)+\lambda_1] + \frac{K_1K_0}{\cos\eta_1}\left\{(1+\lambda_1)\left(t_0 + \frac{\cos\phi_0}{K_0}\right) + \lambda_1\frac{\cos\eta_1}{K_1}\right\} =$$

$$\epsilon[BK_1(1+\lambda_1)+\lambda_1] + \frac{K_1K_0}{\cos\eta_1}\left(\lambda_1\widetilde{W}_0 + \widetilde{W}_0 - \frac{\cos\eta_1}{K_1}\right)$$

which is bounded away from zero if ϵ is small enough. Finally

$$(A+\epsilon B)(-\epsilon+\lambda_1K_1) + \lambda_1(K_0+\epsilon) = \lambda_1\widetilde{W}_0\frac{K_0K_1}{\cos\eta_1} + \epsilon[B(-\epsilon+\lambda_1K_1) - A + \lambda_1] =$$

$$= \frac{1}{\cos\eta_1}\left\{\lambda_1K_0K_1(\widetilde{W}_0 + \ \epsilon[t_0(-\epsilon+\lambda_1) - t_0K_0 - \cos\phi_0\right\} + \epsilon\lambda_1$$

is bounded away from zero if ϵ is small enough.

Then we have proved that pinball billiards bounded by focusing curves satisfying Wojtkowski conditions has dominated splitting. As it has been observed the cardioid (that has a cusp with infinite curvature and $K \le -3/4$) satisfies this condition. Some modifications of it also satisfy this condition, for example

the curve given, in polar coordinates, by $\rho(t) = 1 - \delta\cos t, \quad \arccos\delta^{-1} \le t \le 2\pi - \arccos\delta^{-1}$ with $\delta \gtrsim 1$; or

the curve obtained joining the points $t = \pi/3, 5\pi/3$ of the cardioid $\rho(t) = 1-\cos t, \quad 0 \le t \le 2\pi$ by the vertical tangent line $x = 1/4$ (in this case the cusp 'has disappeared').

5 Slap billiard maps

We study now the following pathological billiard map: The particle moves along straight lines inside the billiard table and it reflects on the boundary along the normal line ("slap" billiard). In other words, the angle contraction is 0.

We assume that the boundary Γ of the billiard table B is given by a C^2 piecewise closed curve; i.e.: Γ is a continuous simple closed curve with a finite number of points $\{c_1, ..., c_n\}$ such that $\Gamma \setminus \{c_1, ..., c_n\}$ is a C^2 curve. In this way, we identify the space of all billiards $\mathcal{B}$ with the set of C^2 pathwise closed curves in the plane, with the distance given by the C^2 distance outside the discontinuity points. In this sense, given two billiard B and $\tilde{B}$, we say that $\tilde{B}$ is close to B if the curve $\tilde{\Gamma}$ is close to Γ; i.e.: $\tilde{\Gamma}$ has finite number of points $\{\tilde{c}_1, ..., \tilde{c}_n\}$ close to $\{c_1, ..., c_n\}$ and $\tilde{\Gamma} \setminus \{\tilde{c}_1, ..., \tilde{c}_n\}$ is a curve C^2 close to $\Gamma \setminus \{c_1, ..., c_n\}$.

Now, given a point x we take the line N_x defined as the line through x with direction given by the normal to Γ at x.

Lemma 9. *There is an open and dense set $\hat{\mathcal{B}}$ in the space of billiards $\mathcal{B}$, such that for any billiard B in $\hat{\mathcal{B}}$ follows that the set of points $x \in \Gamma$ verifying that N_x either*

1. *is tangent to* Γ *or*

2. *goes through a corner,*

is a finite set.

The proof is elementary and is left to the reader. We remark, for illustration, that the curve $y = x^4 \sin^2 \frac{1}{x}, x \neq 0$ does not make part of a billiard in $\hat{\mathcal{B}}$

We take billiards in $\mathcal{B}$ that verify lemma 9. We consider a parametrization of the boundary ∂B of the billiard table B over a compact connected interval and we denote this interval with I_B. We induce a *one-dimensional map* S on I_B where $S(x_0)$ is the intersection of the normal line through x_0 with ∂B (sometimes we denote the map S with S_B to make clear that the map is associated to the table billiard B). This map could have some discontinuity points coming either from corners or tangent lines but due to lemma 9, the set of discontinuities is a finite set. Moreover, the map S is C^1 pathwise continuous map and they are left and right continuous. On the other hand, given $\tilde{B} \in \mathcal{B}$ close to B we can use the same interval I_B as a parametrization of $\tilde{B}$.

Lemma 10. *If S is C^1 at q, then*

$$S'(q) = \frac{t_0 K_0 + 1}{-\cos \eta_1};$$

where η_1 is the angle of incidence of the trajectory at $S(q)$ with the normal direction in this point (Cf. Subsection 3.2).

The next corollary follows immediately from the formula of the derivative of S.

Corollary 1. *If the billiard B has only dispersing or flat components ($K(q) \geq 0$ for any $q \in \Gamma$) then S has no critical points and $|S'| \geq 1$.*

Corollary 2. *If the billiard B has only dispersing or flat components and there are non-parallel components of the boundary then $|S'| > 1$.*

Proof. Since $\cos \eta_1 = 1$ if and only if $\eta_1 = 0$ it follows that $S'(x) = 1$ if and only if the angle of incidence at a point x is equal to the normal on x. From the fact that the dynamics of the billiard holds along the normal direction, it follows that if y is a point such that $S(y) = x$ then the straight line through the point y with direction given by the normal at y (lines that goes through x), coincides with the straight line through the point x with direction given by the normal at x. This last fact holds if and only if the tangent lines to B at the points x and y are parallels.

□

Corollary 3. *If the billiard B has only dispersing or flat components and there are not 2-periodic trajectories joining flat points, then $|S'| > 1$.*

Proof. It follows immediately from the simple fact that there are 2-periodic trajectories joining flat points, if and only if there are parallel components of the boundary. Then, we apply previous corollary. □

Definition 4. *Given a one-dimensional map S we say that a compact invariant set Λ is a transitive attractor if $S_{|\Lambda}$ is transitive, and there is a finite union of closed intervals I contained in I_B such that $S(I) \subset I$ and $\Lambda = \cap_{n \in \mathbb{N}} S^n(I)$. We say that the attractor is maximal invariant if there exists a finite union of open intervals J containing I such that $S_B(J) \subset J$ and $\Lambda = \cap_{n \in \mathbb{N}} S^n(I)$.*

Remark 11. *Observe that by the definition above, the attractors could be a (semi)attracting periodic points. By theorem 2.1 in [MP] follows that any transitive attractor of a one-dimensional expanding map is an interval. However, this attractors are not necessary maximal invariants. See examples in [MP].*

Theorem 12. *If the billiard table B has only dispersing or flat components of the boundary and there are not 2-periodic trajectories joining flat points then $L(S)$ consists of a finite number of expanding attractors.*

Proof. Observe first that there exists $\lambda > 1$ such that $|S'(x)| > \lambda$ for any x which is not a discontinuity point. To prove that any expanding C^1 pathwise continuous maps verifies the thesis of the theorem, see theorem 2.1 in [MP].

□

Now, the following remarks and questions on the one-dimensional dynamics hold immediately:

- It is quite simple to construct slap billiard tables with points that are not reached by any trajectory (the normal of any other point do not touch them). For example, we can consider B being half a circle. The set of these points is isolated. So, to study the global dynamic on slap billiards one must either avoid this points or do no not consider them in the analysis. For other type of example consider the one given by figure 3.
- Critical points could appear for S if $t_0K_0+1 = 0$. In this case the boundary has negative curvature (focusing components): the criticalities are intimately related with the length of the normal lines inside the billiard table.
- As we have mentioned before, the system is expanding if $|t_0K_0 + 1| > \cos\eta_1$. This formula establishes a relation between the length (inside the billiard table) of the normal line at any point with the angle it touches the opposite side of the billiard table. In particular, it is expanding if $t_0K_0 > 0$ or $t_0K_0 < -2$.
- If the components are focusing, as t_0 is smaller than some $\tilde{t}$, if $K_0 < -2/\tilde{t}$ the system is expansive. Are there C^3 curves with an invariant set of points (with respect to the slap billiard map) satisfying this condition? Its limit set is generically hyperbolic, or a rotation or a Cherry map?

5.1 Polygonal billiards.

Now, we consider billiards given by convex polygonal. We also consider the case of regular polygonal (polygonals with all the corner angles being equals). Observe that any of those polygonals satisfy the thesis of lemma 9.

Corollary 4. *If the billiard table B is convex polygonal and there are non-parallel components of the boundary, then S is expanding and $L(S)$ consists of a finite number of expanding attractors.*

Proof. It follows from theorem 12

□

Remark 13. *The set $L(S)$ could be properly contained in I_B. See figure 3. In this case, $L(S)$ is contained in the union of the interval I and proper subintervals contained in II and III.*

Lemma 14. *Any triangular billiards B with corner angle smaller than $\frac{\pi}{2}$ verifies that S is expanding and $L(S) = I_B$.*

Proof. It follows from the fact B has not parallel components (so S is expanding) and that S is Markov: observe that since any corner angle of B is smaller than $\frac{\pi}{2}$, follows that the image of any side of the triangle is the union of the other two sides; so any side, can be subdivided in two such that each component cover other side. See figure 4. □

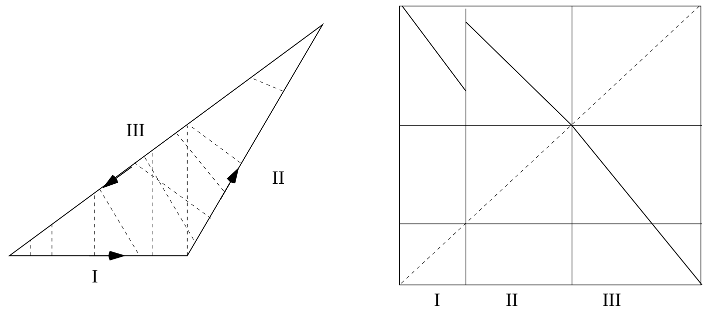

Figure 3: Sketch of S for the billiard table at left.

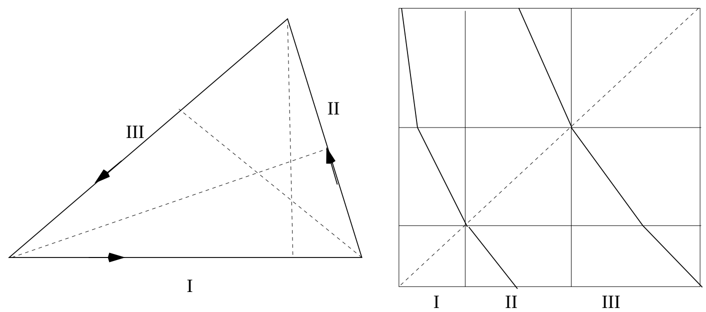

Figure 4: Sketch of S for the billiard table at left.

Corollary 5. *If B is a regular polygonal with odd numbers of sides then S is expanding and therefore contains expanding attractors.*

Proof. The expansion follows from the fact that regular polygons with odds number of sides do not have parallels components. So, $S' > 1$ and it can be applied corollary 4.

$\square$

Remark 15. *The previous corollary is false in the case of polygonals with an even number of sides: they have parallels walls; moreover, $S \circ S$ is the identity map.*

Question 1. *Given a regular polygonal B with odd number of sides. Is it true that $L(S) = I_B$?*

5.2 Slap critical billiards

Now we consider certain type of billiards which their slap billiard maps exhibit criticalities but do not have discontinuities: the map S is smooth, has not discontinuities but there are points where the derivative vanish.

We are taking C^r closed curves ($r \geq 3$). For these billiards, we say that two billiards are C^r close if the curves that bound them are C^r close. For these kind of billiard, we can identify I_B with the circle. Moreover Billiards bounded by C^r closed curves do not have corners, therefore, the discontinuities points that could appear are only coming from pairs of points x, y verifying that

1. $x \in N_y \cap \Gamma$,

2. the segment in the line N_y bounded by x and y is contained inside the table billiard B,

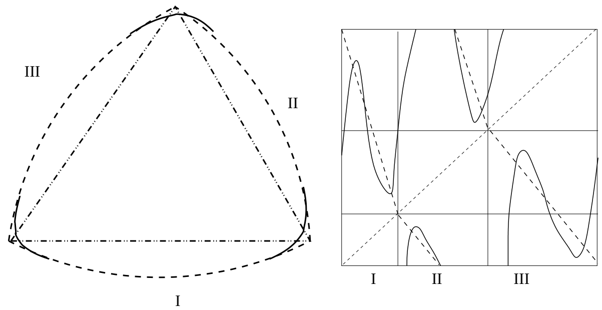

Figure 5: Critical slap billiard maps.

3. $N_y = T_x$,

where N_y denotes the normal line to Γ through y and T_x denotes the tangent line to ∂B through y.

We say that y is a discontinuity value if $y = \lim_{x_n \to x} S(x_n)$ where x is a discontinuity point.

Lemma 16. *Let $I \subset I_B$ such that $K(y) < 0$ for any $y \in I$ then there is neither discontinuity points nor discontinuity values in I.*

Proof. It follows immediately from the fact that for any $x \in I$ holds that a small neighborhood of x in the the tangent line T_x, remains outside the table billiard. $\square$

Now we give robust examples of billiard tables with negative curvature, exhibiting critical points and without discontinuities.

Theorem 17. *There exists a C^r billiard table B ($r \geq 3$) with negative curvature such that the slap billiard map has not discontinuity points and has critical points. Moreover, the same holds for any B' C^r-sufficiently close to B.*

Proof. Let us consider an equilateral triangular billiard B and recall that the map associated to it is Markovian and has five discontinuity points (in particular, each point has two preimages, see figure 4). First we replace each side of the equilateral triangle by a curve with negative curvature close to zero that goes through the vertices of the triangle. Observe that the new billiard map is C^2 close to the one associated to the equilateral triangle, let us call this billiard B_0. Now, for each vertex we take small neighborhood of it, and we replace the boundary inside this neighborhood by a smooth curve with negative curvature that coincide with B_0 in the boundary of the neighborhood. Moreover, this can be done in such a way that the new table billiard is given by a C^2 curve. Therefore, the new table billiard, named B, is given by a C^2 curve with negative curvature that coincide with B_0 outside a small neighborhood of its vertices. In particular, the associated billiard map S_B has not discontinuity points, and since it coincides with S_{B_0} outside a small neighborhood of its discontinuity, it follows that S_B is not one to one. From the fact that it is possible to deform B to a circle billiard through curves of negative curvature, it follows that S_B is homotopic to a rotation by π and therefore, using that S_B is not one to one, it follows that S_B has critical points. See figure 5 (observe that S_B has six critical points, two for each corner point that was smoothed).

$\square$

Now we give robust examples of billiard tables with negative and positive curvature, exhibiting critical points and without discontinuities. From lemma 16, discontinuities only can appear in places where the curvature is positive. To rule out the discontinuity in this case, we have the following immediate lemma.

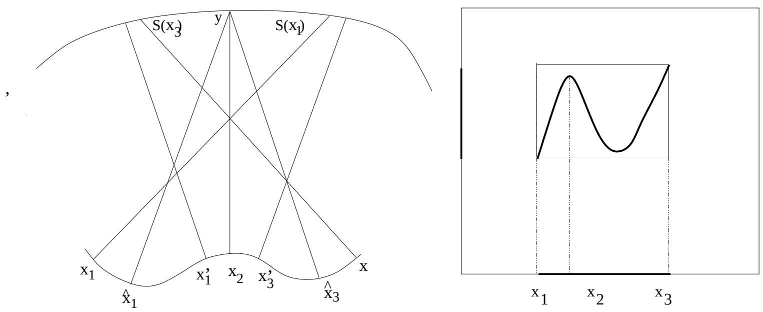

Figure 6: Critical points.

Lemma 18. *Let $J = \{x \in I_B : K(x) > 0\}$ and let us assume that for for any $x \in J$ and $y \in I_B$ such that $x \in N_y \cap \Gamma$ follows that N_y is not parallel to T_x. Then, S has not discontinuity points.*

To guarantee the existence of critical points we get the next lemma.

Lemma 19. *Let B be a table billiard satisfying the hypothesis of lemma 18. Let $I \subset I_B$ such that there are three points $x_1, x_2, x_3 \in I$ verifying:*

1. *$x_1 < x_2 < x_3$ and $S(I) = \hat{I}$ such that $K(x) < 0$ for any $x \in \hat{I}$,*
2. *the normal through the three points intersect in a point inside B,*
3. *$K(x_1) < 0, K(x_3) < 0$ and $K(x_2) > 0$,*

then there is at least two critical points in I, one contained in $[x_1, x_2]$ and the other in $[x_2, x_3]$.

Proof. First observe that from lemma 16 and 18 follows that $S_{|I}$ has not discontinuity points. Therefore, to conclude that there are at least two critical points is enough to show that there is $y \in \hat{I}$ having at least three preimages in I. For that, we take $y = S(x_2)$ and observe that from the fact that the normal through the points x_1, x_2, x_3 intersect in a point inside B, follows that $S(x_1)$ and $S(x_3)$ are in opposite side of y. Since $K(x_1) < 0$ and $K(x_2) > 0$ follows that there exists $x_1' \in [x_1, x_2]$ such that $S(x_1')$ is in the same side of $S(x_3)$; then, there is a point $\hat{x}_1 \in [x_1, x_1']$ such that $S(\hat{x}_1) = y$. Arguing in the same way with the points x_2 and x_3 we get $\hat{x}_3 \in [x_2, x_3]$ such that $S(\hat{x}_3) = y$. See figure 6. To do precisely, the billiard can be obtained from a polygonal billiard of a $W-$shape, after making the corner smooth. □

Remark 20. *It can be assumed that the critical points are isolated and non degenerated. In fact, this can be concluded if the zeros of K' (the derivative of the curvature with respect to the arc length) are isolated and K' is non degenerated at them ($K'' \neq 0$).*

Theorem 21. *There exists a C^r billiard table B ($r \geq 3$) such that the slap billiard map has not discontinuity points and has critical points. Moreover, the same holds for any B' C^r-sufficiently close to B. See figure 7.*

Proof. To get this type of billiards, we consider a closed curve that has two components satisfying the hypothesis of lemma 19. Observe that not discontinuities can appear in I or in $S(I)$. To conclude, we have to join the two curves with twopairs of curves in such a way that no discontinuities are created. To do that, we get these curve in such a way that they satisfy lemma 18. See figure 7. □

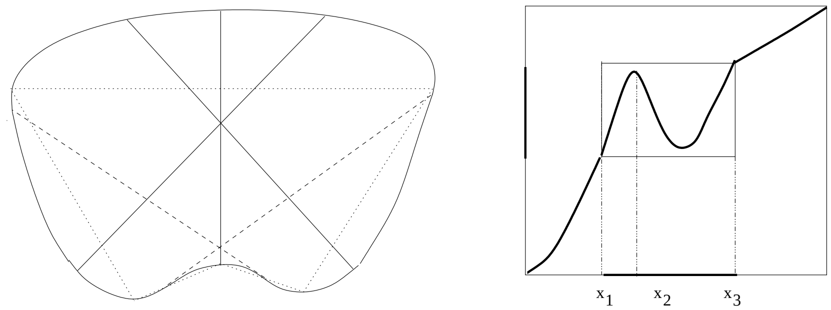

Figure 7: Critical slap billiard map.

5.3 Small perturbations of the slap billiard map.

We consider now a small perturbation of the slap billiard map: after the reflection the trajectory follows not exactly along the perpendicular line, but inside a cone centered on it. Let $f(r,\eta)$ be a C^1 small perturbation of the constant map equal to 0: the exit angle $\phi_1 = -\eta_1 + f(r_1,\eta_1)$ with $f(r,0)=0$ where η_1 is the angle of incidence at q_1. See Subsection3.2 We assume that the partial derivative of f, satisfy $|f_r| << |1 - f_\eta| < \epsilon$ with ϵ small. Let us consider

$$F : \Gamma \times [-\pi/2, \pi/2] \to \Gamma \times [-\pi/2, \pi/2]$$

the billiard map induced. Recall the expression of the derivative of F induced by (B,f) is given by equation 8. Observe that the one-dimensional map S_B can be extended to the space $\Gamma \times [-\pi/2, \pi/2]$, writing $F(r,\eta) = (S_B(r), 0)$. In this sense, the map F keeps invariant the set $\Gamma \times \{0\}$ and the lines $\{r\} \times [-\pi/2, \pi/2]$ works as an invariant foliation. Therefore, provided that f is C^1 sufficiently small, follows that that the two dimensional map F can be interpreted as a small perturbation of the map S_B. Observe that if c is a discontinuity point of S_B, for F small follows that there is a C^1 curve l_c close to $\{c\} \times [-\pi/2, \pi/2]$ containing $(c, 0)$ and formed by discontinuity points of F. Moreover, if c is an isolated discontinuity of S_B then l_c is an isolated discontinuity line of F.

Then, the next results follow:

Theorem 22. *Let B be a billiard table in $\mathcal{B}$ such that S has not critical points. Then, given $f(s,\eta)$ C^1 small, follows that (B,f) has dominated splitting.*

Proof. It holds easily from the results proved in the first section.

□

In a similar way:

Theorem 23. *Let B be a billiard table in $\mathcal{B}$ such that S is expanding. Then, provided that $f(s,\eta)$ is C^1 small, follows that (B,f) is hyperbolic. Moreover, if Λ is a maximal invariant transitive attractor of S_B then it follows that (B,f) has an invariant transitive attractor which is close (in the Hausdorff topology) to Λ.*

Proof. The first part follows immediately from writing explicitly DF and using the results in the first section. For the second part, let us consider the two dimensional map F induced by (B,f) which is close to S_B and let Λ be a maximal invariant transitive attractor of S_B. Let us take the union of open intervals J such that $S_B(J) \subset J$ and that $\Lambda = \cap_{n\in\mathbb{N}} S_B^n(J)$. We can assume that J is sufficiently close to Λ and observe that provided F is close to S_B then it follows that $F(J \times [-\pi/2, \pi/2]) \subset J \times [-\pi/2, \pi/2]$. Moreover, from the fact that DF is hyperbolic and close to DS_B, it follows that F has a C^1 contractive invariant foliation close to the the vertical lines $\{r\} \times [-\pi/2, \pi/2]$ (see for instance chapter 2 in [ArP] for details). Using this foliation of F it is induced a one-dimensional expanding map S_F which is C^1 close to S_B and that $S_F(J) \subset J$. From that, it is concluded that $\Lambda_F = \cap_{n\in\mathbb{N}} F^n(J \times [-\pi/2, \pi/2])$ is a maximal expanding attractor close to Λ.

□

5.3.1 Henon-like billiards.

Let us consider now the type of billiards studied in subsection 5.2: billiards such that S_B does not have discontinuity points but contain critical points. Now, we consider C^r small two dimensional perturbation ($r \geq 3$) of these type of slap critical billiards. In this sense, the two dimensional maps are in the so classic category of Henon-like maps (see [BC, MV, WY]). We wonder if these maps exhibits non-hyperbolic strange attractors.

5.4 One parameter families of billiards.

Let us consider a table billiard B. For each $0 \leq \lambda \leq 1$ let us consider the dynamic $F_{B,\lambda}$ given by assuming that $f(s, \eta) = (1 - \lambda)\eta$. Observe that on one hand $F_{B,1}$ is conservative and on the other hand $F_{B,0}$ is a slap billiard. We wonder about how the dynamics changes from $\lambda = 1$ to $\lambda = 0$.

Remark 24. *Let B be a billiard such that $K \geq 0$ then $F_{B,\lambda}$ has dominated splitting in the limit set for any $\lambda \in (0, 1]$ and $F_{B,0}$ has not critical points.*

So, the whole one parameter family has dominated splitting. However, the dynamics can change dramatically:

Remark 25. *Let B be a triangular billiard with corner angle smaller than $\frac{\pi}{2}$. Then it follows that*

1. *$F_{B,1}$ is parabolic,*
2. *$F_{B,0}$ is expanding and $L(F_{B,0}) = I_B$,*
3. *for λ close to zero, $F_{B,\lambda}$ is hyperbolic.*

6 Open questions

Regarding theorem A in Subsection 2.1 we wonder the following:

Question 2. *If $K \geq 0$ and the periodic trajectories are hyperbolic, is it true that the limit set has a finite spectral decomposition in transitive pieces?*

Question 3. *If $K > 0$, is it true that any hyperbolic attractor exhibits a SRB measure?*

Remark 26. *Observe that for conservative billiards it follows that any point is contained in the limit set. In particular, any point in ∂B is part of the limit set. We wonder if this can be true in the non-conservative case. In this context, we would say that ∂B is contained in the the limit set, if for any $r \in \partial B$ there exists at least one point $\theta \in [-\pi/2, \pi/2]$ such that (r, θ) is in the limit set. With this definition in mind it follows from remark 13 and theorem 23, that not always holds that $L(F)$ contains ∂B.*

In the last Subsection,

Question 4. *Which is the bifurcation process that leads from a non-hyperbolic dynamics to a hyperbolic one?*

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