POLARIZED MINIMAL FAMILIES OF RATIONAL CURVES AND HIGHER FANO MANIFOLDS

CAROLINA ARAUJO AND ANA-MARIA CASTRAVET

ABSTRACT. In this paper we investigate Fano manifolds X whose Chern characters $\mathrm{ch}_k(X)$ satisfy some positivity conditions. Our approach is via the study of *polarized minimal families of rational curves* (H_x, L_x) through a general point $x \in X$. First we translate positivity properties of the Chern characters of X into properties of the pair (H_x, L_x). This allows us to classify polarized minimal families of rational curves associated to Fano manifolds X satisfying $\mathrm{ch}_2(X) \geq 0$ and $\mathrm{ch}_3(X) \geq 0$. As a first application, we provide sufficient conditions for these manifolds to be covered by subvarieties isomorphic to $\mathbb{P}^2$ and $\mathbb{P}^3$. Moreover, this classification enables us to find new examples of Fano manifolds satisfying $\mathrm{ch}_2(X) \geq 0$.

1. INTRODUCTION

Fano manifolds, i.e., smooth complex projective varieties with ample anticanonical class, play an important role in the classification of complex projective varieties. In [Mor79], Mori showed that Fano manifolds are uniruled, i.e., they contain rational curves through every point. Then he studied minimal dominating families of rational curves on Fano manifolds, and used them to characterize projective spaces as the only smooth projective varieties having ample tangent bundle. Since then, minimal dominating families of rational curves have been extensively investigated and proved to be a useful tool in the study of uniruled varieties.

Let X be a smooth complex projective uniruled variety, and $x \in X$ a general point. A *minimal family of rational curves through* x is a *smooth* and *proper* irreducible component H_x of the scheme $\mathrm{RatCurves}^n(X, x)$ parametrizing rational curves on X passing through x. There always exists such a family. For instance, one can take H_x to be an irreducible component of $\mathrm{RatCurves}^n(X, x)$ parametrizing rational curves through x having minimal degree with respect to some fixed ample line bundle on X. While we view X as an abstract variety, H_x comes with a natural polarization L_x, which can be defined as follows (see Section 2 for details). By [Keb02b], there is a *finite morphism* $\tau_x : \ H_x \to \mathbb{P}(T_xX^\vee)$ that sends a point parametrizing a curve smooth at x to its tangent direction at x. We set $L_x = \tau_x^*\mathcal{O}(1)$. We call the pair (H_x, L_x) a *polarized minimal family of rational curves through* x. The image $\mathcal{C}_x$ of τ_x is called the *variety of minimal rational tangents* at x. The natural projective embedding $\mathcal{C}_x \subset \mathbb{P}(T_xX^\vee)$ has been successfully explored to investigate the geometry of Fano manifolds. See [Hwa01] and [Hwa06] for an overview of applications of the variety of minimal rational tangents.

In this paper we view (H_x, L_x) as a *smooth polarized variety*. We start by giving a formula for all the Chern characters of the variety H_x in terms of the

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Chern characters of X and $c_1(L_x)$. This illustrates the general principle that the pair (H_x, L_x) encodes many properties of the ambient variety X. In what follows $\pi_x : U_x \to H_x$ and $\mathrm{ev}_x : U_x \to X$ denote the universal family morphisms introduced in section 2, and the B_j's denote the Bernoulli numbers, defined by the formula $\frac{x}{e^x-1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^j$.

Proposition 1.1. *Let X be a smooth complex projective uniruled variety. Let (H_x, L_x) be a polarized minimal family of rational curves through a general point $x \in X$. For any $k \geq 1$, the k-th Chern character of H_x is given by the formula:*

$$ch_k(H_x) = \sum_{j=0}^{k} \frac{(-1)^j B_j}{j!} c_1(L_x)^j \pi_{x*} \mathrm{ev}_x^* \big(ch_{k+1-j}(X)\big) - \frac{1}{k!} c_1(L_x)^k. \tag{1.1}$$

When k is 1 or 2 this becomes:

$$c_1(H_x) = \pi_{x*} \mathrm{ev}_x^* \big(ch_2(X)\big) + \frac{d}{2} c_1(L_x), \quad \textit{and} \tag{1.2}$$

$$ch_2(H_x) = \pi_{x*} \mathrm{ev}_x^* \big(ch_3(X)\big) + \frac{1}{2} \pi_{x*} \mathrm{ev}_x^* \big(ch_2(X)\big) \cdot c_1(L_x) + \frac{d-4}{12} c_1(L_x)^2. \tag{1.3}$$

Formulas for the first Chern class $c_1(H_x)$ were previously obtained in [Dru06, Proposition 4.2] and [dJS06a, Theorem 1.1]. However, $c_1(L_x)$ appears disguised in those formulas.

Next we turn our attention to Fano manifolds X whose Chern characters satisfy some positivity conditions. In order to state our main theorem, we introduce some notation. See section 2 for details. Given a positive integer k, we denote by $N_k(X)_{\mathbb{R}}$ the real vector space of k-cycles on X modulo numerical equivalence. We denote by $\overline{NE}_k(X) \subset N_k(X)_{\mathbb{R}}$ the closure of the cone generated by effective k-cycles. There is a linear map $\mathrm{ev}_{x*}\pi_x^* : N_k(H_x)_{\mathbb{R}} \to N_{k+1}(X)_{\mathbb{R}}$. A codimension k cycle $\alpha \in A^k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ is *weakly positive* (respectively *nef*) if $\alpha \cdot \beta > 0$ (respectively $\alpha \cdot \beta \geq 0$) for every effective integral k-cycle $\beta \neq 0$. In this case we write $\alpha > 0$ (respectively $\alpha \geq 0$).

One easily checks that the only del Pezzo surface satisfying $\mathrm{ch}_2 > 0$ is $\mathbb{P}^2$. In [AC09], we go through the classification of Fano threefolds, and check that the only ones satisfying $\mathrm{ch}_2 > 0$ are $\mathbb{P}^3$ and the smooth quadric hypersurface in $\mathbb{P}^4$. In higher dimensions, Proposition 1.1 above allows us to translate positivity properties of the Chern characters of X into those of H_x, and classify polarized varieties (H_x, L_x) associated to Fano manifolds X with $\mathrm{ch}_2(X) \geq 0$ and $\mathrm{ch}_3(X) \geq 0$. The following is our main theorem.

Theorem 1.2. *Let X be a Fano manifold. Let (H_x, L_x) be a polarized minimal family of rational curves through a general point $x \in X$. Set $d = \dim H_x$.*

1. *If $ch_2(X) > 0$ (respectively $ch_2(X) \geq 0$), then $-2K_{H_x} - dL_x$ is ample (respectively nef). This necessary condition is also sufficient provided that $\mathrm{ev}_{x*}\pi_x^*\big(\overline{NE}_1(H_x)\big) = \overline{NE}_2(X)$.*
2. *If $ch_2(X) > 0$, then H_x is a Fano manifold with $\rho(H_x) = 1$ except when (H_x, L_x) is isomorphic to one of the following:*
   (a) $\Big(\mathbb{P}^m \times \mathbb{P}^m, p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\Big)$, *with $d = 2m$,*
   (b) $\Big(\mathbb{P}^{m+1} \times \mathbb{P}^m\ ,\ p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\Big)$, *with $d = 2m+1$,*

(c) $\Big(\mathbb{P}_{\mathbb{P}^{m+1}}\Big(\mathcal{O}(2)\oplus\mathcal{O}(1)^{\oplus m}\Big)\ ,\ \mathcal{O}_{\mathbb{P}}(1)\Big)$, *with* $d=2m+1$,

(d) $\Big(\mathbb{P}^m\times Q^{m+1}\ ,\ p_1^*\mathcal{O}(1)\otimes p_2^*\mathcal{O}(1)\Big)$, *with* $d=2m+1$, *or*

(e) $\Big(\mathbb{P}_{\mathbb{P}^{m+1}}\big(T_{\mathbb{P}^{m+1}}\big)\ ,\ \mathcal{O}_{\mathbb{P}}(1)\Big)$, *with* $d=2m+1$.

Morever, each of these exceptional pairs occurs as (H_x,L_x) *for some Fano manifold* X *with* $ch_2(X)>0$.

(3) *If* $ch_2(X)>0$, $ch_3(X)\geq 0$ *and* $d\geq 2$, *then* H_x *is a Fano manifold with* $\rho(H_x)=1$ *and* $ch_2(H_x)>0$.

Remarks 1.3. (i) Fano manifolds X with $\mathrm{ch}_2(X)\geq 0$ were introduced in [dJS06b] and [dJS07]. In [dJS06b] de Jong and Starr described a few examples and many non-examples of such manifolds. Roughly, the only examples of Fano manifolds with $\mathrm{ch}_2(X)>0$ in their list are complete intersections of type $(d_1,\cdots,d_m)$ in $\mathbb{P}^n$, with $\sum d_i^2\leq n$, and the Grassmannians $G(k,2k)$ and $G(k,2k+1)$. Theorem 1.2 explains why, as pointed out in [dJS06b], other examples are difficult to find.

(ii) Eventually, one would hope to classify all Fano manifolds with weakly positive (or even nef) higher Chern characters. Theorem 1.2 is a step in this direction. In fact, many homogeneous spaces X are characterized by their variety of minimal rational tangents $\mathcal{C}_x=\tau_x(H_x)\subset\mathbb{P}(T_xX^\vee)$ among Fano manifolds with Picard number one. This is the case when X is a Hermitian symmetric space or a homogeneous contact manifold [Mok05, Main Theorem], or X is the quotient of a complex simple Lie group by a maximal parabolic subgroup associated to a long simple root [HH08, Main Theorem]. Notice, however, that (H_x,L_x) carries less information than the embedding $\mathcal{C}_x\subset\mathbb{P}(T_xX^\vee)$. For instance, in Example 5.7, X is the 5-dimensional homogeneous space G_2/P, $(H_x,L_x)\cong\big(\mathbb{P}^1,\mathcal{O}(3)\big)$, and $\mathcal{C}_x$ is a twisted cubic in $\mathbb{P}(T_xX^\vee)\cong\mathbb{P}^4$, and thus degenerate.

(iii) In a forthcoming paper we classify polarized varieties (H_x,L_x) associated to Fano manifolds X satisfying $\mathrm{ch}_2(X)\geq 0$. In this case, the list of pairs (H_x,L_x) with $\rho(H_x)>1$ is much longer than the one in Theorem 1.2(2).

By [Mor79], Fano manifolds are covered by rational curves. In [dJS07], de Jong and Starr considered the question whether there is a rational surface through a general point of a Fano manifold X satisfying $\mathrm{ch}_2(X)\geq 0$. They showed that the answer is positive if the pseudoindex i_X of X is at least 3. The condition $i_X\geq 3$ implies that $\dim H_x\geq 1$, and so Theorem 1.2(1) recovers their result. In fact, we can say a bit more:

Theorem 1.4. *Let* X *be a Fano manifold, and* (H_x,L_x) *a polarized minimal family of rational curves through a general point* $x\in X$. *Suppose that* $ch_2(X)\geq 0$ *and* $d=\dim H_x\geq 1$.

(1) *(*[dJS07]*) There is a rational surface through* x.

(2) *If* $ch_2(X)>0$ *and* $(H_x,L_x)\ncong\big(\mathbb{P}^d,\mathcal{O}(2)\big)$, $\big(\mathbb{P}^1,\mathcal{O}(3)\big)$, *then there is a generically injective morphism* $g:(\mathbb{P}^2,p)\to(X,x)$ *mapping lines through* p *to curves parametrized by* H_x.

Suppose moreover that $ch_2(X)>0$, $ch_3(X)\geq 0$ *and* $d\geq 2$.

(3) *There is a rational* 3*-fold through* x, *except possibly if* $(H_x,L_x)\cong\big(\mathbb{P}^2,\mathcal{O}(2)\big)$ *and* $\mathcal{C}_x=\tau_x(H_x)$ *is singular.*

(4) *Let* (W_h,M_h) *be a polarized minimal family of rational curves through a general point* $h\in H_x$. *Suppose that* $(H_x,L_x)\ncong\big(\mathbb{P}^d,\mathcal{O}(2)\big)$ *and* $(W_h,M_h)\ncong$

$\big(\mathbb{P}^k, \mathcal{O}(2)\big)$, $\big(\mathbb{P}^1, \mathcal{O}(3)\big)$. *Then there is a generically injective morphism* $h : (\mathbb{P}^3, q) \to (X, x)$ *mapping lines through* q *to curves parametrized by* H_x.

Remarks 1.5. **(i)** We believe that the exception in Theorem 1.4(3) does not occur. **(ii)** If H_x parametrizes lines under some projective embedding $X \hookrightarrow \mathbb{P}^N$, then the morphisms g and h from Theorem 1.4(2) and (4) map lines of $\mathbb{P}^2$ and $\mathbb{P}^3$ to lines of $\mathbb{P}^N$. Hence, they are isomorphisms onto their images.

This paper is organized as follows. In section 2 we introduce polarized minimal families of rational curves and study some of their basic properties. In section 3 we make a Chern class computation to prove Proposition 1.1. This is a key ingredient to the proof of Theorem 1.2. Theorems 1.2 and 1.4 are proved in section 4. In section 5, we give new examples of Fano manifolds satisfying $\mathrm{ch}_2(X) \geq 0$. In particular, we exhibit Fano manifolds X with $\mathrm{ch}_2(X) > 0$ realizing each of the exceptional pairs in Therorems 1.2.

Notation. Throughout this paper we work over the field of complex numbers. We often identify vector bundles with their corresponding locally free subsheaves. We also identify a divisor on a smooth projective variety X with its corresponding line bundle and its class in $\mathrm{Pic}(X)$. Let E be a vector bundle on a variety X. We denote the Grothendieck projectivization $\mathrm{Proj}_X(Sym(E))$ by $\mathbb{P}(E)$, and the tautological line bundle on $\mathbb{P}(E)$ by $\mathcal{O}_{\mathbb{P}(E)}(1)$, or simply $\mathcal{O}_{\mathbb{P}}(1)$. By a *rational curve* we mean a *proper rational curve*, unless otherwise noted.

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2. Polarized minimal families of rational curves

We refer to [Kol96, Chapters I and II] for the basic theory of rational curves on complex projective varieties. See also [Deb01].

Let X be a smooth complex projective uniruled variety of dimension n. Let $x \in X$ be a *general* point. There is a scheme $\mathrm{RatCurves}^n(X, x)$ parametrizing rational curves on X passing through x. This scheme is constructed as the normalization of a certain subscheme of the Chow variety $\mathrm{Chow}(X)$ parametrizing effective 1-cycles on X. We refer to [Kol96, II.2.11] for details on the construction of $\mathrm{RatCurves}^n(X, x)$. An irreducible component H_x of $\mathrm{RatCurves}^n(X, x)$ is called a *family of rational curves through* x. It can also be described as follows. There is an irreducible open subscheme V_x of the Hom scheme $\mathrm{Hom}(\mathbb{P}^1, X, o \mapsto x)$ parametrizing morphisms $f : \mathbb{P}^1 \to X$ such that $f(o) = x$ and $f_*[\mathbb{P}^1]$ is parametrized by H_x. Then H_x is the quotient of V_x by the natural action of the automorphism group $\mathrm{Aut}(\mathbb{P}^1, o)$. Given a morphism $f : \mathbb{P}^1 \to X$ parametrized by V_x, we use the same symbol $[f]$ to denote the element of V_x corresponding to f, and its image in H_x. Since $x \in X$ is a general point, both V_x and H_x are smooth, and every morphism $f:\mathbb{P}^1\to X$ parametrized by V_x is *free*, i.e., $f^*T_X \ \cong\ \bigoplus_{i=1}^{\dim X}\mathcal{O}_{\mathbb{P}^1}(a_i)$, with all $a_i\geq 0$. From the universal properties of $\mathrm{Hom}(\mathbb{P}^1,X,o\mapsto x)$ and $\mathrm{Chow}(X)$, we get a commutative diagram:

$$\begin{tikzcd} \mathbb{P}^1\times V_x \arrow[r] \arrow[d, shift left] \arrow[rr, bend left, "(t,[f])\mapsto f(t)"] & U_x \arrow[r, "\mathrm{ev}_x"] \arrow[d, "\pi_x"'] & X, \\ V_x \arrow[r] \arrow[u, bend left, "\{o\}\times \mathrm{id}"] & H_x \arrow[u, bend right, "\sigma_x"'] & \end{tikzcd} \tag{2.1}$$

where π_x is a $\mathbb{P}^1$-bundle and σ_x is the unique section of π_x such that $\mathrm{ev}_x\big(\sigma_x(H_x)\big)=x$. We denote by the same symbol both σ_x and its image in U_x, which equals the image of $\{o\}\times V_x$ in U_x.

In [Dru06, Proposition 3.7], Druel gave the following description of the tangent bundle of H_x:

$$T_{H_x}\ \cong\ (\pi_x)_*\Big(\big((\mathrm{ev}_x^*T_X)/T_{\pi_x}\big)(-\sigma_x)\Big), \tag{2.2}$$

where the relative tangent sheaf $T_{\pi_x}=T_{U_x/H_x}$ is identified with its image under the map $d\,\mathrm{ev}_x:T_{U_x}\to \mathrm{ev}_x^*T_X$.

When H_x is proper, we call it a *minimal family of rational curves through x*.

2.1 (Minimal families of rational curves)**.** Let H_x be a minimal family of rational curves through x. For a general point $[f]\in H_x$, we have $f^*T_X\cong\mathcal{O}(2)\oplus\mathcal{O}(1)^{\oplus d}\oplus\mathcal{O}^{\oplus n-d-1}$, where $d=\dim H_x=\deg(f^*T_X)-2\leq n-1$ (see [Kol96, IV.2.9]). Moreover, $d=n-1$ if and only if $X\cong\mathbb{P}^n$ by [CMSB02] (see also [Keb02a]). Let $H_x^{\mathrm{Sing},x}$ denote the subvariety of H_x parametrizing curves that are singular at x. By a result of Miyaoka ([Kol96, V.3.7.5]), if $Z\subset H_x\setminus H_x^{\mathrm{Sing},x}$ is a *proper* subvariety, then $\mathrm{ev}_x\big|_{\pi_x^{-1}(Z)}:\pi_x^{-1}(Z)\to X$ is generically injective. In particular, if $H_x^{\mathrm{Sing},x}=\emptyset$, then ev_x is birational onto its image. By [Keb02b, Theorem 3.3], $H_x^{\mathrm{Sing},x}$ is at most finite, and every curve parametrized by H_x is immersed at x.

2.2 (Polarized minimal families of rational curves)**.** Now we describe a natural polarization L_x associated to a minimal family H_x of rational curves through x. There is an inclusion of sheaves

$$\sigma_x^*\big(T_{\pi_x}\big)\ \hookrightarrow\ \sigma_x^*\,\mathrm{ev}_x^*\,T_X\ \cong\ T_xX\otimes\mathcal{O}_{H_x}. \tag{2.3}$$

By [Keb02b, Theorems 3.3 and 3.4], the cokernel of this map is locally free, and defines a finite morphism $\tau_x:\ H_x\to\mathbb{P}(T_xX^\vee)$. By [HM04], τ_x is birational onto its image. Notice that τ_x sends a curve that is smooth at x to its tangent direction at x. Set $L_x=\tau_x^*\mathcal{O}(1)$. It is an ample and globally generated line bundle on H_x. We call the pair (H_x,L_x) a *polarized minimal family of rational curves through x*.

The following description of L_x from [Dru06, 4.2] is very useful for computations. Set $E_x=(\pi_x)_*\mathcal{O}_{U_x}(\sigma_x)$. Then $U_x\cong\mathbb{P}(E_x)$ over H_x, and under this isomorphism $\mathcal{O}_{U_x}(\sigma_x)$ is identified with the tautological line bundle $\mathcal{O}_{\mathbb{P}(E_x)}(1)$. Notice also that $\sigma_x^*\big(\mathcal{O}_{U_x}(\sigma_x)\big)\cong\sigma_x^*\big(\mathcal{N}_{\sigma_x/U_x}\big)\cong\sigma_x^*\big(T_{\pi_x}\big)$. Therefore (2.3) induces in isomorphism

$$L_x\cong\sigma_x^*\big(T_{\pi_x}\big)^{-1}\cong\sigma_x^*\mathcal{O}_{U_x}(-\sigma_x)\cong\sigma_x^*\mathcal{O}_{\mathbb{P}(E_x)}(-1). \tag{2.4}$$

By pulling back by σ_x, the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}(E_x)}(1) \otimes \left(T_{\pi_x}\right)^{-1} \to \pi_x^* E_x \to \mathcal{O}_{\mathbb{P}(E_x)}(1) \to 0 \tag{2.5}$$

induces an exact sequence

$$0 \to \mathcal{O}_{H_x} \to E_x \to L_x^{-1} \to 0. \tag{2.6}$$

This description of L_x and the projection formula yield the following identities of cycles on U_x:

(i) $\mathrm{ev}_x^*(c_1(X)) = (d+2)(\sigma_x + \pi_x^* c_1(L_x))$,
(ii) $\sigma_x \cdot \mathrm{ev}_x^*(\gamma) = 0$ for any $\gamma \in A^k(X)$, $k \geq 1$, and
(iii) $\sigma_x^2 = -\sigma_x \cdot \pi_x^* c_1(L_x)$,

where, as before, $d = \dim H_x = \deg(f^* T_X) - 2$ for any $[f] \in H_x$.

Lemma 2.3. *Let X be a smooth complex projective uniruled variety. Let (H_x, L_x) be a polarized minimal family of rational curves through a general point $x \in X$. Suppose there is a subvariety $Z \subset H_x$ such that $(Z, L_x|_Z) \cong (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$.*

1. *Then there is a finite morphism $g : (\mathbb{P}^{k+1}, p) \to (X, x)$ that maps lines through p birationally to curves parametrized by H_x.*
2. *If moreover $Z \subset H_x \setminus H_x^{Sing,x}$, then g is generically injective.*

Proof. Let $Z \subset H_x$ be a subvariety such that $(Z, L_x|_Z) \cong (\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(1))$. Set $U_Z := \pi_x^{-1}(Z)$, $\sigma_Z := \sigma_x \cap U_Z$, and $E_Z := E_x|_Z$. By 2.2, U_Z is isomorphic to $\mathbb{P}(E_Z)$ over Z, and under this isomorphism $\mathcal{O}_{U_Z}(\sigma_Z)$ is identified with the tautological line bundle $\mathcal{O}_{\mathbb{P}(E_Z)}(1)$. By (2.6), $E_Z \cong \mathcal{O}_{\mathbb{P}^k} \oplus \mathcal{O}_{\mathbb{P}^k}(-1)$. Thus U_Z is isomorphic to the blowup of $\mathbb{P}^{k+1}$ at a point p, and under this isomorphism σ_Z is identified with the exceptional divisor. Since $\mathrm{ev}_x|_{U_Z} : U_Z \to X$ maps σ_Z to x and contracts nothing else, it factors through a finite morphism $g : \mathbb{P}^{k+1} \to X$ mapping p to x. The lines through p on $\mathbb{P}^{k+1}$ are images of fibers of π_x over Z, and thus are mapped birationally to curves parametrized by $Z \subset H_x$.

If $Z \subset H_x \setminus H_x^{\mathrm{Sing},x}$, then g is generically injective by [Kol96, V.3.7.5]. □

2.4. There is a scheme $\mathrm{RatCurves}^n(X)$ parametrizing rational curves on X. A *minimal dominating family of rational curves on* X is an irreducible component H of $\mathrm{RatCurves}^n(X)$ parametrizing a family of rational curves that sweeps out a dense open subset of X, and satisfying the following condition. For a general point $x \in X$, the (possibly reducible) subvariety $H(x)$ of H parametrizing curves through x is proper. In this case, for each irreducible component $H(x)^i$ of $H(x)$, there is a minimal family H_x^i of rational curves through x parametrizing the same curves as $H(x)^i$. Moreover, H_x^i is naturally isomorphic to the normalization of $H(x)^i$. This follows from the construction of $\mathrm{RatCurves}^n(X)$ and $\mathrm{RatCurves}^n(X,x)$ in [Kol96, II.2.11]. If in addition H is proper, then we say that it is an *unsplit covering family of rational curves*. This is the case, for instance, when the curves parametrized by H have degree 1 with respect to some ample line bundle on X.

We end this section by investigating the relationship between the Chow ring of a Fano manifold and that of its minimal families of rational curves.

Definition 2.5. Let X be a projective variety, and k a non negative integer. We denote by $A_k(X)$ the group of k-cycles on X modulo rational equivalence, and by $A^k(X)$ the k^{th} graded piece of the Chow ring $A^*(X)$ of X. Let $N_k(X)$ (respectively $N^k(X)$) be the quotient of $A_k(X)$ (respectively $A^k(X)$) by numerical equivalence. Then $N_k(X)$ and $N^k(X)$ are finitely generated Abelian groups, and intersection product induces a perfect pairing $N^k(X) \times N_k(X) \to \mathbb{Z}$. For every $\mathbb{Z}$-module B, set $N_k(X)_B := N_k(X) \otimes B$ and $N^k(X)_B := N^k(X) \otimes B$. We denote by $\overline{NE}_k(X) \subset N_k(X)_{\mathbb{R}}$ the closure of the cone generated by effective k-cycles.

Let $\alpha \in N^k(X)_{\mathbb{R}}$. We say that α is

- *ample* if $\alpha = A^k$ for some ample $\mathbb{R}$-divisor A on X;
- *positive* if $\alpha \cdot \beta > 0$ for every $\beta \in \overline{NE}_k(X) \setminus \{0\}$;
- *weakly positive* if $\alpha \cdot \beta > 0$ for every effective integral k-cycle $\beta \neq 0$;
- *nef* if $\alpha \cdot \beta \geq 0$ for every $\beta \in \overline{NE}_k(X)$.

We write $\alpha > 0$ for α weakly positive and $\alpha \geq 0$ for α nef.

Definition 2.6. Let X be a smooth projective uniruled variety, and H_x a minimal family of rational curves through a general point $x \in X$. Let π_x and ev_x be as in (2.1). For every positive integer k, we define linear maps

$$\begin{array}{rcl} T^k : N^k(X)_{\mathbb{R}} & \to & N^{k-1}(H_x)_{\mathbb{R}} \ , \\ \alpha & \mapsto & \pi_{x*}\,\mathrm{ev}_x^*\,\alpha \end{array} \qquad \begin{array}{rcl} T_k : N_k(H_x)_{\mathbb{R}} & \to & N_{k+1}(X)_{\mathbb{R}}. \\ \beta & \mapsto & \mathrm{ev}_{x*}\pi_x^*\beta \end{array}$$

This is possible because ev_x is proper and π_x is a $\mathbb{P}^1$-bundle, and thus flat. We remark that in general these maps are neither injective nor surjective.

Lemma 2.7. *Let X be a smooth projective uniruled variety, and (H_x, L_x) a polarized minimal family of rational curves through a general point $x \in X$.*

1. *Let A be an $\mathbb{R}$-divisor on X, and set $a = \deg f^*A$, where $[f] \in H_x$. Then $T^k(A^k) = a^k c_1(L_x)^{k-1}$.*
2. *T_k maps $\overline{NE}_k(H_x) \setminus \{0\}$ into $\overline{NE}_{k+1}(X) \setminus \{0\}$.*
3. *T^k preserves the properties of being ample, positive, weakly positive and nef.*

Proof. To prove (1), let A be an $\mathbb{R}$-divisor on X, and set $a = \deg f^*A$, where $[f] \in H_x$. Using (2.4) and 2.2(ii), it is easy to see that $\mathrm{ev}_x^* A = a(\sigma_x + \pi_x^* L_x)$ in $N^1(U_x)$. By 2.2(iii) and the projection formula,

$$\begin{aligned} T^k(A^k) = \pi_{x*}\,\mathrm{ev}_x^*(A^k) &= a^k \pi_{x*}\left[\sum_{i=0}^{k}\binom{k}{i}\sigma_x^{k-i}\cdot \pi_x^* c_1(L_x)^i\right] \\ &= a^k \pi_{x*}\left[\left(\sum_{i=0}^{k-1}\binom{k}{i}(-1)^{k-i-1}\right)\sigma_x \cdot \pi_x^* c_1(L_x)^{k-1} + \pi_x^* c_1(L_x)^k\right] \\ &= a^k \pi_{x*}\Big[\sigma_x \cdot \pi_x^* c_1(L_x)^{k-1} + \pi_x^* c_1(L_x)^k\Big] = a^k c_1(L_x)^{k-1}. \end{aligned}$$

Notice that T_k maps effective cycles to effective cycles, inducing a linear map $T_k : \overline{NE}_k(H_x) \to \overline{NE}_{k+1}(X)$. By taking A an ample divisor in (1) above, we see that T_k maps $\overline{NE}_k(H_x) \setminus \{0\}$ into $\overline{NE}_{k+1}(X) \setminus \{0\}$.

By the projection formula, $T^{k+1}(\alpha) \cdot \beta = \alpha \cdot T_k(\beta)$ for every $\alpha \in N^{k+1}(X)_{\mathbb{R}}$ and $\beta \in N_k(H_x)_{\mathbb{R}}$. Together with (1) and (2) above, this implies that T^k preserves the properties of being ample, positive, weakly positive and nef. $\square$

3. A Chern class computation

In this section we prove Proposition 1.1. We refer to [Ful98] for basic results about intersection theory. In particular, $\mathrm{ch}(F)$ denotes the Chern character of the sheaf F, and $\mathrm{td}(F)$ denotes its Todd class. We follow the lines of the proof of [Dru06, Proposition 4.2].

Proof of Proposition 1.1. We use the notation introduced in Section 2.

By Grothendieck-Riemann-Roch

$$\mathrm{ch}\Big(\pi_{x!}\big(\mathrm{ev}_x^* T_X/T_{\pi_x}(-\sigma_x)\big)\Big) =$$
$$\pi_{x*}\Big(\mathrm{ch}\big(\mathrm{ev}_x^* T_X/T_{\pi_x}(-\sigma_x)\big)\cdot \mathrm{td}(T_{\pi_x})\Big) \in A(H_x)_{\mathbb{Q}}.$$

Since $f:\mathbb{P}^1\to X$ is free for any $[f]\in H_x$, $R^1\pi_{x*}\big(\mathrm{ev}_x^* T_X/T_{\pi_x}(-\sigma_x)\big)=0$, and thus $\pi_{x!}\big(\mathrm{ev}_x^* T_X/T_{\pi_x}(-\sigma_x)\big)=\pi_{x*}\big(\mathrm{ev}_x^* T_X/T_{\pi_x}(-\sigma_x)\big)$. It follows from (2.2) that

$$ch_k(H_x)=\mathrm{ch}_k(T_{H_x})=\pi_{x*}\Big(\Big[\mathrm{ch}\big(\mathrm{ev}_x^* T_X/T_{\pi_x}(-\sigma_x)\big)\cdot \mathrm{td}(T_{\pi_x})\Big]_{k+1}\Big).$$

Denote by W_k the codimension k part of the cycle

$$\mathrm{ch}\big(\mathrm{ev}_x^* T_X/T_{\pi_x}(-\sigma_x)\big)\cdot \mathrm{td}(T_{\pi_x}) =$$
$$\big(\mathrm{ev}_x^*\mathrm{ch}(T_X)-\mathrm{ch}(T_{\pi_x})\big)\cdot \mathrm{ch}\big(\mathcal{O}_{U_x}(-\sigma_x)\big)\cdot \mathrm{td}(T_{\pi_x}).$$

Denote by Z_k the codimension k part of the cycle

$$\big(\mathrm{ev}_x^*\mathrm{ch}(T_X)-\mathrm{ch}(T_{\pi_x})\big)\cdot \mathrm{ch}\big(\mathcal{O}_{U_x}(-\sigma_x)\big),$$

Then $\mathrm{ch}_k(H_x)=\pi_{x*}\big(W_{k+1}\big)$, and $W_{k+1}=\sum_{j=0}^{k+1} Z_{k+1-j}\cdot\big[\mathrm{td}(T_{\pi_x})\big]_j$. We have:

$$\mathrm{ev}_x^*\left(\mathrm{ch}(T_X)\right)=\mathrm{ev}_x^*\left(n+c_1(X)+\mathrm{ch}_2(X)+\mathrm{ch}_3(X)+\cdots\right),$$

$$\mathrm{ch}\big(\mathcal{O}_{U_x}(-\sigma_x)\big)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\sigma_x^k,$$

$$\mathrm{ch}(T_{\pi_x})=\sum_{k=0}^{\infty}\frac{1}{k!}c_1(T_{\pi_x})^k,\quad \mathrm{td}(T_{\pi_x})=\sum_{k=0}^{\infty}\frac{(-1)^kB_k}{k!}c_1(T_{\pi_x})^k.$$

From (2.5) and (2.6), $c_1(T_{\pi_x})=2\sigma_x+\pi_x^*c_1(L_x)$. By repeatedly using 2.2(iii), we have the following identities:

(iv) $\pi_x^*c_1(L_x)^i\cdot\sigma_x^j=(-1)^i\sigma_x^{i+j}$, for any $j\geq 1$,
(v) $c_1(T_{\pi_x})^i\cdot\sigma_x^j=\sigma_x^{i+j}$, for any $j\geq1$,
(vi) $c_1(T_{\pi_x})^i=\begin{cases}\pi_x^*c_1(L_x)^i, & \text{if } i \text{ is even}\\ 2\sigma_x^i+\pi_x^*c_1(L_x)^i, & \text{if } i \text{ is odd.}\end{cases}$

Claim 3.1. For any $k\geq1$ we have the following formulas:

$$Z_k=\mathrm{ev}_x^*\mathrm{ch}_k(X)+\frac{(n+1)(-1)^k}{k!}\sigma^k-\frac{1}{k!}\pi_x^*c_1(L_x)^k, \tag{3.1}$$

$$Z_k\cdot\sigma_x=\frac{(n+1)(-1)^k}{k!}\sigma_x^{k+1}-\frac{1}{k!}\sigma_x\cdot\pi_x^*c_1(L_x)^k, \tag{3.2}$$

$$\pi_{x*}Z_k=\pi_{x*}\,\mathrm{ev}_x^*\mathrm{ch}_k(X)-\frac{(n+1)}{k!}c_1(L_x)^{k-1}, \tag{3.3}$$

$$\pi_{x*}\big(Z_k\cdot\sigma_x\big)=\frac{n}{k!}c_1(L_x)^k. \tag{3.4}$$

In addition, $Z_0 = n-1$ (formula (3.1) does not hold for $k=0$).

Proof of Claim 3.1. For $k \geq 1$ we have:

$$Z_k = \sum_{j=0}^{k} \big(\operatorname{ev}_x^* \operatorname{ch}_j(X) - \frac{1}{j!}\pi_x^* c_1(T_{\pi_x})^j \big) \cdot \frac{(-1)^{k-j}}{(k-j)!}\sigma_x^{k-j}.$$

By identity 2.2(ii),

$$Z_k = \operatorname{ev}_x^* \operatorname{ch}_k(X) - \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!}\pi_x^* c_1(T_{\pi_x})^j \cdot \sigma_x^{k-j}.$$

Formula (3.1) follows now from identities (v), (vi) and $\sum_{j=0}^{k} \frac{(-1)^{k-j}}{j!(k-j)!} = 0$. Formula (3.2) follows immediately from (3.1) and 2.2(ii).

Using the identity (iv) and the projection formula, we have

(vii) $\pi_{x*}\sigma_x^k = (-1)^{k-1}c_1(L_x)^{k-1}$, for any $k \geq 1$, and
(viii) $\pi_{x*}\pi_x^*(\gamma) = 0$ for any class $\gamma \in A(H_x)$.

Formulas (3.3) and (3.4) now follow from (vii) and (viii). □

For simplicity, we denote by A_j the coefficient of $c_1(M)^j$ in the formula for the Todd class $\operatorname{td}(M)$ of a line bundle M, i.e., $A_j = \frac{(-1)^j}{j!}B_j$. Recall that $A_0 = 1$, $A_1 = 1/2$, $A_2 = 1/12$, $A_3 = 0$, $A_4 = -1/720$, etc.

We have $W_{k+1} = \sum_{j=0}^{k+1} A_{k+1-j} Z_j \cdot c_1(T_{\pi_x})^{k+1-j}$. Since $A_l = 0$ for all odd $l \geq 3$, by identity (vi) the formula for W_{k+1} becomes:

$$W_{k+1} = \sum_{j=0}^{k+1} A_{k+1-j} Z_j \cdot \pi_x^* c_1(L_x)^{k+1-j} + Z_k \cdot \sigma_x.$$

By the projection formula,

$$\pi_{x*}W_{k+1} = \sum_{j=1}^{k+1} A_{k+1-j}\big(\pi_{x*}Z_j\big) \cdot c_1(L_x)^{k+1-j} + \pi_{x*}\big(Z_k \cdot \sigma_x\big).$$

Using (3.3), (3.4), we have

$$\pi_{x*}W_{k+1} = \sum_{j=1}^{k+1} A_{k+1-j}\pi_{x*} \operatorname{ev}_x^* \operatorname{ch}_j(X) \cdot c_1(L_x)^{k+1-j} +$$
$$-(n+1)\big(\sum_{j=1}^{k+1} \frac{A_{k+1-j}}{j!}\big)c_1(L_x)^k + \frac{n}{k!}c_1(L_x)^k.$$

It is easy to see that the identity $\sum_{l=0}^{m} B_l \binom{m+1}{l} = 0$ implies the identity

$$\sum_{j=1}^{k+1} \frac{A_{k+1-j}}{j!} = \frac{1}{k!}$$

(use $A_l = 0$ for all odd $l \geq 3$) and now (1.1) follows.

To prove (1.2) and (1.3) from (1.1), observe that $\pi_{x*} \operatorname{ev}_x^* c_1(X) = d+2$. □

4. Higher Fano manifolds

We start this section by recalling some results about the index and pseudoindex of a Fano manifold and extremal rays of its Mori cone. We refer to [KM98] for basic definitions and results about the minimal model program.

Definition 4.1. Let X be a Fano manifold. The *index* of X is the largest integer r_X that divides $-K_X$ in $\operatorname{Pic}(X)$. The *pseudoindex* of X is the integer $i_X = \min\big\{ -K_Y \cdot C \bigm| C \subset Y \text{ rational curve } \big\}$.

Notice that $1 \leq r_X \leq i_X$. Moreover $i_X \leq \dim X + 1$, and $i_X = \dim X + 1$ if and only if $X \cong \mathbb{P}^n$ (see 2.1). By [Wiś90], if $\rho(X) > 1$, then $i_X \leq \frac{\dim X}{2} + 1$.

Definition 4.2. Let X be a smooth complex projective variety. Let R be an extremal ray of the Mori cone $\overline{NE}_1(X)$, and let $f : Y \to Z$ be the corresponding contraction. The *exceptional locus* $E(R)$ of R is the closed subset of X where f fails to be a local isomorphism. Given a divisor L on X, we set $L \cdot R = \min\big\{L \cdot C \bigm| C \subset X \text{ rational curve such that } [C] \in R\big\}$. In particular, the *length* of R is $l(R) = -K_X \cdot R = \min\big\{-K_X \cdot C \bigm| C \subset X \text{ rational curve such that } [C] \in R\big\} \geq i_X$.

Let X be a Fano manifold, and R an extremal ray of $\overline{NE}_1(X)$. By the theorem on lengths of extremal rays, $l(R) \leq \dim X + 1$. By [AO05], if $\rho(X) > 1$, then,

$$i_X \ + \ l(R) \ \leq \ \dim E(R) \ + \ 2. \tag{4.1}$$

Lemma 4.3. *Let Y be a d-dimensional Fano manifold. Let L be an ample divisor on Y such that $-2K_Y - dL$ is ample.*

(1) *Suppose that $(Y, L) \not\cong \big(\mathbb{P}^d, \mathcal{O}(2)\big)$, $\big(\mathbb{P}^1, \mathcal{O}(3)\big)$, and let R be any extremal ray of $\overline{NE}_1(Y)$. Then $L \cdot R = 1$.*

(2) *Suppose that $\rho(Y) > 1$. Then (Y, L) is isomorphic to one of the following:*
   (a) $\Big(\mathbb{P}^m \times \mathbb{P}^m, p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\Big)$, *with $d = 2m$,*
   (b) $\Big(\mathbb{P}^{m+1} \times \mathbb{P}^m\ ,\ p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\Big)$, *with $d = 2m+1$,*
   (c) $\Big(\mathbb{P}_{\mathbb{P}^{m+1}}\Big(\mathcal{O}(2) \oplus \mathcal{O}(1)^{\oplus m}\Big)\ ,\ \mathcal{O}_{\mathbb{P}}(1)\Big)$, *with $d = 2m+1$,*
   (d) $\Big(\mathbb{P}^m \times Q^{m+1}\ ,\ p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\Big)$, *with $d = 2m+1$, or*
   (e) $\Big(\mathbb{P}_{\mathbb{P}^{m+1}}\big(T_{\mathbb{P}^{m+1}}\big)\ ,\ \mathcal{O}_{\mathbb{P}}(1)\Big)$, *with $d = 2m+1$.*

Remark 4.4. In (c) above, Y can also be described as the blowup of $\mathbb{P}^{2m+1}$ along a linear subspace of dimension $m-1$. In (e) above, Y can also be described as a smooth divisor of type $(1,1)$ on $\mathbb{P}^{m+1} \times \mathbb{P}^{m+1}$.

Proof of Lemma 4.3. Suppose that $\rho(Y) = 1$. Then $\overline{NE}_1(Y)$ consists of a single extremal ray R, and there is an ample divisor L' on Y such that $\operatorname{Pic}(X) = \mathbb{Z} \cdot [L']$. Let λ be the positive integer such that $L \sim \lambda L'$. If $i_Y = d+1$, then $(Y, L') \cong \big(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1)\big)$, and $-2K_Y - dL \sim \big(d(2-\lambda)+2\big)L'$. Since this is ample, either $\lambda \leq 2$ or $(d, \lambda) = (1, 3)$. If $i_Y \leq d$, then $1 \leq (-2K_Y - dL) \cdot R = 2i_Y - d\lambda(L' \cdot R) \leq d\big(2 - \lambda(L' \cdot R)\big)$. Hence, $\lambda = L' \cdot R = 1$.

From now on we assume that $\rho(Y) > 1$. Then $d > 1$ and $i_Y \geq \frac{d+1}{2}$. Moreover, by [Wiś90], $r_Y \leq i_Y \leq \frac{d}{2} + 1$. Let R be any extremal ray of $\overline{NE}_1(Y)$. We claim that $L \cdot R = 1$. Indeed, if $L \cdot R \geq 2$, then $l(R) = d+1$, contradicting (4.1).

Suppose that $d = 2m$ is even. Then $i_Y = m + 1$. Set $A = -K_Y - mL$. By assumption A is ample. For any extremal ray $R \subset \overline{NE}_1(Y)$, (4.1) implies that $l(R) = m + 1$, and thus $A \cdot R = 1 = L \cdot R$. Hence, $A \equiv L$, and so $A \sim L$ since Y is Fano. In particular $-K_Y \sim (m+1)L$, and thus $r_Y = m + 1$. By [Wiś90, Theorem B], this implies that $Y \cong \mathbb{P}^m \times \mathbb{P}^m$.

Now suppose that $d = 2m + 1$ is odd. Then $i_Y = m + 1$. Set $A' = -2K_Y - (2m+1)L$. By assumption A' is ample. Let R be an extremal ray of $\overline{NE}_1(Y)$. Then $l(R) \geq m+1$, and $\dim E(R) \leq 2m+1$. By (4.1), there are three possibilities:

(a) $l(R) = m + 2$, $E(R) = Y$, and equality holds in (4.1);
(b) $l(R) = m + 1$, $\dim E(R) = 2m$, and equality holds in (4.1); or
(c) $l(R) = m + 1$, $E(R) = Y$, and equality in (4.1) fails by 1.

In [AO05], Andreatta and Occhetta classify the cases in which equality holds in (4.1), assuming $\dim Y - 1 \leq \dim E(R) \leq \dim Y$. They show that in this case either Y is a product of projective spaces, or a blowup of $\mathbb{P}^{2m+1}$ along a linear subspace of dimension at most $m - 1$. From this we see that in case (a) we must have $Y \cong \mathbb{P}^{m+1} \times \mathbb{P}^m$, while in case (b) Y must be isomorphic to the blowup of $\mathbb{P}^{2m+1}$ along a linear subspace of dimension $m - 1$.

From now on we assume that every extremal ray R of $\overline{NE}_1(Y)$ falls into case (c) above, which implies that $A' \cdot R = 1 = L \cdot R$. Thus $A' \sim L$, $-K_Y \sim (m + 1)L$, and thus $r_Y = m + 1$. By [Wiś91], this implies that either $Y \cong \mathbb{P}^m \times Q^{m+1}$, or $Y \cong \mathbb{P}_{\mathbb{P}^{m+1}}\big(T_{\mathbb{P}^{m+1}}\big)$. $\square$

Proof of Theorem 1.2. Let X be a Fano manifold with $\mathrm{ch}_2(X) \geq 0$. Let (H_x, L_x) be a polarized minimal family of rational curves through a general point $x \in X$. Set $d = \dim H_x$. By Proposition 1.1,

$$c_1(H_x) = \pi_{x*}\,\mathrm{ev}_x^*\left(\mathrm{ch}_2(X)\right) + \frac{d}{2}c_1(L_x).$$

By Lemma 2.7, $\pi_{x*}\,\mathrm{ev}_x^*$ preserves the properties of being weakly positive and nef. Thus $-K_{H_x}$ is ample and $-2K_{H_x} - dL_x$ is nef. Since H_x is Fano, $\pi_{x*}\,\mathrm{ev}_x^*\big(\mathrm{ch}_2(X)\big)$ is ample if and only if it is weakly positive. Hence, $\mathrm{ch}_2(X) > 0$ implies that $-2K_{H_x} - dL_x$ is ample. If $\mathrm{ev}_{x*}\pi_x^*\big(\overline{NE}_1(H_x)\big) = \overline{NE}_2(X)$, and $-2K_{H_x} - dL_x$ is ample (respectively nef), then clearly $\mathrm{ch}_2(X)$ is positive (respectively nef). This proves the first part of the theorem.

The second part follows from Lemma 4.3(2). Examples of Fano manifolds X with $\mathrm{ch}_2(X) > 0$ realizing each of the exceptional pairs are given in section 5.

Finally, suppose that $\mathrm{ch}_2(X) > 0$, $\mathrm{ch}_3(X) \geq 0$ and $d \geq 2$. We already know from part (1) that $-2K_{H_x} - dL_x$ is ample. We want to prove that $\mathrm{ch}_2(H_x) > 0$ and $\rho(H_x) = 1$. For that purpose we may assume that $(H_x, L_x) \not\cong \big(\mathbb{P}^d, \mathcal{O}(2)\big)$. Let $R \subset \overline{NE}_1(H_x)$ be an extremal ray. By Lemma 4.3(1), there is a rational curve $\ell \subset H_x$ such that $R = \mathbb{R}_{\geq 0}[\ell]$ and $L_x \cdot \ell = 1$. Moreover,

$$\pi_{x*}\,\mathrm{ev}_x^*\left(2\,\mathrm{ch}_2(X)\right) \cdot [\ell] = 2\,\mathrm{ch}_2(X) \cdot \mathrm{ev}_{x*}\pi_x^*[\ell] = \left(c_1(T_X)^2 - 2c_2(T_X)\right) \cdot \mathrm{ev}_{x*}\pi_x^*[\ell]$$

is a positive integer, and thus ≥ 1. Therefore $\eta := \pi_{x*}\,\mathrm{ev}_x^*\big(\mathrm{ch}_2(X)\big) - \frac{1}{2}c_1(L_x) \in N^1(H_x)_{\mathbb{Q}}$ is nef. We rewrite formula (1.3) of Proposition 1.1 as

$$\mathrm{ch}_2(H_x) = \pi_{x*}\,\mathrm{ev}_x^*\left(\mathrm{ch}_3(X)\right) + \frac{1}{2}\eta \cdot c_1(L_x) + \frac{d-1}{12}c_1(L_x)^2.$$

Since $\pi_{x*}\operatorname{ev}_x^*$ preserves the properties of being nef, we conclude that $\operatorname{ch}_2(H_x)$ is positive. By 5.8, none of the exceptional pairs (H_x, L_x) from part (2) satisfy $\operatorname{ch}_2(H_x) > 0$. Hence, $\rho(H_x) = 1$. $\square$

Lemma 4.5. *Let X be a Fano manifold. Let (H_x, L_x) be a polarized minimal family of rational curves through a general point $x \in X$. Set $d = \dim H_x$.*

(1) *Suppose that $ch_2(X) > 0$, $d \geq 1$ and $(H_x, L_x) \not\cong \big(\mathbb{P}^d, \mathcal{O}(2)\big)$, $\big(\mathbb{P}^1, \mathcal{O}(3)\big)$. Then any minimal dominating family of rational curves on H_x parametrizes smooth rational curves of L_x-degree equal to 1.*

(2) *Suppose that $ch_2(X) > 0$, $ch_3(X) \geq 0$, $d \geq 2$ and $(H_x, L_x) \not\cong \big(\mathbb{P}^d, \mathcal{O}(2)\big)$. Let (W_h, M_h) be a polarized minimal family of rational curves through a general point $h \in H_x$. Suppose that $(W_h, M_h) \not\cong \big(\mathbb{P}^k, \mathcal{O}(2)\big)$, $\big(\mathbb{P}^1, \mathcal{O}(3)\big)$. Then there is an isomorphism $g : \mathbb{P}^2 \to S \subset H_x$ mapping a point $p \in \mathbb{P}^2$ to $h \in H_x$, sending lines through p to curves parametrized by W_h, and such that $g^*L_x \cong \mathcal{O}_{\mathbb{P}^2}(1)$.*

Proof. Suppose that $\operatorname{ch}_2(X) > 0$, $d \geq 1$ and $(H_x, L_x) \not\cong \big(\mathbb{P}^d, \mathcal{O}(2)\big)$, $\big(\mathbb{P}^1, \mathcal{O}(3)\big)$. Let W be a minimal dominating family of rational curves on H_x. We will show that the curves parametrized by W have L_x-degree equal to 1. If $\rho(H_x) > 1$, then this can be checked directly from the list in Theorem 1.2(2). So we assume $\rho(H_x) = 1$. Let $\ell \subset H_x$ be a curve parametrized by W. Then, as in 2.1, $-K_{H_x} \cdot \ell \leq d+1$, and $-K_{H_x} \cdot \ell = d+1$ if and only if $H_x \cong \mathbb{P}^d$. By Theorem 1.2(1), $-K_{H_x} \cdot \ell > \frac{d}{2} L_x \cdot \ell$. If $L_x \cdot \ell > 1$, then $(H_x, L_x) \cong \big(\mathbb{P}^d, \mathcal{O}(2)\big)$ or $\big(\mathbb{P}^1, \mathcal{O}(3)\big)$, contradicting our assumptions. We conclude that $L_x \cdot \ell = 1$. The generically injective morphism $\tau_x : H_x \to \mathbb{P}(T_x X^\vee)$ defined in 2.2 maps curves parametrized by W to lines. So all curves parametrized by W are smooth.

Now suppose we are under the assumptions of the second part of the lemma. By Theorem 1.2(3), H_x is a Fano manifold with $\operatorname{ch}_2(H_x) > 0$ and $\rho(H_x) = 1$. If $d = 2$, then $H_x \cong \mathbb{P}^2$, and by our assumptions $L_x \cong \mathcal{O}(1)$. Now suppose $d = 3$. Recall that the only Fano threefolds satisfying $\operatorname{ch}_2 > 0$ are $\mathbb{P}^3$ and the smooth quadric hypersurface $Q^3 \subset \mathbb{P}^4$ ([AC09]). The polarized minimal family of rational curves through a general point of Q^3 is isomorphic to $\big(\mathbb{P}^1, \mathcal{O}(2)\big)$. So our assumptions imply that $(H_x, L_x) \cong \big(\mathbb{P}^3, \mathcal{O}(1)\big)$, and the conclusion of the lemma is clear. Finally assume that $d \geq 4$. By the first part of the lemma, $L_x \cdot \ell = 1$ for any curve ℓ parametrized by W_h, and $W_h^{\mathrm{Sing},h} = \emptyset$. Theorem 1.2(1) implies that $i_{H_x} > \frac{d}{2} \geq 2$, and thus $\dim W_h \geq i_{H_x} - 2 \geq 1$. By the first part of the lemma, now applied to the variety H_x, W_h is covered by smooth rational curves of M_h-degree equal to 1. By Lemma 2.3, applied to the variety H_x, there is a generically injective morphism $g : (\mathbb{P}^2, p) \to (H_x, h)$ mapping lines through p to curves on H_x parametrized by W_h. Since these curves have L_x-degree equal to 1, they are mapped to lines by the generically injective morphism $\tau_x : H_x \to \mathbb{P}(T_x X^\vee)$. We conclude that the composition $\tau_x \circ g : \mathbb{P}^2 \to \mathbb{P}(T_x X^\vee)$ is an isomorphism onto its image and $(\tau_x \circ g)^*\mathcal{O}_{\mathbb{P}}(1) \cong \mathcal{O}_{\mathbb{P}^2}(1)$. This proves the second part of the lemma. $\square$

Proof of Theorem 1.4. Let the notation and assumptions be as in Theorem 1.4.

Part (1) was proved in [dJS07]. It also follows from Theorem 1.2(1): the conditions $\operatorname{ch}_2(X) \geq 0$ and $d \geq 1$ imply that H_x is a positive dimensional Fano manifold, and thus covered by rational curves. For any rational curve $\ell \subset H_x$,

$S = \mathrm{ev}_x\left(\pi_x^{-1}(\ell)\right)$ is a rational surface on X through x. Notice that S is covered by rational curves parametrized by H_x.

For part (2), assume $\mathrm{ch}_2(X) > 0$ and $(H_x, L_x) \ncong \left(\mathbb{P}^d, \mathcal{O}(2)\right),\ \left(\mathbb{P}^1, \mathcal{O}(3)\right)$. If $d = 1$, then $(H_x, L_x) \cong \left(\mathbb{P}^1, \mathcal{O}(1)\right)$. In this case the morphism $\tau_x : H_x \to \mathbb{P}(T_x X^\vee)$ is an isomorphism onto its image, and thus $H_x^{\mathrm{Sing},x} = \emptyset$ by [Ara06, Corollary 2.8]. If $d > 1$, let W be a minimal dominating family of rational curves on H_x. By Lemma 4.5(1), the curves parametrized by W are smooth and have L_x-degree equal to 1. Moreover, since $H_x^{\mathrm{Sing},x}$ is at most finite, a general curve parametrized by W is contained in $H_x \setminus H_x^{\mathrm{Sing},x}$ by [Kol96, II.3.7]. Part (2) now follows from Lemma 2.3.

From now on assume that $\mathrm{ch}_2(X) > 0$, $\mathrm{ch}_3(X) \geq 0$ and $d \geq 2$. Then H_x is a Fano manifold with $\rho(H_x) = 1$ and $\mathrm{ch}_2(H_x) > 0$ by Theorem 1.2(3). If $d = 2$, then $H_x \cong \mathbb{P}^2$ and $U_x = \mathbb{P}(E_x)$ is a rational 3-fold. Hence, $\mathrm{ev}_x(U_x)$ is a rational 3-fold through x except possibly if ev_x fails to be birational onto its image. This can only occur if $\mathcal{C}_x$ is singular by [Ara06, Corollary 2.8], in which case $L_x \cong \mathcal{O}(2)$. Now suppose $d \geq 3$. We claim that there is a rational surface $S \subset H_x \setminus H_x^{\mathrm{Sing},x}$. From this it follows that $\mathrm{ev}_x\big|_{\pi_x^{-1}(S)}$ is generically injective and $\mathrm{ev}_x\left(\pi_x^{-1}(S)\right)$ is a rational 3-fold through x. If $d = 3$, then $H_x \cong \mathbb{P}^3$ or $Q^3 \subset \mathbb{P}^4$, and we can find a rational surface $S \subset H_x \setminus H_x^{\mathrm{Sing},x}$. Now assume $d \geq 4$, and let W_h be a minimal family of rational curves through a general point $h \in H_x$. Then W_h is a Fano manifold and $\dim W_h \geq i_{H_x} - 2 \geq 1$. As in the proof of part (1), now applied to H_x, each rational curve on W_h yields a rational surface S on H_x through h. Recall that $H_x^{\mathrm{Sing},x}$ is at most finite. If $S \cap H_x^{\mathrm{Sing},x} \neq \emptyset$, then there is a rational curve on H_x parametrized by W_h meeting $H_x^{\mathrm{Sing},x}$. If this holds for a general point $h \in H_x$, then there is a point $h_0 \in H_x^{\mathrm{Sing},x}$ that can be connected to a general point of H_x by a curve parametrized by a suitable minimal dominating family of rational curves on H_x. But this implies that a curve from this minimal dominating family has $-K_{H_x}$-degree equal to $d+1$, and so we must have $H_x \cong \mathbb{P}^d$ (see 2.1). Since $d > 2$, we can find a rational surface $S' \subset H_x \setminus H_x^{\mathrm{Sing},x}$. This proves part (3).

Finally, suppose we are under the assumptions of part (4). By Lemma 4.5(2), H_x is covered by surfaces S such that $(S, L_x|_S) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$. Exactly as in the proof of part (3) above, we can take such a surface $S \subset H_x \setminus H_x^{\mathrm{Sing},x}$. Part (4) now follows from Lemma 2.3. □

5. Examples

In this section we discuss examples of Fano manifolds X with $\mathrm{ch}_2(X) \geq 0$. Theorem 1.2 provides a new way of checking positivity of $\mathrm{ch}_2(X)$, enabling us to find new examples. Examples 5.1 and 5.2 below appear in [dJS06b]. Example 5.3 does not appear explicitly in [dJS06b], but it can be inferred from [dJS06b, Theorem 1.1(3)]. Examples 5.4, 5.5, 5.6 and 5.7 are new.

5.1 (Complete Intersections)**.** Let X be a complete intersection of type $(d_1, \ldots, d_c)$ in $\mathbb{P}^n$. Standard Chern class computations show that $\mathrm{ch}_k(X) > 0$ (respectively ≥ 0) if and only if $\sum d_i^k \leq n$ (respectively $\leq n+1$). See for instance [dJS06b, 2.1 and 2.4].

Let $x \in X$ be a general point, and let H_x be the variety of lines through x on X. Then H_x is a complete intersection of type $(1, 2, \ldots d_1, \ldots, 1, 2, \ldots d_c)$ in $\mathbb{P}^{n-1}$, and $L_x \cong \mathcal{O}(1)$. The condition from Theorem 1.2(1) of $-2K_{H_x} - dL_x$ being ample (respectively nef) is clearly equivalent to $\sum d_i^2 \leq n$ (respectively $\leq n+1$).

5.2 (Grassmannians)**.** Let $X = G(k,n)$ be the Grassmannian of k-dimensional linear subspaces of an n-dimensional vector space V, with $2 \leq k \leq \frac{n}{2}$. As computed in [dJS06b, 2.2], the second Chern class of X is given by

$$\mathrm{ch}_2(X) = \frac{n+2-2k}{2}\sigma_2 - \frac{n-2-2k}{2}\sigma_{1,1},$$

where σ_2 and $\sigma_{1,1}$ are the usual Schubert cycles of codimension 2. Recall that $\overline{NE}_2(X)$ is generated by the dual Schubert cycles $\sigma_{1,1}^*$ and σ_2^*. Thus $\mathrm{ch}_2(X) > 0$ (respectively ≥ 0) if and only if $2k \leq n \leq 2k+1$ (respectively $2k \leq n \leq 2k+2$).

Given $x \in X$, let H_x be the variety of lines through x on X under the Plücker embedding. As explained in [Hwa01, 1.4.4], $(H_x, L_x) \cong \big(\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}, p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\big)$. Indeed, if x parametrizes a linear subspace $[W]$, then a line through x corresponds to subspaces U and U' of V, of dimension $k-1$ and $k+1$, such that $U \subset W \subset U'$. So there is a natural identification $H_x \cong \mathbb{P}(W) \times \mathbb{P}(V/W)^*$.

The condition of $-2K_{H_x} - dL_x$ being ample (respectively nef) is clearly equivalent to $2k \leq n \leq 2k+1$ (respectively $2k \leq n \leq 2k+2$). Notice also that the map $T_1 : \overline{NE}_1(H_x) \to \overline{NE}_2(X)$ sends lines on fibers of p_1 and p_2 to the dual Schubert cycles σ_2^* and $\sigma_{1,1}^*$. In particular it is surjective.

Exceptional pairs (a), (b) in Theorem 1.2(2) occur in this case.

5.3 (Hyperplane sections of Grassmannians)**.** Let X be a general hyperplane section of the Grassmannian $G(k,n)$ under the Plücker embedding, where $2 \leq k \leq \frac{n}{2}$. Let $x \in X$ be a general point, and H_x the variety of lines through x on X. Then H_x is a smooth divisor of type $(1,1)$ in $\mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$ and L_x is the restriction to H_x of $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$. Thus $-2K_{H_x} - dL_x$ is ample (respectively nef) if and only if $n = 2k$ (respectively $2k \leq n \leq 2k+1$). In these cases $T_1 : \overline{NE}_1(H_x) \to \overline{NE}_2(X)$ is surjective, and thus Theorem 1.2(1) applies. We conclude that $\mathrm{ch}_2(X) > 0$ (respectively ≥ 0) if and only if $n = 2k$ (respectively $2k \leq n \leq 2k+1$).

This example occurs as the exceptional case (e) in Theorem 1.2(2).

5.4 (Orthogonal Grassmannians)**.** We fix Q a nondegenerate symmetric bilinear form on the n-dimensional vector space V, and k an integer satisfying $2 \leq k < \frac{n}{2} - 1$. Let $X = OG(k,n)$ be the subvariety of the Grassmannian $G(k,n)$ parametrizing linear subspaces that are isotropic with respect to Q. Then X is a Fano manifold of dimension $\frac{k(2n-3k-1)}{2}$ and $\rho(X) = 1$. Notice that X is the zero locus in $G(k,n)$ of a global section of the vector bundle $\mathrm{Sym}^2(\mathcal{S}^*)$, where $\mathcal{S}^*$ is the universal quotient bundle on $G(k,n)$. Using this description and the formula for $\mathrm{ch}_2\big(G(k,n)\big)$ described in 5.2, standard Chern class computations show that

$$\mathrm{ch}_2(X) = \frac{n-1-3k}{2}\sigma_2 - \frac{n-3-3k}{2}\sigma_{1,1},$$

where we denote by the same symbols σ_2 and $\sigma_{1,1}$ the restriction to X of the corresponding Schubert cycles on $G(k,n)$.

Given $x \in X$, let H_x be the variety of lines through x on X under the Plücker embedding. We claim that $(H_x, L_x) \cong \big(\mathbb{P}^{k-1} \times Q^{n-2k-2}, p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)\big)$. Indeed, if x parametrizes a linear subspace $[W]$, then a line through x on X corresponds to a pair $(U, U') \in \mathbb{P}(W) \times \mathbb{P}(V/W)^*$ such that $U' \subset U^\perp$ and $Q(v,v) = 0$ for any $v \in U'$. This is equivalent to the condition that $U' \subset W^\perp$ and $Q(v,v) = 0$ for any $v \in U'$.

The form Q induces a nondegenerate quadratic form on $W^\perp/W$, which defines a smooth quadric Q^{n-2k-2} in $\mathbb{P}(W^\perp/W)^* \cong \mathbb{P}^{n-2k-1}$. The condition then becomes $U' \subset W^\perp$ and $[U'/W] \in Q^{n-2k-2}$, proving the claim. Thus $-2K_{H_x} - dL_x$ is ample (respectively nef) if and only if $n = 3k+2$ (respectively $3k+1 \le n \le 3k+3$). In these cases there are lines on fibers of p_1 and p_2 contained in $H_x \subset \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$, and thus the composite map $\overline{NE}_1(H_x) \to \overline{NE}_2(X) \hookrightarrow \overline{NE}_2\big(OG(k,n)\big)$ is surjective. Thus $T_1 : \overline{NE}_1(H_x) \to \overline{NE}_2(X)$ is surjective, and Theorem 1.2(1) applies. We conclude that $\mathrm{ch}_2(X) > 0$ (respectively ≥ 0) if and only if $n = 3k+2$ (respectively $3k+1 \le n \le 3k+3$).

The exceptional pair (d) in Theorem 1.2(2) occurs in this case.

5.5 (Symplectic Grassmannians)**.** We fix ω a non-degenerate antisymmetric bilinear form on the n-dimensional vector space V, n even, and k an integer satisfying $2 \le k \le \frac{n}{2}$. Let $X = SG(k,n)$ be the subvariety of the Grassmannian $G(k,n)$ parametrizing linear subspaces that are isotropic with respect to ω. Then X is a Fano manifold of dimension $\frac{k(2n-3k+1)}{2}$ and $\rho(X) = 1$. Notice that X is the zero locus in $G(k,n)$ of a global section of the vector bundle $\wedge^2(\mathcal{S}^*)$, where $\mathcal{S}^*$ is the universal quotient bundle on $G(k,n)$. Using this description and the formula for $\mathrm{ch}_2\big(G(k,n)\big)$ described in 5.2, standard Chern class computations show that

$$\mathrm{ch}_2(X) = \frac{n+3-3k}{2}\sigma_2 - \frac{n+1-3k}{2}\sigma_{1,1},$$

where we denote by the same symbols σ_2 and $\sigma_{1,1}$ the restriction to X of the corresponding Schubert cycles on $G(k,n)$.

Given $x \in X$, let $H_x \subset \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$ be the variety of lines through x on X under the Plücker embedding. By [Hwa01, 1.4.7], $(H_x, L_x) \cong \big(\mathbb{P}_{\mathbb{P}^{k-1}}(\mathcal{O}(2) \oplus \mathcal{O}(1)^{n-2k}), \mathcal{O}_{\mathbb{P}}(1)\big)$. When $n = 2k$ this becomes $(H_x, L_x) \cong \big(\mathbb{P}^{k-1}, \mathcal{O}(2)\big)$. When $n > 2k$, H_x can also be described as the blow-up of $\mathbb{P}^{n-k-1}$ along a linear subspace $\mathbb{P}^{n-2k-1}$, and L_x as $2H - E$, where H is the hyperplane class in $\mathbb{P}^{n-k-1}$ and E is the exceptional divisor. Thus $-2K_{H_x} - dL_x$ is ample (respectively nef) if and only if $n = 2k$ or $n = 3k-2$ (respectively $n = 2k$ or $3k-3 \le n \le 3k-1$). In these cases $T_1 : \overline{NE}_1(H_x) \to \overline{NE}_2(X)$ is surjective. Indeed, if $n = 2k$, then $b_4(X) = 1$. If $3k-3 \le n \le 3k-1$, then there are lines on fibers of p_1 and p_2 contained in $H_x \subset \mathbb{P}^{k-1} \times \mathbb{P}^{n-k-1}$, and thus the composite map $\overline{NE}_1(H_x) \to \overline{NE}_2(X) \hookrightarrow \overline{NE}_2\big(OG(k,n)\big)$ is surjective. So Theorem 1.2(1) applies, and we conclude that $\mathrm{ch}_2(X) > 0$ (respectively ≥ 0) if and only if $n = 2k$ or $n = 3k-2$ (respectively $n = 2k$ or $3k-3 \le n \le 3k-1$).

When m is even, the exceptional pair (c) in Theorem 1.2(2) occurs for $X = SG(m+2, 3m+4)$. The exceptional pair $(H_x, L_x) \cong (\mathbb{P}^d, \mathcal{O}(2))$ in Theorem 1.4(2) occurs for $X = SG(d+1, 2d+2)$.

5.6 (A two-orbit variety)**.** We fix ω an antisymmetric bilinear form of maximum rank $n-1$ on the n-dimensional vector space V, n odd, and k an integer satisfying $2 \le k < \frac{n}{2}$. Let X be the subvariety of the Grassmannian $G(k,n)$ parametrizing linear subspaces that are isotropic with respect to ω. Then X is a Fano manifold of dimension $\frac{k(2n-3k+1)}{2}$ and $\rho(X) = 1$. Note that X is not homogeneous.

The same argument presented in 5.5 above, taking $x \in X$ a general point, shows that $\mathrm{ch}_2(X) > 0$ (respectively $\mathrm{ch}_2(X) \ge 0$) if and only if $n = 3k-2$ (respectively $3k-3 \le n \le 3k-1$). When m is odd, the exceptional pair (c) in Theorem 1.2(2) occurs for such X, with $k = m+2$ and $n = 3m+4$.

5.7 (The 5-dimensional homogeneous space G_2/P)**.** Let X be the 5-dimensional homogeneous space G_2/P. Then X is a Fano manifold with $\rho(X) = 1$, and $(H_x, L_x) \cong \big(\mathbb{P}^1, \mathcal{O}(3)\big)$, as explained in [Hwa01, 1.4.6]. Since $b_4(X) = 1$, the map $T_1 : \overline{NE}_1(H_x) \to \overline{NE}_2(X)$ is surjective, and thus Theorem 1.2(1) applies. We conclude that $\mathrm{ch}_2(X) > 0$. The exceptional pair $(H_x, L_x) \cong (\mathbb{P}^1, \mathcal{O}(3))$ in Theorem 1.4(2) occurs in this case.

5.8 (Non-Examples)**.** By [dJS06b, Theorem 1.2], the following smooth projective varieties do not satisfy $\mathrm{ch}_2(X) > 0$.

- Products $X \times Y$, with $\dim X, \dim Y > 0$.
- Projective space bundles $\mathbb{P}(E)$, with $\dim X > 0$ and $\operatorname{rank} E \geq 2$.
- Blowups of $\mathbb{P}^n$ along smooth centers of codimension 2.

By Theorem 1.2(1), if X a Fano manifold and H_x is not Fano, then $\mathrm{ch}_2(X)$ is not nef. This is the case, for instance, when X is the moduli space of rank 2 vector bundles with fixed determinant of odd degree on a smooth curve C of genus ≥ 2. In this case H_x is the family of Hecke curves through $x = [E] \in X$, which are conics with respect to the ample generator of $\mathrm{Pic}(X)$. As explained in [Hwa01, 1.4.8], $H_x \cong \mathbb{P}_C(E)$, which is not Fano.

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Carolina Araujo: IMPA, Estrada Dona Castorina 110, Rio de Janeiro, RJ 22460-320, Brazil
E-mail address: caraujo@impa.br

Ana-Maria Castravet: Department of Mathematics, University of Arizona, 617 N Santa Rita Ave, Tucson, AZ 85721-0089, USA
E-mail address: noni@math.arizona.edu