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$\mathbb{Z}^d$-ACTIONS WITH PRESCRIBED TOPOLOGICAL AND ERGODIC PROPERTIES

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Abstract. We extend constructions of Hahn-Katznelson [7] and Pavlov [10] to $\mathbb{Z}^d$-actions on symbolic dynamical spaces with prescribed topological and ergodic properties. More specifically, we describe a method to build $\mathbb{Z}^d$-actions which are (totally) minimal, (totally) strictly ergodic and have positive topological entropy.

1. Introduction

Ergodic theory studies statistical and recurrence properties of measurable transformations T acting in a probability space $(X, \mathcal{B}, \mu)$, where μ is a measure invariant by T, that is, $\mu(T^{-1}A) = \mu(A)$, for all $A \in \mathcal{B}$. It investigates a wide class of notions, such as ergodicity, mixing and entropy. These properties, in some way, give qualitative and quantitative aspects of the randomness of T. For example, ergodicity means that T is indecomposable in the metric sense with respect to μ and entropy is a concept that counts the exponential growth rate for the number of statistically significant distinguishable orbit segments.

In most cases, the object of study has topological structures: X is a compact metric space, $\mathcal{B}$ is the Borel σ-algebra of X, μ is a Borel measure probability and T is a homeomorphism of X. In this case, concepts such as minimality and topological mixing give topological aspects of the randomness of T. For example, minimality means that T is indecomposable in the topological sense, that is, the orbit of every point is dense in X.

A natural question arises: how do ergodic and topological concepts relate to each other? How do ergodic properties forbid topological phenomena and vice-versa? Are metric and topological indecomposability equivalent? This last question was answered negatively in [5] via the construction of a minimal diffeomorphism of the torus $\mathbb{T}^2$ which preserves area but is not ergodic.

Another question was raised by W. Parry: suppose T has a unique Borel probability invariant measure and that (X, T) is a minimal transformation. Can (X, T) have positive entropy? The difficulty in answering this at the time was the scarcity of a wide class of minimal and uniquely ergodic transformations. This was solved affirmatively in [7], where F. Hahn and Y. Katznelson developed an inductive method of constructing symbolic dynamical systems with the required properties. The principal idea of the paper was the weak law of large numbers.

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Later, works of Jewett and Krieger (see [11]) proved that every ergodic measure-preserving system $(X, \mathcal{B}, \mu, T)$ is metrically isomorphic to a minimal and uniquely ergodic homeomorphism on a Cantor set and this gives many examples to Parry's question: if an ergodic system $(X, \mathcal{B}, \mu, T)$ has positive metric entropy and $\Phi : (X, \mathcal{B}, \mu, T) \to (Y, \mathcal{C}, \nu, S)$ is the metric isomorphism obtained by Jewett-Krieger's theorem, then (Y, S) has positive topological entropy, by the variational principle.

It is worth mentioning that the situation is quite different in smooth ergodic theory, once some regularity on the transformation is assumed. A. Katok showed in [9] that every $C^{1+\alpha}$ diffeomorphism of a compact surface can not be minimal and simultaneously have positive topological entropy. More specifically, he proved that the topological entropy can be written in terms of the exponential growth of periodic points of a fixed order.

Suppose that T is a mesure-preserving transformation on the probability space $(X, \mathcal{B}, \mu)$ and $f : X \to \mathbb{R}$ is a measurable function. A successful area in ergodic theory deals with the convergence of averages $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^k x\right)$, $x \in X$, when n converges to infinity. The well known *Birkhoff's Theorem* states that such limit exists for almost every $x \in X$ whenever f is an L^1-function. Several results have been (and still are) proved when, instead of $\{1, 2, \ldots, n\}$, average is made along other sequences of natural numbers. A remarkable result on this direction was given by J. Bourgain [3], where he proved that if $p(x)$ is a polynomial with integer coefficients and f is an L^p-function, for some $p > 1$, then the averages $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^{p(k)} x\right)$ converge for almost every $x \in X$. In other words, convergence fails to hold for a negligible set with respect to the measure μ. In [1], V. Bergelson asked if this set is also negligible from the topological point of view. It turned out, by a result of R. Pavlov [10], that this is not true. He proved that, for every sequence $(p_n)_{n \geq 1} \subset \mathbb{Z}$ of zero upper-Banach density, there exist a totally minimal, totally uniquely ergodic and topologically mixing transformation (X, T) and a continuous function $f : X \to \mathbb{R}$ such that $n^{-1} \cdot \sum_{k=1}^{n} f\left(T^{p_k} x\right)$ fails to converge for a residual set of $x \in X$.

Suppose now that (X, T) is totally minimal, that is, (X, T^n) is minimal for every positive integer n. Pavlov also proved that, for every sequence $(p_n)_{n \geq 1} \subset \mathbb{Z}$ of zero upper-Banach density, there exists a totally minimal, totally uniquely ergodic and topologically mixing continuous transformation (X, T) such that $x \notin \overline{\{T^{p_n} \, ; \, n \geq 1\}}$ for an uncountable number of $x \in X$.

In this work, we extend the results of Hahn-Katznelson and Pavlov, giving a method of constructing (totally) minimal and (totally) uniquely ergodic $\mathbb{Z}^d$-actions with positive topological entropy. We carry out our program by constructing closed shift invariant subsets of a sequence space. More specifically, we build a sequence of finite configurations $(\mathcal{C}_k)_{k \geq 1}$ of $\{0, 1\}^{\mathbb{Z}^d}$, $\mathcal{C}_{k+1}$ being essentially formed by the concatenation of elements in $\mathcal{C}_k$ such that each of them occurs statistically well-behaved in each element of $\mathcal{C}_{k+1}$, and consider the set of limits of shifted $\mathcal{C}_k$-configurations as $k \to +\infty$. The main results are

Theorem 1. *There exist totally strictly ergodic $\mathbb{Z}^d$-actions $(X, \mathcal{B}, \mu, T)$ with arbitrarily large positive topological entropy.*

We should mention that this result is not new, because Jewett-Krieger's Theorem is true for $\mathbb{Z}^d$-actions [12]. This formulation emphasizes to the reader that the constructions, which may be used in other settings, have the additional advantage of controlling topological entropy.

Theorem 2. *Given a set $P \subset \mathbb{Z}^d$ of zero upper-Banach density, there exist a totally strictly ergodic $\mathbb{Z}^d$-action $(X, \mathcal{B}, \mu, T)$ and a continuous function $f : X \to \mathbb{R}$ such that the ergodic averages*

$$\frac{1}{|P \cap (-n,n)^d|} \sum_{g \in P \cap (-n,n)^d} f\left(T^g x\right)$$

fail to converge for a residual set of $x \in X$. In addition, $(X, \mathcal{B}, \mu, T)$ can have arbitrarily large topological entropy.

The above theorem has a special interest when P is an *arithmetic set* for which classical ergodic theory and Fourier analysis have established almost-sure convergence. This is the case (also proved in [3]) when

$$P = \{(p_1(n), \ldots, p_d(n))\,;\, n \in \mathbb{Z}\},$$

where $p_1, \ldots, p_d$ are polynomials with integer coefficients: for any $f \in L^p$, $p > 1$, the limit

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(T^{(p_1(k), \ldots, p_d(k))} x\right)$$

exists almost-surely. Note that P has zero upper-Banach density whenever one of the polynomials has degree greater than 1.

Theorem 3. *Given a set $P \subset \mathbb{Z}^d$ of zero upper-Banach density, there exists a totally strictly ergodic $\mathbb{Z}^d$-action $(X, \mathcal{B}, \mu, T)$ and an uncountable set $X_0 \subset X$ for which $x \notin \overline{\{T^{p_n} x\,;\, n \geq 1\}}$, for every $x \in X_0$. In addition, $(X, \mathcal{B}, \mu, T)$ can have arbitrarily large topological entropy.*

Yet in the arithmetic setup, Theorem 3 is the best topological result one can expect. Indeed, Bergelson and Leibman proved in [2] that if T is a minimal $\mathbb{Z}^d$-action, then there is a residual set $Y \subset X$ for which $x \in \overline{\{T^{(p_1(n), \ldots, p_d(n))} x\,;\, n \in \mathbb{Z}\}}$, for every $x \in Y$.

2. Preliminaries

We begin with some notation. Consider a metric space X, $\mathcal{B}$ its Borel σ-algebra and a G group with identity e. Throughout this work, G will denote $\mathbb{Z}^d$, $d > 1$, or one of its subgroups.

2.1. **Group actions.**

Definition 4. A G-action on X is a measurable transformation $T : G \times X \to X$, denoted by (X, T), such that

(i) $T(g_1, T(g_2, x)) = T(g_1 g_2, x)$, for every $g_1, g_2 \in G$ and $x \in X$.
(ii) $T(e, x) = x$, for every $x \in X$.

In other words, for each $g \in G$, the restriction

$$\begin{array}{cccc} T^g & : & X & \longrightarrow & X \\ & & x & \longmapsto & T(g,x) \end{array}$$

is a bimeasurable transformation on X such that $T^{g_1 g_2} = T^{g_1} T^{g_2}$, for every $g_1, g_2 \in G$. When G is abelian, $(T^g)_{g \in G}$ forms a commutative group of bimeasurable transformations on X. For each $x \in X$, the *orbit* of X with respect to T is the set

$$\mathcal{O}_T(x) \doteq \{T^g x\,;\, g \in G\}.$$

If F is a subgroup of G, the restriction $T|_F : F \times X \to X$ is clearly a F-action on X.

Definition 5. We say that (X,T) is *minimal* if $\mathcal{O}_T(x)$ is dense in X, for every $x \in X$, and *totally minimal* if $\mathcal{O}_{T|_F}(x)$ is dense in X, for every $x \in X$ and every subgroup $F < G$ of finite index.

Remind that the *index* of a subgroup F, denoted by $(G : F)$, is the number of elements of the quocient group G/F. The above definition extends the notion of total minimality of $\mathbb{Z}$-actions. In fact, a $\mathbb{Z}$-action (X,T) is totally minimal if and only if $T^n : X \to X$ is a minimal transformation, for every $n \in \mathbb{Z}$.

Consider the set $\mathcal{M}(X)$ of all Borel probability measures in X. A probability $\mu \in \mathcal{M}(X)$ is *invariant under* T or simply T*-invariant* if

$$\mu\left(T^g A\right) = \mu(A),\ \forall\, g \in G,\ \forall\, A \in \mathcal{B}.$$

Let $\mathcal{M}_T(X) \subset \mathcal{M}(X)$ denote the set of all T-invariant probability measures. Such set is non-empty whenever G is amenable, by a Krylov-Bogolubov argument applied to any Fϕlner sequence of G.

Definition 6. A G *measure-preserving system* or simply G*-mps* is a quadruple $(X,\mathcal{B},\mu,T)$, where T is a G-action on X and $\mu \in \mathcal{M}_T(X)$.

We say that $A \in \mathcal{B}$ is T-invariant iff $T^g A = A$, for all $g \in G$.

Definition 7. The G-mps $(X,\mathcal{B},\mu,T)$ is *ergodic* if it has only trivial invariant sets, that is, if $\mu(A) = 0$ or 1 whenever A is a measurable set invariant under T.

Definition 8. The G-action (X,T) is *uniquely ergodic* if $\mathcal{M}_T(X)$ is unitary, and *totally uniquely ergodic* if, for every subgroup $F < G$ of finite index, the restricted F-action $(X,T|_F)$ is uniquely ergodic.

Definition 9. We say that (X,T) is *strictly ergodic* if it is minimal and uniquely ergodic, and *totally strictly ergodic* if, for every subgroup $F < G$ of finite index, the restricted F-action $(X,T|_F)$ is strictly ergodic.

The result below was proved in [13] and states the pointwise ergodic theorem for $\mathbb{Z}^d$-actions.

Theorem 10. *Let $(X,\mathcal{B},\mu,T)$ be a $\mathbb{Z}^d$-mps. Then, for every $f \in L^1(\mu)$, there is a T-invariant function $\tilde{f} \in L^1(\mu)$ such that*

$$\lim_{n\to+\infty} \frac{1}{n^d} \sum_{g\in[0,n)^d} f\left(T^g x\right) = \tilde{f}(x)$$

for μ-almost every $x \in X$. In particular, if the action is ergodic, $\tilde{f}$ is constant and equal to $\int f d\mu$.

Above, $[0,n)$ denotes the set $\{0,1,\ldots,n-1\}$, $[0,n)^d$ the d-dimensional cube $[0,n)\times\cdots\times[0,n)$ of $\mathbb{Z}^d$ and by a T-invariant function we mean that $f \circ T^g = f$, for every $g \in G$. These averages allow the characterization of unique ergodicity. Let $C(X)$ denote the space of continuous functions from X to $\mathbb{R}$.

Proposition 11. *Let (X,T) be a $\mathbb{Z}^d$-action on the compact metric space X. The following items are equivalent.*

(a) (X,T) is uniquely ergodic.

(b) For every $f \in C(X)$ and $x \in X$, the limit

$$\lim_{n\to+\infty} \frac{1}{n^d} \sum_{g\in[0,n)^d} f\left(T^g x\right)$$

exists and is independent of x.

(c) For every $f \in C(X)$, the sequence of functions

$$f_n = \frac{1}{n^d} \sum_{g\in[0,n)^d} f \circ T^g$$

converges uniformly in X to the constant function $\tilde{f} = \int f d\mu$.

Proof. The implications (c)⇒(b)⇒(a) are obvious. It remains to prove (a)⇒(c). By contradiction, suppose f_n does not converge uniformly to $\tilde{f}$ for some $f \in C(X)$. This means that there exist $\varepsilon > 0$, $n_i \to \infty$ and $x_i \in X$ such that

$$\left| f_{n_i}(x_i) - \int f d\mu \right| \geq \varepsilon.$$

For each i, let $\nu_i \in \mathcal{M}(X)$ be the probability measure associated to the linear functional $\Theta_i : C(X) \to \mathbb{R}$ defined by

$$\Theta_i(\varphi) = \frac{1}{{n_i}^d} \sum_{g\in[0,n_i)^d} \varphi(T^g x_i)\,,\ \ \varphi \in C(X).$$

Restricting to a subsequence, if necessary, we assume that $\nu_i \to \nu$ in the weak-star topology. Because the cubes $A_i = [0, n_i)^d$ form a Fϕlner sequence in $\mathbb{Z}^d$, $\nu \in \mathcal{M}_T(X)$. In fact, for each $h \in \mathbb{Z}^d$,

$$\begin{aligned}
\left|\int \left(\varphi \circ T^h\right) d\nu - \int \varphi d\nu\right| &= \lim_{i\to\infty} \frac{1}{{n_i}^d} \left| \sum_{g\in A_i+h} \varphi(T^g x_i) - \sum_{g\in A_i} \varphi(T^g x_i) \right| \\
&\leq \max_{x\in X} |\varphi(x)| \cdot \lim_{i\to\infty} \frac{\# A_i \Delta (A_i + h)}{\# A_i} \\
&= 0.
\end{aligned}$$

But

$$\left|\int f d\nu - \int f d\mu\right| = \lim_{i\to\infty} \left|\int f d\nu_i - \int f d\mu\right| = \lim_{i\to\infty} \left| f_{n_i}(x_i) - \int f d\mu \right| \geq \varepsilon$$

and so $\nu \neq \mu$, contradicting the unique ergodicity of (X, T). □

2.2. **Subgroups of $\mathbb{Z}^d$.** Let $\mathcal{F}$ be the set of all subgroups of $\mathbb{Z}^d$ of finite index. This set is countable, because each element of $\mathcal{F}$ is generated by d linearly independent vectors of $\mathbb{Z}^d$. Consider, then, a subset $(F_k)_{k\in\mathbb{N}}$ of $\mathcal{F}$ such that, for each $F \in \mathcal{F}$, there exists $k_0 > 0$ such that $F_k < F$, for every $k \geq k_0$. For this, just consider an enumeration of $\mathcal{F}$ and define F_k as the intersection of the first k elements. Such intersections belong to $\mathcal{F}$ because

$$(\mathbb{Z}^d : F \cap F') \leq (\mathbb{Z}^d : F) \cdot (\mathbb{Z}^d : F'),\ \forall\, F, F' < \mathbb{Z}^d.$$

Restricting them, if necessary, we assume that $F_k = m_k \cdot \mathbb{Z}^d$, where $(m_k)_{k\geq 1}$ is an increasing sequence of positive integers. Observe that $(\mathbb{Z}^d : F_k) = {m_k}^d$. Such sequence will be fixed throughout the rest of the paper.

Definition 12. Given a subgroup $F < \mathbb{Z}^d$, we say that two elements $g_1, g_2 \in \mathbb{Z}^d$ are *congruent modulo* F if $g_1 - g_2 \in F$ and denote it by $g_1 \equiv_F g_2$. The set $\bar{F} \subset \mathbb{Z}^d$ is a *complete residue set modulo* F if, for every $g \in \mathbb{Z}^d$, there exists a unique $h \in \bar{F}$ such that $g \equiv_F h$.

Every complete residue set modulo F is canonically identified to the quocient $\mathbb{Z}^d/F$ and has exactly $(\mathbb{Z}^d : F)$ elements.

2.3. **Symbolic spaces.** Let $\mathcal{C}$ be a finite alphabet and consider the set $\Omega(\mathcal{C}) = \mathcal{C}^{\mathbb{Z}^d}$ of all functions $x : \mathbb{Z}^d \to \mathcal{C}$. We endow $\mathcal{C}$ with the discrete topology and $\Omega(\mathcal{C})$ with the product topology. By Tychonoff's theorem, $\Omega(\mathcal{C})$ is a compact metric space. We are not interested in a particular metric in $\Omega(\mathcal{C})$. Instead, we consider a basis of topology $\mathcal{B}_0$ to be defined below.

Consider the family $\mathcal{R}$ of all finite d-dimensional cubes $A = [r_1, r_1 + n) \times \cdots \times [r_d, r_d + n)$ of $\mathbb{Z}^d$, $n \geq 0$. We say that A has length n and is centered at $g = (r_1, \ldots, r_d) \in \mathbb{Z}^d$.

Definition 13. A *configuration* or *pattern* is a pair $b_A = (A, b)$, where $A \in \mathcal{R}$ and b is a function from A to $\mathcal{C}$. We say that b_A is *supported in* A *with encoding function* b.

Let $\Omega_A(\mathcal{C})$ denote the space of configurations supported in A and $\Omega^*(\mathcal{C})$ the space of all configurations in $\mathbb{Z}^d$:

$$\Omega^*(\mathcal{C}) \doteq \{b_A \,;\, b_A \text{ is a configuration}\}.$$

Given $A \in \mathcal{R}$, consider the map $\Pi_A : \Omega(\mathcal{C}) \to \Omega_A(\mathcal{C})$ defined by the restriction

$$\begin{array}{rcrcl} \Pi_A(x) & : & A & \longrightarrow & \mathcal{C} \\ & & g & \longmapsto & x(g) \end{array}$$

In particular, $\Pi_{\{g\}}(x) = x(g)$. We use the simpler notation $x|_A$ to denote $\Pi_A(x)$.

Definition 14. If $A \in \mathcal{R}$ is centered at g, we say that $x|_A$ is a *configuration of* x *centered at* g or that $x|_A$ *occurs in* x *centered at* g.

For $A_1, A_2 \in \mathcal{R}$ such that $A_1 \subset A_2$, let $\pi_{A_1}^{A_2} : \Omega_{A_2} \to \Omega_{A_1}$ be the restriction

$$\begin{array}{rcrcl} \pi_{A_1}^{A_2}(b) & : & A_1 & \longrightarrow & \mathcal{C} \\ & & g & \longmapsto & b(g) \end{array}$$

As above, when there is no ambiguity, we denote $\pi_{A_1}^{A_2}(b)$ simply by $b|_{A_1}$. It is clear that the diagram below commutes.

$$\begin{tikzcd} \Omega(\mathcal{C}) \arrow[r, "\Pi_{A_2}"] \arrow[dr, "\Pi_{A_1}"'] & \Omega_{A_2}(\mathcal{C}) \arrow[d, "\pi_{A_1}^{A_2}"] \\ & \Omega_{A_1}(\mathcal{C}) \end{tikzcd}$$

These maps will help us to control the patterns to appear in the constructions of Section 3.

By a cylinder in $\Omega(\mathcal{C})$ we mean the set of elements of $\Omega(\mathcal{C})$ with some fixed configuration. More specifically, given $b_A \in \Omega^*(\mathcal{C})$, the *cylinder generated by* b_A is the set

$$\mathrm{Cyl}(b_A) \doteq \{x \in \Omega(\mathcal{C}) \,;\, x|_A = b_A\}.$$

The family $\mathcal{B}_0 := \{\mathrm{Cyl}(b_A) \,|\, b_A \in \Omega^*(\mathcal{C})\}$ forms a clopen set of cylinders generating $\mathcal{B}$. Consequently, the set $C_0 = \{\chi_B \,;\, B \in \mathcal{B}_0\}$ of cylinder characteristic functions is dense in $C(\Omega(\mathcal{C}))$. Let μ be the probability measure defined by

$$\mu(\mathrm{Cyl}(b_A)) = |\mathcal{C}|^{-|A|},\ \forall\, b_A \in \Omega^*(\mathcal{C})\,,$$

and extended to $\mathcal{B}$ by Caratheódory's Theorem. Above, $|\cdot|$ denotes the number of elements of a set.

Consider the $\mathbb{Z}^d$-action $T : \mathbb{Z}^d \times \Omega(\mathcal{C}) \to \Omega(\mathcal{C})$ defined by

$$T^g(x) = (x(g+h))_{h\in\mathbb{Z}^d}\,,$$

also called the *shift action*. Given $B = \mathrm{Cyl}(b_A)$ and $g \in \mathbb{Z}^d$, let $B+g$ denote the cylinder associated to $b_{A+g} = (\tilde{b}, A+g)$, where $\tilde{b} : A+g \to \{0,1\}$ is defined by $\tilde{b}(h) = b(h-g)$, $\forall\, h \in A+g$. With this notation,

$$\chi_B \circ T^g = \chi_{B+g}\,. \tag{2.1}$$

In fact,

$$\begin{array}{lrcl} & \chi_B(T^g x) & = & 1 \\ \Longleftrightarrow & T^g x & \in & B \\ \Longleftrightarrow & x(g+h) & = & b(h),\ \forall\, h \in A \\ \Longleftrightarrow & x(h) & = & \tilde{b}(h),\ \forall\, h \in A+g \\ \Longleftrightarrow & x & \in & B+g. \end{array}$$

Definition 15. A *subshift* of $(\Omega(\mathcal{C}), T)$ is a $\mathbb{Z}^d$-action (X,T), where X is a closed subset of $\Omega(\mathcal{C})$ invariant under T.

2.4. **Topological entropy.** For each subset X of $\Omega(\mathcal{C})$ and $A \in \mathcal{R}$, let

$$\Omega_A(\mathcal{C}, X) = \{x|_A \,;\, x \in X\}$$

denote the set of configurations supported in A which occur in elements of X and $\Omega(\mathcal{C}, X)$ the space of all configurations in $\mathbb{Z}^d$ occuring in elements of X,

$$\Omega^*(\mathcal{C}, X) = \bigcup_{A\in\mathcal{R}} \Omega_A(\mathcal{C}, X).$$

Definition 16. The *topological entropy* of the subshift (X,T) is the limit

$$h(X,T) = \lim_{n\to+\infty} \frac{\log |\Omega_{[0,n)^d}(\mathcal{C}, X)|}{n^d}\,, \tag{2.2}$$

which always exists and is equal to $\inf_{n\in\mathbb{N}} \frac{1}{n^d} \cdot \log |\Omega_{[0,n)^d}(\mathcal{C}, X)|$.

2.5. **Frequencies and unique ergodicity.**

Definition 17. Given configurations $b_{A_1} \in \Omega_{A_1}(\mathcal{C})$ and $b_{A_2} \in \Omega_{A_2}(\mathcal{C})$, the *set of ocurrences of* b_{A_1} *in* b_{A_2} is

$$S(b_{A_1}, b_{A_2}) \doteq \{g \in \mathbb{Z}^d \,;\, A_1 + g \subset A_2 \text{ and } \pi^{A_2}_{A_1+g}(b_{A_2}) = b_{A_1+g}\}.$$

The *frequency of* b_{A_1} *in* b_{A_2} is defined as

$$\mathrm{fr}(b_{A_1}, b_{A_2}) \doteq \frac{|S(b_{A_1}, b_{A_2})|}{|A_2|}\,.$$

Given $F \in \mathcal{F}$ and $h \in \mathbb{Z}^d$, the *set of ocurrences of* b_{A_1} *in* b_{A_2} *centered at* h *modulo* F is

$$S(b_{A_1}, b_{A_2}, h, F) \doteq \{g \in S(b_{A_1}, b_{A_2})\,;\, A_1 + g \text{ is centered at a vertex } \equiv_F h\}$$

and the *frequency of* b_{A_1} *in* b_{A_2} *centered at* h *modulo* F is the quocient

$$\mathrm{fr}(b_{A_1}, b_{A_2}, h, F) \doteq \frac{|S(b_{A_1}, b_{A_2}, h, F)|}{|A_2|}.$$

Observe that if $\bar{F} \subset \mathbb{Z}^d$ is a complete residue set modulo F, then

$$\mathrm{fr}(b_{A_1}, b_{A_2}) = \sum_{g \in \bar{F}} \mathrm{fr}(b_{A_1}, b_{A_2}, g, F).$$

To our purposes, we rewrite Proposition 11 in a different manner.

Proposition 18. *A subshift* (X, T) *is uniquely ergodic if and only if, for every* $b_A \in \Omega^*(\mathcal{C})$ *and* $x \in X$,

$$\mathrm{fr}(b_A, x) \doteq \lim_{n \to +\infty} \mathrm{fr}\left(b_A, x|_{[0,n)^d}\right)$$

exists and is independent of x.

Proof. By approximation, condition (b) of Proposition 11 holds for $C(X)$ if and only if it holds for $C_0 = \{\chi_B\,;\, B \in \mathcal{B}_0\}$. If $f = \chi_{\mathrm{Cyl}(b_A)}$, (2.1) implies that

$$\begin{aligned}
\lim_{n \to +\infty} f_n(x) &= \lim_{n \to +\infty} \frac{1}{n^d} \sum_{g \in [0,n)^d} f\left(T^g x\right)\\
&= \lim_{n \to +\infty} \frac{1}{n^d} \sum_{g \in [0,n)^d} \chi_{\mathrm{Cyl}(b_{A+g})}(x)\\
&= \lim_{n \to +\infty} \frac{1}{n^d} \sum_{\substack{g \in [0,n)^d \\ A+g \subset [0,n)^d}} \chi_{\mathrm{Cyl}(b_{A+g})}(x)\\
&= \lim_{n \to +\infty} \mathrm{fr}\left(b_A, x|_{[0,n)^d}\right)\\
&= \mathrm{fr}(b_A, x),
\end{aligned}$$

where in the third equality we used that, for a fixed $A \in \mathcal{R}$,

$$\lim_{n \to +\infty} \frac{|\{g \in [0,n)^d\,;\, A + g \not\subset [0,n)^d\}|}{n^d} = 0.$$

$\square$

Corollary 19. *A subshift* (X, T) *is totally uniquely ergodic if and only if, for every* $b_A \in \Omega^*(\mathcal{C})$, $x \in X$ *and* $F \in \mathcal{F}$,

$$\mathrm{fr}(b_A, x, F) \doteq \lim_{n \to +\infty} \mathrm{fr}\left(b_A, x|_{[0,n)^d}, 0, F\right)$$

exists and is independent of x.

So, unique ergodicity is all about constant frequencies. We'll obtain this via the Law of Large Numbers, equidistributing ocurrences of configurations along residue classes of subgroups.

2.6. **Law of Large Numbers.** Intuitively, if A is a subset of $\mathbb{Z}^d$, each letter of $\mathcal{C}$ appears in $x|_A$ with frequency approximately $1/|\mathcal{C}|$, for almost every $x \in \Omega(\mathcal{C})$. This is what the Law of Large Number says. For our purposes, we state this result in a slightly different way. Let $(X, \mathcal{B}, \mu)$ be a probability space and $A \subset \mathbb{Z}^d$ infinite. For each $g \in A$, let $\mathbb{X}_g : X \to \mathbb{R}$ be a random variable.

Theorem 20. *(Law of Large Numbers) If $(\mathbb{X}_g)_{g\in A}$ is a family of independent and identically distributed random variables such that $\mathbb{E}[\mathbb{X}_g] = m$, for every $g \in A$, then the sequence $\left(\overline{\mathbb{X}}_n\right)_{n\geq 1}$ defined by*

$$\overline{\mathbb{X}}_n = \frac{\sum_{g\in A\cap[0,n)^d} \mathbb{X}_g}{|A \cap [0,n)^d|}$$

converges in probability to m, that is, for any $\varepsilon > 0$,

$$\lim_{n\to+\infty} \mu\left(\left|\overline{\mathbb{X}}_n - m\right| < \varepsilon\right) = 1.$$

Consider the probability measure space $(X, \mathcal{B}, \mu)$ defined in Subsection 2.3. Fixed $w \in \mathcal{C}$, let $\mathbb{X}_g : \Omega(\mathcal{C}) \to \mathbb{R}$ be defined as

$$\begin{aligned} \mathbb{X}_g(x) &= 1\ , \quad \text{if } x(g) = w \\ &= 0\ , \quad \text{if } x(g) \neq w. \end{aligned} \tag{2.3}$$

It is clear that $(\mathbb{X}_g)_{g\in\mathbb{Z}^d}$ are independent, identically distributed and satisfy

$$\mathbb{E}[\mathbb{X}_g] = \int_X \mathbb{X}_g(x) d\mu(x) = \frac{1}{|\mathcal{C}|}\ ,\ \forall\, g \in \mathbb{Z}^d.$$

In addition,

$$\overline{\mathbb{X}}_n(x) = \frac{\sum_{g\in[0,n)^d} \mathbb{X}_g(x)}{n^d} = \frac{\left|S\left(w, x|_{[0,n)^d}\right)\right|}{n^d} = \mathrm{fr}\left(w, x|_{[0,n)^d}\right),$$

which implies the

Corollary 21. *Let $w \in \mathcal{C}$, $g \in \mathbb{Z}^d$, $F \in \mathcal{F}$ and $\varepsilon > 0$.*

(a) The number of elements $b \in \Omega_{[0,n)^d}(\mathcal{C})$ such that

$$\left|\mathrm{fr}(w, b) - \frac{1}{|\mathcal{C}|}\right| < \varepsilon$$

is assymptotic to $|\mathcal{C}|^{n^d}$ as $n \to +\infty$.

(b) The number of elements $b \in \Omega_{[0,n)^d}(\mathcal{C})$ such that

$$\left|\mathrm{fr}(w, b, g, F) - \frac{1}{|\mathcal{C}| \cdot (\mathbb{Z}^d : F)}\right| < \varepsilon$$

is assymptotic to $|\mathcal{C}|^{n^d}$ as $n \to +\infty$.

(c) The number of elements $b \in \Omega_{[0,n)^d}(\mathcal{C})$ such that

$$\left|\mathrm{fr}(w, b, g, F) - \frac{1}{|\mathcal{C}| \cdot (\mathbb{Z}^d : F)}\right| < \varepsilon$$

for every $w \in \mathcal{C}$ and $g \in \mathbb{Z}^d$ is assymptotic to $|\mathcal{C}|^{n^d}$ as $n \to +\infty$.

Proof. (a) The required number is equal to

$$|\mathcal{C}|^{n^d}\cdot\mu\left(\left\{x\in\Omega(\mathcal{C})\,;\,\left|\mathrm{fr}\left(w,x|_{[0,n)^d}\right)-|\mathcal{C}|^{-1}\right|<\varepsilon\right\}\right)$$

and is asymptotic to $|\mathcal{C}|^{n^d}$, as the above μ-measure converges to 1.
(b) Take $A=F+g$ and $(\mathbb{X}_h)_{h\in A}$ as in (2.3). For any $x\in\Omega(\mathcal{C})$,

$$\begin{aligned}\overline{\mathbb{X}}_n(x) &= \frac{\left|S\left(w,x|_{[0,n)^d},g,F\right)\right|}{|A\cap[0,n)^d|}\\ &= \mathrm{fr}\left(w,x|_{[0,n)^d},g,F\right)\cdot(\mathbb{Z}^d:F)+o(1)\,,\end{aligned}$$

because $\left|A\cap[0,n)^d\right|$ is asymptotic to $n^d/(\mathbb{Z}^d:F)$. This implies that for n large

$$\left|\mathrm{fr}\left(w,x|_{[0,n)^d},g,F\right)-\frac{1}{|\mathcal{C}|\cdot(\mathbb{Z}^d:F)}\right|<\varepsilon\iff\left|\overline{\mathbb{X}}_n(x)-\frac{1}{|\mathcal{C}|}\right|<\varepsilon\cdot(\mathbb{Z}^d:F)$$

and then Theorem 20 guarantees the conclusion.
(c) As the events are independent, this follows from (b). □

3. Main Constructions

Let $\mathcal{C}=\{0,1\}$. In this section, we construct subshifts (X,T) with topological and ergodic prescribed properties. To this matter, we build a sequence of finite non-empty sets of configurations $\mathcal{C}_k\subset\Omega_{A_k}(\mathcal{C})$, $k\geq 1$, such that:

(i) $A_k=[0,n_k)^d$, where $(n_k)_{k\geq1}$ is an increasing sequence of positive integers.
(ii) $n_1=1$ and $\mathcal{C}_1=\Omega_{A_1}(\mathcal{C})\cong\{0,1\}$.
(iii) $\mathcal{C}_k$ is the concatenation of elements of $\mathcal{C}_{k-1}$, possibly with additional lines of zeroes and ones.

Given such sequence $(\mathcal{C}_k)_{k\geq1}$, we consider $X\subset\Omega(\mathcal{C})$ as the set of limits of shifted $\mathcal{C}_k$-patterns as $k\to+\infty$, that is, $x\in X$ if there exist sequences $(w_k)_{k\geq1}$, $w_k\in\mathcal{C}_k$, and $(g_k)_{k\geq1}\subset\mathbb{Z}^d$ such that

$$x=\lim_{k\to+\infty}T^{g_k}w_k.$$

The above limit has an abuse of notation, because T acts in $\Omega(\mathcal{C})$ and $w_k\notin\Omega(\mathcal{C})$. Formally speaking, this means that, for each $g\in\mathbb{Z}^d$, there exists $k_0\geq1$ such that

$$x(g)=w_k(g+g_k),\ \forall\,k\geq k_0.$$

By definition, X is invariant under T and, for any k, every $x\in X$ is an infinite concatenation of elements of $\mathcal{C}_k$ and lines of zeroes and ones.

If $\mathcal{C}_k\subset\Omega_{A_k}(\{0,1\})$ and $A\in\mathcal{R}$, $\Omega_A(\mathcal{C}_k)$ is identified in a natural way to a subset of $\Omega_{n_kA}(\{0,1\})$. In some situations, to distinguish this association, we use small letters for $\Omega_A(\mathcal{C}_k)$ and capital letters for $\Omega_{n_kA}(\{0,1\})$[1]. In this situation, if $w\in\Omega_A(\mathcal{C}_k)$ and $g\in A$, the pattern $w(g)\in\mathcal{C}_k$ occurs in $W\in\Omega_{n_kA}(\{0,1\})$ centered at n_kg. In other words, if $w_k\in\mathcal{C}_k$, then

$$S(w_k,W,n_kg,F)=n_k\cdot S(w_k,w,g,F). \tag{3.1}$$

In each of the next subsections, $(\mathcal{C}_k)_{k\geq1}$ is constructed with specific combinatorial and statistical properties.

[1]For example, $w\in\Omega_A(\mathcal{C}_k)$ and $W\in\Omega_{n_kA}(\{0,1\})$ denote the "same" element.

3.1. **Minimality.** The action (X,T) is minimal if and only if, for each $x,y \in X$, every configuration of x is also a configuration of y. For this, suppose $\mathcal{C}_k \subset \Omega_{A_k}(\{0,1\})$ is defined and non-empty.

By the Law of Large Numbers, if l_k is large, every element of $\mathcal{C}_k$ occurs in almost every element of $\Omega_{[0,l_k)^d}(\mathcal{C}_k)$ (in fact, by Corollary 21, each of them occurs approximately with frequency $1/|\mathcal{C}_k| > 0$). Take any subset $\mathcal{C}_{k+1}$ of $\Omega_{[0,l_k)^d}(\mathcal{C}_k)$ with this property and consider it as a subset of $\Omega_{[0,n_{k+1})^d}(\{0,1\})$, where $n_{k+1} = l_k n_k$.

Let us prove that (X,T) is minimal. Consider $x,y \in X$ and $x|_A$ a finite configuration of x. For large k, $x|_A$ is a subconfiguration of some $w_k \in \mathcal{C}_k$. As y is formed by the concatenation of elements of $\mathcal{C}_{k+1}$, every element of $\mathcal{C}_k$ is a configuration of y. In particular, w_k (and then $x|_A$) is a configuration of y.

3.2. **Total minimality.** The action (X,T) is totally minimal if and only if, for each $x,y \in X$ and $F \in \mathcal{F}$, every configuration $x|_A$ of x centered[2] at 0 also occurs in y centered at some $g \in F$. To guarantee this for every $F \in \mathcal{F}$, we inductively control the ocurrence of subconfigurations centered in finitely many subgroups of $\mathbb{Z}^d$.

Consider the sequence $(F_k) \subset \mathcal{F}$ defined in Subsection 2.2. By induction, suppose $\mathcal{C}_k \subset \Omega_{A_k}(\{0,1\})$ is non-empty satisfying (i), (ii), (iii) and the additional assumption

(iv) $\gcd(n_k, m_k) = 1$ (observe that this holds for $k = 1$).

Take l_k large and $\tilde{\mathcal{C}}_{k+1} \subset \Omega_{[0,l_k m_{k+1})^d}(\mathcal{C}_k)$ non-empty such that

(v) $S(w_k, w|_{[0,l_k m_{k+1}-1)^d}, g, F_k) \neq \emptyset$, for every triple $(w_k, w, g) \in \mathcal{C}_k \times \tilde{\mathcal{C}}_{k+1} \times \mathbb{Z}^d$.

Considering $w|_{[0,l_k m_{k+1}-1)^d}$ as an element of $\Omega_{[0,l_k m_{k+1} n_k - n_k)^d}(\{0,1\})$, (3.1) implies that

$$S(w_k, W|_{[0,l_k m_{k+1} n_k - n_k)^d}, n_k g, F_k) \neq \emptyset,\ \forall\, (w_k, w, g) \in \mathcal{C}_k \times \tilde{\mathcal{C}}_{k+1} \times \mathbb{Z}^d.$$

As $\gcd(n_k, m_k) = 1$, the set $n_k\mathbb{Z}^d$ runs over all residue classes modulo F_k and so (the restriction to $[0, l_k m_{k+1} n_k - n_k)^d$ of) every element of $\tilde{\mathcal{C}}_{k+1}$ contains every element of $\mathcal{C}_k$ centered at every residue class modulo F_k.

Obviously, $\mathcal{C}_{k+1}$ must not be equal to $\tilde{\mathcal{C}}_{k+1}$, because m_{k+1} divides $l_k m_{k+1} n_k$. Instead, we take $n_{k+1} = l_k m_{k+1} n_k + 1$ and insert positions B_i, $i = 1, 2, \ldots, d$, next to faces of the cube $[0, l_k m_{k+1} n_k)^d$. These are given by

$$B_i = \{(r_1, \ldots, r_d) \in A_{k+1}\,;\, r_i = l_k m_{k+1} n_k - n_k\}.$$

There is a natural surjection $\Phi : \Omega_{A_{k+1}}(\{0,1\}) \to \Omega_{[0,n_{k+1}-1)^d}(\{0,1\})$ obtained removing the positions $B_1, \ldots, B_d$. More specifically, if

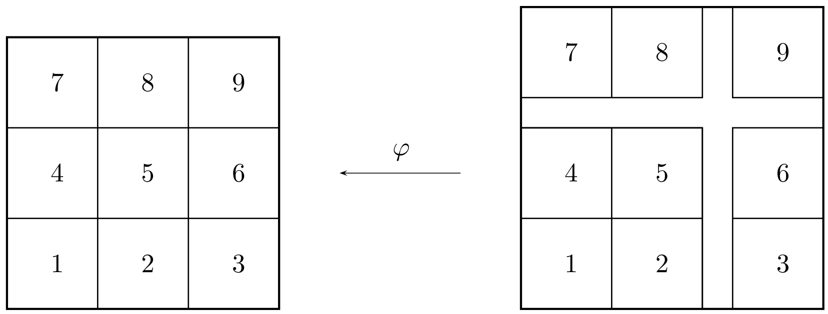
$$\begin{aligned} \delta(r) &= 0\,, \text{ if } r < l_k m_{k+1} n_k - n_k\,, \\ &= 1\,, \text{ otherwise} \end{aligned}$$

and $\Delta(r_1, \ldots, r_d) = (\delta(r_1), \ldots, \delta(r_d))$, the map Φ is given by

$$\Phi(W)(g) = W\left(g + \Delta(g)\right),\ \forall\, (r_1, \ldots, r_d) \in [0, n_{k+1} - 1)^d.$$

We conclude the induction step taking $\mathcal{C}_{k+1} = \Phi^{-1}(\tilde{\mathcal{C}}_{k+1})$.

[2]Because of the T-invariance of X, we can suppose that $x|_A$ is centered in $0 \in \mathbb{Z}^d$. In fact, instead of x, y, we consider $T^g x, T^g y$.



$\varphi(w_{k+1}) \in \tilde{\mathcal{C}}_{k+1}$ $\qquad$ $w_{k+1} \in \mathcal{C}_{k+1}$

By definition, w_{k+1} and $\varphi(w_{k+1})$ coincide in $[0, n_{k+1} - n_k - 1)^d$, for every $w_{k+1} \in \mathcal{C}_{k+1}$. This implies that every element of $\mathcal{C}_k$ appears in every element of $\mathcal{C}_{k+1}$ centered at every residue class modulo F_k.

Let us prove that (X, T) is totally minimal. Fix elements $x, y \in X$, a subgroup $F \in \mathcal{F}$ and a pattern $x|_A$ of x centered in $0 \in \mathbb{Z}^d$. By the definition of X, $x|_A$ is a subconfiguration of some $w_k \in \mathcal{C}_k$, for k large enough such that $F_k < F$. As y is built concatenating elements of $\mathcal{C}_{k+1}$, w_k occurs in y centered in every residue class modulo F and the same happens to $x|_A$. In particular, $x|_A$ occurs in y centered in some $g \in F$, which is exactly the required condition.

3.3. **Total strict ergodicity.** In addition to the ocurrence of configurations in every residue class of subgroups of $\mathbb{Z}^d$, we also control their frequency. Consider a sequence $(d_k)_{k\geq 1}$ of positive real numbers such that $\sum_{k\geq 1} d_k < +\infty$. Assume that $\mathcal{C}_1, \ldots, \mathcal{C}_{k-1}, \mathcal{C}_k$ are non-empty sets satisfying (i), (ii), (iii), (iv) and

(vi) For every $(w_{k-1}, w_k, g) \in \mathcal{C}_{k-1} \times \mathcal{C}_k \times \mathbb{Z}^d$,

$$\mathrm{fr}(w_{k-1}, w_k, g, F_{k-1}) \in \left(\frac{1 - d_{k-1}}{{m_{k-1}}^d \cdot |\mathcal{C}_{k-1}|} \;,\; \frac{1 + d_{k-1}}{{m_{k-1}}^d \cdot |\mathcal{C}_{k-1}|} \right).$$

Before going to the inductive step, let us make an observation. Condition (vi) also controls the frequency on subgroups F such that $F_{k-1} < F$. In fact, if $\bar{F}_{k-1}$ is a complete residue set modulo F_{k-1},

$$\mathrm{fr}(w_{k-1}, w_k, g, F) = \sum_{\substack{h \in \bar{F}_{k-1} \\ h \equiv_F g}} \mathrm{fr}(w_{k-1}, w_k, h, F_{k-1}) \tag{3.2}$$

and, as $\left|\{h \in \bar{F}_{k-1} \,;\, h \equiv_F g\}\right| = (F : F_{k-1})$,

$$\mathrm{fr}(w_{k-1}, w_k, g, F) \in \left(\frac{1 - d_{k-1}}{(\mathbb{Z}^d : F) \cdot |\mathcal{C}_{k-1}|} \;,\; \frac{1 + d_{k-1}}{(\mathbb{Z}^d : F) \cdot |\mathcal{C}_{k-1}|} \right). \tag{3.3}$$

We proceed the same way as in the previous subsection: take l_k large and $\tilde{\mathcal{C}}_{k+1} \subset \Omega_{[0, l_k m_{k+1})^d}(\mathcal{C}_k)$ non-empty such that

$$\mathrm{fr}(w_k, \tilde{w}_{k+1}, g, F_k) \in \left(\frac{1 - d_k}{{m_k}^d \cdot |\mathcal{C}_k|} \;,\; \frac{1 + d_k}{{m_k}^d \cdot |\mathcal{C}_k|} \right) \tag{3.4}$$

for every $(w_k, \tilde{w}_{k+1}, g) \in \mathcal{C}_k \times \tilde{\mathcal{C}}_{k+1} \times \mathbb{Z}^d$. Note that the non-emptyness of $\tilde{\mathcal{C}}_{k+1}$ is guaranteed by Corollary 21. Also, let $n_{k+1} = l_k m_{k+1} n_k + 1$ and $\mathcal{C}_{k+1} = \Phi^{-1}(\tilde{\mathcal{C}}_{k+1})$.

Fix $b_A \in \Omega^*(\mathcal{C})$. Using the big-$O$ notation, we have

$$\mathrm{fr}(b_A, W_{k+1}, g, F) - \mathrm{fr}(b_A, W_{k+1}|_{[0, n_{k+1} - n_k - 1)^d}, g, F) = O(1/l_k)\,, \tag{3.5}$$

because these two frequencies differ by the frequency of b_A in $[n_{k+1}-n_k-1, n_{k+1})^d$ and

$$\frac{(n_k+1)^d}{{n_{k+1}}^d} = \left(\frac{n_k+1}{l_k m_{k+1} n_k + 1}\right)^d = O(1/l_k).$$

The same happens to $\mathrm{fr}(w_k, w_{k+1}, g, F)$ and $\mathrm{fr}(w_k, \varphi(w_{k+1}), g, F)$, because $\Delta(g) = 0$ for all $g \in [0, n_{k+1}-n_k-1)^d$. To simplify citation in the future, we write it down:

$$\mathrm{fr}(w_k, w_{k+1}, g, F) - \mathrm{fr}(w_k, \varphi(w_{k+1}), g, F) = O(1/l_k). \tag{3.6}$$

These estimates imply we can assume, taking l_k large enough, that

$$\mathrm{fr}(w_k, w_{k+1}, g, F_k) \in \left(\frac{1-d_k}{{m_k}^d \cdot |\mathcal{C}_k|}\ ,\ \frac{1+d_k}{{m_k}^d \cdot |\mathcal{C}_k|}\right),\ \forall\, (w_k, w_{k+1}, g) \in \mathcal{C}_k \times \mathcal{C}_{k+1} \times \mathbb{Z}^d.$$

We make a calculation to be used in the next proposition. Fix $b_A \in \Omega^*(\mathcal{C})$ and $F \in \mathcal{F}$. The main (and simple) observation is: if b_A occurs in $W_k \in \mathcal{C}_k$ centered at g and w_k occurs in $\varphi(w_{k+1}) \in \tilde{\mathcal{C}}_{k+1}$ centered at $h \in [0, l_k m_{k+1}-1)^d$, then b_A occurs in $W_{k+1} \in \mathcal{C}_{k+1}$ centered at $g + n_k h$. This implies that, if $\bar{F}$ is a complete residue set modulo F, the cardinality of $S(b_A, W_{k+1}|_{[0,n_{k+1}-n_k-1)^d}, g, F)$ is equal to

$$\begin{aligned}
&\sum_{\substack{h\in\bar{F}\\ w_k\in\mathcal{C}_k}} \sum_{\substack{w_k \text{ occurring in} \\ w_{k+1}|_{[0,l_k m_{k+1}-1)^d} \\ \text{at a vertex } \equiv_F h}} |S(b_A, W_k, g - n_k h, F)| + T \\
= &\sum_{\substack{h\in\bar{F}\\ w_k\in\mathcal{C}_k}} \left|S(w_k, w_{k+1}|_{[0,l_k m_{k+1}-1)^d}, h, F)\right| \cdot |S(b_A, W_k, g - n_k h, F)| + T,
\end{aligned}$$

where T denotes the number of ocurrences of b_A in $W_{k+1}|_{[0,n_{k+1}-n_k-1)^d}$ not entirely contained in a concatenated element of $\mathcal{C}_k$. Observe that[3]

$$0 \le T \le d \cdot l_k m_{k+1} \cdot n^{d-1} \cdot n_{k+1} < dn^{d-1} \cdot \frac{{n_{k+1}}^2}{n_k}\ ,$$

where n is the length of b_A. Dividing $\left|S(b_A, W_{k+1}|_{[0,n_{k+1}-n_k-1)^d}, g, F)\right|$ by ${n_{k+1}}^d$ and using (3.5), (3.6), we get

$$\begin{aligned}
\mathrm{fr}(b_A, W_{k+1}, g, F) &= \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \sum_{\substack{h\in\bar{F}\\ w_k\in\mathcal{C}_k}} \mathrm{fr}(w_k, w_{k+1}, h, F)\mathrm{fr}(b_A, W_k, g - n_k h, F) \\
&\quad + O(1/l_{k-1})\,.
\end{aligned} \tag{3.7}$$

We wish to show that $\mathrm{fr}(b_A, x, F)$ does not depend on $x \in X$. For this, define

$$\begin{aligned}
\alpha_k(b_A, F) &= \min\left\{\mathrm{fr}(b_A, W_k, g, F)\,;\, W_k \in \mathcal{C}_k, g \in \mathbb{Z}^d\right\} \\
\beta_k(b_A, F) &= \max\left\{\mathrm{fr}(b_A, W_k, g, F)\,;\, W_k \in \mathcal{C}_k, g \in \mathbb{Z}^d\right\}.
\end{aligned}$$

The required property is a direct consequence[4] of the next result.

Proposition 22. *If $b_A \in \Omega^*(\mathcal{C})$ and $F \in \mathcal{F}$, then*

$$\lim_{k\to+\infty} \alpha_k(b_A, F) = \lim_{k\to+\infty} \beta_k(b_A, F).$$

[3]For each line parallel to a coordinate axis $e_i\mathbb{Z}$ between two elements of $\mathcal{C}_k$ in W_{k+1} or containing a line of ones, there is a rectangle of dimensions $n \times \cdots \times n \times n_{k+1} \times n \times \cdots \times n$ in which b_A is not entirely contained in a concatenated element of $\mathcal{C}_k$.

[4]In fact, just take the limit in the inequality $\alpha_k(b_A, F) \le \mathrm{fr}(b_A, x|_{A_k}, 0, F) \le \beta_k(b_A, F)$.

Proof. By (3.2), if l is large such that $F_l < F$, then

$$(F : F_l) \cdot \alpha_k(b_A, F_l) \le \alpha_k(b_A, F) \le \beta_k(b_A, F) \le (F : F_l) \cdot \beta_k(b_A, F_l).$$

This means that we can assume $F = F_l$. We estimate $\alpha_{k+1}(b_A, F)$ and $\beta_{k+1}(b_A, F)$ in terms of $\alpha_k(b_A, F)$ and $\beta_k(b_A, F)$, for $k \ge l$. As b_A and F are fixed, denote the above quantities by α_k and β_k. Take $W_{k+1} \in \mathcal{C}_{k+1}$. By (3.7),

$$\begin{aligned}
\mathrm{fr}(b_A, W_{k+1}, g, F) &\ge \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \alpha_k \cdot \sum_{\substack{h \in \bar{F} \\ w_k \in \mathcal{C}_k}} \mathrm{fr}(w_k, w_{k+1}, h, F) + O(1/l_{k-1}) \\
&= \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \alpha_k + O(1/l_{k-1})
\end{aligned}$$

and, as W_{k+1} and g are arbitrary, we get

$$\alpha_{k+1} \ge \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \alpha_k + O(1/l_{k-1})\,. \tag{3.8}$$

Equality (3.7) also implies the upper bound

$$\begin{aligned}
\mathrm{fr}(b_A, W_{k+1}, g, F) &\le \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \beta_k \cdot \sum_{\substack{h \in \bar{F} \\ w_k \in \mathcal{C}_k}} \mathrm{fr}(w_k, w_{k+1}, h, F) + O(1/l_{k-1}) \\
&= \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \beta_k + O(1/l_{k-1}) \\
\Longrightarrow \qquad \beta_{k+1} &\le \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d \cdot \beta_k + O(1/l_{k-1})\,. \qquad (3.9)
\end{aligned}$$

Inequalities (3.8) and (3.9) show that α_{k+1} and β_{k+1} do not differ very much from α_k and β_k. The same happens to their difference. Consider $w_1, w_2 \in \mathcal{C}_{k+1}$ and $g_1, g_2 \in \mathbb{Z}^d$. Renaming $g - n_k h$ by h in (3.7) and considering n_{-k} the inverse of n_k modulo m_l, the difference $\mathrm{fr}(b_A, W_1, g_1, F) - \mathrm{fr}(b_A, W_2, g_2, F)$ is at most

$$\sum_{\substack{h \in \bar{F} \\ w_k \in \mathcal{C}_k}} \mathrm{fr}(b_A, W_k, h, F)\left|\mathrm{fr}(w_k, w_1, n_{-k}(g_1 - h), F) - \mathrm{fr}(w_k, w_2, n_{-k}(g_2 - h), F)\right|$$
$$+O(1/l_{k-1}) \cdot$$

From (3.4),

$$\begin{aligned}
\mathrm{fr}(b_A, W_1, g_1, F) - \mathrm{fr}(b_A, W_2, g_2, F) &\le \frac{2d_k}{{m_l}^d \cdot |\mathcal{C}_k|} \sum_{\substack{h \in \bar{F} \\ w_k \in \mathcal{C}_k}} \mathrm{fr}(b_A, W_k, h, F) \\
&\quad + O(1/l_{k-1}) \\
&\le 2d_k + O(1/l_{k-1})\,,
\end{aligned}$$

implying that

$$0 \;\le\; \beta_{k+1} - \alpha_{k+1} \;\le\; 2d_k + O(1/l_{k-1})\,. \tag{3.10}$$

In particular, $\beta_k - \alpha_k$ converges to zero as $k \to +\infty$. The proposition will be proved if β_k converges. Let us estimate $|\beta_{k+1} - \beta_k|$. On one side, (3.9) gives

$$\beta_{k+1} - \beta_k \;\le\; O(1/l_{k-1})\,. \tag{3.11}$$

On the other, by (3.8) and (3.10),

$$\begin{aligned}\beta_{k+1}-\beta_k &\ge \alpha_{k+1}-\beta_k\\ &\ge \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d\cdot\alpha_k-\beta_k+O(1/l_{k-1})\\ &\ge \left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d\cdot[\beta_k-2d_{k-1}-O(1/l_{k-2})]-\beta_k+O(1/l_{k-1})\end{aligned}$$

which, together with (3.11), implies that

$$\begin{aligned}|\beta_{k+1}-\beta_k| &\le 2d_{k-1}+\beta_k\cdot\left[1-\left(\frac{n_{k+1}-1}{n_{k+1}}\right)^d\right]+O(1/l_{k-2})\\ &= 2d_{k-1}+O(1/l_{k-2})\,.\end{aligned}$$

As $\sum d_k$ and $\sum 1/l_k$ both converge, $(\beta_k)_{k\ge1}$ is a Cauchy sequence, which concludes the proof. □

From now on, we consider (X,T) as the dynamical system constructed as above. Note that we have total freedom to choose $\mathcal{C}_k$ with few or many elements. This is what controls the entropy of the system.

3.4. **Proof of Theorem 1.** By (2.2), the topological entropy of the $\mathbb{Z}^d$-action (X,T) satisfies

$$h(X,T)\ge\lim_{k\to+\infty}\frac{\log|\mathcal{C}_k|}{{n_k}^d}\,.$$

Consider a sequence $(\nu_k)_{k\ge1}$ of positive real numbers. In the construction of $\mathcal{C}_{k+1}$ from $\mathcal{C}_k$, take l_k large enough such that

(vii) $\nu_k\cdot n_{k+1}\ge1$.

(viii) $|\tilde{\mathcal{C}}_{k+1}|\ge|\mathcal{C}_k|^{(l_km_{k+1})^d\cdot(1-\nu_k)}$.

These inequalities imply

$$\begin{aligned}\frac{\log|\mathcal{C}_{k+1}|}{{n_{k+1}}^d} &\ge \frac{\log|\tilde{\mathcal{C}}_{k+1}|}{{n_{k+1}}^d}\\ &\ge \frac{(l_km_{k+1})^d\cdot(1-\nu_k)\cdot\log|\mathcal{C}_k|}{{n_{k+1}}^d}\\ &\ge (1-\nu_k)^{d+1}\cdot\frac{\log|\mathcal{C}_k|}{{n_k}^d}\end{aligned}$$

and then

$$\frac{\log|\mathcal{C}_k|}{{n_k}^d}\ge\prod_{i=1}^{k-1}(1-\nu_i)^{d+1}\cdot\frac{\log|\mathcal{C}_1|}{{n_1}^d}=\prod_{i=1}^{k-1}(1-\nu_i)^{d+1}\cdot\log 2\,.$$

If $\nu\in(0,1)$ is given and $(\nu_k)_{k\ge1}$ are chosen also satisfying

$$\lim_{k\to+\infty}\prod_{i=1}^{k}(1-\nu_i)^{d+1}=1-\nu\,,$$

we obtain that $h(X,T)\ge(1-\nu)\log2>0$. If, instead of $\{0,1\}$, we take $\mathcal{C}$ with more elements and apply the construction verifying (i) to (viii), the topological entropy of the $\mathbb{Z}^d$-action is at least $(1-\nu)\log|\mathcal{C}|$. We have thus proved Theorem 1.

4. Proof of Theorems 2 and 3

Given a finite alphabet $\mathcal{C}$, consider a configuration $b_{A_1} : A_1 \to \mathcal{C}$ and any $A_2 \subset A_1$ such that $|A_2| \leq \varepsilon |A_1|$. If $b_{A_2} : A_2 \to \mathcal{C}$, the element $w \in \Omega_{A_1}(\mathcal{C})$ defined by

$$\begin{aligned} w(g) &= b_{A_1}(g)\,, \text{ if } g \in A_1 \backslash A_2\,, \\ &= b_{A_2}(g)\,, \text{ if } g \in A_2 \end{aligned}$$

has frequencies not too different from b_{A_1}, depending on how small ε is. In fact, for any $c \in \mathcal{C}$,

$$|S(c, b_{A_1}, g, F)| - |A_2| \leq |S(c, w, g, F)| \leq |S(c, b_{A_1}, g, F)| + |A_2|$$

and then $|\mathrm{fr}(c, b_{A_1}, g, F) - \mathrm{fr}(c, w, g, F)| \leq \varepsilon$.

Definition 23. The *upper-Banach density* of a set $P \subset \mathbb{Z}^d$ is equal to

$$d^*(P) = \limsup_{n_1,\ldots,n_d \to +\infty} \frac{|P \cap [r_1, r_1 + n_1) \times \cdots \times [r_d, r_d + n_d)|}{n_1 \cdots n_d}\,.$$

Consider a set $P \subset \mathbb{Z}^d$ of zero upper-Banach density. We will make l_k grow quickly such that any pattern of $P \cap (A_k + g)$ appears as a subconfiguration in an element of $\mathcal{C}_k$. Let's explain this better. Consider the d-dimensional cubes $(A_k)_{k \geq 1}$ that define (X, T). For each $k \geq 1$, let $\tilde{A}_k \subset A_k$ be the region containing concatenated elements of $\mathcal{C}_{k-1}$. Inductively, they are defined as $\tilde{A}_1 = \{0\}$ and

$$\tilde{A}_{k+1} = \bigcup_{g \in [0, l_k m_{k+1})^d} \left(\tilde{A}_k + n_k g + \Delta(n_k g)\right)\,,\ \forall\, k \geq 1.$$

Lemma 24. *If $P \subseteq \mathbb{Z}^d$ has zero upper-Banach density, there exists a totally strictly ergodic $\mathbb{Z}^d$-action (X, T) with the following property: for any $k \geq 1$, $g \in \mathbb{Z}^d$ and $b : P \cap (A_k + g) \to \{0, 1\}$, there exists $w_k \in \mathcal{C}_k$ such that*

$$w_k(h - g) = b(h)\,,\ \forall\, h \in P \cap (A_k + g).$$

Proof. We proceed by induction on k. The case $k = 1$ is obvious, since $\mathcal{C}_1 \cong \{0, 1\}$. Suppose the result is true for some $k \geq 1$ and consider $b : P \cap (A_{k+1} + g_0) \to \{0, 1\}$. By definition, any 0,1 configuration on $A_{k+1} \backslash \tilde{A}_{k+1}$ is admissible, so that we only have to worry about positions belonging to $\tilde{A}_{k+1}$. For each $g \in [0, l_k m_{k+1})^d$, let

$$b^g : P \cap \left(\tilde{A}_k + n_k g + \Delta(n_k g) + g_0\right) \to \{0, 1\}$$

be the restriction of b to $P \cap \left(\tilde{A}_k + n_k g + \Delta(n_k g) + g_0\right)$. If $\varepsilon > 0$ is given and l_k is large enough,

$$\begin{aligned} \frac{\left|P \cap \left(\tilde{A}_{k+1} + g_0\right)\right|}{\left|\tilde{A}_{k+1} + g_0\right|} &< \frac{\varepsilon}{(2n_k)^d} \\ \Longrightarrow \qquad \left|P \cap \left(\tilde{A}_{k+1} + g_0\right)\right| &< \varepsilon \cdot (l_k m_{k+1})^d\,, \end{aligned}$$

for any $g_0 \in \mathbb{Z}^d$. This implies that $P \cap \left(\tilde{A}_k + n_k g + \Delta(n_k g) + g_0\right)$ is non-empty for at most $\varepsilon \cdot (l_k m_{k+1})^d$ values of $g \in [0, l_k m_{k+1})^d$. For each of these, the inductive hypothesis guarantees the existence of $w^g \in \mathcal{C}_k$ such that

$$w^g(h - n_k g - \Delta(n_k g) - g_0) = b^g(h)\,,\ \forall\, h \in P \cap \left(\tilde{A}_k + n_k g + \Delta(n_k g) + g_0\right).$$

Take any element $z \in \mathcal{C}_{k+1}$ and define $\tilde{z} \in \Omega_{A_{k+1}}(\{0,1\})$ by

$$\begin{aligned} \tilde{z}(h) &= w^g\left(h - n_k g - \Delta(n_k g)\right), \text{ if } h \in \tilde{A}_k + n_k g + \Delta(n_k g) \\ &= b(h) \qquad\qquad\qquad\qquad\quad, \text{ if } h \in A_{k+1} \backslash \tilde{A}_{k+1} \\ &= z(h) \qquad\qquad\qquad\qquad\quad, \text{ otherwise.} \end{aligned}$$

If $\varepsilon > 0$ is sufficiently small, $\tilde{z} \in \mathcal{C}_{k+1}$. By its own definition, $\tilde{z}$ satisfies the required conditions. ☐

The above lemma is the main property of our construction. It proves the following stronger statement.

Corollary 25. *Let (X,T) be the $\mathbb{Z}^d$-action obtained by the previous lemma. For any $b : P \to \{0,1\}$, there is $x \in X$ such that $x|_P = b$. Also, given $x \in X$, $A \in \mathcal{R}$ and $b : P \to \{0,1\}$, there are $\tilde{x} \in X$ and $n \in \mathbb{N}$ such that $\tilde{x}|_A = x|_A$ and $\tilde{x}(g) = b(g)$ for all $g \in P\backslash(-n,n)^d$.*

Proof. The first statement is a direct consequence of Lemma 24 and a diagonal argument. For the second, remember that x is the concatenation of elements of $\mathcal{C}_k$ and lines of zeroes and ones, for every $k \geq 1$. Consider $k \geq 1$ sufficiently large and $z_k \in \mathcal{C}_k$ such that $x|_A$ occurs in z_k. For any $z \in \mathcal{C}_{k+1}$, there is $g \in \mathbb{Z}^d$ such that $z|_{A_k+g} = z_k$. Constructing $\tilde{z}$ from z making all substitutions described in Lemma 24, except in the pattern $z|_{A_k+g}$, we still have that $\tilde{z} \in \mathcal{C}_{k+1}$. ☐

4.1. **Proof of Theorem 2.** Consider $f : X \to \mathbb{R}$ given by $f(x) = x(0)$. Then

$$\frac{1}{|P \cap (-n,n)^d|} \sum_{g \in P \cap (-n,n)^d} f\left(T^g x\right) = \mathrm{fr}\left(1, x|_{P \cap (-n,n)^d}\right).$$

For each $n \geq 1$, consider the sets

$$\begin{aligned} \Lambda_n &= \bigcup_{k \geq n} \left\{x \in X \,;\, \mathrm{fr}\left(1, x|_{P \cap (-k,k)^d}\right) < 1/n\right\} \\ \Lambda^n &= \bigcup_{k \geq n} \left\{x \in X \,;\, \mathrm{fr}\left(1, x|_{P \cap (-k,k)^d}\right) > 1 - 1/n\right\}. \end{aligned}$$

Fixed k and n, the sets $\left\{x \in X \,;\, \mathrm{fr}(1, x|_{P \cap (-k,k)^d}) < 1/n\right\}$ and is clearly open, so that the same happens to Λ_n. It is also dense in X, as we will now prove. Fix $x \in X$ and $\varepsilon > 0$. Let $k_0 \in \mathbb{N}$ be large enough so that $d(x,y) < \varepsilon$ whenever $x|_{(-k_0,k_0)^d} = y|_{(-k_0,k_0)^d}$. Take $y \in X$ such that $y|_{(-k_0,k_0)^d} = x|_{(-k_0,k_0)^d}$ and $y(g) = 0$ for all $g \in P\backslash(-n,n)^d$ as in Corollary 25. As $\mathrm{fr}(1, y|_{(-k,k)^d})$ approaches to zero as k approaches to infinity, $y \in \Lambda_n$, proving that Λ_n is dense in X. The same argument show that Λ^n is a dense open set. Then

$$X_0 = \bigcap_{n \geq 1} \left(\Lambda_n \cap \Lambda_n\right)$$

is a countable intersection of dense open sets, thus residual. For each $x \in X_0$,

$$\begin{aligned} \liminf_{n \to +\infty} \frac{1}{|P \cap (-n,n)^d|} \sum_{g \in P \cap (-n,n)^d} f\left(T^g x\right) &= 0 \\ \limsup_{n \to +\infty} \frac{1}{|P \cap (-n,n)^d|} \sum_{g \in P \cap (-n,n)^d} f\left(T^g x\right) &= 1, \end{aligned}$$

which concludes the proof of Theorem 2.

4.2. **Proof of Theorem 3.** Choose an infinite set $G = \{g_i\}_{i\geq 1}$ in $\mathbb{Z}^d$ disjoint from P such that $P' = G \cup P \cup \{0\}$ also has zero upper-Banach density and let (X,T) be the $\mathbb{Z}^d$-action given by Lemma 24 with respect to P', that is: for every $b : P' \to \{0,1\}$, there exists $x^b \in X$ such that $x^b|_{P'} = b$. Consider

$$X_0 = \left\{x^b \in X\,;\, b(0) = 0 \text{ and } b(g) = 1,\ \forall\, g \in P\right\}.$$

This is an uncountable set (it has the same cardinality of $2^G = 2^{\mathbb{N}}$) and, for every $x^b \in X_0$ and $g \in P$, the elements $T^g x^b$ and x^b differ at $0 \in \mathbb{Z}^d$, implying that $x^b \notin \overline{\{T^g x^b\,;\, g \in P\}}$. This concludes the proof.

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