Pricing rules and Arrow Debreu ambiguous valuation*

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Abstract

This paper considers pricing rules of single-period securities markets with finitely many states and without arbitrage opportunities. Our main result characterize those pricing rules C that are super-replication prices of a frictionless incomplete asset structure. This characterization relies on the equivalence between the sets of *frictionless securities* and *undominated securities* priced by C. The former captures securities without bid-ask spreads while the second captures the class of securities where, if some of its delivers is replaced by a higher payoff, then the resulting security is characterized by a higher value priced by C.

We also analyze the special case of pricing rules revealing securities markets admitting a structure of basic assets paying one in some event and nothing otherwise. In this case we show that any security can be priced against a capacity. This *risk-neutral capacity,* or *Arrow-Debreu ambiguous state price*, can be viewed as a generalization for incomplete markets of Arrow Debreu price valuation, and the corresponding pricing rule is determined by an integral w.r.t. a subadditive capacity. For instance, a special class of Choquet integral is related to frictionless incomplete markets of Arrow securities and a riskless asset.

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1 Introduction

Since the Arrow's Role of Securities seminal paper, the general equilibrium theory has been the fundamental scope for the study of some fundamental issues in economic theory. As an important branch of this theory that highlights the role of uncertainty in economics, the financial general equilibrium models assume that the price of assets satisfies equilibrium conditions in a setting where many agents demand assets profiles in accordance with their preferences and their endowments, providing the foundations for the study of financial markets through a fundamental result says that for an economy with financial markets satisfying mild conditions, at equilibrium, financial markets must not offer arbitrage opportunities for any agent. For instance, in a two period economy it implies the impossibility, at equilibrium, to realize positive net financial returns in the second period without spending at the initial period some amount of money in the asset market. Furthermore, the fundamental theorem of asset pricing for frictionless complete markets[1] enforce linear pricing rule: the cost of replication of any security is given by the mathematical expectation of its payoffs stream under the unique state contingent price or risk neutral probability obtained by the no-arbitrage principle.

Nowadays, a widely accept paradigm says that complete markets assumption becomes the exception rather than the rule in the study of financial markets. Thus, since market incompleteness says that not all securities admit a perfect hedge, the studies of securities markets reveals a new and important aspect when compared to complete market case: in many cases the seller of a security should consider a superhedging strategy[2] in order to protect against any possible claims of the buyer of such security[3]. Hence, in a financial economy where agents can trade a finite and potential limited number of frictionless securities, the pricing rule gives the minimum cost of getting a payoff equal to (or larger than) a given contingent claim in any state of nature, which is also known as the super-replication price. Importantly, by no-arbitrage and assuming the presence of a fair risk-free security, the super-replication price of any security can be determinate by its supremum expected value with respect to all risk-neutral probabilities.

Another prominent problem in the study of financial markets is the possibility of frictions affecting tradeable assets. Among others, frictions includes

[1] Recall that a financial market is complete if the trading of basic securities reproduce any financial payoff stream, otherwise the financial market is incomplete.

[2] A superhedging strategy or super-replication is a portfolio strategy which generates payoffs across the states that are at least as large as the underlying security.

[3] Some results show that this is typically the case for some important classes of securities markets, for example, a well known result from Ross (1976) says that whenever the payoff of every *call* or *put option* can be replicated, the securities market must be complete. Also, Aliprantis and Tourky (2002) showed that if the number of securities is less than half the number of states of the world, then generically we have the absence of perfect replication of *any* option. Hence, the approach of finding the value of an option by reference to the prices of the primitive securities breaks down for any option. In another way, Baptista (2007) showed that (generically) if every *risk binary contingent claim* is non attainable then every option is non attainable. For further results, see Polyrakis and Xanthos (2010).

bid-ask spreads, short sales constraints and short selling costs, and differences between borrowing and lending rates. In such cases, the market might be complete and we still have more than one underlying risk-neutral probability and the pricing rule is also given by the supremum over the expected values with respect to all risk-neutral probabilities. A consequence is that any normalized pricing rule should satisfy a set of mild and intuitively conditions as essentially obtained by a well known representation theorem for a functional described by a set of probability measures[4]. Next section assumes such conditions as a primitive for pricing rules, as done by Jouini and Kallal (2001), and discusses its intuitive appeal.

It is quite immediate that given a pricing rule there are many candidates for the corresponding underlying type of financial structure. So, given a non-linear pricing rule, how to identify the type of market imperfection related to it? Our main result identifies the case of pricing rules related to frictionless incomplete markets by finding a special property for pricing rules avoiding frictions in all tradeable securities.

For our main result characterizing those pricing rules C that are super-replication prices of some frictionless incomplete asset structure, we established an equivalence between the set of *frictionless securities* and *undominated securities* priced by C. The set of *frictionless securities* priced by C is defined as

$$F_C := \{Y : C(Y) + C(-Y) = 0\},$$

and the set of *undominated securities* priced by C is defined as

$$L_C := \{Y : X > Y \Rightarrow C(X) > C(Y)\}.$$

While a frictionless security can be bought and sold without any frictions, undominated securities have the property that if a payoff assigned to a state by the security is replaced by a bigger payoff, then the resulting security has a strictly superior super-replication price[5]. So, for an undominated security, there is no gain that can be added while maintaining its super-replication price. On other hand, for a dominated security X, there is some Y paying never less than X and delivering more in at least one state of nature with same price, *i.e.*, $C(Y) = C(X)$. So, if an agent purchase X instead of Y then she/he is discarding the positive contingent wealth sure in the event where the first security reveals a worse performance than the second one. Hence, our main result says that C reveals a frictionless securities market if, and only if, every security such that every payoff can not be improved without additional cost is frictionless.

We also analyze the special case of pricing rules revealing securities markets admitting a structure of basic assets paying one in some event and nothing

[4] See, for instance, Huber (1981), Gilboa and Schmeidler (1989), and Chateauneuf (1991). The same characterization is the key for the representation of coherent risk measures as introduced by Artzner et al. (1999).

[5] Formally, this definition captures the pricing rule's domain of monotonicity. Also, given an arbitrary pricing rule, any frictionless security is an undominated security.

otherwise, *i.e.*, a structure of *bets*. In this case we show that any asset can be priced against a capacity. This *risk-neutral capacity*, or *Arrow Debreu ambiguous state price*, can be viewed as a generalization for incomplete markets of risk neutral probabilities or Arrow Debreu state prices, and the corresponding pricing rule is determined by an integral w.r.t. such non-additive probability. More precisely, general markets of bets are revealed through pricing rules given by a Lehrer integral and the special case of partition markets (*i.e.*, markets of bets where basic assets induces a partition of state space) are revealed through pricing rules given by a Choquet Integral.

2 Framework

We consider a single-period economy where the uncertainty is modeled by a finite state space $S = \{s_1, ..., s_n\}$. A mapping $X : S \to \mathbb{R}$ is a security that gives the right to $X(s)$ units of consumption or wealth in the second period in each state of nature $s \in S$.

We denote by $C : \mathbb{R}^S \to \mathbb{R}$ a pricing rule, *i.e.*, agents have to pay $C(X)$ units of initial wealth in order to guarantee at least $X(s)$ units of wealth in each state $s \in S$. Following well-known works in the literature, we shall make the following assumptions concerning a pricing rule

Definition 1 *A pricing rule satisfies :*
i) C is sublinear, i.e.,

$$C(\lambda X) = \lambda C(X), \text{ and}$$

$$C(X + Y) \leq C(X) + C(Y),$$

for all $X, Y \in \mathbb{R}^S$ and all non-negative real number λ;
ii) C is arbitrage free, i.e., $C(X) > 0$ for any nonzero security $X \geq 0$;
iii) C is normalized, i.e., $C(1_S) = 1$;
iv) C is monotonic, i.e., $C(X) \geq C(Y)$ for all $X, Y \in \mathbb{R}^S$ s.t. $X \geq Y$;
v) C is constant additive, i.e.,

$$C(X + k1_S) = C(X) + k,$$

for all $X \in \mathbb{R}^S$ and all real number k.

Such properties are usual and have been proposed by Jouini (2000), Jouini and Kallal (2001) and Castagnoli et. al. (2002), among others. The assumption (i) means that the price of a security is proportional to the quantity purchased and that it is less expensive to purchase a portfolio of securities than to purchase each security separately. We note that subadditivity implies that

$$C(X) \geq -C(-X),$$

that is, the price at which X can be bought is larger than or equal to the price at which it can be sold. The assumption (ii) captures the absence of arbitrage opportunities by imposing that there are no free security that are nonnegative in every state of nature and strictly positive in at least one. Assumption (iii) means that the riskless asset can be bought and sold without any frictions and that riskless rate is equal to zero. The assumption (iv) is a natural condition saying that any investor will not pay more for less. Finally, the assumption (v) means either to purchase a portfolio composed by a security and the riskless asset or to purchase each of these securities separately.

We recall that in a given financial economy, where in order to transfer wealth from the initial date to the future agents can trade a finite number of securities, the induced pricing rule C reveals for any security X its minimum cost $C(X)$ of getting a payoff equal to (or larger than) the delivers promised by X across the states of nature. The pricing rule C is also referred as a *super-replication price* of its underlying securities market[6].

In terms of representation, every pricing rule can be computed by the following representation[7]:

Theorem 2 *For any pricing rule satisfying conditions (i)-(v) there is a closed and convex set $\mathcal{Q}$ of probability measures, where at least one element is strictly positive, such that for any security X*

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X)$$

With such representation in mind, given a pricing rule C, its *extended* set of risk-neutral probabilities $\mathcal{Q}$ is the closure of the usual set of risk-neutral probabilities, knowing also as the set of "underlying" linear pricing rules. By this representation we are motivated to adopt a definition saying that the probability measure $P \in \mathcal{Q}$ "prices" X if its satisfies $C(X) = E_P(X)$.

3 Pricing Rules and Frictionless Securities Markets

The usual way in the literature that obtains pricing rules starts from a given financial market without arbitrage opportunities, and by considering the notion of super-replication obtains a functional form for the super-replication price describing the market pricing rule that, of course, satisfies the five conditions as given in the definition of pricing rules. It seems useful to review some cases of market structure and its induced pricing rule:

(i) if markets are complete and frictionless then the set $\mathcal{Q}$ has only one element, *i.e.*, C is well know linear pricing rule of a complete market;

[6] We note that values $-C(-X)$ and $C(X)$ can also be interpreted as arbitrage bounds on the price of X. Indeed, the normative argument is that investors would not pay more than $C(X)$ for X and would not sell it for less than $-C(-X)$, because in both cases a better outcome can be research through securities trading.

[7] This result can be derived from Huber (1981).

(*ii*) If markets are incomplete, without any other imperfection, then $\mathcal{Q}$ is the closure of the set of risk neutral probabilities or, taking into account a multiperiod framework, the set of martingale measures of the traded securities normalized price (Jouini and Kallal, 1995);

(*iii*) If the traded securities can be bought at a price (the ask) that is potentially higher than the price (the bid) at which they can be sold, then $\mathcal{Q}$ is the closure of the set of martingale measures of any price between the normalized bid and ask price (Jouini and Kallal, 1995; see also, Bensaid et. all 1992 and Baccara et. all 2006).

(*iv*) If agents are subjective to short sales constraints then the set $\mathcal{Q}$ is the closure of the set of supermartingale measures of the traded securities normalized price (Dybvig and Ross 1986; Jouini and Kallal, 1995).

Such cases illustrate how different market structures share a common form of pricing rules and attest the naturalness of its general definition followed by us as we saw proposed by Jouini and Kallal (1991). Such generality reveals an interesting identification problem, in fact, for a given pricing rule it is possible that there are many candidates for its underlying market structure type. Of course, if we take a linear pricing rule it is quite immediate that the underlying market must be complete and frictionless. On the other hand, in the case of a non-linear pricing rule seems problematic regarding the identification of the respective market imperfection related to it. Traditionally, in a competitive market the observed price reveal the whole pertinent information to agents. One question that seems interesting to us is whether the knowledge of the pricing rule can reveal the type of incompleteness or else if exist some kind of friction for tradeable securities in the market.

Our main result characterizes those pricing rules C that are super-replication prices of a frictionless incomplete securities structure with the riskless bond. We perform this resulting by adding a new condition to the list of necessary properties (i)-(v) shared by all financial pricing rule[8].

3.1 Frictionless and Unambiguously Priced Securities

Consider a pricing rule C, the possible lack of additivity is related to the possibility of frictions in the financial market. For instance, there is friction for a security X if the buying price C is not the same as its selling price $-C(-X)$, and in this case the subadditivity captures the natural intuition that its selling pricing $C(X)$ may be greater its buying price $-C(-X)$. Thus, the set of *frictionless securities* is defined by

$$F_C := \left\{ X \in \mathbb{R}^S : C(X) + C(-X) = 0 \right\}.$$

The fact that $X \in F_C$ means that the security X can be bought and sold without any frictions when priced by C. Thanks to its basic properties, any pricing rule induces a collection of frictionless securities with a structure of linear space, in fact:

[8] See Appendix, Part A, for a summary about frictionless securities market.

Lemma 3 *Let C be a pricing rule, the set of frictionless securities F_C is a linear subspace.*

We already know that given a pricing rule C there is a unique convex and closed set probabilities $\mathcal{Q}$ related to it that reveals the set of linear pricing rules compatible with the underlying market. Due to the possibility of multiple linear pricing rules related to the original non-linear price rule, for several securities it is possible to find many expected values for different choices of the linear pricing operator. On the other hand, many securities may be immune to the existence of multiple linear pricing, which motivates to define a security X as an *unambiguously priced security* if for all linear pricing rules $P, Q \in \mathcal{Q}$

$$E_P(X) = E_Q(X).$$

Such condition means that all linear pricing rules agree about the price of X, *i.e.*, any risk-neutral probability $P \in \mathcal{Q}^+$ "prices" the security X. A simple and interesting result says that

Lemma 4 *Given a pricing rule C, a security X is frictionless if, and only if, X is unambiguously priced.*

3.2 Pricing Rules and Undominated Securities

Given a pricing rule C, its set of *undominated securities* is defined by[9]

$$L_C := \left\{ X \in \mathbb{R}^S : Y > X \ \Rightarrow C(Y) > C(X) \right\}.$$

A undominated security X is a security with the property that if some payoff assigned to a state by the claim is replaced by a better payoff, then the resulting security is strictly more expensive than the original one.

On other hand, for a dominated security X, by definition, there is Y such that $Y > X$ and $C(Y) = C(X)$: It means that if an agent purchase X instead of Y as above then she/he is discarding the wealth $Y(s) - X(s)$ in each state of the event $\{Y > X\}$. We note that all frictionless security X is undominated: in fact, for a pricing rule C with a set of multiple linear pricing rules $\mathcal{Q}$ that contains a strictly positive probability P_0, if $Y > X$ since X is unambiguously priced we obtain that[10]

$$C(Y) \geq E_{P_0}(Y) > E_{P_0}(X) = C(X).$$

3.3 Main Result

Our result that characterizes frictionless incomplete market says that

[9] In the context of decision theory under uncertainty, Lehrer (2007) provided a representation for preferences using a similar notion called *fat-free acts*.

[10] Or, in another way, given a security X s.t. $C(X) = -C(-X)$, for any $Y > X$ we obtain that $C(Y) - C(X) = C(Y) + C(-X) \geq C(Y - X) > 0$.

Theorem 5 *The pricing rule C is a super-replication price of a frictionless incomplete market of securities if, and only if, $F_C = L_C$.*

The main intuition of this result is that for a given pricing rule that the investor would observe in the financial market, the underlying complete or incomplete market structure will not exhibit friction in any tradeable security if, and only if, any undominated security is unambiguously priced or, equivalently, frictionless. Also, in this case, the presence of any non-linearity reveals that the corresponding financial market is incomplete. In another way, taking into account the viewpoint of a price taker investor choosing between securities priced by a non-linear pricing rule as in our main result, the choice of any friction security will make the investor suboptimal in the sense that it is available a security that improves the former in at least one contingence[11].

Now, we present some examples showing how our result can reveals when the underlying market is incomplete and without frictions, and also the corresponding set of tradeable securities.

Example 6 *Consider the pricing rule $C : \mathbb{R}^3 \to \mathbb{R}$ defined by*

$$C(X) = \max\{E_{P_1}(X), E_{P_2}(X)\},$$

where $P_1 = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ and $P_2 = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$. We note that, for all security $X = (x_1, x_2, x_3)$

$$C(X) = \max\left\{\alpha x_1 + \left(\frac{3}{4} - \alpha\right) x_2 + \frac{1}{4} x_3 : \alpha \in \left[\frac{1}{4}, \frac{1}{2}\right]\right\}.$$

It is simple to see that $F_C = \left\{X \in \mathbb{R}^3 : x_1 = x_2\right\}$ and $X = (1, 2, 0) \in L_C$ with bid-ask $1/4$. Hence, C is not a super-replication price of a frictionless incomplete market.

An interesting fact is that the pricing rule in Example 6 is a special case of *insurance functional* as studied by Castagnoli, Maccheroni and Marinacci (2002). So, in this case, the underlying insurance market must admit frictions for some tradeable securities.

Example 7 *Consider $C : \mathbb{R}^3 \to \mathbb{R}$ defined by*

$$C(X) = \begin{cases} x_3, & \text{if } x_1 + x_2 - 2x_3 < 0 \\ \frac{1}{2}(x_1 + x_2), & \text{if } x_1 + x_2 - 2x_3 \geq 0 \end{cases}.$$

We note that, for all security X

$$C(X) = \max\left\{\alpha x_1 + \alpha x_2 + (1 - 2\alpha) x_3 : \alpha \in \left[0, \frac{1}{2}\right]\right\}$$

We note that case $F_C = L_C = \left\{X \in \mathbb{R}^3 : x_1 + x_2 - 2x_3 = 0\right\}$. Hence, C is the super-replication price of the incomplete market where, e.g.*, basic assets are given by $(1, 1, 1), (2, 0, 1)$, both with price 1.*

[11] Of course, this reasoning supposes an investor that prefers always increase his/her wealth any future contingency.

Example 8 *Given a probability* $Q \in \Delta^{+}$*, let* $C : \mathbb{R}^S \to \mathbb{R}$ *be a pricing rule defined by*

$$C(X) = (1-\varepsilon) E_Q(X) + \varepsilon \max X(S),$$

that we call a epsilon-contamination pricing rule*.*

In fact, for all security X

$$C(X) = \max_{P \in (1-\varepsilon)\{Q\}+\varepsilon\Delta} E_P(X).$$

In this case $F_C = span\{1_S\}$ *and* $L_C = \mathbb{R}^S$*. Hence,* C *is not a super-replication price of a frictionless incomplete market. We note that for all security* X*, its bid-ask is given by*

$$BA(X) := C(X) + C(-X) = \varepsilon(\max X - \min X).$$

4 Markets of Bets

Arrow (1953) introduced the notion of contingent markets where agents can trade promises concerning the future uncertainty realizations. A wide class of assets used is known as Arrow securities characterized by a promise on a particular state of nature $s \in S$, *i.e.*, in a financial market the set of possible Arrow securities is given by $\mathbb{A} := \left\{\{s\}^* : s \in S\right\}$[12]. Given an event A, the $\{0,1\}$-security A^* is also often called a *bet on (the event) A*.

Definition 9 *We say that the mapping* $C : \mathbb{R}^S \to \mathbb{R}$ *is the super-replication price of a frictionless market of* $\{0,1\}$*-securities without arbitrage opportunities if* C *satisfies the conditions of Lemma 17*[13] *and there is a collection of events* $B_1, ..., B_m$ *such that* $X_j = B_j^*$ *for any* $j \in \{1, ..., m\}$*.*

For instance, of course, the simplest example is given by a complete securities market revealed by a pricing rule C such that there is a probability $P \in \Delta^{+}$ where

$$C(X) = E_P(X).$$

In such case the value $P(\{s\}) := p_s$ is the Arrow-Debreu state price of the contingence $s \in S$. Also, in this case the underlying markets can be constructed by choosing the whole collection of simple bets $1_{\{s\}}$ with respective prices p_s.

In this section we characterize the class of frictionless incomplete markets of bets. Before, we need to recall some mathematical notation and definition about nonadditive measure and integration.

[12] Of course, markets with only Arrow securities is a very particular case of markets with $\{0,1\}$-securities.

[13] See Appendix, Part A.

4.1 Capacities and non additive integration

A capacity is a set-function $\mu : 2^S \to [0,1]$ such that: (i) $\mu(\emptyset) = 0$ and $\mu(S) = 1$; and (ii) $A \supseteq B \Rightarrow \mu(A) \geq \mu(B)$. We say that a capacity μ is concave if for all $A, B \in 2^S$

$$\mu(A \cup B) + \mu(A \cap B) \leq \mu(A) + \mu(B).$$

Of course, all concave capacity is subadditive, in the sense that for all disjoint event $A, B \in 2^S$

$$\mu(A \cup B) \leq \mu(A) + \mu(B),$$

but the converse is not true[14]. The case of convex and subadditive capacities follows an analogous way by taking the reverse inequalities.

The set of *unambiguous events* induced by the capacity μ is defined by[15]

$$\mathcal{E}_\mu := \left\{A \in 2^S : \mu(A) + \mu(A^c) = 1\right\},$$

which defines the linear subspace

$$F_\mu := span\left\{A^* : A \in \mathcal{E}_\mu\right\}.$$

We note that by Lemma 4, B is a an unambiguous event if and only if the bet B^* is frictionless.

Another important concept related to a capacity μ is its *acore* defined by

$$acore(\mu) := \left\{P \in \Delta : P(A) \leq \mu(A), \forall A \in 2^S\right\}.$$

The outer capacity of μ, denoted by μ^*, is defined by:

$$E \in 2^S \mapsto \mu^*(A) = \min\left\{\mu(B) : B \in \mathcal{E}_\mu \text{ and } A \subset B\right\},$$

So, given a capacity μ, since $\mu^* \geq \mu$ clearly $acore(\mu) \subset acore(\mu^*)$.

A capacity μ is a-exact if $acore(\mu) \neq \emptyset$ and for all event $E \subset S$

$$\mu(A) = \max\left\{P(A) : P \in acore(\mu)\right\}.$$

We note that, given a pricing rule C, the induced *price of bets*

$$\mu_C(A) := C(A^*) \text{ for any } E \subset S,$$

is a subadditive capacity.

The capacity μ has no-gap if for every event $A \subset S$ and for every positive measure $\tau : 2^S \to [0,1]$ that satisfies $\mu \geq \tau$, there is p in the acore of μ such that $p \geq \tau$. Note that it is not imposed that $\tau(S) = 1$.

The "concave integral" was proposed and characterized by Lehrer (2009) for capacities, which differs from the well-known Choquet integral when the capacity is not convex. In a similar way, Lehrer integral can be defined as a

[14] See, for instance, Schmeidler (1972) and Chateauneuf and Jaffray (1989).

[15] Of course, an event B is unambiguous iff the corresponding bet B^* is unambiguously priced.

"convex integral". A special goal of this paper is the case of pricing rules that are characterized by a convex Lehrer integral. For the next definition, we use the convention saying that a *contingent claim* X is a security with non negative payoffs.

Definition 10 *Let C be a pricing rule over the set of contingent claims $\mathbb{R}_+^S$, then C is a Lehrer integral if*

$$C(X) = (\mathcal{L}) \int X d\mu_C \text{ for all } X \in \mathbb{R}_+^S$$

where,

$$(\mathcal{L}) \int X d\mu_C := \min \left\{ \sum \alpha_i \mu_C(A_i) : \sum \alpha_i A_i^* = X, \alpha_i \geq 0 \right\}.$$

In this case,

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X).$$

The last condition in the definition of the Lehrer integral follows from Azrieli and Lehrer (2007) because every pricing rule is supposed to be constant additive, and hence the underlying capacity has no-gap[16]. Also, following the Remark 3 of Lehrer (2009), if a pricing rule over contingent claims $X \in \mathbb{R}_+^S$ is given by a Lehrer integral then the constant additivity property enables us to extend the domain of the "Lehrer pricing rule" from the non-negative securities to all securities. In fact, for instance, given $X \in \mathbb{R}^S$ with $\min_S X < 0$, then $X - (\min_S X) S^* \geq 0$ and

$$C(X) := C\left(X - \left(\min_S X\right) S^*\right) + \min_S X.$$

Now, we recall the definition of Choquet integral (Choquet, 1954):

Definition 11 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be a pricing rule, then C is a Choquet integral if*

$$C(X) = (\mathcal{C}) \int X d\mu_C \text{ for all } X \in \mathbb{R}_+^S$$

where,

$$(\mathcal{C}) \int X d\mu_C := \int_{-\infty}^{0} \left[\mu_C(\{X \geq t\}) - 1\right] dt + \int_{0}^{\infty} \mu_C(\{X \geq t\}) dt.$$

We note that by Schmeidler (1986), if the capacity μ_C is concave then

$$(\mathcal{C}) \int X d\mu_C = \max_{P \in acore(\mu_C)} E_P(X),$$

which says that in this case the Choquet Integral coincides with Lehrer Integral for non-negative random variables.

[16] In fact, this result was established for core of capacities with large core (the dual concept of no-gap) is also presented in Lehrer (2009).

4.2 Pricing Rules of Incomplete Markets of Bets

Next result will explain why non additive measures and integration is useful in some important characterization of incomplete markets. Suppose that a pricing rule C is given by a non additive integral of Lehrer or Choquet, in this case we call the subadditive capacity μ_C by a *risk-neutral capacity* or an *ambiguous state price*. Essentially, the main idea of the fundamental Arrow Debreu valuation in complete markets is that for every security its arbitrage-free price is the (usual) integral of the state-payoff weighted by its unique state price or risk-neutral probability. By extending the possibilities of Arrow Debreu valuation through non additive probabilities, this paper also shows that, for an interesting class of incomplete market, the super-replication price of every security can be computed as an integral of the state-payoff weighted by its unique ambiguous state price or risk-neutral capacity. In fact, this paper identify the class of ambiguous state prices related to frictionless market of bets and shows how in this case the pricing rule is simple.

The following result characterize the case of frictionless securities markets admitting a structure of $\{0,1\}$-assets and shows that the complete markets pricing rule given by an expected value with respected to the unique risk-neutral probability can be restarted in a market of bets by considering a expected value, in the sense of Lehrer (2009), with respect to a *unique risk-neutral capacity.*

Theorem 12 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be a pricing rule, then (i) is equivalent to (ii):*

(i) C is a super-replication price of a frictionless incomplete market of bets;

(ii) C is a pricing rule such that,

(a) μ_C is an a-exact capacity whose the $acore\,(\mu_C)$ contains a strictly positive probability,

(b) $acore\,(\mu_C) = acore\,(\mu_C^*)$,

(c) For any non-negative security X,

$$C(X) = (\mathcal{L}) \int X d\mu_C.$$

Also, $F_{\mathcal{E}_{\mu_C}}$ is the set of attainable claims and $acore\,(\mu_C)$ is the set $\mathcal{Q}$ of extended risk-neutral probabilities of the underlying market.

This theorem provides a complete characterization of the class of non-additive probabilities that can be viewed as an ambiguous state price of a frictionless market of bets. In fact, a capacity μ is a risk-neutral capacities of a frictionless market of bets without arbitrage opportunities if, and only if, μ is an a-exact capacity, its $acore\,(\mu_C)$ contains a strictly positive probability, and $acore\,(\mu) = acore\,(\mu^*)$. By our main Theorem 5 we can interpreted the condition that establishing the equality between the "acores" as a property which guarantees that every frictionless bets can not be dominated by some security X.

Next, an example that gives a case of pricing rules of a frictionless incomplete market with no structure of assets given by $\{0,1\}$-securities.

Example 13 *We consider again the functional as in the Example 7, where for all security X,*

$$C(X) = \max\left\{\alpha x_1 + \alpha x_2 + (1-2\alpha)x_3 : \alpha \in \left[0, \frac{1}{2}\right]\right\}$$

We already proved that C is a pricing rule of a frictionless incomplete market. Note that for all non empty event $E \neq \emptyset$,

$$\mu_C(E) \in \left\{\frac{1}{2}, 1\right\} \text{ with } \mu_C(E) = \frac{1}{2} \text{ iff } A \in \{\{s_1\}, \{s_2\}\},$$

which implies that $\mathcal{E}_{\mu_C} = \{\emptyset, S\}$, hence for any $E \neq \emptyset$, we have that $\mu_C^*(A) = 1$ and $acore(\mu_C^*) = \Delta$. Since $\delta_{\{s_1\}} \notin acore(\mu_C)$ we obtain that

$$acore(\mu_C) \neq acore(\mu_C^*).$$

Hence, any underlying market of the pricing rule C is not a market of bets.

4.3 Frictionless Partition Markets

This section studies a special case of market of bets. Given a frictionless incomplete market of securities whose basic assets are given by the collection of securities $(B_1^*, ..., B_n^*)$, with respective vector of prices $(q_1, ..., q_n)$. We say that such market is a *frictionless partition market of securities* if $\{B_k\}_{k=1}^n$ is a partition of the state space S.

Next result characterizes in a interesting way pricing rules for partition markets:

Theorem 14 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then the following assertions are equivalent:*

(i) C is a super-replication price of a frictionless partition market of securities;

(ii) There is a strictly positive probability P and a partition $\{B_j\}_{j=1}^n$ of S and such that $C(X) = \sum_{j=1}^n P(B_j) \max_{s \in B_j} X(s)$, for all $X \in \mathbb{R}^S$;

(iii) μ_C is concave, $\mu_C = \mu_C^*$, there is a strictly positive probability $P_0 \in acore(\mu_C)$, and $C(X) = (\mathcal{C}) \int X d\mu_C$, for all $X \in \mathbb{R}^S$;

In any case, the set of attainable claims is generated by the P-atoms of the "Boolean algebra" $\mathcal{E}_{\mu_C}$ and the extended set of all risk neutral probabilities is given by $acore(\mu_C)$.

Theorem 14 states that in any frictionless partition market the price valuation is given by a Choquet pricing with respect to a concave risk-neutral capacity. Also, given a risk-neutral capacity μ of a frictionless market of bets, by the last Subsection that characterizes Lehrer pricing rules of frictionless market of securities, we have that a bet A^* is frictionless if, and only if, there is no security $X \geq A^*$ such that

$$(\mathcal{L}) \int X d\mu = \mu(A).$$

The additional condition $\mu = \mu^*$ characterizing Choquet pricing rules of frictionless securities markets jointly with our main Theorem 5 give us that a bet A^* is frictionless if, and only if, there is no bet B^* such that $B \supset A$ and $\mu(B) = \mu(A)$.

Next, an example that gives a case of pricing rules of a frictionless market of bets with no partition structure of basic bets.

Example 15 *Consider a capacity μ over the power algebra generated by the state space $S = \{s_1, s_2, s_3, s_4\}$ defined by*[17]

$$\begin{aligned}
\mu_1 &= \mu_4 = \mu_{12} = \mu_{34} = \frac{1}{2}, \\
\mu_2 &= \mu_3 = \mu_{23} = 1 - \mu_{14} = \frac{1}{3}, \\
\mu_{13} &= \mu_{123} = \mu_{234} = \frac{5}{6}, \\
\mu_{24} &= \mu_{124} = \mu_{134} = 1.
\end{aligned}$$

We note that μ is a-exact and $acore(\mu) = acore(\mu^*)$ contains a strictly positive probability. So, by our Lehrer pricing rule characterization, the price functional

$$C(X) := \min \left\{ \sum \alpha_i \mu(A_i) : \sum \alpha_i A_i^* = X, \alpha_i \geq 0 \right\}, \text{ for all } X \geq 0,$$

defines a Lehrer pricing rule of a frictionless market of bets. On the other hand, since $\mu_{123}^* = 1$, i.e., $\mu \neq \mu^*$ every possible basic structure of bets can not induces a partition of the state space. For instance, a possible underlying market is given by the assets $(1,1,0,0)$, $(0,1,1,0)$, and S^* pricing by the risk-netral capacity μ.

One interesting feature of this class of pricing rules is that the set of attainable securities related to the underlying market is that if two securities is attainable then the minimum between then should be attainable too. In a more precisely way,

Corollary 16 *A pricing rule revealing frictionless securities markets with the bond C is a Choquet integral if and only if its underlying set of attainable securities is a Riesz space.*

We note that the class of super-replication prices of frictionless that can be written as a Choquet integral is linked to financial markets where *derivative markets* (in the sense of Aliprantis, Brown and Werner (2000)) are complete[18].

[17] For simplicity, we denote for each $A \subset S$, $\mu(A) = \mu_{i:i \in A}$.

[18] A *ABW-derivative security* is any security that has the same payoff in states in which the payoffs of all basic assets are the same.

A restatement of the result due to Ross (1976), provided by Aliprantis, Brown and Werner (2000), says that derivative markets are complete if and only if the vector space of attainable securities is a Riesz subspace. Hence, by the previous proposition we have that *frictionless Choquet pricing rules* describe the super-replication prices in markets where derivative markets are complete. For instance, note that in the Example 15 the Arrow security $\{s_2\}^*$ is not replicated and the basic securities $\{s_1, s_2\}^*$ and $\{s_2, s_3\}^*$ induces through the "min" operator the Arrow security paying related to the scenario two. On the other hand, given a partition $\{B_j\}_{j=1}^n$ of the state space S and a strictly positive probability P inducing the risk neutral capatity

$$\mu(A) = \sum_{l:B_l \cap A} P(B_l).$$

There is no no-replicated bet A^* given through a security induced by the minimum between two bets in this partition market[19].

4.4 Arrow Debreu Ambiguous Valuation

In both results about markets of bets, an integral with respect to a capacity plays a fundamental role. The capacities characterized in the study of market of bets captures the whole information in the market and "prices" any security, in fact, for any security X

$$C(X) = \text{"Integral" of } X \text{ w.r.t } \mu_C.$$

Also, for all event $A \subset S$, its "price of bet" is given by $C(A^*) = \mu_C(A)$, and, by considering $A = \{s\}$, we have computed the vector of *Arrow Debreu ambiguous state prices* $(\mu_s)_{s \in S}$ defined by $\mu_s := \mu_C(\{s\})$. Clearly, this information might not be enough because the subadditivity of μ_C implies that the knowledge of each state price μ_s only gives a upper bound on the prices of bets on events[20]. Hence, in general, it is important to known the price of every bet, and in fact, the Lehrer pricing rule for market of bets says, for instance, that for any contingent claim $X \geq 0$,

$$C(X) = \min\left\{\sum \alpha_i \mu_C(A_i) : \sum \alpha_i A_i^* = X, \alpha_i \geq 0\right\}.$$

Which means that in order to pricing a claim X, it is enough to consider the unique non additive *event price* $\{\mu_C(A)\}_{A \subset S}$ and find the cheapest portfolio of bets that "replicates" X.

Thus, the notion of non-linear market evaluation extends the usual way of pricing in complete market setting to incomplete markets of bets without arbitrage opportunities by taking the introduced Arrow Debreu ambiguous valuation

[19] In fact, as we saw in the Theorem 14, in this class of markets the family of the unambiguous events (unambiguously priced bets) form a algebra (Riesz linear subspace) of subsets.

[20] For instance, given the state prices μ_s and μ_{s^*} then $\mu_{ss^*} \leq \mu_s + \mu_{s^*}$.

through an non-additive integral with respect to an *ambiguous state price* or *risk-neutral capacity* μ_C.

A simple example of pricing rule that captures the previous intuition is given by the functional

$$C_A(X) = \sum_{s \in E_0} X(s)\, Q(\{s\}) + Q(E_o^c) \underset{s \in E_o^c}{max} X(s),$$

where $Q \in \Delta^+$. Note that the cost of betting on the event E is given by the following concave capacity,

$$\mu_{C_A}(E) = \begin{cases} Q(E), \ E \subseteq E_0 \\ Q(E \cap E_0) + Q(E_0^c), \text{ otherwise.} \end{cases}$$

One possible underlying market of securities revealed by this pricing rule is the potential incomplete market of Arrow securities and one bond with the assets $1_S, \left(1_{\{s_k\}}\right)_{k=1,\ldots,K}$ and corresponding prices $1, (q_k)_{k=1,..,K}$, where E_0 is the set of all unambiguous states and $q_k = Q(\{s_k\})$. Hence, C_A can be viewed as an *Arrow ambiguous pricing rule* of an incomplete market of Arrow securities.

This results about Choquet pricing shows that even if without transaction costs, the valuation in incomplete financial markets can be achieved through the notion of Choquet integration. In fact, in a sense, this result comes in contrast with Bettzüze et. all (2000) that analyzed a general equilibrium model with transaction costs satisfying mild conditions, and showed that the Choquet non-linear pricing approach, as proposed by Chateauneuf et. all (1996) for the case of transactions costs, typically does impose restrictive pricing conditions that are incompatible with non-linear equilibrium prices. A positive aspect of this limitation is that Choquet pricing can distinguish special market characteristics that are beyond the condition of a perfect market. For instance, from our results we can say that every incomplete market given through a basic structure of assets with the riskless bond and only Arrow securities can be constructed by a special case of Choquet pricing. In the case of transaction costs, Choquet pricing also have special implications, as showed by Castagnoli et all (2004)[21], and this topic represents a start point of our goal for future research about pricing rules for markets with frictions.

[21] They have shown that Choquet pricing rules can represent strong frictionalities, in the sense that that the existence of any frictionless tradeable security makes the whole market frictionless. The *weakness* of this result in a set-up of two period economy with finitely many states is the required existence of a fully revealing security, *i.e.,* a security which all available information is summarized by its contingent payments.

5 Appendix

5.1 Part A : Frictionless securities markets and its pricing rules

Arrow (1953) proposed the approach of contingent markets with the presence of a complete securities market and used the results from Arrow and Debreu (1954) as well as McKenzie (1954) for the existence of equilibrium. Magill and Quinzii (1996) and Magill and Shafer (1991) are basics references for the case of general equilibrium analysis of incomplete markets, and such works provided a list of the main contributions in this field. In another way, Föllmer and Shied (2004) provided a treatment of the basics results in frictionless incomplete markets following the lines of finance theory.

Next, we describe the case of a securities market without assuming completeness and avoiding the possibility of frictions in any tradeable security. Formally, a pricing rule C is a no-arbitrage super-replication price of a frictionless incomplete market if we have the following conditions:

- There is a finite number of assets $X_j \in \mathbb{R}^S$, $0 \leq j \leq m$, with respective prices $q_j \in \mathbb{R}$, where $X_0 = S^* := (1, ..., 1)$ is the *riskless bond* with the price normalization $q_0 = 1$. We note that there is only possible deviation from the standard frictionless complete markets set up given by the possibility of incomplete markets when the set of attainable claims, denoted by $F := span\{X_0, X_1, ..., X_m\}$, is a proper subspace of $\mathbb{R}^S$.

- This collection of assets and prices characterize a market of securities denoted by

  $$\mathcal{M} = (X_j, q_j;\ 0 \leq j \leq m),$$

  which is supposed to be without arbitrage opportunities, *i.e.*, for all portfolio $\theta \in \mathbb{R}^{m+1}$,

  $$\begin{aligned} \sum_{j=0}^{m} \theta_j X_j &> 0 \Rightarrow \sum_{j=0}^{m} \theta_j q_j > 0, \\ \sum_{j=0}^{m} \theta_j X_j &= 0 \Rightarrow \sum_{j=0}^{m} \theta_j q_j = 0. \end{aligned}$$

  Recall that a financial market $\mathcal{M} = (X_j, q_j;\ 0 \leq j \leq m)$ offers no-arbitrage opportunity if and only if there is a strictly positive probability[22] $P_0 \in \Delta$ such that $E_{P_0}(X_j) = q_j$, $0 \leq j \leq m$. Also, given the financial market $\mathcal{M}$, we denoted by

  $$\mathcal{Q}_{\mathcal{M}} = \{P \in \Delta^+ : E_P(X_j) = q_j, \forall j \in \{0, ..., m\}\},$$

[22] Note that P_0 strictly positive means that $P_0(\{s\}) > 0$ for any $s \in S$. The collection of strictly positive probabilities is denoted by Δ^+. Also, we are denoting $E_P(X)$ as the integral of the random variable X w.r.t. the probability P.

the set of *risk-neutral probabilities (or martingale measures)*

- Finally, C is the super-replication price of the frictionless securities market $\mathcal{M}$, *i.e.*, for all security $X \in \mathbb{R}^S$

$$C(X) = inf \left\{ \sum_j \theta_j q_j : \sum_j \theta_j X_j \geq X \right\}.$$

  Any $Y = \sum_j \theta_j X_j \geq X$ gives a corresponding super-replication strategy $\theta \in \mathbb{R}^{m+1}$ for the security X and, in our case, the existence of superhedging strategies for all security follows from the existence of the riskless bond.

It is worth noticing that for a frictionless securities market $\mathcal{M}$ offering no-arbitrage opportunity, the super-replication prices satisfies[23]

$$C(X) = \sup_{P \in \mathcal{Q}_{\mathcal{M}}} E_P(X), \text{ for all } X \in \mathbb{R}^S.$$

Hence, by taking the closure of the set of risk neutral probabilities $\mathcal{Q} := \overline{\mathcal{Q}_{\mathcal{M}}}$, we have

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X), \text{ for all } X \in \mathbb{R}^S.$$

Building on the well-known properties discussed above, a trivial Lemma about super-replication prices is naturally derived:

Lemma 17 *The mapping $C : X \in \mathbb{R}^S \to C(X) \in \mathbb{R}$ is the super-replication price of a frictionless securities market without arbitrage opportunities if, and only if:*

1) *There exist $X_0, X_1, ..., X_m \in \mathbb{R}^S$ with $X_0 = S^*$ and a strictly positive probability P_0 such that: $E_{P_0}(X_j) = C(X_j) = -C(-X_j)$, $0 \leq j \leq m$;*

2) *Denoting $\mathcal{Q} := \{P \in \Delta : E_P(X_j) = C(X_j),\ 0 \leq j \leq m\}$, then*

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X), \textit{ for all } X \in \mathbb{R}^S.$$

So, in this case C is the super-replication price of the market $\mathcal{M} = \{X_j, q_j\}_{j=0}^m$, where $q_j := E_{P_0}(X_j)$.

This Lemma 17 summarizes the structure of a frictionless incomplete market with the bond revealed by a pricing rule C. Point 1) says that each basic security is free of arbitrage and pricing by C without frictions. Point 2) gives that any security has its super-replication cost computed through the set of risk neutral probabilities $\mathcal{Q}$.

[23] See, for instance, Föllmer and Shied (2004).

5.2 Part B: Proofs of the results in the main text

Theorem 2:

By a result due to Huber (1981), conditions i), iii), iv) v) is necessary and suficient for the existence of a closed and convex set $\mathcal{Q}$ of probability measures such that

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X).$$

Now if the pricing rule C is strictly positive then $C\left(\{s_i\}^*\right) > 0, \forall i \in \{1, ..., n\}$. Hence, for every state $s_i \in S$ there is a probability $P_i \in \mathcal{Q}$ such that $E_{P_i}\left(\{s_i\}^*\right) > 0$. Since $\mathcal{Q}$ is convex we obtain that it is possible to find a strictly positive probability in $\mathcal{Q}$. For the converse, by assumption there is a strictly positive probability $P_0 \in \mathcal{K}$, hence if $X > 0$

$$C(X) \geq E_{P_0}(X) \geq \max_{s \in S} P_0(\{s\}) X(s) > 0.$$

⊠

Lemma 17:

This lemma is trivial by considering the well know results in finance theory as discussed in the main text.⊠

Lemma 3:

In fact, for the proof we need only sublinearity. First, consider $Y \in F_C$ and $\lambda \in \mathbb{R}_+$, since C is positively homogeneous we have that $C(\lambda Y) = \lambda C(Y)$ and $C(\lambda(-Y)) = \lambda C(-Y)$, then $C(\lambda Y) + C(-\lambda Y) = 0$, *i.e.*, $\lambda Y \in F_C$. If $\lambda < 0$, follows from the definition that $-Y \in F_C$ and then $(-\lambda)(-Y) \in F_C$, *i.e.*, $\lambda Y \in F_C$.

Now, if $Y, Z \in F_C$, since C is sub-additive

$$\begin{aligned} C(Y+Z) &\leq C(Y) + C(Z), \text{ and} \\ C(-(Y+Z)) &\leq C(-Y) + C(-Z), \end{aligned}$$

hence, adding these two inequalities

$$0 = C(0) \leq C(Y+Z) + C(-(Y+Z)) \leq 0,$$

i.e., $Y + Z \in F_C$. ⊠

Lemma 4:

Since for all security X, $C(X) = \max_{P \in \mathcal{Q}} E_P(X)$, if X is frictionless then

$$\max_{P \in \mathcal{Q}} E_P(X) = -\max_{P \in \mathcal{Q}} E_P(-X)$$

which is equivalent to

$$\max_{P \in \mathcal{Q}} E_P(X) = \min_{P \in \mathcal{Q}} E_P(X),$$

and since $P \to E_P(X)$ is continuous and $\mathcal{Q}$ is compact then $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{Q}$. For the converse, if X is such that $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{Q}$, and the same is true for $-X$. Hence, $C(X) = E_P(X)$ and $C(-X) = E_P(-X)$ for any $P \in Q$, which entails $C(X) = -C(-X)$.

Theorem 5:

We need some auxiliary results.

For a given frictionless securities market we recall the following simple and important result:

Lemma 18 *Consider a market* $\mathcal{M} = \{X_j, q_j, 0 \le j \le m\}$ *with no arbitrage opportunity, a security* $X \in F := span\{X_j\}_{0 \le j \le m}$ *if, and only if,* $E_P(X) = E_Q(X)$ *for all* $P, Q \in \mathcal{Q}_{\mathcal{M}} = \{P \in \Delta^+ : E_P(X_j) = q_j, \forall j \in \{0, ..., m\}\}$.

Proof of Lemma 18[24]**:**

That $X \in F$ implies that all risk measures agree is obvious. In order to prove the reverse implication, assume that $X \notin F$ and $P(X) = Q(X)$ for any $P, Q \in \mathcal{Q}_{\mathcal{M}}$.

First, we note that:

$$S(X) := \inf\left\{\sum_{j=0}^{m} \theta_j q_j : \sum_{j=0}^{m} \theta_j X_j \ge X\right\} = \min\left\{\sum_{j=0}^{m} \theta_j q_j : \sum_{j=0}^{m} \theta_j X_j \ge X\right\}.$$

In fact, by the no arbitrage condition there is a strictly positive probability P_0 such that such that $S(Y) = E_{P_0}(Y)$ for any $Y \in F$. For any $n \in \{1, 2, ...\}$ consider the attainable claim Y^n such that $E_{P_0}(Y^n) \le S(X) + n^{-1}$. Hence, for any $s \in S$

$$Y^n(s) \le P_0(\{s\})^{-1}\left(S(X) + n^{-1}\right) \le (S(X) + 1) \max_{s \in S} P_0(\{s\})^{-1} =: k$$

therefore $Y^n \le kS^*$ for any $n \ge 1$. Clearly,

$$S(X) = \inf\{S(Y) : X \le Y \ \le kS^* \text{and } Y \in F\},$$

and since $\{Y \in F : X \le Y \ \le kS^*\}$ is compact and S is continuous (since it is a maximum over a set of probabilities then this functional is Lipschitz continuous) we obtain that the min can be substituted to inf in the definition of C.

Hence, given $X \in \mathbb{R}^S \backslash F$ there is $Y_0 \in F$ such that $Y_0 > X$ and $S(X) = E_{P_0}(Y_0)$. So, we have that $E_{P_0}(Y_0) > E_{P_0}(X)$. Now, since

$$S(X) = \sup_{P \in \mathcal{Q}_{\mathcal{M}}} E_P(X),$$

and have supposed that $E_P(X) = E_Q(X)$ for any $P, Q \in \mathcal{Q}_{\mathcal{M}}$, it turns out that $E_{P_0}(X) = S(X)$, hence

$$C(X) = E_{P_0}(Y_0) > E_{P_0}(X) = C(X),$$

[24] For sake of completeness we give a proof of this result. For the case of a general state space see, for instance, Föllmer and Shied (2004), chapter 1. We also note that $E_P(X) = E_Q(X)$ for all $P, Q \in \mathcal{Q}_{\mathcal{M}}$ if, and only if, $E_P(X) = E_Q(X)$ for all $P, Q \in \overline{\mathcal{Q}_{\mathcal{M}}}$.

a contradiction. ⊠

Given a pricing rule C, its induced set of probabilities that agree about the expected value of every frictionless securities is given by,

$$Q_C := \{P \in \Delta : \ E_P(Y) = C(Y), \text{ for any } Y \in F_C\}.$$

An useful characterization of pricing rules of a frictionless securities market says

Lemma 19 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then (i) is equivalent to (ii):*
(i) C is pricing rule of a frictionless securities market;
(ii) C is a strictly positive linear form on F_C and

$$C(X) = \max_{P \in \mathcal{Q}_C} E_P(X).$$

Theorem 20 *Furthermore, under (i) and (ii) F_C is the set of attainable claims and $\mathcal{Q}_C$ is the set of extended risk-neutral probabilities of the underlying market.*

Proof of Lemma 19:

$(i) \Rightarrow (ii)$

By our assumption, there are $X_0, \ X_1, ..., \ X_m \in \mathbb{R}^S$ with $X_0 = S^*$ and a strictly positive probability P_0 on 2^Ssuch that $E_{P_0}(X_j) = C(X_j) = -C(-X_j)$, $0 \le j \le m$. Moreover, $\forall X \in \mathbb{R}^S$

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

where $\mathcal{Q}_{\mathcal{M}} = \{P \in \Delta : E_P(X_j) = C(X_j) =: q_j; \ 0 \le j \le m\}$.

Now, note that no-arbitrage principle implies that C is a strictly positive linear form on F; actually, by non arbitrage condition, there is a strictly positive probability P_0 such that $\forall Y \in F$, $C(Y) = E_{P_0}(Y)$.

Also, we note that if $C : \mathbb{R}^S \to \mathbb{R}$ is an super-replication price of the market

$$M = \{X_j, \ q_j; \ 0 \le j \le m\}$$

then $F_C = F$. In fact, Since $E_P(X) = C(X)$ for any $X \in F$ and for any $P \in \mathcal{Q}$ clearly $F \subset F_C$. Conversely, let $X \in F_C$, since for any $P \in \mathcal{Q}$,

$$E_P(X) \le C(X) \ \text{ and } E_P(-X) \le C(-X),$$

and

$$E_P(X) + E_P(-X) = 0 = C(X) + C(-X),$$

we obtain that $E_P(X) < C(X)$ is impossible for any $P \in \mathcal{Q}$, *i.e.*, for all $X \in F_C$ the mapping $P \mapsto \Phi_X(P) := E_P(X)$ is constant over $\mathcal{Q}$, and by Lemma 18, $X \in F$. Hence C is a strictly positive linear form on F_C.

Since $\mathcal{Q}_C$ is a nonempty, closed and convex set of probabilities, it remains to show that $\mathcal{Q}_C = \overline{\mathcal{Q}_{\mathcal{M}}} = \{P \in \Delta : E_P(X_j) = q_j, \forall j \in \{0, ..., m\}\}$. If $P \in \overline{\mathcal{Q}_{\mathcal{M}}}$we know that $C(Y) = E_P(Y)$ for any $Y \in F$, since $F = F_C$ we obtain that $P \in$

$\mathcal{Q}_C$. Now, $P \in \mathcal{Q}_C$ says that $C(Y) = E_P(Y)$ for any $Y \in F_C$. Again, since $F = F_C$ entails that

$$F_C = span(X_0, ..., X_m),$$

in particular, $C(X_j) = E_P(X_j)$ for any $j \in \{0, 1, ..., m\}$, *i.e.*, $P \in \mathcal{Q}$.

$(ii) \Rightarrow (i)$

Since $S^* \in F_C$, let us consider $X_0,\ X_1, ..., X_m$, with $X_0 = S^*$, a basis of the linear subspace F_C. We intend to show that C is a super-replication price for the family of securities $X_0,\ X_1, ..., X_m$.

By our assumption the restriction $C\mid_{F_C}$ of C on the linear subspace F_C of the Euclidian space $\mathbb{R}^S$ is a strictly positive linear form, hence it admits a strictly positive linear extension $\overline{C}\mid_{F_C}$ on $\mathbb{R}^S$ (see, for instance, Gale (1960)). Clearly, it is true that $\overline{C}\mid_{F_C}(S^*) = 1$, therefore there is a strictly positive probability P_0 on $(S, 2^S)$ such that $E_{P_0}(X) = \overline{C}\mid_{F_C}(X)$, for any $X \in \mathbb{R}^S$; in particular, $E_{P_0}(X_j) = \overline{C}\mid_{F_C}(X_j) = C(X_j)$, $0 \leq j \leq m$. So, the condition 1) of Lemma 17 is satisfied. Recalling that $F = span(X_0, ..., X_m)$, by our construction F_C is the set of attainable claims. The proof of (ii) implies (i) will be completed if we prove that C satisfies condition 2) of Lemma 17, or equally, that $\mathcal{Q}_C = \mathcal{Q}$, where $\mathcal{Q}$ is the set of risk neutral probabilities. By definition,

$$\mathcal{Q}_C := \{P \in \Delta :\ E_P(Y) = C(Y),\text{ for any } Y \in F_C\},$$

which is nonempty because we saw that there is a strictly positive probability $P_0 \in \mathcal{Q}_C$.

Since for any $j \in \{0, 1, ..., m\}$ the security X_j is frictionless, we obtain that every probability $P \in \mathcal{Q}_C$ is a risk-neutral probability for the market $\mathcal{M} = \{X_j, q_j := C(X_j);\ 0 \leq j \leq m\}$[25]. It remains to prove that every risk-neutral probability belongs to $\mathcal{Q}_C$. In fact, let $P \in \mathcal{Q}$ and $Y \in F_C$, *i.e.*,

$$E_P(X_j) = C(X_j),\ 0 \leq j \leq m,$$

and there are $\lambda_0, \lambda_1, ..., \lambda_m \in \mathbb{R}$ such that

$$Y = \sum_{j=0}^{m} \lambda_j X_j.$$

Since the restriction $C\mid_{F_C}$ of C on the linear subspace F_C is a linear mapping,

$$\begin{aligned} E_P(Y) &= E_P\left(\sum_{j=0}^{m} \lambda_j X_j\right) = \sum_{j=0}^{m} \lambda_j E_P(X_j) = \\ \sum_{j=0}^{m} \lambda_j E_P(X_j) &= \sum_{j=0}^{m} \lambda_j C(X_j) = C\left(\sum_{j=0}^{m} \lambda_j X_j\right) = C(Y). \end{aligned}$$

[25] By the existence of the strictly positive probability P_0, the financial market $\mathcal{M}$ is a market of securities with no-arbitrage opportunity.

henceforth,

$$E_P(Y) = C(Y) \text{ , for any } Y \in F_C,$$

this entails that $P \in \mathcal{Q}_C$, which completes the proof. ⊠

Proof of Theorem 5:

($\Rightarrow$) We want to show that $L_C = F_C$. In fact, we saw in the main text that for all pricing rule C it is true that $F_C \subset L_C$.

Now, suppose that $X \in L_C$ then by definition $Y > X \Rightarrow C(Y) > C(X)$. Since C is a super-replication price of a frictionless market of securities $F_C = F$. So, by supposing that $X \notin F_C = F$, since

$$\begin{aligned} C(X) &= \min\{C(Y) : Y \geq X \text{ and } Y \in F_C\} \\ &\overset{(X \notin F_C)}{=} \min\{C(Y) : Y > X \text{ and } Y \in F_C\}, \end{aligned}$$

there is $Z \in F_C$ such $Z > X$ and $C(Z) = C(X)$, a contradiction.

($\Leftarrow$) Since C is pricing rule we know that there is a nonempty, closed and convex set $\mathcal{K} \subset \Delta$ such that for any $X \in \mathbb{R}^S$,

$$C(X) = \max_{P \in \mathcal{K}} E_P(X).$$

By Lemma 19 it is enough to show that C is strictly positive and $\mathcal{K} = \mathcal{Q}_C$.

The inclusion $\mathcal{K} \subset \mathcal{Q}_C$ is simple: Consider $P \in \mathcal{K}$, if $P \notin \mathcal{Q}_C$ then there is $X \in F_C$ such that $E_P(X) < C(X) = -C(-X)$, hence $E_P(-X) > C(-X) = \max_{P \in \mathcal{K}} E_P(-X)$, a contradiction.

So, we need to show that $\mathcal{Q}_C \subset \mathcal{K}$, or equally that $\mathcal{K} \subsetneq \mathcal{Q}_C$ is impossible. Assume that there is $P_1 \in \mathcal{Q}_C$ such that $P_1 \notin \mathcal{K}$. Then through the classical strict separation theorem (see, for instance, Dunford and Schwartz (1958)) there is a security X_0 such that

$$E_{P_1}(X_0) > \max_{P \in \mathcal{K}} E_P(X_0) = C(X_0).$$

If we prove that there is $Y \in F_C,\ Y \geq X_0$ such that $C(X_0) = C(Y)$, this will entail a contradiction, since

$$E_{P_1}(X_0) > C(X_0) = C(Y) = E_{P_1}(Y) \geq E_{P_1}(X_0).$$

So it is enough to show that for any security X, setting

$$E_X := \left\{Y \in \mathbb{R}^S : Y \geq X \text{ and } C(Y) = C(X)\right\},$$

there is $Y \in F_C \cap E_X$.

This result is obvious if $X \in F_C$, so let us assume that $X \notin F_C$. Recall that, since C is a pricing rule, the multiple probabilities set $\mathcal{K}$ contains at least a strictly positive probability P_0.

Let us now prove that E_X is bounded from above, otherwise there would exist a sequence $\{Y_k\}_{k \geq 1}$, $Y_k \in E_X$, $\forall k \geq 1$ and $s_0 \in S$ such that $\lim_k Y_k(s_0) = +\infty$.

But

$$\begin{aligned}\lim_k C(Y_k) &\geq \lim_k E_{P_0}(Y_k) = \lim_k \sum_{s\in S} P_0(s)\, Y_k(s) \\ &\geq \sum_{s\neq s_0} P_0(s)\, X(s) + \lim_k P_0(s_0)\, Y_k(s_0) = \infty,\end{aligned}$$

contradicting $C(Y_k) = C(X)$, $\forall k \geq 1$.

Let us now show that E_X has a maximal element for the partial preorder $\geq$ on $\mathbb{R}^S$. Thanks to Zorn´s lemma we just need to prove that every chain $(Y_\lambda)_{\lambda\in\Phi}$ in E_X has an upper bound. Define Y by

$$Y(s) := \sup_{\lambda\in\Phi} Y_\lambda(s), \forall s \in S,$$

E_X bounded from above implies that $Y \in \mathbb{R}^S$. It remains to check that $C(Y) = C(X)$, let $\varepsilon > 0$ be given, and let $s_i \in S$, hence there is $\lambda_i \in \Phi$ such that $Y(s_i) \leq Y_{\lambda_i}(s_i) + \varepsilon$, since $(Y_\lambda)_{\lambda\in\Phi}$ is a chain there is $n \geq 1$ and $\widetilde{\lambda} \in \{\lambda_1, ..., \lambda_n\}$ such that $Y_{\widetilde{\lambda}} \leq Y \leq Y_{\widetilde{\lambda}} + \varepsilon$, therefore $C\left(Y_{\widetilde{\lambda}}\right) \leq C(Y) \leq C\left(Y_{\widetilde{\lambda}}\right) + \varepsilon$, since $C\left(Y_{\widetilde{\lambda}}\right) = C(X)$ it turns out that $C(Y) = C(X)$. Let now Y_0 be a maximal element of E_X, the proof will be completed if we show that $Y_0 \in F_C$. From the hypothesis $F_C = L_C$, it is enough to show that $Y_0 \in L_C$. Let Y_1 be an arbitrary security such that $Y_1 > Y_0$, since Y_0 is a maximal element in E_X , it comes that $Y_1 \notin E_X$, but $Y_1 > X$, therefore $C(Y_1) > C(X) = C(Y_0)$, so $Y_0 \in L_C$ which completes the proof. ⊠

Theorem 12:

First, we will show a result connecting a pricing rule C to the induced set of probabilities

$$\mathcal{Q}_{\mu_C} = \left\{P \in \Delta : P(A) = \mu_C(A) \text{ for all } A \in \mathcal{E}_{\mu_C}\right\}.$$

Of course, $\mathcal{Q}_{\mu_C} \subset \mathcal{Q}_C$.

Theorem 21 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then (i) is equivalent to (ii):*

(i) C is super-replication price of a market of bets;

(ii) There is a strictly positive probability P_0 belonging to $\mathcal{Q}_{\mu_C}$ and for any contingent claim X,

$$C(X) = \max_{P\in\mathcal{Q}_{\mu_C}} E_P(X).$$

Furthermore, under (i) and (ii) $F_{\mathcal{E}_{\mu_C}}$ is the set of attainable claims and $\mathcal{Q}_{\mu_C}$ is the set of risk-neutral probabilities of the underlying market

Proof of Theorem 21:

$(i) \Rightarrow (ii)$ Our assumption says that there exist $B_0, B_1, ..., B_m \in 2^S$ with $B_0 = S$ and a strictly positive probability P_0 on 2^S such that $P_0(B_j) = C\left(B_j^*\right)$, for any $j \in \{0, 1, ..., m\}$ and $\forall X \in \mathbb{R}^S$,

$$C(X) = \max_{P\in\mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : P(B_j) = C(B_j^*),\ 0 \le j \le m\}$.

Let us now prove that there is a strictly positive probability P_0 belonging $\mathcal{Q}_{\mu_C}$. From Lemma 18 we know that if $B^* \in F$ then $P(B) = P_0(B)$ for any $P \in Q$, hence $\mu_C(B) = P_0(B)$. Also, it is easy to see that Lemma 18 implies that $B \in \mathcal{E}_{\mu_C}$ if and only if $B^* \in F$, which turns out that $P_0(B) = \mu_C(B)$, $\forall B \in \mathcal{E}_{\mu_C}$.

Now we need to show that $\mathcal{Q} = \mathcal{Q}_{\mu_C}$. Note that by Lemma 19 and by the fact that $\{B^* : B \in \mathcal{E}_{\mu_C}\} \subset F_C$, we obtain that $\mathcal{Q} = \mathcal{Q}_C \subset \mathcal{Q}_{\mu_C}$. For the other inclusion, taking $P \in \mathcal{Q}_{\mu_C}$ and B_j let us show that $P(B_j) = C(B_j^*)$. Since B_j^* is an attainable security (in fact, it is a basic asset) we know that $B_j \in \mathcal{E}_{\mu_C}$, hence $P \in \mathcal{Q}_{\mu_C}$ so $\mathcal{Q}_{\mu_C} \subset \mathcal{Q}$.

So $\mathcal{Q}_{\mu_C}$ is actually the set of risk-neutral probabilities of the initial market. It remains to prove that $F_{\mathcal{E}_{\mu_C}} = F$. In fact, by $B \in \mathcal{E}_{\mu_C} \Leftrightarrow B^* \in F$, we obtain that $F_{\mathcal{E}_{\mu_C}} = span\{B^* : B \in \mathcal{E}_{\mu_C}\} = span\{B^* : B^* \in F\} = F$.

$(ii) \Rightarrow (i)$ Since $B_0 = S^*$, let us consider the finite family of all unambiguous events $B_0, B_1, ..., B_m$. By assumption there is a strictly positive probability P_0 such that $P(B_j) = C(B_j^*),\ 0 \le j \le m$. The proof will be completed if we show that $\mathcal{Q} = \mathcal{Q}_{\mu_C}$ and $F_{\mathcal{E}_{\mu_C}} = F$, where $\mathcal{Q}$ and F refer to the previous defined market of $\{0,1\}$-securities $\mathcal{M} = (B_0^*, B_1^*, ..., B_m^*; 1, \mu_C(B_1^*), ..., \mu_C(B_m^*))$. But this is straightforward by the equality $\mathcal{E}_{\mu_C} = \{B_0, B_1, ..., B_m\}$. ⊠

Another important lemma for Theorem 12 is given by

Theorem 22 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then (i) is equivalent to (ii):*
(i) C is a super-replication price of a frictionless market of bets;
(ii) C satisfies,
(a) $acore(\mu_C)$ contains a strictly positive probability P_0,
(b) $acore(\mu_C) = acore(\mu_C^*)$,
(c) For any contingent claim X,

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X).$$

Furthermore, under (i) and (ii) $F_{\mathcal{E}_{\mu_C}}$ is the set of attainable claims and $acore(\mu_C)$ is the set of extended risk-neutral probabilities of the underlying market

Proof of Theorem 22:

$(i) \Rightarrow (ii)$ By our assumption says that C is a super-replication price of a frictionless securities market of bets $\{B_j^*\}_{j=0}^m$, for all $A \subset S$

$$\mu_C(A) = C(A^*) = \max_{P \in \mathcal{Q}} P(A),$$

hence μ_C is an anti-exact capacity and the $acore(\mu_C)$ contains at least one strictly positive probability, namely P_0.

Let us now show that

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X),\ \forall X \in \mathbb{R}^S.$$

Note that it is enough to show that $\mathcal{Q} = acore\,(\mu_C)$:

Consider $P \in acore\,(\mu_C)$, hence $P\,(B_j) \leq \mu_C\,(B_j)\,, 0 \leq j \leq m$. Since B_j^* is attainable by assumption, then B_j is an unambiguous event, i.e., $\mu_C\,(B_j) + \mu_C\,(B_j^c) = 1$. Also, $P\,(B_j^c) \leq \mu_C\,(B_j^c)\,, 0 \leq j \leq m$ and then

$$P\,(B_j) + P\,(B_j^c) = 1 = \mu_C\,(B_j) + \mu_C\,(B_j^c) = 1,$$

allows us to obtain $P\,(B_j) = \mu_C\,(B_j)\,, 0 \leq j \leq m$, *i.e.*, $P \in \mathcal{Q}$.

Now, setting $P \in Q$ and $A \subset S$, since our assumption says that

$$\mu_C\,(A) = \max_{P \in \mathcal{Q}} P\,(A)\,,$$

clearly $P\,(A) \leq \mu_C\,(A)$, *i.e.*, $P \in acore\,(\mu_C)$.

For (b) it is enough to show that $acore\,(\mu_C^*) \subset acore\,(\mu_C)$, or else from the previous identity saying $\mathcal{Q} = acore\,(\mu_C)$ it is enough to show that $acore\,(\mu_C^*) \subset \mathcal{Q}$. So let $P \in acore\,(\mu_C^*)$ and let B_j be a basic bet. By definition of μ_C^*, one has $\mu_C^*\,(B_j) = \mu_C\,(B_j)$ therefore $P\,(B_j) \leq \mu_C^*\,(B_j)$ implies $P\,(B_j) \leq \mu_C\,(B_j)$; as we notice before $B_j \in \mathcal{E}_{\mu_C}$, hence $\mu_C^*\,(B_j) = \mu_C\,(B_j)$ and $P\,(B_j^c) \leq \mu_C^*\,(B_j^c)$ implies $P\,(B_j^c) \leq \mu_C\,(B_j^c)$ from $P\,(B_j) + P\,(B_j^c) = 1 = \mu_C\,(B_j) + \mu_C\,(B_j^c)$, it turns out that $P\,(B_j) = \mu_C\,(B_j)$.

$(ii) \Rightarrow (i)$ We need to prove that there exist $B_0,\ B_1, ...,\ B_m \in 2^S$ with $B_0 = S$ and a strictly positive probability P_0 on 2^S such that $P_0\,(B_j) = C\,(B_j^*)\,,$ for any $j \in \{0, 1, ..., m\}$ and $\forall\ X \in \mathbb{R}^S$,

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

where $\mathcal{Q} = \{P \in \Delta : P\,(B_j) = C\,(B_j^*)\,,\ 0 \leq j \leq m\}$.

Note that C is well defined since $acore\,(\mu_C) \neq \emptyset$ (by assumption (a)) and compact, moreover for any $A \subset S$

$$\mu_C\,(A) = C\,(A^*) = \max_{P \in acore(\mu_C)} P\,(A)\,.$$

Clearly $B_0 := S \in \mathcal{E}_{\mu_C}$, and $\mathcal{E}_{\mu_C}$ is formed with a finite number of events $B_0, B_1, ..., B_m$. Note that for any $B \in \mathcal{E}_{\mu_C}$ and for any $P \in acore\,(\mu_C)$ it is true that $P\,(B) = \mu_C\,(B)$: actually $P \in acore\,(C)$ implies that $P\,(B) \leq \mu_C\,(B)$, $P\,(B^c) \leq \mu_C\,(B^c)$ and $P\,(B) + P\,(B^c) = 1 = \mu_C\,(B_j) + \mu_C\,(B_j^c)$, gives the desired equality (note that its implies that $\mathcal{Q} \supset acore\,(\mu_C)$). Since, by hypothesis there is a strictly positive probability $P_0 \in acore\,(\mu_C)$, it turns out that the first requirement is satisfied. So the formula

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

holds for any $X \in \mathbb{R}^S$ if and only if $\mathcal{Q} = acore\,(\mu_C)$. Just above we proved that $\mathcal{Q} \supset acore\,(\mu_C)$. By our assumption (b) we only have to show that $\mathcal{Q} \subset acore\,(\mu_C^*)$. Let $P \in \mathcal{Q}$ and $A \subset S$, from the definition of μ_C^* we have that there is $B \in \mathcal{E}_{\mu_C}$ such that $A \subset B$ and $\mu_C^*\,(A) = \mu_C\,(B)$, hence

$$P\,(A) \leq P\,(B) = C\,(B) = \mu_C^*\,(A)\,,$$

i.e., $P \in acore\,(\mu_C^*)$.

Furthermore, under (i) and (ii) $acore\,(\mu_C)$ is the set of extended risk-neutral probabilities and by Theorem 21 $F_{\mathcal{E}_{\mu_C}}$ is the set of attainable claims. ⊠

Proof of Theorem 12:

It follows from the previous Lemma 22 because any pricing rule is supposed to be constant additive, and in this case by Proposition 2 in Lehrer (2009), given a contingent claim $X \in \mathbb{R}_+^S$ it follows that

$$(\mathcal{L}) \int X d\mu_C = \max_{P \in acore(\mu_C)} E_P\,(X) = C\,(X)\,.$$

⊠

Theorem 14:

This theorem needs some previous results. We will see that the possibility of pricing rules of frictionless securities markets given by a Choquet integral is related to some strong condition on the set of attainable securities. For that we present the next well known definition,

Definition 23 *A Riesz subspace of* $\mathbb{R}^S$ *is a linear subspace* F *of* $\mathbb{R}^S$ *such that* $X, Y \in F$ *implies that* $X \vee Y \in F$ *and* $X \wedge Y \in F$.

Lemma 24 *If a pricing rule of a frictionless securities market* C *is a Choquet integral then the induced capacity* μ_C *is concave and the subspace* F *of attainable securities is a Riesz-space.*

Proof of Lemma 24:

First, we note that from Proposition 3 given by Schmeidler (1986) we have that if C is a subadditive Choquet integral with respect to the capacity μ_C then μ_C is a concave capacity.

Let us now prove that F is a Riesz space:

Let $X, Y \in F$, then by Lemma 18 we have that for any $P \in \mathcal{Q}$, $E_P\,(X) + E_P\,(Y) = C(X) + C\,(Y)$. Since C is a Choquet Integral with respect to a concave capacity, it turns out that[26]

$$C\,(X) + C\,(Y) \geq C\,(X \vee Y) + C\,(X \wedge Y)\,.$$

Therefore, using the previous equality

$$E_P\,(X \vee Y) + E_P\,(X \wedge Y) = E_P\,(X) + E_P\,(Y) \geq C\,(X \vee Y) + C\,(X \wedge Y)\,.$$

But $E_P\,(X \vee Y) \leq C\,(X \vee Y)$ and $E_P\,(X \wedge Y) \leq C\,(X \wedge Y)$ for any $P \in \mathcal{Q}$. Hence, $E_P\,(X \vee Y) = C\,(X \vee Y)$ and $E_P\,(X \wedge Y) = C\,(X \wedge Y)$ for any $P \in \mathcal{Q}$ which implies by Lemma 18 that $X \vee Y$ and $X \wedge Y$ belongs to F. ⊠

Another important lemma is,

[26] See, for instance, Huber (1981) pages 260 and 261.

Lemma 25 [27] *Let F be a Riesz subspace of $\mathbb{R}^n$ containing the unit vector $1_{\mathbb{R}^n} = (1,...,1) \in \mathbb{R}^n$ then F is a "partition" linear subspace of $\mathbb{R}^n$,* i.e., *up to a permutation:*

$$x \in F \text{ iff } x = (x_1, ...x_1, ..., x_j, ..., x_j, ..., x_m, ..., x_m).$$

Proof: The proof is by induction on the cardinality $\#S$ of $S \geq 1$. Clearly the result is true if $\#S = 1$, now assume that the result is true for $\#S = k$ and let us show that it remains true for $\#S = k+1$.

So let F be a subspace of $\mathbb{R}^{k+1}$ containing $1_{\mathbb{R}^{k+1}}$, and let G be defined by[28]:

$$G := \left\{ y = (x_1, ..., x_k) \in \mathbb{R}^k : \exists x_{k+1} \text{ s.t. } (y, x_{k+1}) \in F \right\}.$$

It is straightforward to check that G is a Riesz-subspace of $\mathbb{R}^k$ containing $1_{\mathbb{R}^k}$, therefore by the induction hypothesis and up to a permutation $y \in G$ is equivalent to $y = (x_1, ..., x_1, ...x_j, ..., x_j, ..., x_m, ..., x_m)$ where $x_j \in \mathbb{R}$, $1 \leq j \leq m$. Clearly, if $x \in F$ then $x \in \widetilde{G} \oplus \widetilde{H}$ the direct sum of the linear subspaces of $\mathbb{R}^{k+1}$ given by

$$\begin{aligned} \widetilde{G} &= \left\{ (y, 0) \in \mathbb{R}^{k+1} : y \in G \right\} \\ \widetilde{H} &= \left\{ (0, ..., 0, x_{k+1}) \in \mathbb{R}^{k+1} : x_{k+1} \in \mathbb{R} \right\}. \end{aligned}$$

Therefore, $\dim F \leq \dim \widetilde{G} \oplus \widetilde{H} = m + 1$. It is also immediate to see that $\dim F \geq m$: in fact, $y \in G$ is equivalent to

$$y = \sum_{j=1}^{m} x_j V_j^*,$$

where each $V_j^* \in \mathbb{R}^k$, *i.e.*, $V_j \subset \{1, ..., k\}$, and $\{V_1^*, ..., V_m^*\}$ is a basis of G. Let $z_j \in \mathbb{R}$ be such that $\left(V_j^*, z_j\right) \in F$, $1 \leq j \leq m$; it is immediate to see that $\{\{V_1^*\}, ..., \{V_m^*\}\}$ linearly independent in G implies $\{\{V_1^*, z_1\}, ..., \{V_m^*, z_m\}\}$ linearly independent in F, hence $\dim F \geq m$.

Two cases have to be examined:

1) $\dim F = m + 1$: Clearly since $F \subset \widetilde{G} \oplus \widetilde{H}$, this implies that $F = \widetilde{G} \oplus \widetilde{H}$ and F is a "partition" space.

2) $\dim F = m$: In such a case since $\left\{ W_j^* := \left\{ V_j^*, z_j \right\}, 1 \leq j \leq m \right\}$ is linearly independent in F, $\left\{ W_j^* :, 1 \leq j \leq m \right\}$ is a basis of F. Hence, we obtain that $x \in F$ if and only if there are x_j, $1 \leq j \leq m$ such that $x = \sum_{j=1}^{m} x_j W_j^*$, in

[27] For sake of completeness we give a direct proof of this result, which in fact has been obtained independently by Polyrakis (1996, 1999).

[28] For $y = (x_1, ..., x_k) \in \mathbb{R}^k$ and $x_{k+1} \in \mathbb{R}$ we use the following notation:

$$(y, x_{k+1}) := (x_1, ..., x_k, x_{k+1}) \in \mathbb{R}^{k+1}.$$

particular,

$$x_{k+1} = \sum_{j=1}^{m} x_j z_j, \qquad (\Gamma)\,.$$

So, it remains to show that there is $j_0 \in \{1, ..., m\}$ such that for any $x \in F$ it is possible to write $x = \sum_{j=1}^{m} x_j V_j^* + x_{j_0}$. Note that is enough to show that all the z_j´s are equal to zero except $z_{j_0} = 1$. Since $1_{\mathbb{R}^{k+1}} \in F$ by the above property (Γ), we obtain that $\sum_{j=1}^{m} z_j = 1$.

Now take $j \neq i,\ j, i \in \{1, ..., m\}$. Since F is a Riesz space, $W_j^*, W_i^* \in F$ implies that $W_j^* \wedge W_i^* \in F$, but $W_j^* \wedge W_i^* = \left((V_j \cap V_i)^*, z_j \wedge z_i\right)$ and $V_j \cap V_i = \emptyset$, hence by property (Γ) we obtain that $0 = \sum_{j=1}^{m} x_j z_j = z_j \wedge z_i$, therefore $z_j \geq 0$. On the other hand, the Riesz space structure implies also that $W_j^* \vee W_i^* \in F$, but $W_j^* \vee W_i^* = (1_{\mathbb{R}^{k+1}}, z_j \vee z_i)$, hence by property (Γ) we obtain that $z_j \vee z_i = z_j + z_i$. Summing up, we have

$$\begin{aligned} \sum_{j=1}^{m} z_j &= 1, \text{ therefore for any } j \neq i,\ j, i \in \{1, ..., m\} : \\ z_j \wedge z_i &= 0 \text{ and } \ z_j \vee z_i = z_j + z_i; \end{aligned}$$

this implies that there is a unique $j_0 \in \{1, ..., m\}$ such that $z_{j_0} = 1$ and for any $j \in \{1, ..., m\} \setminus \{j_0\}$ it is true that $j = 0$, the desired result. ⊞

Theorem 14 is a special case of the more general and technical Lemma below

Theorem 26 *Let $C : \mathbb{R}^S \to \mathbb{R}$ be given, then the following assertions are equivalent:*

(i) C is an pricing rule of a frictionless securities markets which is a Choquet integral;

(ii) C is a pricing rule of a frictionless partition market;

(iii) There is a strictly positive probability P_0 and a partition $B_1, ..., B_j, ..., B_m$ of S such that $\forall X \in \mathbb{R}^S$

$$C(X) = \sum_{j=1}^{m} P(B_j) \max_{s \in B_j} X(s);$$

(iv) μ_C is concave, $\mu_C = \mu_C^*$, there is at least a strictly positive probability $P_0 \in acore\,(\mu_C)$, and $\forall\ X \in \mathbb{R}^S$

$$C(X) = \max_{P \in acore(\mu_C)} E_P(X),$$

(v) C satisfies,

(a) $\mathcal{E}_{\mu_C}$ *is an algebra*[29]*,*
(b) There is a strictly positive probability P_0 *belonging to* $\mathcal{Q}_{\mu_C}$*,*
(c) For any contingent claim X*,*

$$C(X) = \max_{P \in \mathcal{Q}_{\mu_C}} E_P(X).$$

In any case, the set of attainable claims is generated by the P_0*-atoms*[30] *of the Boolean algebra* $\mathcal{E}_{\mu_C}$ *and the set of all risk neutral probabilities is given by* $acore(\mu_C)$*.*

$(i) \Rightarrow (ii)$ By Lemma 24 we know that the set of attainable securities F is a Riesz subspace of $\mathbb{R}^S$ containing the riskless bond S^*. Therefore, by Lemma 25 we obtain that F is a "partition" linear subspace of $\mathbb{R}^S$, hence C is the super-replication price of a "partition" market of $\{0,1\}$-securities without arbitrage opportunities.

$(ii) \Rightarrow (iii)$ By assumption we have a partition $\{B_1, ..., B_m\}$ of the state space S and a strictly positive probability P_0 such that $P_0(B_j) = C(B_j^*)$ for any $j \in \{1, ..., m\}$. Recall that,

$$\mathcal{Q} = \{P \in \Delta : P(B_j) = P_0(B_j), 1 \leq j \leq m\}$$

and

$$C(X) = \max_{P \in \mathcal{Q}} E_P(X),$$

hence since $E_P(X) = \sum_{j=1}^{m} \sum_{s \in B_j} P(\{s\}) X(s)$. Now, denote by Q the risk neutral probability such that for any $j \in \{1, ..., m\}$,

$$Q(B_j) = Q(\{\widehat{s} \in B_j : X(\widehat{s}) = \max X(B_j)\}).$$

Hence,

$$\begin{aligned} C(X) &= \max_{P \in \mathcal{Q}} \left\{ \sum_{j=1}^{m} \sum_{s \in B_j} P(\{s\}) X(s) \right\} = \\ &= \sum_{j=1}^{m} \max_{P \in \mathcal{Q}} \left\{ \sum_{s \in B_j} P(\{s\}) X(s) \right\} = \sum_{j=1}^{m} Q(B_j) \max X(B_j). \end{aligned}$$

Which allows us to write,

$$C(X) = \sum_{j=1}^{m} P_0(B_j) \max_{s \in B_j} X(s).$$

[29] A family $\mathcal{E}$ of subsets of S is called an algebra if $\mathcal{E}$ contains S, it is closed for (finite) intersection and complement.

[30] Let $\mathcal{E}$ a Boolean algebra of subsets of S and P a probability measure over E, we say that an event $E \in \mathcal{E}$ is a P-atom if $P(E) > 0$ and for any $F \in \mathcal{E}$ such that $F \subset E$, $P(F) = P(E)$ or $P(F) = 0$. If P is strictly positive on the finite Boolean algebra $\mathcal{E}$, E is a P-atom iff $P(E) > 0$ and if $F \subset E$ and $F \neq \emptyset$ then $F \notin \mathcal{E}$.

$(iii) \Rightarrow (i)$ By our assumption we have that there is a strictly positive probability P_0 and a partition $B_1, ..., B_j, ..., B_m$ of S and such that $\forall X \in \mathbb{R}^S$

$$C(X) = \sum_{j=1}^{m} P(B_j) \max_{s \in B_j} X(s).$$

Hence,

$$\mu_C(A) = \sum_{k \in \{j : B_j \cap A \neq \emptyset\}} P_0(B_j),$$

and it is well know that

$$C(X) = \int X d\mu_C,$$

which completes this part of the proof.

Note that we proved that $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

$(ii) \Rightarrow (iv)$ From Theorem 25, it remains to prove that C is concave and that $\mu_C = \mu_C^*$. Take $A \subset S$, since $(ii) \Leftrightarrow (iii)$, it comes from (iii) that

$$\mu_C(A) = \sum_{k \in \{j : B_j \cap A \neq \emptyset\}} P_0(B_j);$$

since $P_0(B_j) > 0$ and $\sum P_0(B_j) = 1$, as it is well-known μ_C is a plausibility function (*i.e.*, the dual of a belief function), hence μ_C is concave.

It remains to show that $\mu_C^* \leq \mu_C$. From Nehring (1999), we know that μ_C is concave, which implies that $\mathcal{E}_{\mu_C}$ is a Boolean algebra; let us show that it entails that μ_C^* is concave: Let A_1, A_2 be subsets of S, by definition of μ_C^* there exist $B_1 \supset A_1$ and $B_2 \supset A_2$, $B_i \in \mathcal{E}_{\mu_C}$ such that $\mu_C^*(A_i) = \mu_C(B_i)$, $i = 1, 2$. Hence, $\mu_C^*(A_1) + \mu_C^*(A_2) = \mu_C(B_1) + \mu_C(B_2) \geq \mu_C(B_1 \cup B_2) + \mu_C(B_1 \cap B_2)$. Since $B_1 \cup B_2, B_1 \cap B_2 \in \mathcal{E}_{\mu_C}$, $B_1 \cup B_2 \supset A_1 \cup A_2$ and $B_1 \cap B_2 \supset A_1 \cap A_2$, it turns out that $\mu_C^*(A_1) + \mu_C^*(A_2) \geq \mu_C^*(A_1 \cup A_2) + \mu_C^*(A_1 \cap A_2)$. Let $A \subset S$, μ_C^* concave implies that there is a probability $P \in acore(\mu_C^*)$, but Theorem **??** guarantees that $acore(\mu_C) = acore(\mu_C^*)$ hence $P \in acore(\mu_C)$, therefore:

$$\mu_C^*(A) = P(A) \leq \mu_C(A),$$

which completes this part of the proof.

$(iv) \Rightarrow (v)$ Note that (a) comes from μ_C concave and the previously quoted result of Nehring (1999).

$(v) \Rightarrow (ii)$ By hypothesis, there is a strictly positive probability $P_0 \in \mathcal{Q}_{\mu_C}$ and $\mathcal{E}_{\mu_C}$ is a Boolean algebra. Let $\{B_1, ..., B_m\}$ be the collection of P_0-atoms of the Boolean algebra $\mathcal{E}_{\mu_C}$, hence $\{B_1, ..., B_m\}$ is a partition of S. Of course, $P_0(B_j) = C(B_j^*)$, for any $j \in \{1, ..., m\}$ and $\mathcal{Q} \supset \mathcal{Q}_{\mu_C}$. For $\mathcal{Q} \supset \mathcal{Q}_{\mu_C}$, note that if $P \in \Delta$ is such that $P(B_j) = \mu_C(B_j)$ for any $j \in \{1, ..., m\}$ then if $B \in \mathcal{E}_{\mu_C}$ and $B \notin \{B_1, ..., B_m\}$ hence there is $\Lambda \subset \{1, ..., m\}$ such that $B = \cup_{j \in \Lambda} B_j$, therefore $P(B) = \sum_{j \in \Lambda} P(B_j) = \sum_{j \in \Lambda} \mu_C(B_j) = \mu_C(\cup_{j \in \Lambda} B_j) = \mu_C(B)$. Hence, C is a pricing rule of a "partition market". ⊠

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