On diagonal subdifferential operators in nonreflexive Banach spaces

A. N. Iusem* B. F. Svaiter†

July 20, 2011

Abstract

Consider a real-valued bifunction f defined on $C \times C$, where C is a closed and convex subset of a Banach space X, which is concave in its first argument and convex in its second one. We study its subdifferential with respect to the second argument, evaluated at pairs of the form (x, x), and the subdifferential of $-f$ with respect to its first argument, evaluated at the same pairs. We prove that if f vanishes whenever both arguments coincide, these operators are maximal monotone, under rather undemanding continuity assumptions on f. We also establish similar results under related assumptions on f, e.g. monotonicity and convexity in the second argument. These results were known for the case in which the Banach space is reflexive and $C = X$. Here we use a different approach, based upon a recently established sufficient condition for maximal monotonicity of operators, in order to cover the nonreflexive and constrained case ($C \neq X$). Our results have consequences in terms of the reformulation of equilibrium problems as variational inequality ones.

Key words: Equilibrium problem, maximal monotone operator, diagonal subdifferential, convex representations.

Mathematical Subject Classification (2000): 90C47,49J35.

*Instituto de Matématica Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, RJ, CEP 22460-320, Brazil, `iusp@impa.br`. The work of this author author was partially supported by CNPq grant no. 301280/86.

†Instituto de Matématica Pura e Aplicada, (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, RJ, CEP 22460-320, Brazil, `benar@impa.br`. The work of this author author was partially supported by CNPq grants no. 474944/2010-7, 303583/2008-8 and FAPERJ grant E-26/110.821/2008.

1 Introduction

Let X be a Banach space and X^* its topological dual. Consider a function $f : X \times X \to \mathbb{R}$ which is concave in its first argument and convex in its second one. We will be concerned in this paper with two set-valued operators related to the bifunction f, namely $R_f, S_f : X \to \mathcal{P}(X^*)$, defined as:

$$R_f(x) = \partial_1(-f)(x, x), \tag{1}$$

$$S_f(x) = \partial_2 f(x, x), \tag{2}$$

where ∂_1, ∂_2 denote the subdifferentials with respect to the first and second argument respectively.

We will refer to R_f, S_f as *diagonal subdifferential* operators. Observe that neither R_f nor S_f are subdifferentials of convex functions: at each point x each one of them coincides with the subdifferential of a certain convex function evaluated at x, but the functions themselves change with x. More precisely, $S_f(x)$ is the subdifferential of the convex function $f(x, \cdot)$ evaluated at x. Similarly, $R_f(x)$ is the subdifferential of the convex function $-f(\cdot, x)$ evaluated at x. In fact both R_f and S_f may fail to be monotone operators, unless additional assumptions are imposed upon f. The basic one seems to be that $f(x, x) = 0$ for all $x \in X$. We will also consider the case in which the domain of f is a set of the form $C \times C$, where $C \subset X$ is closed and convex, and satisfies also an additional technical condition. In this case the definitions of S_f, R_f require some technical adjustments.

The motivation for studying these operators arises from the so called *equilibrium problem*, which we describe next. Given X, C and f as above (possibly with additional and/or slightly different assumptions on f, some of which will be detailed later on), the equilibrium problem $\mathrm{EP}(f, C)$ consists of finding $\hat{x} \in C$ such that $f(\hat{x}, x) \geq 0$ for all $x \in C$. See [2], [10] and [9] for definitions and properties of equilibrium problems pertinent to the subject of this paper.

Under the additional assumption that $f(x, x) = 0$ for all $x \in C$, the convexity of $f(x, \cdot)$ implies easily that $\hat{x}$ solves $\mathrm{EP}(f, C)$ if and only $\hat{x}$ minimizes the marginal function $f(\hat{x}, \cdot)$ on the feasible set C, which happens if and only if $\hat{x}$ is a zero of the sum of the subdifferential of this objective function and the normalized cone N_C of C, i.e. a zero of $S_f + N_C$. Equivalently, $\hat{x}$ is a solution of the variational inequality problem $\mathrm{VIP}(S_f, C)$. It is well known that variational inequality problems are substantially easier to solve when the involved operator is maximal monotone. Thus, the study of conditions under which S_f is maximal monotone has a significant impact on the theory of equilibrium problems. We remind here that a set-valued operator $T : X \to \mathcal{P}(X^*)$ is *monotone* if $\langle u_1 - u_2, x_1 - x_2 \rangle \geq 0$ for all $(x_1, u_1), (x_2, u_2) \in G(T)$, where the *graph* $G(T)$ of T is defined as $G(T) = \{(x, u) \in X \times X^* : u \in T(x)\}$. T is said to be *maximal monotone* if it is monotone and $G(T) = G(T')$ for all monotone operator $T' : X \to \mathcal{P}(X^*)$ such that $G(T) \subset G(T')$.

We will prove in this paper that R_f and S_f are maximal monotone under some further assumptions on the behavior of f as a function of its two arguments, like for instance vanishing on the diagonal of $C \times C$.

We will also study the monotonicity properties of R_f and S_f under a different set of assumptions on f. We will drop the concavity of $f(\cdot, y)$, imposing instead stronger joint assumptions on f: it must vanish on the diagonal and be a *monotone bifunction*, meaning that $f(x,y)+f(y,x)\le 0$ for all $(x,y)\in C\times C$. In this case the monotonicity of S_f is almost immediate, but that of R_f is to some extent unexpected: Since $-f(\cdot,y)$ may fail to be convex, in principle $R_f(x)$ could be empty for some or even for all values of x. We will prove nevertheless that R_f is indeed maximal monotone, and that its graph contains the graph of S_f. S_f, while trivially monotone, may fail to be maximal monotone, in the absence of further assumptions on f as a function of its first argument.

Similarly, it will be proved that S_f is maximal monotone when $-f$ is monotone and it vanishes on the diagonal, without requiring convexity of $f(x,\cdot)$, and that $G(S_f)\supset G(R_f)$, while R_f in this case is trivially monotone, but not necessarily maximal monotone. Some rather undemanding semi-continuity assumptions on f are also needed for all these results.

For the case in which X is reflexive and $C=X$, most of these these results were established in [8]. In particular, it was proved that both S_f and R_f are maximal monotone under the first set of assumptions, and that the same happens with S_f when $f(x,\cdot)$ is convex, $f(\cdot,y)$ is continuous, f is monotone and it vanishes on the diagonal of $X\times X$, and with R_f when $f(\cdot,y)$ is concave, $f(x,\cdot)$ is continuous, $-f$ is monotone and it vanishes on the diagonal of $X\times X$. In this reference, reflexivity of X was heavily used, both for establishing the monotonicity of S_f and R_f and the maximality. The latter was proved using a classical result by Rockafellar (see Theorem 4.4.7 in [4]), which states that a monotone operator T such that $T+J$ is onto (where $J:X\to X^*$ is the duality operator), is maximal monotone. This result holds in smooth reflexive Banach spaces, and can be extended to nonsmooth reflexive Banach spaces, using a renormalization procedure proposed in [1]. The technique used in [8] for proving monotonicity of S_f and R_f requires local boundedness of these operators. Since the image of a point in the boundary of the domain through a maximal monotone operator always contains a halfline, the results in [8] cover only the unconstrained case, i.e. when $C=X$, that is to say, f is finite on the whole $X\times X$, so that the domain of S_f and R_f is also the whole space X, and henceforth the boundary of their domains is empty.

In this paper we get rid of these limitations, through the use of a completely different approach for establishing maximal monotonicity, based upon the characterizations of maximal monotone operators through convex functions defined on $X\times X^*$. We describe next the fundamentals of this theory.

Let X be a Banach space and X^*, X^{**} its topological dual and bi-dual, respectively. $\langle\cdot,\cdot\rangle$ will denote the duality coupling in $X\times X^*$ and $X^*\times X^{**}$ respectively, i.e., $\langle x,x^*\rangle = x^*(x)$, $\langle x^*,x^{**}\rangle = x^{**}(x^*)$ for all $x\in X, x^*\in X^*, x^{**}\in X^{**}$. In 1988, S. Fitzpatrick associated to each point-to-set operator $T:X\to\mathcal{P}(X^*)$ the function $\varphi_T: X\times X^*\to\mathbb{R}\cup\{+\infty\}$ defined as

$$\varphi_T(x,x^*) = \sup_{(y,y^*)\in G(T)}\{\langle x-y, y^*-x^*\rangle\} + \langle x,x^*\rangle, \tag{3}$$

called the *Fitzpatrick function* of T, see [7]. It is easy to check that φ_T is convex and lower semicontinuous and that $\varphi_T(x,x^*)\ge\langle x,x^*\rangle$ for all $(x,x^*)\in G(T)$. Also, T is maximal monotone if and only if

$$G(T)=\{(x,x^*)\in X\times X^* : \varphi_T(x,x^*)=\langle x,x^*\rangle\}; \tag{4}$$

indeed, it follows easily from (3) that $\varphi_T(x, x^*) = \langle x, x^* \rangle$ if and only if

$$\langle x - y, x^* - y^* \rangle \geq 0 \;\; \forall (y, y^*) \in G(T). \tag{5}$$

It is an elementary consequence of the respective definitions that monotonicity of T is equivalent to the validity of (5) for all pairs $(x, x^*) \in G(T)$, and that maximality holds if and only if all pairs (x, x^*) satisfying (5) belong to $G(T)$. Thus maximal monotonicity of T is equivalent to the equality in (4). The Fitzpatrick function was independently rediscovered in [5] and [14]. Note that, in view of (4), φ_T fully characterizes the operator T.

The introduction of φ_T naturally leads to the consideration of other functions that represent a maximal monotone operator T in a similar way, i.e. convex and lower semicontinuous functions $h : X \times X^*$ such that $h(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$, with equality iff $(x, x^*) \in G(T)$. Calling $\mathcal{F}_T$ such family of functions, it was proved in [7] that φ_T is the smallest member of the family, i.e. $\varphi_T(x, x^*) \leq h(x, x^*)$ for all $(x, x^*) \in X \times X^*$ and all $h \in \mathcal{F}_T$. The family $\mathcal{F}_T$, and more generally the theory of convex representation of monotone operators, was analyzed in several papers; see e.g. [3], [11], [13], [15], [16] and [17]. We will be concerned here with two results within this theory. We recall that given a function $\phi : X \to \mathbb{R} \cup \{+\infty\}$, its *convex conjugate* $\phi^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined as $\phi^*(x^*) = sup_{x \in X}\{\langle x, x^* \rangle - \phi(x)\}$. It has been proved in [6] that if X is reflexive, T is maximal monotone and h belongs to the above defined family $\mathcal{F}_T$, then it also holds that $h^*(x^*, x) \geq \langle x, x^* \rangle$ for all $(x^*, x) \in X^* \times X = X^* \times X^{**}$. Furthermore, this inequality, together with $h(x, x^*) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$ (embedded in the definition of $\mathcal{F}_T$), are enough to establish maximal monotonicity of the operator whose graph is the set where the latter inequality holds as an equality. More precisely, the result, presented in Theorem 3.1 of [6], is the following:

Theorem 1. *Let X be a reflexive Banach space, and $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous function (in the strong-weak* topology), satisfying*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle \;\; \forall (x, x^*) \in X \times X^*.$$

Consider the operator $T : X \to \mathcal{P}(X^*)$ whose graph $G(T)$ is given by

$$G(T) = \{(x, x^*) \in X \times X^* : h(x, x) = \langle x, x^* \rangle\}.$$

Then,

i) T is maximal monotone.

ii) $G(T) = \{(x, x^*) \in X \times X^* : h^*(x^*, x) = \langle x, x^* \rangle\}$.

We will use the following extension of Theorem 1 to nonreflexive Banach spaces, which has been proved in Theorem 3.1 and Corollary 3.2 of [12]. We will consider X as a subset of X^{**} through the natural immersion: a point $x \in X$ is seen as an element of X^{**}, i.e. a continuous linear functional defined on X^*, whose value $x(x^*)$ at a point $x^* \in X^*$ is given by $x^*(x)$. We will denote as $D_1(h)$ the projection of the domain of h onto X, i.e.

$$D_1(x) = \{x \in X : \text{there exists } x^* \in X^* : \text{such that } h(x,x^*) < +\infty\}.$$

Theorem 2. *Let X be an arbitrary Banach space, and $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ a proper, convex and lower semicontinuous function (in the strong-weak * topology), satisfying*

$$h(x,x^*) \geq \langle x, x^* \rangle \quad \forall (x,x^*) \in X \times X^* \tag{6}$$

$$h^*(x^*,x) \geq \langle x^*, x \rangle \quad \forall (x^*,x) \in X^* \times X \subset X^* \times X^{**}. \tag{7}$$

Assume also that for some $x^0 \in D_1(h)$, the set $\cup_{\lambda>0}\lambda[D_1(h) - x^0]$ is a closed subspace of X. Consider the operator $T : X \to \mathcal{P}(X^*)$ whose graph $G(T)$ is given by

$$G(T) = \{(x,x^*) \in X \times X^* : h(x,x^*) = \langle x, x^* \rangle\}.$$

Then

i) T is maximal monotone.

ii) $G(T) = \{(x,x^*) \in X \times X^* \subset X^{**} \times X^* : h^*(x^*,x) = \langle x, x^* \rangle\}$.

We mention that the assumption on $D_1(h)$ holds when the affine hull of the projection of the domain of h onto X is closed, and this projection has nonempty relative interior. It holds automatically when the domain of h itself has nonempty interior.

We remark that the sufficient condition for maximal monotonicity given by Theorems 1, 2 turns out to be a quite powerful tool. One could think at first view that it is a mere refinement of Fitzpatrick's result, which states a similar conclusion using φ_T instead of a general h. It is the case, nevertheless, that the Fitzpatrick function does not help much for proving that a given operator is maximal monotone, because, as shown above, the equivalence between the maximal monotonicity of T and the equality related to φ_T, holding precisely on the graph of T, is too immediate, so that both statements are equally hard to prove. On the other hand, Theorems 1 and 2 above allow us to use a large array of possible functions h, for some of which proving the inequalities above might be a far easier task than establishing in a direct way the maximal monotonicity of T. This fact is enhanced by our proofs in the following sections. We will find certain h's which adequately fit the operators S_f, R_f , for each set of assumptions on the bifunction f, and such that the corresponding inequalities are quite easy to establish (we emphasize that these h's *are not* the Fitzpatrick functions of these operators). Maximal monotonicity of the operators will then be an immediate consequence of Theorem 2. Comparison of this proofline with the one adopted for proving quite weaker similar results in [8] leads to a categorical corroboration of the strength of the theory of convex representation of monotone operators in general, and of Theorems 1 and 2 in particular.

2 Maximal monotonicity of the diagonal subdifferential operators for a concave-convex f

We start by introducing the concave-convex property of the bifunction f in a formal way. Let X be an arbitrary Banach space and C a non-empty closed and convex subset of X. We consider the following two assumptions on $f : C \times C \to \mathbb{R}$.

A1) $f(\cdot, y) : C \to \mathbb{R}$ is concave and upper semicontinuous for all $y \in C$.

A2) $f(x, \cdot) : C \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in C$.

We define now, for each $x \in C$, the functions $f_x, f^x : X \to \mathbb{R} \cup \{+\infty\}$ as

$$f_x(y) = \begin{cases} f(x,y) & \text{if } y \in C \\ +\infty & \text{otherwise,} \end{cases} \tag{8}$$

$$f^x(y) = \begin{cases} -f(y,x) & \text{if } y \in C \\ +\infty & \text{otherwise.} \end{cases} \tag{9}$$

It follows from (A1), (A2), (8) and (9) that both f_x and f^x are *closed convex* functions for all $x \in C$, and are *proper* if and only if $x \in C$. We define now the operators $R_f, S_f : X \to \mathcal{P}(X^*)$:

$$R_f(x) = \begin{cases} \partial f^x(x) & \text{if } x \in C \\ \emptyset & \text{otherwise,} \end{cases} \tag{10}$$

$$S_f(x) = \begin{cases} \partial f_x(x) & \text{if } x \in C \\ \emptyset & \text{otherwise.} \end{cases} \tag{11}$$

We remark now that under just (A1) and (A2), the operators R_f, S_f may fail to be monotone, as shown in the following example, taken from [8]: Consider $X = C = \mathbb{R}^n$, and an indefinite $A \in \mathbb{R}^{n\times n}$, i.e. such that there exist $\tilde{x}, \hat{x} \in \mathbb{R}^n$ satisfying $\tilde{x}^t A \tilde{x} > 0$, $\hat{x}^t A \hat{x} < 0$. Define $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as $f(x, y) = x^t A y$, so that $R_f(x) = -Ax, S_f(x) = A^t x$. It is immediate that both (A1) and (A2) hold, but the indefiniteness of A implies that neither R_f nor S_f is monotone.

An additional condition related to the joint behavior of f in its two arguments is needed, and an appropriate one is:

A3) $f(x, x) = 0$ for all $x \in C$.

Now we introduce two additional properties of the bifunction f, to be used in Section 3. We recall that a bifunction $f : C \times C \to \mathbb{R}$ is said to be *monotone* if

$$f(x, y) + f(y, x) \leq 0$$

for all $(x, y) \in C \times C$.

The announced monotonicity assumptions on f are:

A4) f is monotone.

A5) $-f$ is monotone.

At this point, it is convenient to formalize a certain symmetry relation between R_f and S_f. To any bifunction $f : C \times C \to \mathbb{R}$ we associate the bifunction $g : C \times C \to \mathbb{R}$ defined as $g(x, y) = -f(y, x)$. The connections between R_f, S_f, R_g and S_g are encapsulated in the following proposition.

Proposition 1. *i) f satisfies (A1) iff g satisfies (A2),*

ii) f satisfies (A2) iff g satisfies (A1),

iii) f satisfies (A3) iff g satisfies (A3),

iv) f satisfies (A4) iff g satisfies (A5),

v) f satisfies (A5) iff g satisfies (A4),

vi) $R_f = S_g$, $S_f = R_g$.

Proof. Elementary, cf. Proposition 2 in [8]. □

We will prove now that, assuming (A1), (A2) and (A3), the operators R_f and S_f are maximal monotone. A similar result can be found in Theorem 4.1 of [8], but only for the reflexive and unconstrained cases, i.e. when X is reflexive and $C = X$.

Theorem 3. *Assume that either X is reflexive or $\cup_{\lambda>0}\lambda[C - x^0]$ is a closed subspace of X for some $x^0 \in C$. If (A1), (A2) and (A3) hold then both R_f and S_f are maximal monotone, and $R_f = S_f$.*

Proof. We prove first that R_f is maximal monotone. Define $h : X \times X^* \to \mathbb{R} \cup \{+\infty\}$ as

$$h(x, x^*) = \begin{cases} f_x^*(x^*) & \text{if } x \in C \\ +\infty & \text{otherwise,} \end{cases} \tag{12}$$

where f_x^* denotes the convex conjugate of the function f_x defined in (10). We intend to apply Theorems 1 or 2 with this h, and so we must check the validity of the assumptions of these theorems. Since $D_1(h) = C$ by (12), if X is non-reflexive, our hypothesis on C implies the assumption on $D_1(h)$ made in Theorem 2.

We move on now to convexity and lower semicontinuity of h. We remark that we need to show that these properties hold for h as a function of its two arguments. In view of (12), (8), (9) and the definition of the convex conjugation,

$$h(x,x^*) = \sup_{y\in X}\{\langle y,x^*\rangle - f_x(y)\}$$
$$= \sup_{y\in C}\{\langle y,x^*\rangle - f(x,y)\} = \sup_{y\in C}\{\langle y,x^*\rangle + f^y(x)\},$$

for all $x \in C, x^* \in X^*$, using the fact that $f_x(y) = +\infty$ for $y \notin C$ in the second equality. Therefore

$$h(x,x^*) = \sup_{y\in C}\{\langle y,x^*\rangle + f^y(x)\} \tag{13}$$

for all $x \in X, x^* \in X^*$. Since the supremum of convex and lower semicontinuous functions is convex and lower semicontinuous, in view of (13) it suffices to check that the function $\psi_y : X \times X^*$, defined as $\psi_y(x,x^*) = \langle y,x^*\rangle + f^y(x)$, is convex and lower semicontinuous for all $y \in C$. Since f^y is convex and lower semicontinuous by (A1) and (9), the joint lower semicontinuity of ψ_y in its two arguments, in the strong-weak* topology, follows now from the facts that $x^* \mapsto \langle y,\cdot\rangle$ is continuous in the weak* topology of X^*, and that ψ_y is separable.

We must show now that the values of both h and its conjugate h^* remain above the duality coupling, i.e. that (6), (7) hold. We start with (6), which holds trivially if $x \notin C$, because in such a case, according to (12), the left hand side of (6) is $+\infty$. If $(x,x^*) \in C \times X^*$, using (12), (8) and A3) we conclude that

$$h(x,x^*) = \sup_{y\in X}\langle y,x^*\rangle - f_x(y) \geq \langle x,x^*\rangle - f_x(x) = \langle x,x^*\rangle - f(x,x) = \langle x,x^*\rangle.$$

It follows that

$$h(x,x^*) \geq \langle x,x^*\rangle, \quad \forall (x,x^*) \in X \times X^*. \tag{14}$$

Now we must check the inequality in (7). Note that, for $(x^*,x^{**}) \in X^* \times X^{**}$

$$h^*(x^*,x^{**}) = \sup_{(z,y^*)\in X\times X^*}\{\langle z,x^*\rangle + \langle y^*,x^{**}\rangle - h(z,y^*)\} =$$

$$\sup_{z\in X}\left\{\langle z,x^*\rangle + \sup_{y^*\in X^*}\{\langle y^*,x^{**}\rangle - h(z,y^*)\}\right\}. \tag{15}$$

Since $h(z,y^*) = +\infty$ for $z \notin C$ by (12), the inner supremum in (15) takes the value $-\infty$ for $z \notin C$, so that we might assume that the outer supremum is taken over $z \in C$, in which case we have, again by (12), $h(z,y^*) = f_z^*(y^*)$. Thus, for $z \in C$,

$$\sup_{y^*\in X^*}\{\langle y^*,x^{**}\rangle - h(z,y^*)\} = \sup_{y^*\in X^*}\{\langle y^*,x^{**}\rangle - f_z^*(y^*)\} = (f_z^*)^*(x^{**}) = f_z^{**}(x^{**}). \tag{16}$$

We conclude from (15) and (16) that

$$h^*(x^*, x^{**}) = \sup_{z \in C}\{\langle z, x^*\rangle + f_z^{**}(x^{**})\}. \tag{17}$$

Since f_x is convex and lower semicontinuous, an elementary property of the convex conjugation guarantees that the restriction of f_z^{**} to X (seen as a subset of X^{**}), coincides with f_z, so that, taking into account (8), (9) and the definition of convex conjugation, (17) becomes, for points $x^{**} = x \in X$,

$$h^*(x^*, x) = \sup_{z \in C}\{\langle z, x^*\rangle + f_z(x)\}. \tag{18}$$

Again, if $x \notin C$, then we conclude from (8) that $f_z(x) = \infty$ for all $z \in C$ and $h^*(x^*, x) = \infty > \langle x, x^*\rangle$. If $x \in C$, then we get from (18), taking $x = z$,

$$h^*(x^*, x) \geq \langle x, x^*\rangle + f_x(x) = \langle x, x^*\rangle + f(x, x) = \langle x, x^*\rangle, \tag{19}$$

using A3 in the second equality. It follows from (19) that (7) holds. We have finished checking the assumptions of Theorems 1 and 2, and so their conclusions are valid in our setting, namely, the operator T whose graph $G(T)$ is given by $G(T) = \{(x, x^*) \in X \times X^* : h(x, x^*) = \langle x, x^*\rangle\}$ is maximal monotone, and also

$$G(T) = \{(x, x^*) \in X^* \times X \subset X^* \times X^{**} : h^*(x^*, x) = \langle x, x^*\rangle\} \subset C \times X^*. \tag{20}$$

The only remaining task consists of verifying that $T = R_f$.

Using (18), (8) and (9) we have

$$h^*(x^*, x) = \begin{cases} (f^x)^*(x^*) & \text{if } x \in C \\ +\infty & \text{otherwise,} \end{cases}$$

Therefore, in view of (7) and (20), (x, x^*) belongs to $G(T)$ if and only if

$$x \in C, \quad \langle x, x^*\rangle \geq h^*(x^*, x) = (f^x)^*(x^*) = \sup_{y \in X}\{\langle y, x^*\rangle - f^x(y)\}$$

which is equivalent to

$$x \in C, \quad \langle x, x^*\rangle \geq \langle y, x^*\rangle - f^x(y) = \langle y, x^*\rangle - f^x(y) + f^x(x), \tag{21}$$

for all $y \in X$, using (A3) in the equality. Now, (21) is equivalent to

$$x \in C, \quad f^x(y) \geq \langle y - x, x^*\rangle + f^x(x)$$

for all $y \in X$, which occurs if and only if $x^* \in \partial f^x(x) = R_f(x)$. We have established that $T = R_f$ and hence R_f is maximal monotone by Theorem 1 or 2.

In order to prove that $S_f = R_f$, in view of (12) and (6), $(x, x^*) \in G(T)$ if and only if

$$x \in C, \;\; \langle x, x^* \rangle \geq h(x, x^*) = f_x^*(x^*) = \sup_{y \in X} \{ \langle y, x^* \rangle - f_x(y) \}$$

which is equivalent to

$$x \in C, \;\; \langle x, x^* \rangle \geq \langle y, x^* \rangle - f_x(y) = \langle y, x^* \rangle - f_x(y) + f_x(x), \tag{22}$$

for all $y \in X$, using (A3) in the equality. Now, (22) is equivalent to

$$x \in C, \;\; f_x(y) \geq \langle y - x, x^* \rangle + f_x(x)$$

for all $y \in X$, which occurs if and only if $x^* \in \partial f_x(x) = S_f(x)$. □

3 Maximal monotonicity of the diagonal subdifferential operators for a monotone f

We recall that a bifunction $f : C \times C \to \mathbb{R}$ is said to be *monotone* if $f(x, y) + f(y, x) \leq 0$ for all $(x, y) \in C \times C$.

In this section we will relax assumptions (A1) or (A2), demanding instead monotonicity of either f or $-f$ (i.e., assumptions (A4) and (A5)), while keeping (A3). Working under these assumptions, we can relax the concavity-convexity hypotheses on f: we will need only convexity of $f(x, \cdot)$, i.e. (A2), for proving monotonicity of S_f and maximal monotonicity of R_f, and just concavity of $-f(\cdot, y)$, i.e. (A1), for monotonicity of R_f and maximal monotonicity of S_f.

We mention that these sets of assumptions, namely (A1), (A3) and (A5), or (A2), (A3) and (A4), are independent of (A1), (A2) and (A3). We present two examples, taken from [8], both of them with $X = C = \mathbb{R}^n$. Take $A, B \in \mathbb{R}^{n \times n}$ positive semidefinite, but such that $A - B$ is indefinite. Define

$$f(x, y) = -x^t A x + y^t B y + x^t (A - B) y. \tag{23}$$

This f satisfies (A1), (A2) and (A3), but neither (A4) nor (A5), because neither f nor $-f$ is monotone: note that $f(x, y) + f(y, x) = (x - y)^t (B - A)(x - y)$, which is neither positive nor negative for all $x, y \in X$, due to the indefiniteness of $A - B$. A non-quadratic example with the same properties is obtained by taking $\bar{f}(x, y) = f(x, y) - \phi(x) + \phi(y)$, with f as in (23), where $\phi : X \to \mathbb{R}$ is an arbitrary convex function.

Consider now $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x, y) = \sum_{j=1}^{n} x_j^3 (y_j - x_j).$$

This f satisfies (A2), (A3) and (A4) (note that $f(x,y)+f(y,x)=\sum_{j=1}^{n}(x_j^3-y_j^3)(y_j-x_j)\le 0$), but (A1) fails, because f is not concave in x for all y. The bifunction $-f$, with f as in this example, satisfies (A1), (A3) and (A5), but not (A2).

We have the following results on monotonicity of R_f, S_f, assuming monotonicity properties of f.

Theorem 4. *Assume that either X is reflexive or $\cup_{\lambda>0}\lambda[C-x^0]$ is a closed subspace of X for some $x^0\in C$.*

i) If f satisfies (A2), (A3) and (A4) then R_f is maximal monotone and S_f is monotone.

ii) If f satisfies (A1), (A3) and (A5) then S_f is maximal monotone and R_f is monotone.

Proof. i) We intend to use again Theorems 1 or 2, but now the function h defined in (12) does not work, because it is not convex any more as a function of x, due to the absence of (A1) among the assumptions of this item, which entails lack of concavity of $f(\cdot,y)$. Thus, we define instead $\hat{h}:X\times X^*\to\mathbb{R}\cup\{+\infty\}$ as

$$\hat{h}(x,x^*)=\begin{cases}(f^x)^*(x^*) & \text{if } x\in C\\ +\infty & \text{otherwise,}\end{cases} \tag{24}$$

with f^x as in (9). We remark that although f^x may fail to be convex, there is no problem with taking its convex conjugate. Convex conjugates of nonconvex functions might not be proper, but this does not occur when the nonconvex function is proper and is bounded below by a proper closed convex function, which is precisely the case in our situation, because, since $f(x,y)+f(y,x)\le 0$ for all $x,y\in C\times C$ by (A4), we have that

$$f^x(y)\ge f_x(y) \tag{25}$$

for all $y\in X$, with f_x as in (8), and f_x is indeed convex, by virtue of (A2). Now we check the assumptions of Theorem 2. As in Theorem 3, the assumption on $D_1(h)$ is taken care by our hypothesis on C. Convexity and lower semicontinuity of $\hat{h}$ are proved also like in Theorem 3, but we repeat the argument, so that the use of (A2), instead of (A1), becomes fully transparent.

For any $x\in C$,

$$\begin{aligned}\hat{h}(x,x^*)&=\sup_{y\in X}\{\langle y,x^*\rangle-f^x(y)\}\\ &=\sup_{y\in C}\{\langle y,x^*\rangle-f^x(y)\}=\sup_{y\in C}\{\langle y,x^*\rangle+f(y,x)\}.\end{aligned} \tag{26}$$

Therefore

$$\hat{h}(x,x^*)=\sup_{y\in C}\{\langle y,x^*\rangle+f_y(x)\}, \tag{27}$$

for all $x \in X, x^* \in X^*$. Using (A2) we conclude that, for each $y \in C$, the function $\rho_y : X \times X^* \to \mathbb{R}$ defined as $\rho_y(x, x^*) = \langle y, x^* \rangle + f_y(x)$ is convex and lower-semicontinuous in the strong×weak* topology of $X \times X^*$. It folows from (27) that $\hat{h}$ is also convex and lower-semicontinuous in this topology.

To prove that $\hat{h}$ satisfies (6), first note that in view of (24), it suffices to verify this inequality for $x \in C$. If $x \in C$, then taking $y = x$ in (26) we conclude that

$$h(x, x^*) \geq \langle x, x^* \rangle + f(x, x) = \langle x, x^* \rangle$$

where the equality follows from (A3). Therefore, $\hat{h}$ satisfies (6).

For (7), note that

$$\hat{h}^*(x^*, x^{**}) = \sup_{z \in X} \left\{ \langle z, x^* \rangle + \sup_{y^* \in X^*} \{ \langle y^*, x^{**} \rangle - \hat{h}(z, y^*) \} \right\}.$$

As in the proof of Theorem 3 we assume that the outer supremum is taken over $z \in C$, in which case we have, again by (24), $\hat{h}(z, y^*) = (f^z)^*(y^*)$. Thus, for $z \in C$,

$$\sup_{y^* \in X^*} \{ \langle y^*, x^{**} \rangle - \hat{h}(z, y^*) \} = \sup_{y^* \in X^*} \{ \langle y^*, x^{**} \rangle - (f^z)^*(y^*) \} = (f^z)^{**}(x^{**}). \tag{28}$$

We conclude from (15) and (28) that

$$\hat{h}^*(x^*, x^{**}) = \sup_{z \in X} \{ \langle z, x^* \rangle + (f^z)^{**}(x^{**}) \}. \tag{29}$$

At this point the situation differs from the proof of Theorem 3. As mentioned above, f^z in not convex in general, and hence it does not hold that $(f^z)^{**} = f^z$. We invoke instead the following elementary property of convex conjugation: if $\phi_1, \phi_2 : X \to \mathbb{R} \cup \{+\infty\}$ are such that $\phi_1(x) \leq \phi_2(x)$ for all $x \in X$, then $\phi_1^{**}(x) \leq \phi_2^{**}(x)$ for all $x \in X$. In view of (25), we can apply this result to f_x, f^x, concluding that

$$\langle z, x^* \rangle + (f^z)^{**}(x^{**}) \geq \langle z, x^* \rangle + f_z^{**}(x^{**}),$$

so that we get from (29)

$$\hat{h}^*(x^*, x^{**}) \geq \sup_{z \in X} \{ \langle z, x^* \rangle + f_z^{**}(x^{**}) \} \tag{30}$$

for all $(x^*, x^{**}) \in X^* \times X^{**}$. Since f_z is convex and lower semicontinuous by (A2) and (8), we have, as before, $f_z = f_z^{**}$ in X, so that (30) becomes, for the restriction of $\hat{h}^*$ to $X^* \times X \subset X^* \times X^{**}$,

$$\hat{h}^*(x^*, x) \geq \sup_{z \in X} \{ \langle z, x^* \rangle + f_z(x) \}$$

and therefore, taking $z = x$ and using (A3), we get $\hat{h}^*(x^*, x) \geq \langle x, x^* \rangle$ for all $(x, x^*) \in X \times X^*$, thus establishing (7). It follows that the conclusion of Theorem 1(i) or Theorem 2(i) holds and hence the operator $\hat{T}$, whose graph $G(\hat{T})$ is given by

$$G(\hat{T}) = \{(x, x^*) \in X \times X^* : \hat{h}(x, x^*) = \langle x, x^* \rangle\}, \tag{31}$$

is maximal monotone. We proceed to identify the operator $\hat{T}$. In view of (31) and (24), x^* belongs to $\hat{T}(x)$ if and only if

$$x \in C, \quad \langle x, x^* \rangle \geq \hat{h}(x, x^*) = \sup_{y \in X} \{\langle y, x^* \rangle - f^x(y)\}$$

which is equivalent to

$$x \in C, \quad \langle x, x^* \rangle \geq \langle y, x^* \rangle - f^x(y)$$

for all $y \in X$, which happens if and only if

$$\langle y - x, x^* \rangle \leq f^x(y) = f^x(y) - f^x(x), \tag{32}$$

using (9) and (A3) in the last equality. Now, a point $x^* \in X^*$ satisfies (32) precisely when it is a subgradient of f^x at x, i.e. when it belongs to $R_f(x)$. We have proved that R_f coincides with $\hat{T}$, and it is therefore maximal monotone.

We look now at S_f. A rather immediate proof of its monotonicity can be found in Theorem 1 of [8]. We give next an alternative proof, better fitted to our previous argument. We claim that

$$G(S_f) \subset G(R_f), \tag{33}$$

and we proceed to prove the claim. Note first that monotonicity of f implies that $f_x(y) \leq f^x(y)$ for all $y \in X$, and that $f_x(x) = f^x(x) = 0$ by (A3). Take now $u \in S_f(x)$. In view of the definition of S_f, for all $y \in X$,

$$\langle u, y - x \rangle \leq f_x(y) - f_x(x) = f_x(y) \leq f^x(y) = f^x(y) - f^x(x),$$

implying that $u \in R_f(x)$ and establishing the claim. Since we have proved that R_f is maximal monotone, monotonicity of S_f follows from (33).

ii) We consider, as before, the bifunction $g : C \times C \to \mathbb{R}$ defined as $g(x, y) = -f(y, x)$. In view of items (i), (iii) and (v) of Proposition 1, we get that g satisfies (A2), (A3) and (A4), so that, by virtue of item (i) of this theorem, R_g is maximal monotone, and hence we get maximal monotonicity of S_f from Proposition 1(vi). An argument similar to the one used to prove the inclusion in (33) establishes that

$$G(R_f) \subset G(S_f), \tag{34}$$

which entails the monotonicity of R_f.

□

We mention that in the absence of further assumptions, S_f may fail to be maximal monotone under the hypotheses of Theorem 4(i), and the same holds for R_f under the hypotheses of Theorem 4(ii). Consider the following example. Take $X = C = \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ nondecreasing (but possibly discontinuous). Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $f(x,y) = \phi(x)(y-x)$. It is rather elementary to prove that f satisfies (A2), (A3) and (A4). Taking in particular

$$\phi(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0, \end{cases}$$

a simple algebra shows that

$$S_f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

Note that S_f is not maximal monotone (in particular its graph is not closed). For this example, one has

$$R_f(x) = \begin{cases} 0 & \text{if } x < 0 \\ [0,1] & \text{if } x = 0 \\ 1 & \text{if } x \geq 0, \end{cases}$$

which is indeed maximal monotone, in agreement with the result of Theorem 4(i). Note also that the inclusion in (33) holds. A similar counterexample to the maximal monotonicity of R_f under (A1), (A3) and (A5) can be easily constructed.

We mention that in the reflexive and unconstrained case (i.e., X reflexive and $X = C$), Theorem 4 of [8] establishes maximal monotonicity of S_f under (A2), (A3) and (A4), assuming additionally continuity of $f(\cdot, y)$, and of R_f under (A1), (A3) and (A5), assuming additionally continuity of $f(x, \cdot)$. Note that the additional continuity assumption does not hold for the example above, because ϕ is discontinuous. We conjecture that these results hold also in our constrained and nonreflexive setting. In view of (33) and (34), it suffices to prove that these continuity assumptions imply that $S_f = R_f$.

We also comment that [8] contains no monotonicity results for R_f under (A2), (A3) and (A4), or for S_f under (A1), (A3) and (A5).

References

[1] Asplund, E. Averaged norms. *Israel Journal of Mathematics* **5** (1967) 227-233.

[2] Blum, E., Oettli, W. From optimization and variational inequalities to equilibrium problems. *The Mathematics Student* **63** (1994) 123-145.

[3] Borwein, J.M. Maximal monotonicity via convex analysis. *Journal of Convex Analysis* **13** (2006) 561-586.

[4] Burachik, R.S., Iusem, A.N. *Set-Valued Mappings and Enlargements of Monotone Operators*. Springer, Berlin (2007).

[5] Burachik, R.S., Svaiter, B.F. Maximal monotone operators, convex functions and a special family of enlargements. *Set-Valued Analysis* **10** (2002) 297-316.

[6] Burachik, R.S., Svaiter, B.F. Maximal monotonicity, conjugation and the duality product. *Proceedings of the American Mathematical Society* **131** (2003) 2379-2383.

[7] Fitzpatrick, S. Representing monotone operators by convex functions. In *Miniconference on Functional Analysis and Optimization*, Proceedings of the Centre for Mathematical Analysis of the Australian National University, Canberra, **20** (1988) 59-65.

[8] Iusem, A.N. On the maximal monotonicity of diagonal subdifferential operators. *Journal of Convex Analysis* **18** (2011) 489-503.

[9] Iusem, A.N., Kassay, G., Sosa, W. On certain conditions for the existence of solutions of equilibrium problems. *Mathematical Programming* **116** (2009) 259-273.

[10] Iusem, A.N., Sosa, W. New existence results for equilibrium problems. *Nonlinear Analysis* **52** (2003) 621-635.

[11] Marques Alves, M., Svaiter, B.F. Brøndsted-Rockafellar property and maximality of monotone operators representable by convex functions in non-reflexive Banach spaces. *Journal of Convex Analysis* **15** (2008) 693-706.

[12] Marques Alves, M., Svaiter, B.F. Maximal monotonicity, conjugation and the duality product in non-reflexive Banach spaces. *Journal of Convex Analysis* **17** (2010) 553-563.

[13] Martínez-Legaz, J.E., Svaiter, B.F. Monotone operators representable by lower semicontinuous convex functions. *Set-Valued Analysis* **13** (2005) 21-46.

[14] Martínez-Legaz, J.E., Théra, M. A convex representation of maximal monotone operators. *Journal of Nonlinear and Convex Analysis* **2** (2001) 243-247.

[15] Reich, S., Simons, S. Fenchel duality, Fitzpatrick functions and the Kirszbraun-Valentine extension theorem. *Proceedings of the American Mathematical Society***133** (2005) 2657-2660.

[16] Simons, S. The Fitzpatrick function and nonreflexive spaces. *Journal of Convex Analysis* **13** (2006) 861-881.

[17] Svaiter, B.F. Fixed points in the family of convex representations of a maximal monotone operator. *Proceedings of the American Mathematical Society* **131** (2003) 3851-3859.