Projections onto convex sets on the sphere

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Abstract

In this paper some concepts of convex analysis are extended in an intrinsic way from the Euclidean space to the sphere. In particular, relations between convex sets in the sphere and pointed convex cones are presented. Several characterizations of the usual projection onto a Euclidean convex set are extended to the sphere and an extension of Moreau's theorem for projection onto a pointed convex cone is exhibited.

Keywords: Sphere, pointed convex cone, convex set in the sphere, projection onto a pointed convex cone.

1 Introduction

It is natural to extend the concepts and techniques of Optimization from the Euclidean space to the Euclidean sphere. This has been done frequently before. The motivation of this extension is either of purely theoretical nature or aims at obtaining efficient algorithms; see [2, 5, 14, 17, 18, 20]. Indeed, many optimization problems are naturally posed on the sphere, which has a specific underlining algebraic structure that could be exploited to greatly reduce the cost of obtaining the solutions. Besides the theoretical interest, constrained optimization problems on the sphere also have a wide range of applications in many different areas of study such as numerical multilinear algebra (see, e.g., [11]), solid mechanics (see, e.g., [6]), signal processing (see, e.g., [12, 16]) and quantum mechanics (see, e.g.,[1]).

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The aim of this paper is to extend some concepts and techniques of convex analysis of the Euclidean space to the Euclidean sphere in an intrinsic way. First of all, we will establish a relation between convex sets in the sphere and pointed convex cones. Using this relation many results related to convex cones obtained by Iusem and Seeger [8, 9] can be stated intrinsically in the sphere context. We will extend the concept of projection of a point onto a convex set to the sphere and we will study some intrinsic properties of it. In particular, we will present several characterizations of the projection onto a convex set which extend the usual characterizations of the projection onto Euclidean convex sets to the sphere. In particular, we will extend Moreau's theorem for projections onto convex cones (see [10]) to the sphere.

The structure of this paper is as follows. In Section 2 we recall some notations, definitions and basic properties about the geometry of the sphere used throughout the paper. In Section 3 we present some properties of the convex sets on the sphere. In Section 4 we present some properties of the projection onto convex sets.We conclude this paper by making some final remarks in Section 5.

2 Basics results about the sphere

In this section we recall some notations, definitions and basic properties about the geometry of the sphere used throughout the paper. They can be found in many introductory books on Riemannian and Differential Geometry, for example in [3], [4] and [13].

Let $\langle , \rangle$ be the *Euclidean inner product*, with corresponding *norm* denoted by $\| \ \|$. Throughout the paper the *n-dimensional Euclidean sphere* and its *tangent hyperplane at a point* p are denoted by

$$\mathbb{S}^n := \{p = (p_1, \ldots, p_{n+1}) \in \mathbb{R}^{n+1} : \|p\| = 1\}$$

and

$$T_p\mathbb{S}^n := \{v \in \mathbb{R}^n \ : \ \langle p, v\rangle = 0\},$$

respectively. Let I be the $(n + 1) \times (n + 1)$ identity matrix. The *projection onto the tangent hyperplane* $T_p\mathbb{S}^n$ is the linear mapping defined by

$$I - pp^T : \mathbb{R}^{n+1} \to T_p\mathbb{S}^n, \tag{1}$$

where p^T denotes the transpose of the vector p.

The *intrinsic distance on the sphere* between two arbitrary points $p, q \in \mathbb{S}^n$ is defined by

$$d(p, q) := \arccos\langle p, q\rangle. \tag{2}$$

It can be shown that the intrinsic distance $d(p, q)$ between two arbitrary points $p, q \in \mathbb{S}^n$ is obtained by minimizing the *arc length functional* ℓ,

$$\ell(c) := \int_a^b \|c'(t)\|dt,$$

over the set of all piecewise continuously differentiable curves $c : [a,b] \to \mathbb{S}^n$ joining p to q, i.e., such that $c(a) = p$ and $c(b) = q$. Moreover, d is a distance in $\mathbb{S}^n$ and $(\mathbb{S}^n, d)$ is a complete metric space, so that $d(,p,q) \geq 0$ for all $p, q \in \mathbb{S}^n$, and $d(p,q) = 0$ if and only if $p = q$. It is easy to check also that $d(p,q) \leq \pi$ for all $p, q \in \mathbb{S}^n$, and $d(p,q) = \pi$ if and only if $p = -q$.

The intersection curve of a plane though the origin of $\mathbb{R}^{n+1}$ with the sphere $\mathbb{S}^n$ is called a *geodesic*. A geodesic segment $\gamma : [a,b] \to \mathbb{S}^n$ is said to be *minimal* if its arc length is equal the intrinsic distance between its end points, i.e., if $\ell(\gamma) := \arccos\langle \gamma(a), \gamma(b)\rangle$. We say that γ is a *normalized geodesic* if $\|\gamma'\| = 1$. If $p, q \in \mathbb{S}^n$ are such that $q \neq p$ and $q \neq -p$, then the unique *segment of minimal normalized geodesic from to p to q* is

$$\gamma_{pq}(t) = \left(\cos t - \frac{\langle p,q\rangle \sin t}{\sqrt{1-\langle p,q\rangle^2}}\right) p + \frac{\sin t}{\sqrt{1-\langle p,q\rangle^2}}\, q, \qquad t \in [0,\ d(p,q)]. \tag{3}$$

Let $p \in \mathbb{S}^n$ and $v \in T_p\mathbb{S}^n$ such that $\|v\| = 1$. The minimal segment of geodesic connecting p to $-p$, starting at p with velocity v at p is given by

$$\gamma_{p\{-p\}}(t) := \cos(t)\, p + \sin(t)\, v, \qquad t \in [0,\ \pi]. \tag{4}$$

The *exponential mapping* $\exp_p : T_p\mathbb{S}^n \to \mathbb{S}^n$ is defined by $\exp_p v = \gamma_v(1)$, where γ_v is the geodesic defined by its initial position p, with velocity v at p. Hence,

$$\exp_p v := \begin{cases} \cos(\|v\|)\, p + \sin(\|v\|)\, \dfrac{v}{\|v\|}, & v \in T_p\mathbb{S}^n/\{0\}, \\ p, & v = 0. \end{cases} \tag{5}$$

It is easy to prove that $\gamma_{tv}(1) = \gamma_v(t)$ for any values of t. Therefore, for all $t \in \mathbb{R}$ we have

$$\exp_p tv := \begin{cases} \cos(t\|v\|)\, p + \sin(t\|v\|)\, \dfrac{v}{\|v\|}, & v \in T_p\mathbb{S}^n/\{0\}, \\ p, & v = 0. \end{cases} \tag{6}$$

We will also use the expression above for denoting the geodesic starting at $p \in \mathbb{S}^n$ with velocity $v \in T_p\mathbb{S}^n$ at p. The *inverse of the exponential mapping* is given by

$$\exp_p^{-1} q := \begin{cases} \dfrac{\arccos\langle p,q\rangle}{\sqrt{1-\langle p,q\rangle^2}} \left(I - pp^T\right) q, & q \notin \{p, -p\}, \\ 0, & q = p. \end{cases} \tag{7}$$

It follows from (2) and (7) that

$$d_q(p) = \|\exp_q^{-1} p\|, \qquad p, q \in \mathbb{S}^n. \tag{8}$$

We end this section by stating some more or less standard notation. We denote the *open* and the *closed ball* with radius $\delta > 0$ and center in $p \in \mathbb{S}^n$ by $B_\delta(p) := \{q \in \mathbb{S}^n : d(p,q) < \delta\}$ and $\bar{B}_\delta(p) := \{q \in \mathbb{S}^n : d(p,q) \leq \delta\}$ respectively. Let $A \subset \mathbb{S}^n$. We denote by $\mathrm{span}A$ the *subspace of* $\mathbb{R}^{n+1}$ *generated by the set* A. A point $p \in A$ is in the set $\mathrm{ri}A$, the *relative interior* of A, if there exists $\delta > 0$ such that $B_\delta(p) \cap \mathrm{span}A \subset A$. The set $\mathrm{rb}A = A \backslash \mathrm{ri}A$ denotes the *relative boundary* of A. For each $q \in \mathbb{S}^n$ we will denote as $d_q : \mathbb{S}^n \to \mathbb{R}$ the mapping $d(\cdot, q)$, with d as in (2) i.e.

$$d_q(p) := \arccos\langle p, q\rangle. \tag{9}$$

3 Convex sets on the sphere

In this section we present some properties of the convex sets of the sphere. It is worth to remark that the convex sets on the sphere $\mathbb{S}^n$ are closely related to the pointed convex cones in the Euclidean space $\mathbb{R}^{n+1}$.

Definition 1. The set $C \subseteq \mathbb{S}^n$ is said to be *convex* if for any p, $q \in C$ all the minimal geodesic segments joining p to q are contained in C.

The next result is a quite intuitive property of convex sets on the sphere $\mathbb{S}^n$; it states that proper convex sets of the sphere cannot contain both a point and its opposite.

Proposition 1. Let $C \subseteq \mathbb{S}^n$ be a convex set. If there exists $p \in C$ such that $-p \in C$ then $C = \mathbb{S}^n$.

Proof. Assume that $p, -p \in C$. It is sufficient to prove that $\mathbb{S}^n \backslash \{p, -p\} \subset C$. Take $q \in \mathbb{S}^n \backslash \{p, -p\}$. Then, a minimal geodesic segment from p to $-p$ is given by:

$$\gamma_{p\{-p\}}(t) = \cos(t)\, p + \sin(t)\, \frac{\exp_p^{-1} q}{\|\exp_p^{-1} q\|}, \qquad t \in [0,\ \pi], \tag{10}$$

using (4) in order to obtain the right hand side of (10). Since C is convex, $p, -p \in C$ and $\gamma_{p\{-p\}}$ is a minimal geodesic segment from p to $-p$, we have that $\gamma_{p\{-p\}}(t) \in C$ for all $t \in [0,\ \pi]$. As $q \in \mathbb{S}^n \backslash \{p, -p\}$, we obtain $d_p(q) < \pi$. Hence, by combining (7), (9) and (10), we have

$$q = \gamma_{p\{-p\}}\left(d_p(q)\right) \in C.$$

Therefore, the inclusion $\mathbb{S}^n \subset C$ follows and the result holds. $\square$

For each set $A \subset \mathbb{S}^n$, let K_A be the *cone spanned by* A, namely,

$$K_A := \{tp \, : \, p \in A, \ t \in [0, +\infty)\}. \tag{11}$$

Clearly, K_A is the smallest cone which contains A. In the next result we relate a convex set with the cone spanned by it, but first we need another definition. A convex cone $K \subset \mathbb{R}^{n+1}$ is said to be *pointed* if $K \cap (-K) \subseteq \{0\}$, or equivalently, if K does not contain straight lines through the origin.

Proposition 2. Let $C \subsetneq \mathbb{S}^n$ be nonempty. The set C is convex if and only if the cone K_C is convex and pointed.

Proof. Assume that $C \subsetneq \mathbb{S}^n$ is a nonempty convex set. Let $\hat{p}_1, \hat{p}_2 \in K_C$. For proving that K_C is convex it suffices to prove that

$$q = \hat{p}_1 + \hat{p}_2 \in K_C.$$

The definition of K_C implies that there exist $p_1, p_2 \in C$ and $t_1, t_2 \in [0, +\infty)$ such that $\hat{p}_1 = t_1 p_1$ and $\hat{p}_2 = t_2 p_2$. Take

$$\gamma_{p_1p_2}(t) = \left(\cos t - \frac{\langle p_1, p_2 \rangle \sin t}{\sqrt{1 - \langle p_1, p_2 \rangle^2}} \right) p_1 + \frac{\sin t}{\sqrt{1 - \langle p_1, p_2 \rangle^2}}\, p_2, \qquad t \in [0,\ d_{p_1}(p_2)], \tag{12}$$

the minimal normalized segment of geodesic from p_1 to p_2. Using (9) and (12), we get after some algebra that

$$d_{p_1}\left(\frac{q}{\|q\|}\right) \leq d_{p_1}(p_2), \qquad \gamma_{p_1p_2}\left(d_{p_1}\left(\frac{q}{\|q\|}\right)\right) = \frac{q}{\|q\|}.$$

Since C is convex and $d_{p_1}(p_2) \geq d_{p_1}(q/\|q\|)$, we have $\gamma_{p_1p_2}(d_{p_1}(q/\|q\|)) \in C$, which, together with the last equality and (11), implies that $q = \|q\|\gamma_{p_1p_2}(d_{p_1}(q/\|q\|)) \in K_C$. Thus, K_C is convex. Suppose by contradiction that K_C is not pointed, so that $K \cap -K \neq \{0\}$. Thus, there exists $p \neq 0$ such that $p, -p \in K_C$. Therefore, (11) implies that $p/\|p\|, -p/\|p\| \in C$, which entails, in view of Proposition 1, that $C = \mathbb{S}^n$. Hence, as $C \neq \mathbb{S}^n$ by assumption, we get a contradiction, establishing that K_C is pointed.

Now, assume that the cone K_C is convex and pointed. First note that $C = K_C \cap \mathbb{S}^n$. Take $p_1, p_2 \in C$ with $p_2 \neq p_1$. We must prove that the minimal geodesic segment from p_1 to p_2 is contained in C. As $p_1, p_2 \in K_C$ and K_C is pointed, we conclude that $p_2 \neq -p_1$. Thus, $|\langle p_1, p_2 \rangle| < 1$ and from (9) we have that $0 < d_{p_1}(p_2) < \pi$. Let

$$[0,\ d_{p_1}(p_2)] \ni t \mapsto \gamma_{p_1p_2}(t) = \alpha(t)p_1 + \beta(t)p_2,$$

be the minimal normalized geodesic segment from p_1 to p_2, where

$$\alpha(t) = \cos t - \frac{\langle p_1, p_2 \rangle \sin t}{\sqrt{1 - \langle p_1, p_2 \rangle^2}}, \qquad \beta(t) = \frac{\sin t}{\sqrt{1 - \langle p_1, p_2 \rangle^2}}. \tag{13}$$

Since $\gamma_{p_1p_2}(t) \in \mathbb{S}^n$, $p_1, p_2 \in K_C$ and K_C is convex, for proving that $\gamma_{p_1p_2}(t) \in C$ for all $t \in [0,\ d_{p_1}(p_2)]$, it suffices to prove that $\alpha(t) \geq 0$ and $\beta(t) \geq 0$ for all $t \in [0,\ d_{p_1}(p_2)]$. First, note that $\beta(t) \geq 0$ for all $t \in [0,\ d_{p_1}(p_2)]$, since $d_{p_1}(p_2) < \pi$. Thus, it remains to be proved that $\alpha(t) \geq 0$ for all $t \in [0,\ d_{p_1}(p_2)]$. As $d_{p_1}(p_2) < \pi$ and $0 \leq t \leq d_{p_1}(p_2)]$, we conclude, using (9), that

$$0 \leq t < \pi, \qquad \langle p_1, p_2 \rangle \leq \cos t. \tag{14}$$

Now we consider two cases: $0 \le t \le \pi/2$ and $\pi/2 \le t < \pi$. Assume first that $0 \le t \le \pi/2$. If $\langle p_1, p_2\rangle \le 0$, then we are done, because $\cos t$ and $\sin t$ are nonnegative in this interval. Assume now that $\langle p_1, p_2\rangle > 0$. In this case, it is easy to see, using (14), that

$$-\langle p_1, p_2\rangle \sin t \ge -\langle p_1, p_2\rangle \sqrt{1-\langle p_1, p_2\rangle^2}. \tag{15}$$

Note that a combination of (14) with (15) implies that $\alpha(t) \ge 0$ for all $t \in [0,\ \pi/2]$. Now, assume that $\pi/2 \le t < \pi$. In this interval, we conclude, using (14), that

$$-1 < \langle p_1, p_2\rangle \le \cos t \le 0,$$

which implies that $\sqrt{1-\langle p_1, p_2\rangle^2} \le \sin t$. Therefore, we have

$$\langle p_1, p_2\rangle \le \cos t \le 0, \qquad \sqrt{1-\langle p_1, p_2\rangle^2} \le \sin t.$$

Hence, these three inequalities, together with the second inequality in (13), imply that $\alpha(t) \ge 0$ for all $t \in [\pi/2, \pi)$, completing the proof. □

Remark 1. The convex sets on the sphere are intersections of the sphere with pointed convex cones. Indeed, it follows easily from Proposition 2, that if $K \subset \mathbb{R}^{n+1}$ is a pointed convex cone, then $C = K \cap \mathbb{S}^n$ is a convex set and $K = K_C$.

Let $C \subset \mathbb{S}^n$ with $C \neq \mathbb{S}^n$ be a nonempty convex set. The *polar set* of the set C is intrinsically defined by

$$C^{\perp} := \left\{ q \in \mathbb{S}^n \ : \ d_p(q) \ge \frac{\pi}{2}, \forall\, p \in C \right\}. \tag{16}$$

Since the function $[-1,1] \ni t \mapsto \arccos(t)$ is decreasing, it is easy to conclude from (9) that

$$C^{\perp} := \{ q \in \mathbb{S}^n \ : \ \langle p, q\rangle \le 0, \forall\, p \in C\}. \tag{17}$$

Let $K^{\perp} := \{y \in R^{n+1} : \langle x, y\rangle \le 0,\ \forall\ x \in K\}$ be the *polar cone* of the cone K and $K_{C^{\perp}}$ be the cone spanned by $C^{\perp}$, as defined in (11). The next proposition is an immediate consequence of (11), together with the definition and properties of the polar cone.

Proposition 3. Let $C \subset \mathbb{S}^n$ with $C \neq \mathbb{S}^n$ be a nonempty convex set with nonempty (intrinsic) interior. The polar set $C^{\perp}$ of C satisfies the following properties:

(i) $K_{C^{\perp}} = K_C^{\perp}$, where $K_C^{\perp}$ is the polar cone of the cone K_C;

(ii) $K_C^{\perp}$ is pointed. As a consequence, $C^{\perp}$ is convex;

(iii) if C is closed then $C^{\perp}$ is closed and $C^{\perp\perp} = C$.

Let $C \subset \mathbb{S}^n$ with $C \neq \mathbb{S}^n$ be a nonempty closed convex set. The *dual set* of C is intrinsically defined by

$$C^* := \left\{ q \in \mathbb{S}^n : d_p(q) \leq \frac{\pi}{2}, \; \forall \, p \in C \right\}. \tag{18}$$

Since the function $[-1, 1] \ni t \mapsto \arccos(t)$ is decreasing, it is easy to conclude from (9) that

$$C^* := \{ q \in \mathbb{S}^n : \langle p, q \rangle \geq 0, \; \forall \, p \in C \}. \tag{19}$$

Equations (17) and (19) imply that $C^* = -C^{\perp}$. Let $K^* := \{ y \in R^{n+1} : \langle x, y \rangle \geq 0, \; \forall \, x \in K \}$ be the *dual cone* of the cone K and K_{C^*} be the cone spanned by the set C^*, as defined in (11). The next proposition is an immediate consequence of (11), together with the definition and properties of the dual cone.

Proposition 4. Let $C \subset \mathbb{S}^n$ be a convex set with nonempty interior. The dual set C^* of C satisfies the following properties:

(i) $K_{C^*} = K_C^*$, where K_C^* is the dual cone of the cone K_C;

(ii) K_C^* is pointed. As a consequence, C^* is convex;

(iii) if C is closed then C^* is closed and $C^{**} = C$.

We define a hemisphere of the sphere as a certain sub-level of the intrinsic distance from a fixed point. More precisely, the *open hemisphere* and the *closed hemisphere* with *pole* $p \in \mathbb{S}^n$ are defined by

$$S_p^n := \{ q \in \mathbb{S}^n : d_p(q) < \pi/2 \}$$

and

$$\bar{S}_p^n := \{ q \in \mathbb{S}^n : d_p(q) \leq \pi/2 \},$$

respectively.

Remark 2. Note that the open and the closed hemispheres with pole $p \in \mathbb{S}^n$ are the intersections of the open and the closed half-spaces defined by p with the sphere $\mathbb{S}^n$, namely $\{ q \in \mathbb{R}^{n+1} : \langle p, q \rangle > 0 \}$ and $\{ q \in \mathbb{R}^{n+1} : \langle p, q \rangle \geq 0 \}$ respectively. Indeed, since the function $[-1, 1] \ni t \mapsto \arccos(t)$ is decreasing, it is easy to conclude from (9) that

$$\{ q \in \mathbb{S}^n : d_p(q) < \pi/2 \} = \{ q \in \mathbb{S}^n : \langle p, q \rangle > 0 \}, \qquad \{ q \in \mathbb{S}^n : d_p(q) \leq \pi/2 \} = \{ q \in \mathbb{S}^n : \langle p, q \rangle \geq 0 \}.$$

Open hemispheres are convex sets, since open half-spaces are pointed convex cones. On the other hand, closed hemispheres are not convex sets, because closed half-spaces are not pointed.

Corollary 1. If $C \subsetneq \mathbb{S}^n$ is a closed convex set, then there exist $p \in \mathbb{S}^n$ such that $C \subset S_p^n$.

Proof. First note that, since $C \subseteq \mathbb{S}^n$ is closed, (11) implies that K_C is also closed. Since $C \subseteq \mathbb{S}^n$ is convex, it follows from Proposition 2 that K_C is a convex cone. Let $H = \{q \in \mathbb{R}^{n+1} : \langle v, q\rangle = 0\}$ be a supporting hyperplane to the cone K_C at its vertex 0, for some $v \in \mathbb{R}^{n+1}$. Hence, as $C = K_C \cap \mathbb{S}^n$ is closed, it is easy to see that $C \subset H^+ = \{q \in \mathbb{R}^{n+1} : \langle v, q\rangle > 0\}$ or $C \subset H^- = \{q \in \mathbb{R}^{n+1} : \langle v, q\rangle < 0\}$. So, from Remark 2, we obtain that $C \subset \mathbb{S}^n_p$ or $C \subset \mathbb{S}^n_{-p}$, where $p = v/\|v\|$, completing the proof. $\square$

Example 1. If we drop the assumption that C is closed in Corollary 1, then we can only prove that there exist $p \in \mathbb{S}^n$ such that $C \subset \bar{S}^n_p$. Indeed, there is no open hemisphere that contains the convex set

$$C = \{p = (x, y, z) \in S^2 \,:\, z \geq 0,\, y = \sqrt{1 - x^2},\, -1 < x \leq 1\}.$$

However, C is contained in the hemisphere $\{p = (x, y, z) \in S^2 \,:\, z \geq 0\}$.

Example 2. In the Euclidean space the closure of a convex set is convex, but in the sphere this result does not hold. Indeed, the set

$$C = \{p = (x, y, z) \in S^2 \,:\, z > 0\},$$

is convex, but its closure $\bar{C} = \{p = (x, y, z) \in S^2 \,:\, z \geq 0\}$ is not convex.

4 Projection onto convex sets on the sphere

In this section we present some properties of the projection onto convex sets on the sphere. In particular, we will extend Moreau's theorem for projections onto convex cones to the sphere.

Let $C \subset \mathbb{S}^n$ be a closed convex set. The *projection mapping* $P_C(\,.\,) : \mathbb{S}^n \to \mathbb{P}(C)$ *onto the set* C is defined by

$$P_C(p) := \{\bar{p} \in C : d_p(\bar{p}) \leq d_p(q), \forall\, q \in C\}\,, \tag{20}$$

that is, it is the set of minimizers of the function $C \ni q \mapsto d_p(q)$. The minimal value of the function $C \ni q \mapsto d_p(q)$ is called the *distance of* p *from* C and it is denoted by $d_C(p)$. Hence, using this new notation, and equations (16) and (18), we can rewrite the polar and dual of C as

$$C^{\perp} := \left\{p \in \mathbb{S}^n \,:\, d_C(p) \geq \frac{\pi}{2}\right\}.$$

and

$$C^{*} := \left\{p \in \mathbb{S}^n \,:\, d_C(p) \leq \frac{\pi}{2}\right\},$$

respectively.

The next result is an immediate consequence of the definitions of the intrinsic distance and the projection.

Proposition 5. Let $C \subset \mathbb{S}^n$ be a nonempty closed convex set, $p \in \mathbb{S}^n$ and $\bar{p} \in C$. Then, $\bar{p} \in P_C(p)$ if and only if $\langle p, q \rangle \leq \langle p, \bar{p} \rangle$ for all $q \in C$.

Proof. Since the function $[-1, 1] \ni t \mapsto \arccos(t)$ is decreasing the result follows from (9) and (20). $\square$

An immediate consequence of Proposition 5 is the montonicity of the projection mapping, stated as follows:

Corollary 2. Let $C \subset \mathbb{S}^n$ be a nonempty closed convex set. Then the projection mapping $P_C(\,.\,) : \mathbb{S}^n \to \mathbb{P}(C)$ onto the set C satisfies

$$\langle \bar{p} - \bar{q}, p - q \rangle \geq 0, \qquad \forall\ \bar{p} \in P_C(p),\ \forall\ \bar{q} \in P_C(q).$$

Proof. Take $p, q \in \mathbb{S}^n$, $\bar{p} \in P_C(p)$ and $\bar{q} \in P_C(q)$. Since $\bar{p}, \bar{q} \in C$, it follows from Proposition 5 that $\langle p, \bar{q} \rangle \leq \langle p, \bar{p} \rangle$ and $\langle q, \bar{p} \rangle \leq \langle q, \bar{q} \rangle$. Hence $\langle p, \bar{q} - \bar{p} \rangle \leq 0$ and $\langle q, \bar{p} - \bar{q} \rangle \leq 0$, which easily implies the desired inequality. $\square$

The next result is an important property of the projection onto the set C. See a more general interpretation of this result in [15].

Proposition 6. Let $C \subset \mathbb{S}^n$ with $C \neq \mathbb{S}^n$ be a nonempty closed convex set. Consider $p \in \mathbb{S}^n$ and $\bar{p} \in C$. If $\bar{p} \in P_C(p)$, then

$$\left\langle \left(I - \bar{p}\bar{p}^T\right) p, \left(I - \bar{p}\bar{p}^T\right) q \right\rangle \leq 0, \qquad \forall\, q \in C,$$

or equivalently, $\left(I - \bar{p}\bar{p}^T\right) p = p - \langle p, \bar{p} \rangle \bar{p} \in K_{C^\perp}$.

Proof. If $p = \bar{p}$, $q = \bar{p}$ or $q = -\bar{p}$, then the inequality trivially holds. Assume that $p \neq \bar{p}$ and $\bar{p} \in P_C(p)$. Take $q \in C \backslash \{\bar{p}\}$ and note that the convexity of C implies that $-\bar{p} \notin C$. Let

$$[0, 1] \ni t \mapsto \exp_{\bar{p}}(t \exp_{\bar{p}}^{-1} q) = \cos(t d_{\bar{p}}(q))\, \bar{p} + \frac{\sin(t d_{\bar{p}}(q))}{d_{\bar{p}}(q)} \exp_{\bar{p}}^{-1} q, \tag{21}$$

be the minimal geodesic from $\bar{p}$ to q. Since $\bar{p} \in P_C(p)$, it follows from the definition of the projection in (20) that $d_p(\bar{p}) \leq d_p(\exp_{\bar{p}}(t \exp_{\bar{p}}^{-1} q))$. Hence, by combining (9), (8) and (21), we conclude that

$$\arccos\langle p, \bar{p} \rangle \leq \arccos \left\langle p\,,\, \cos(t d_{\bar{p}}(q))\, \bar{p} + \frac{\sin(t d_{\bar{p}}(q))}{d_{\bar{p}}(q)} \exp_{\bar{p}}^{-1} q \right\rangle, \qquad \forall\, t \in [0, 1].$$

Since the function $[-1, 1] \ni s \mapsto \arccos(s)$ is decreasing, we obtain from (21) that

$$\left\langle p\,,\, \cos(t d_{\bar{p}}(q))\, \bar{p} + \frac{\sin(t d_{\bar{p}}(q))}{d_{\bar{p}}(q)} \exp_{\bar{p}}^{-1} q \right\rangle \leq \langle p, \bar{p} \rangle, \qquad \forall\, t \in [0, 1].$$

After some algebra, we conclude from the previous inequality that

$$\frac{\sin(td_{\bar{p}}(q))}{td_{\bar{p}}(q)} \left\langle p, \exp_{\bar{p}}^{-1} q \right\rangle \leq \frac{1-\cos(td_{\bar{p}}(q))}{td_{\bar{p}}(q)} d_{\bar{p}}(q) \langle p, \bar{p} \rangle, \qquad \forall\, t \in [0,1].$$

Taking the limit in the latter inequality as t tends to zero, we get that $\langle p, \exp_{\bar{p}}^{-1} q \rangle \leq 0$, which, in view of (7), yields

$$\frac{\arccos\langle \bar{p}, q\rangle}{\sqrt{1-\langle \bar{p}, q\rangle^2}} \left\langle p, \left(I - \bar{p}\bar{p}^T\right) q \right\rangle \leq 0.$$

As $\arccos\langle \bar{p}, q\rangle > 0$ and $\langle -\bar{p}\bar{p}^T p, \left(I - \bar{p}\bar{p}^T\right) q \rangle = 0$, the desired inequality follows. Using (11) and the definition of the polar set, the equivalence of the inequality and the inclusion in the statement of the proposition is trivial . □

Proposition 7. Let $C \subset \mathbb{S}^n$ with $C \neq \mathbb{S}^n$ be a nonempty closed convex set. Consider $p \in \mathbb{S}^n$ and $\bar{p} \in C$ and assume that $\langle p, \bar{p} \rangle > 0$. The following statements are equivalent:

i) $\bar{p} \in P_C(p)$

ii) $\left\langle \left(I - \bar{p}\bar{p}^T\right) p, \left(I - \bar{p}\bar{p}^T\right) q \right\rangle \leq 0$, for all $\quad q \in C$.

iii) $\left(I - \bar{p}\bar{p}^T\right) p = p - \langle p, \bar{p}\rangle \bar{p} \in K_{C^\perp}$.

Moreover, $P_C(p)$ is a singleton.

Proof. Proposition 6 establishes that if $\bar{p} \in P_C(p)$, then the inequality holds, and hence item (i) implies item (ii).

Take $p \in \mathbb{S}^n$ and $\bar{p} \in C$ such that $\langle p, \bar{p} \rangle > 0$. It is easy to check that the inequality in the statement of the the proposition is equivalent to

$$\langle p, q\rangle \leq \langle \bar{p}, q\rangle \langle p, \bar{p}\rangle, \qquad \forall\, q \in C. \tag{22}$$

Since $\langle \bar{p}, q\rangle \leq 1$ for all $q \in C$ and $\langle p, \bar{p}\rangle > 0$, the previous inequality becomes $\langle p, q\rangle \leq \langle p, \bar{p}\rangle$. As the function $[-1,1] \ni t \mapsto \arccos(t)$ is decreasing, this inequality implies that $\arccos(\langle p, \bar{p}\rangle) \leq \arccos\langle p, q\rangle$ for all $q \in C$, or equivalently that $d_p(\bar{p}) \leq d_p(q)$ for all $q \in C$, which entails that $\bar{p} \in P_C(p)$. Hence, item (ii) implies item (i).

The equivalence between items (ii) and (iii) follows from (11) and the definition of the polar set.

Let $\bar{p}, \hat{p} \in P_C(p)$. The first statement entails the equivalence of the inequality in item (ii) with (22). Since $\bar{p}, \hat{p} \in C$, (22) implies that

$$\langle p, \bar{p}\rangle \leq \langle \bar{p}, \bar{p}\rangle \langle p, \bar{p}\rangle, \qquad \langle p, \hat{p}\rangle \leq \langle \bar{p}, \hat{p}\rangle \langle p, \bar{p}\rangle.$$

A simple combination of the two previous inequalities implies that $\langle p, \bar{p}\rangle \leq \langle \bar{p}, \hat{p}\rangle^2 \langle p, \bar{p}\rangle$. As $\langle p, \bar{p}\rangle > 0$, we obtain that $1 \leq \langle \bar{p}, \hat{p}\rangle^2$, which implies that $\hat{p} = \bar{p}$ or $\hat{p} = -\bar{p}$. Since $\bar{p} \in C$ and C is convex, we conclude that $-\bar{p} \notin C$. Therefore, $\hat{p} = \bar{p}$ and $P_C(p)$ is a singleton set. □

Proposition 8. Let $C \subset \mathbb{S}^n$ with $C \neq \mathbb{S}^n$ be a nonempty closed convex set. Take $p \in \mathbb{S}^n$. If $u = P_{K_C}(p)$ and $u \neq 0$, then

$$\left\langle p, \frac{u}{\|u\|} \right\rangle \geq \|u\| > 0, \qquad \frac{u}{\|u\|} = P_C(p),$$

where $P_{K_C}(p)$ denotes the usual orthogonal projection onto the closed convex cone K_C.

Proof. Since $C \subset \mathbb{S}^n$ is a nonempty closed convex set, Proposition 2 implies that K_C is a closed convex cone. Take $q \in C$. As $u = P_{K_C}(p)$ and $\|u\|q \in K_C$, we have

$$0 \geq \langle p - u, \|u\|q - u \rangle = \|u\| \langle p, q \rangle - \|u\| \langle u, q \rangle - \langle p, u \rangle + \|u\|^2.$$

It is easy to show that the last inequality is equivalent to

$$\left\langle p, \frac{u}{\|u\|} \right\rangle \geq \langle p, q \rangle + \|u\| \left[1 - \left\langle \frac{u}{\|u\|}, q \right\rangle \right].$$

So, the latter inequality implies that $\langle p, u/\|u\| \rangle \geq \langle p, q \rangle$ for all $q \in C$. Hence, we conclude from Proposition 5 that $u/\|u\| \in P_C(p)$.

Since $u = P_{K_C}(p)$, we have $0 \geq \langle p - u, v - u \rangle$ for all $v \in K_C$. Taking $v = 0$, this inequality becomes $\langle p, u \rangle \geq \|u\|^2 > 0$. So, the inequalities in the statement of the proposition hold. As $\langle p, u/\|u\| \rangle > 0$, we conclude, using Proposition 7, that $P_C(p)$ is a singleton and $u/\|u\| = P_C(p)$. □

Proposition 9. Let $C \subset \mathbb{S}^n$ with $C \neq \mathbb{S}^n$ be a nonempty closed convex set. Take $p, u \in \mathbb{S}^n$. Assume that $d_p(u) \leq \pi/2$. Then, $P_C(p) = u$ if and only if

$$P_{K_C}(p) = \cos(d_p(u))\, u, \tag{23}$$

where $P_{K_C}(p)$ denotes the usual orthogonal projection onto the closed convex cone K_C.

Proof. It is well known that, for a closed convex set D of a Hilbert space $(H, \langle \cdot, \cdot \rangle)$, it holds that $y = P_D(x)$ if and only if

$$\langle x - y, y - z \rangle \geq 0 \;\forall\, z \in D, \tag{24}$$

where $P_D(x)$ denotes the projection of x onto D with respect to the norm generated by the scalar product $\langle \cdot, \cdot \rangle$ of H (see, e.g., formula (1.2) of [19] or Theorem III-3.1.1 in [7]). Note now that the following equality holds for all $q \in \mathbb{R}^{n+1} \backslash \{0\}$:

$$\langle p - \langle p, u \rangle u\, ,\, q - \langle p, u \rangle\, u \rangle = \|q\| \left\langle \left(I - uu^T\right) p\, ,\, \left(I - uu^T\right) \frac{q}{\|q\|} \right\rangle. \tag{25}$$

Assume that $P_C(p) = u$ and $d_p(u) \leq \pi/2$. Then, $\cos(d_p(u)) \geq 0$. As $P_C(p) = u$, we obtain that $u \in K_C$ and, we conclude, taking into account the inequality $\cos(d_p(u)) \geq 0$ and (9), that

$$\langle p, u \rangle u = \cos(d_p(u))\, u \in K_C.$$

Now, take $q \in K_C \backslash \{0\}$. Since $P_C(p) = u$, it follows from Proposition 6 that both sides of equation (25) are nonpositive, so that, since $\langle p, u\rangle u \in K_C$, taking $q = 0$ in $\langle p - \langle p, u\rangle u\,,\, q - \langle p, u\rangle\, u\rangle$, we conclude that $\langle p - \langle p, u\rangle u\,,\, q - \langle p, u\rangle\, u\rangle \leq 0$ for all $q \in K_C$. Thus, it follows from (24) that (23) holds.

Now, assume that (23) holds. Take $q \in C$. It follows from (24) and (9) that both sides in equation (25) are nonpositive. Since $C \subset K_C$ and $\langle p, u\rangle = \cos(d_p(u)) > 0$, we conclude, applying Proposition 7, that $P_C(p) = u$. □

The next theorem is called *Moreau's theorem* and its proof can be found in [10].

Theorem 1. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex cone and $K^\perp$ its polar cone. For $x, y, z \in \mathbb{R}^n$ the following statements are equivalent:

(i) $z = x + y, \quad x \in K,\; y \in K^\perp, \quad \langle x, y\rangle = 0;$

(ii) $P_K(z) = x,\; P_{K^\perp}(z) = y.$

The following theorem extends Moreau's theorem on projections onto convex cones to the sphere.

Theorem 2. Let $C \subset \mathbb{S}^n$ with $C \neq \mathbb{S}^n$ be a nonempty closed convex set such that the interior of K_C is nonempty. Let $C^\perp \subset \mathbb{S}^n$ be the polar of C, and take $p, u, v \in \mathbb{S}^n$. Assume that $d_p(u) < \pi/2$ and $d_p(v) < \pi/2$. Then, the following statements are equivalent:

(i) $p = \cos(d_p(u))\, u + \cos(d_p(v))\, v, \quad u \in C, \quad v \in C^\perp, \quad d(u, v) = \pi/2.$

(ii) $P_C(p) = u, \qquad P_{C^\perp}(p) = v.$

Proof. First note that Proposition 2 implies that K_C is a pointed closed convex cone, and Proposition 3 implies that $C^\perp$ is a closed convex set, $K_{C^\perp}$ is a closed pointed convex cone and $K_{C^\perp} = K_C^\perp$. Assume that item (i) holds. The definitions of K_C and $K_C^\perp$ imply that $C \subset K_C$, $C^\perp \subset K_C^\perp$. Since $u \in C$, $v \in C^\perp$, $d_p(u) < \pi/2$ and $d_p(v) < \pi/2$, we have $\cos(d_p(u))\, u \in K_C$ and $\cos(d_p(v))\, v \in K_C^\perp$. By (9) we also have $\langle u, v\rangle = \cos(d(u, v)) = \cos(\pi/2) = 0$. Hence, since

$$p = \cos(d_p(u))\, u + \cos(d_p(v))\, v, \quad u \in C, \quad v \in C^\perp, \quad \langle u, v\rangle = 0,$$

we conclude from Theorem 1 that $\cos(d_p(u))\, u = P_{K_C}(p)$ and $\cos(d_p(v))\, v = P_{K_C^\perp}(p)$, where P_{K_C} and $P_{K_C^\perp}$ denote the usual orthogonal projections onto the cones K_C and $K_C^\perp$, respectively with respect to the norm generated by the canonical scalar product of $\mathbb{R}^n$. So, taking into account that $K_{C^\perp} = K_C^\perp$, it follows from Proposition 9 that $u = P_C(p)$ and $v = P_{C^\perp}(p)$. Therefore, item (i) implies item (ii).

For the converse, assume that item (ii) holds. From Proposition 9, we have $\cos(d_p(u))\, u = P_{K_C}(p)$ and $\cos(d_p(v))\, v = P_{K_C^\perp}(p)$. Hence, we conclude, using Theorem 1, that

$$p = \cos(d_p(u))\, u + \cos(d_p(v))\, v, \quad \cos(d_p(u))\, u \in K_C, \quad \cos(d_p(v))\, v \in K_C^\perp,$$
$$\langle \cos(d_p(u))\, u,\; \cos(d_p(u))\, v\rangle = 0.$$

Since $\cos(d_p(u))\, u > 0$ (from $d_p(u) < \pi/2$) and $\cos(d_p(v))\, v > 0$ (from $d_p(v) < \pi/2$), the previous equality yields $\langle u, v\rangle = 0$ and the inclusions above imply that $u \in K_C$ and $v \in K_C^{\perp}$. Since $u \in K_C$, $v \in K_C^{\perp}$ and $K_{C^{\perp}} = K_C^{\perp}$, the definitions of K_C and $K_C^{\perp}$ imply that $u \in C$ and $v \in C^{\perp}$. So, item (i) holds, completing the proof. □

Corollary 3. Let $C = \mathbb{R}^{n+1}_+ \cap \mathbb{S}^n$. If $p \notin -\mathbb{R}^{n+1}_+ = \{-q \,:\, q \in \mathbb{R}^{n+1}_+\}$, then

$$P_C(p) = \frac{p^+}{\|p^+\|}, \qquad p^+ = (p_1^+, \ldots, p_{n+1}^+), \quad p_i^+ = \max\{u_i, 0\}, \quad i = 1, \ldots, n+1,$$

where $\mathbb{R}^{n+1}_+ = \{p = (p_1, \ldots, p_{n+1}) \in \mathbb{R}^{n+1} \,:\, p_i \geq 0,\ i = 1, \ldots n+1\}$.

Proof. Note first that $K_C = \mathbb{R}^{n+1}_+$ is a pointed convex cone and its polar is $K_C^{\perp} = -\mathbb{R}^{n+1}_+$. Since $p \notin K_C^{\perp}$, we obtain that $P_{K_C}(p) = p^+$ and $p^+ \neq 0$. The result follows applying Proposition 8. □

The *intrinsic diameter* of a nonempty closed convex set $C \subset \mathbb{S}^n$ is defined as the maximum of the intrinsic distance between two points of the set C; that is,

$$\operatorname{diam}(C) := \sup\{d(p, q) : p, q \in C\}, \tag{26}$$

where d is the intrinsic distance on the sphere as defined in (2).

Proposition 2 gives us a connection between convex sets in the sphere and pointed convex cones. As a consequence, the intrinsic diameter of a convex set is related to the maximal angle of a cone (see, [8], [9]). More precisely, let $C \subset \mathbb{S}^n$ be a nonempty compact convex set and let K_C be the cone spanned by C, as defined in (11). Using the notation of [8], we have $\theta_{\max}(K_C) = \operatorname{diam}(C)$, where $\theta_{\max}(K_C)$ denotes the *maximal angle between unit vectors in the cone* K_C. Therefore, many results related to the maximal angle in pointed convex cones, for example those established in [8, 9], can be reformulated as results about the diameter of compact convex sets in the sphere and viceversa. For stating some of these results we need some definitions. The next definition is equivalent to the definition of antipodal pair of a convex cone given by Iusem and Seeger in [8, 9].

Definition 2. Let $C \subset \mathbb{S}^n$ be a nonempty closed convex set. The pair $(u, v) \in \mathbb{S}^n \times \mathbb{S}^n$ is called an antipodal pair of C if $u, v \in C$ and $d(u, v) = \operatorname{diam}(C)$.

The next result is Lemma 2.1 of [8], where it was called of *Principle of the Relative Boundary*. Here we will given a intrinsic proof of this result.

Proposition 10. Let $C \subset \mathbb{S}^n$ be a nonempty closed convex set. If the pair $(u, v) \in \mathbb{S}^n \times \mathbb{S}^n$ with $u \neq v$ is an antipodal pair of C then $u, v \in \operatorname{rb}C$, where $\operatorname{rb}C$ is the relative boundary of C.

Proof. Since C is convex we have $u \notin \{-v, v\}$. Therefore, we can define the minimal geodesic

$$[0, \pi) \ni t \mapsto \gamma(t) = \exp_v(t \exp_v^{-1} u).$$

Let $\mathrm{span}\{u,v\}$ be the plane of $\mathbb{R}^{n+1}$ generated by u,v. As $u,v\in C$, we conclude that $\mathrm{span}\{u,v\}\subset \mathrm{span}C$. Now, assume by contradiction that $u\in \mathrm{ri}C$. Take δ such that $0<\delta<d(u,v)$ and $B_\delta(u)\cap \mathrm{span}C\subset C$. Since $\mathrm{span}\{u,v\}\subset \mathrm{span}C$, we obtain that $B_\delta(u)\cap\mathrm{span}\{u,v\}\subset C$. Taking into account that γ is a geodesic segment, $\gamma(0)=v$ and $\gamma(1)=u$, we have $A=\{\gamma(t)\,:\,t\in[0,\pi)\}\subset \mathrm{span}\{u,v\}$. Hence, $B_\delta(u)\cap A\subset C$, which implies that $\{\gamma(t) : t\in[0,1+\delta/d(v,u))\}\subset C$, because the set C is convex and γ is a minimizing geodesic. Thus, $d(v,\gamma(\bar t))=\bar t d(v,u)>d(v,u)$ for $1<\bar t<1+\delta/d(v,u)$, which, according to Definition 2, is a contradiction. Hence, $u\in\mathrm{rb}C$. A similar argument can be used for proving that $v\in\mathrm{rb}C$. $\square$

The next theorem follows easily from Theorem 4.1 of [8], using (11) and (19).

Theorem 3. Let $C\subset\mathbb{S}^n$ be a closed convex set with nonempty interior such that $C\neq\mathbb{S}^n$. If (u,v) is an antipodal pair of C, then it holds that:

$$\frac{u-\langle u,v\rangle v}{\sqrt{1-\langle u,v\rangle^2}}\in C^*,\qquad \frac{v-\langle u,v\rangle u}{\sqrt{1-\langle u,v\rangle^2}}\in C^*. \tag{27}$$

Proposition 11. Let $C\subset\mathbb{S}^n$ be a closed convex set with nonempty interior such that $C\neq\mathbb{S}^n$. Assume that (u,v) is an antipodal pair of C. Then, it holds that:

(i) If $C\subset C^*$, then

$$P_C\left(\frac{u-\langle u,v\rangle v}{\sqrt{1-\langle u,v\rangle^2}}\right)=u,\qquad P_C\left(\frac{v-\langle u,v\rangle u}{\sqrt{1-\langle u,v\rangle^2}}\right)=v.$$

(ii) If $C^*\subset C$, then

$$P_{C^*}(u)=\frac{u-\langle u,v\rangle v}{\sqrt{1-\langle u,v\rangle^2}},\qquad P_{C^*}(v)=\frac{v-\langle u,v\rangle u}{\sqrt{1-\langle u,v\rangle^2}}.$$

Proof. We prove the first equality in item (i). Assume that $C\subset C^*$. In view of Proposition 5, it suffices to establish the following inequality:

$$\left\langle \frac{u-\langle u,v\rangle v}{\sqrt{1-\langle u,v\rangle^2}}\,,\,u\right\rangle\geq\left\langle \frac{u-\langle u,v\rangle v}{\sqrt{1-\langle u,v\rangle^2}}\,,\,q\right\rangle,\qquad \forall\, q\in C. \tag{28}$$

It is easy to check that the previous inequality is equivalent to the following one:

$$1-\langle u,v\rangle^2-\langle u,q\rangle+\langle u,v\rangle\langle q,v\rangle\geq 0,\qquad \forall\, q\in C. \tag{29}$$

Since $C\subset C^*$ and $u,v\in C$, we have $\langle u,v\rangle\geq 0$. As (u,v) is an antipodal pair of C, we conclude that $\langle q,v\rangle\geq\langle u,v\rangle$ for all $q\in C$. Hence,

$$1-\langle u,v\rangle^2-\langle u,q\rangle+\langle u,v\rangle\langle q,v\rangle\geq 1-\langle u,v\rangle^2-\langle u,q\rangle+\langle u,v\rangle^2=1-\langle u,q\rangle\geq 0,\qquad \forall\, q\in C,$$

which implies that (29) holds. As (29) is equivalent to (28), (28) also holds. The proof of the second equality in item (ii) is analogous.

Now, we prove the first equality in item (ii). In order to simplify our notation, define

$$\bar{u} = \frac{u - \langle u, v\rangle v}{\sqrt{1 - \langle u, v\rangle^2}}, \qquad \bar{v} = \frac{v - \langle u, v\rangle u}{\sqrt{1 - \langle u, v\rangle^2}}. \tag{30}$$

Since $C \neq \mathbb{S}^n$ is a convex set with nonempty interior, we have $|\langle u, v\rangle| < 1$. It follows easily that $\langle u, \bar{u}\rangle = \sqrt{1 - \langle u, v\rangle^2} > 0$. Hence, in view of Proposition 7, it suffices to prove that

$$\left\langle \left(I - \bar{u}\bar{u}^T\right) u, \left(I - \bar{u}\bar{u}^T\right) q \right\rangle \leq 0, \qquad \forall\, q \in C^*.$$

Using the definition of $\bar{u}$ in (30), a simple algebra shows that the previous inequality is equivalent to

$$\langle u, v\rangle \langle v, q\rangle \leq 0, \qquad \forall\, q \in C^*.$$

In view of Theorem 3, we have $\bar{u}, \bar{v} \in C^*$. Since $C^* \subset C$, $v \in C$ and $\bar{u}, \bar{v} \in C^*$, we have $\langle \bar{u}, \bar{v}\rangle \geq 0$ and $\langle v, q\rangle \geq 0$ for all $q \in C$. Using the fact that $\langle \bar{u}, \bar{v}\rangle \geq 0$, it is easy to conclude that $\langle u, v\rangle \leq 0$. Hence, the inequality above holds, and so does the first equality in item (ii). A similar argument establishes the second equality in item (ii). $\square$

5 Final remarks

In this paper we study some basics intrinsic properties of convex sets and projections onto convex sets on the sphere, and we touch only slightly the convexity theory in this new context. We expect that the results of this paper become a first step towards a more general theory, including algorithms for solving convex optimization problems on the sphere. We forsee further progress in this topic in the nearby future.

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