ROBUST TRANSITIVITY FOR ENDOMORPHISMS

CRISTINA LIZANA† AND ENRIQUE PUJALS‡

Abstract. We address the problem about under what conditions an endomorphism having a dense orbit, verifies that a sufficiently close perturbed map also exhibits a dense orbit. In this direction, we give sufficient conditions, that cover a large class of examples, for endomorphisms on the $n-$dimensional torus to be robustly transitive: the endomorphism must be volume expanding and any large connected arc must contain a point such that its future orbit belong to an expanding region.

1. Introduction

One goal in dynamics is to look for conditions that guarantee that certain phenomena is robust under perturbations, that is, under which hypothesis some main feature of a dynamical system is shared by all nearby systems. In particular, we are interested in the hypotheses under which an endomorphism is robust transitive (see definitions 1.5 and 4.1).

In the diffeomorphism case, there are many examples of robust transitive systems. The best known is the transitive Anosov diffeomorphism. In the nonhyperbolic context, the first example was given by Shub in $\mathbb{T}^4$ in 1971 (see [Shu71]); another example is the Mañé's Derived from an Anosov in $\mathbb{T}^3$ (see [Mañ78]); Bonatti and Díaz [BD96] gave a geometrical construction that produce partially hyperbolic robust transitive systems and these constructions were generalized by Bonatti and Viana providing robust transitive diffeomorphisms with dominated splitting which are not partially hyperbolic (see [BV00]). All those examples are adapted (and some new ones are extended) to the case of endomorphisms (see section 5).

On the other hand, any C^1-robust transitive diffeomorphism exhibits a dominated splitting (see [BDP03]). This is no longer true for endomorphisms (see example 1 in section 5.1). Therefore, for endomorphisms, conditions that imply robust transitivity cannot hinge on the existence of splitting.

The first question that arises is what necessary condition a robust transitive endomorphism has to verify. Adapting some parts of the proof in [BDP03] it is shown in Theorem 2, section 4, that for endomorphisms not exhibiting a dominated splitting (in a robust way, see definition 4.2), volume expanding is a C^1 necessary condition. However, volume expanding is not a sufficient condition that guarantees robust transitivity for a local diffeomorphism, as it follows considering an expanding endomorphism times an irrational rotation (this system is volume expanding and transitive but not robust transitive, see remark 15 for more details). Hence, we need an extra condition (that persists by perturbations and does not depend on the existence of any type of splitting) that allow us to conclude robustness. The

Date: March 14, 2012.
† This work was partially supported by TWAS-CNPq and Universidad de Los Andes.

extra hypothesis that we require can be formulated as follows: *any arc of diameter large enough have a point such that its forward iterates remain in some expanding region* (see Main Theorem below).

Before introducing the Main Theorem, we recall some definitions and we introduce some notation that we use throughout this work.

An *endomorphism* of a differentiable manifold M is a differentiable function $f : M \to M$ of class C^r with $r \geq 1$. Let us denote by $E^r(M)$ $(r \geq 1)$ the space of C^r-endomorphisms of M endowed with the usual C^r topology. A *local diffeomorphism* is an endomorphism $f : M \to M$ such that given any point $x \in M$, there exists an open set V in M containing x such that f from V to $f(V)$ is a diffeomorphism.

Definition 1.1. *We say that a map $f \in E^1(M)$ is* volume expanding *if there exists $\sigma > 1$ such that $|det(Df)| > \sigma$.*

Observe that volume expanding endomorphisms are local diffeomorphisms.

If $L : V \to W$ is a linear isomorphism between normed vector spaces, we denote by $m\{L\}$ the *minimum norm* of L, i.e. $m\{L\} = \|L^{-1}\|^{-1}$.

Definition 1.2. *We say that a set $\Lambda \subset M$ is a* forward invariant set *for $f \in E^r(M)$ if $f(\Lambda) \subset \Lambda$ and it is* invariant *for f if $f(\Lambda) = \Lambda$.*

Definition 1.3. *We say that a map $f \in E^1(M)$ is* expanding *in U, a subset of M, if there exists $\lambda > 1$ such that $\min_{x \in U}\{m\{D_x f\}\} > \lambda$. It is said that a compact invariant set Λ is an* expanding set *for an endomorphism f if $f\,|_\Lambda$ is an expanding map.*

Definition 1.4. *Let U be an open set in $\mathbb{T}^n$. Denote by $\widetilde{U}$ the lift of U restricted to a fundamental domain of $\mathbb{R}^n$. Define the* diameter *of U by*

$$diam(U) = \max\{dist(x, y) : x, y \in \widetilde{U}\}.$$

Define the internal diameter *of U^c by*

$$diam_{int}(U^c) = \min_{k \in \mathbb{Z}^n \setminus \{0\}} dist(\widetilde{U}, \widetilde{U} + k),$$

where $dist(A, B) := \inf\{\max_{1 \leq i \leq n} |x_i - y_i| : x = (x_1, \ldots, x_n) \in A, y = (y_1, \ldots, y_n) \in B\}$.

Related to the last definition, observe that if $diam(U) < 1$ then, translating the frame $\mathbb{Z}^n$, we can assume that $\widetilde{U}$ is contained in the interior of $[0, 1]^n$ and in particular, $diam_{int}(U^c) > 0$.

Definition 1.5. *Let Λ be an invariant set for an endomorphism $f : M \to M$. It is said that Λ is* topologically transitive *(or transitive) if there exists a point $x \in \Lambda$ such that its forward orbit $\{f^k(x)\}_{k \geq 0}$ is dense in Λ. We say that f is* topologically transitive *if $\{f^k(x)\}_{k \geq 0}$ is dense in M for some $x \in M$.*

The following lemma is a more useful characterization of transitivity.

Lemma 1. *Let $f : M \to M$ be a continuous map of a locally compact separable metric space M into itself. The map f is topologically transitive if and only if for any two nonempty open sets $U, V \subset M$, there exists a positive integer $N = N(U, V)$ such that $f^N(U) \cap V$ is nonempty.*

Proof. See for instance [KH95, pp.29]. ∎

Instead of transitivity we may assume the density of the pre-orbit of any point. Observe that this implies transitivity, but the reciprocal assertion does not necessarily hold. In fact, it is enough to have a dense subset of the manifold such that every point in this set has dense pre-orbit to obtain transitivity. On the contrary of diffeomorphisms case, for endomorphisms, to have just one point with dense pre-orbit is not enough to guarantee transitivity. We leave the details to the reader, it is not hard to construct an example having some points with dense pre-orbit but non-transitive.

Let us state the main theorem of the present work.

Main Theorem *Let $f \in E^r(\mathbb{T}^n)$ be a volume expanding map ($n \geq 2, r \geq 1$) such that $\{w \in f^{-k}(x) : k \in \mathbb{N}\}$ is dense for every $x \in \mathbb{T}^n$ and satisfying the following properties:*

1. *There is an open set U_0 in $\mathbb{T}^n$ such that $f|_{U_0^c}$ is expanding and diam$(U_0)<1$.*
2. *There exists $0 < \delta_0 < diam_{int}(U_0^c)$ and there exists an open neighborhood U_1 of $\overline{U}_0$ such that for every arc γ in U_0^c with diameter larger than δ_0, there is a point $y \in \gamma$ such that $f^k(y) \in U_1^c$ for any $k \geq 1$.*
3. *Moreover, for every $z \in U_1^c$, there exists $\bar{z} \in U_1^c$ such that $f(\bar{z}) = z$.*

Then, for every g C^r-close enough to f, $\{w \in g^{-k}(x) : k \in \mathbb{N}\}$ is dense for every $x \in \mathbb{T}^n$. In particular, f is C^r-robust transitive.

We would like to say a few words about the hypotheses of the Main Theorem. The first hypothesis states that there exists a set U_0 (not necessarily connected) where f fails to be expanding (if U_0 is empty, then f is expanding and the thesis follows from standard arguments for expanding maps), however, U_0 is contained in a ball of radius one and in the complement of it, f is expanding. The second hypothesis states that for any large connected arc in the expanding region, there is a point that its forward iterates remains in the expanding region. We assume $n = dim\mathbb{T}^n$ greater or equal 2, since in dimension 1 if a map is volume expanding, then it is an expanding map.

A class of systems that verifies the hypotheses of the Main Theorem is a certain type of maps isotopic to expanding endomorphisms. More precisely, we call those maps as "*Derived from Expanding*", the reason to use this name is inspired on the *Derived from Anosov* (see [Mañ78]) which are maps isotopic to an Anosov but they are not Anosov. In particular, Derived from Expanding maps that satisfies the hypotheses of Main Theorem are robustly transitive. In examples 1 and 2 in section 5, we show that there exist Derived from Expanding maps satisfying the hypotheses of Main Theorem. We want to point out that in the hypotheses of the Main Theorem it is not assumed that f is isotopic to an expanding map.

Some questions that arises from the above discussion are: *if a map satisfies the hypotheses of the Main Theorem, then is this map isotopic to an expanding endomorphism? Are robust transitive endomorphisms without dominated splitting (in a robust way) isotopic to expanding endomorphisms?*

We suggest to the reader that before entering into the proof of Main Theorem, to give a glance to section 2.1 in order to gain some insight about the proof. We want to highlight that this theorem as it is enunciated, does not assume the existence of a tangent bundle splitting (neither it assumes the lack of a dominated splitting) and it covers examples of robust transitive endomorphisms without any dominated splitting (recall example 1 in section 5). The Main Theorem can be re-casted in terms of the geometrical properties, see Main Theorem Revisited in section 2.8. In section 3, we adapt the Main Theorem for the case that the endomorphism has partially hyperbolic splitting, this is given in Theorem 1 and the proof is an adaptation of [PS06b].

In section 5, we provide examples satisfying the main results. Those satisfying the Main Theorem are done in such a way that they do not have any dominated splitting and they are Derived from Expanding endomorphisms. For this case we provide two type of examples: ones are built through bifurcation of periodic points and the other are "far from" expanding endomorphisms (see examples 1 and 2). In examples 3 and 4, we show that there are open sets of endomorphisms that satisfy Theorem 1. Those examples are partially hyperbolic and they are not isotopic to expanding endomorphisms.

2. Proof of the Main Result

Before starting the proof, we state a series of remarks that could help to understand the hypotheses of the Main Theorem and in subsection 2.1 we provide a sketch of the proof, pointing out the main details and the general strategy.

Remark 1. *As we say in the introduction, the condition diam*$(U_0) < 1$ *implies that we can assume that the closure of* $\widetilde{U}_0$ *is contained in the interior of* $[0,1]^n$, *where* $\widetilde{U}_0$ *is the lift of* U_0 *restricted to* $[0,1]^n$. *Note that* U_0 *do not need to be simply connected and could have finitely many connected components. Actually the important fact is that the closure of the convex hull of the lift of* U_0 *restricted to* $[0,1]^n$ *is still contained in* $(0,1)^n$. *Observe,* $diam_{int}(U_0^c) = diam_{int}(\mathfrak{U}_0^c)$, *where* $\mathfrak{U}_0$ *is the convex hull of* $\widetilde{U}_0$.

The Main Theorem is formulated for the $n-$dimensional torus. Some of the examples provided in section 5 are isotopic to expanding endomorphisms. Taking into account [Shu69], we may formulate the following:

Conjecture 1. *The Main Theorem holds, at least, for any manifold supporting expanding endomorphisms.*

Remark 2. *Using hypothesis (3) of the Main Theorem, given any point* $x \in U_1^c$, *we can construct a sequence* $\{x_k\}_{k\geq 0}$ *such that* $x_0 = x$, $x_k \in U_1^c$ *and* $f(x_{k+1}) = x_k$ *for every* $k \geq 0$. *We call this sequence by* inverse path.

Remark 3. *The hypothesis of diameter less than 1 and hypothesis (3) are technical. This means that they are necessary conditions for the present proof of our result, but we do not know if there exist weaker conditions that imply the thesis of our theorem.*

Remark 4. *Observe that* $\Lambda_0 := \bigcap_{n\geq 0} f^{-n}(U_0^c)$ *is an expanding set. Moreover, from hypothesis (2) follows that given any arc* γ *in* U_0^c *with diameter greater than* δ_0, *there exists a point* $x \in \gamma$ *such that* $f^k(x)$ *is not in* U_1 *for any* $k \geq 1$. *Therefore,* $\gamma \cap \Lambda_0 \neq \emptyset$ *and in particular* Λ_0 *is not trivial.*

Definition 2.1. *Let Λ be an expanding set for $f \in E^1(M)$. If there is an open neighborhood V of Λ such that $\Lambda = \bigcap_{k\geq 0} f^{-k}(\overline{V})$ then Λ is said to be* locally maximal *(or isolated) set. V is called the* isolating block *of Λ.*

All previous remark can be summarized and extended in the next observation.

Remark 5. *Let us denote $\Lambda_1 := \bigcap_{n\geq 0} f^{-n}(U_1^c)$. This set has the following properties:*

1. *Λ_1 is an expanding set.*
2. *By hypothesis (2) of the Main Theorem, given any arc γ in U_0^c with diameter greater than δ_0, there exists a point $x \in \gamma$ such that $f(x) \in \Lambda_1$.*
3. *Since the hypothesis $0 < \delta_0 < diam_{int}(U_0^c)$ is an open condition, we may take U_1 an open neighborhood of $\overline{U_0}$ such that $\delta_0 < diam_{int}(U_1^c) < diam_{int}(U_0^c)$. Then for every arc γ in U_1^c with diameter greater than δ_0 holds that $\gamma \cap \Lambda_1$ is non empty.*
4. *Λ_1 is invariant, i.e. $f(\Lambda_1) = \Lambda_1$. It is clear that Λ_1 is forward invariant. So let us prove that $\Lambda_1 \subset f(\Lambda_1)$. Pick a point $x \in \Lambda_1$ and consider the sequence $\{x_k\}_{k\geq 0}$ given by remark (2). Let us show that $x_k \notin W$ for any $k \geq 0$, where $W = \cup_{n\geq 0} f^{-n}(U_1) = \Lambda_1^c$. If this is not true, there exist $k \geq 0$ and $n_k \geq 0$ such that $f^{n_k}(x_k) \in U_1$. First, observe that remark (2) implies that $f^n(x_k) = x_{k-n}$ for $0 \leq n \leq k$. In particular, $f^k(x_k) = x_0$ if $k \geq 0$. And $f^n(x_k) = f^{n-k}(f^k(x_k)) = f^{n-k}(x_0)$ for $n > k \geq 0$. Therefore, if $-k \leq -n_k \leq 0$, then $f^{n_k}(x_k) = x_{k-n_k}$. Since every x_k belongs to U_1^c, we obtain that $f^{n_k}(x_k)$ belongs to U_1^c which is a contradiction because it was supposed that $f^{n_k}(x_k) \in U_1$. If $-n_k < -k < 0$, then $f^{n_k}(x_k) = f^{n_k-k}(x_0)$. Since $x_0 \in \Lambda_1$, every positive iterate of x_0 by f belongs to U_1^c, thus $f^{n_k}(x_k) \in U_1^c$, which contradicts the fact that $f^{n_k}(x_k) \in U_1$. Thus, $x_k \in \Lambda_1$ for every $k \geq 0$.*
5. *In section 2.2, we prove that this set is locally maximal or it is contained in an expanding locally maximal set.*

2.1. **Sketch of the Proof of Main Theorem.** Observe that if f satisfies the hypotheses of the Main Theorem, then it satisfies the following property denoted as *internal radius growing* (I.R.G.) property:

> *There exists R_0 depending on the initial system such that given any open set U, there exist $x \in U$ and $K \in \mathbb{N}$ verifying that $f^K(U)$ contains a ball of a fixed radius R_0 centered in $f^K(x)$.*

In fact, since f is volume expanding, then the lift of f is a diffeomorphism in the universal covering space $\mathbb{R}^n$. In consequence, given any open set $U \subset \mathbb{T}^n$, volume expanding implies that the diameter of the iterates by f of U grows on the covering space, (see Lemma 6 for details). Then, for some $N > 0$, the diameter of $f^N(U) \cap U_0^c$ is greater or equal to δ_0 (the constant in the second hypothesis). Then we can pick an arc in $f^N(U) \cap U_0^c$ of sufficiently large diameter and using the second hypothesis we get that there exists a point in $f^N(U)$ such that its forward orbit remain in the expanding region. Therefore, the internal radius of $f^{k+N}(U)$ grows as k grows and the I.R.G. property follows.

Hence, if we have that g also verifies the I.R.G. property, then the Main Theorem is proved: since every pre-orbit by f is dense in the manifold, given $0 < \varepsilon < R_0$, for g $\varepsilon/2-$close to f, the pre-orbit of every point by g are $\varepsilon-$dense (see subsection

2.7), then $g^K(U)$ intersects $\{w \in g^{-n}(z) : n \in \mathbb{N}\}$ for any z. Therefore, taking pre-images by g, we get that U intersects $\{w \in g^{-n}(z) : n \in \mathbb{N}\}$ for any z.

Therefore, the aim is to show that for every g sufficiently close to f, g verifies the I.R.G. property, in other words we want to show that the I.R.G. property is robust. In order to prove this statement, we use a geometrical approach:

(1) Since the initial map f is volume expanding, then the perturbed map g is also volume expanding. So, its lift is also a diffeomorphism in the universal covering space $\mathbb{R}^n$.
(2) Hypothesis (2) implies that there is an expanding subset Λ_f that "separates", meaning that a nice class of arcs in U_0^c intersect this set (see remarks 4 and 5 and lemma 4 in section 2.4).
(3) The set Λ_f can be chosen as locally maximal (see lemma 2 in section 2.2).
(4) Hence the set Λ_f has a continuation: for g nearby, $f\mid_{\Lambda_f}$ is conjugate (see definition 2.4) to $g\mid_{\Lambda_g}$, and this conjugation is extended to a neighborhood of Λ_f and Λ_g (see propositions 1 and 2 in section 2.3).
(5) Therefore, the topological property of separation persists: for a nice class of arcs, every arc intersects Λ_g following that the I.R.G. property holds for g (see lemma 5 in section 2.4).

∎

2.2. **Existence of an Expanding Locally Maximal Set for f.** In the present subsection (lemma 2) we show that Λ_1 (as defined in remark 5) is either locally maximal or is contained in a locally maximal one. The third hypothesis in the Main Theorem is essential to prove this fact (see remark 6 for a discussion about this issue).

Lemma 2. *Either Λ_1 is a locally maximal set or there exists Λ^* an expanding locally maximal set for f such that $\Lambda_1 \subset \Lambda^*$ and Λ^* verifies that every arc γ in U_0^c with diameter greater than δ_0 has a point such that the image by f belongs to Λ^*. Moreover, every arc γ in U_1^c with diameter greater than δ_0 intersects Λ^*.*

Proof. We may divide the proof in two cases:

Case I. $\Lambda_1 \cap \partial U_1 = \emptyset$.

Let us observe that $\Lambda_1 \cap \partial U_1 = \emptyset$ implies that Λ_1 is contained in the open neighborhood $V = int(U_1^c)$. Then V is an isolating block for Λ_1, therefore Λ_1 is locally maximal.

Case II. $\Lambda_1 \cap \partial U_1 \neq \emptyset$.

In this case, V fails to be an isolating neighborhood. To overcome this situation, we extend V in a proper way and we show that the extension is now an isolating neighborhood of the respective maximal invariant set. Choose $\varepsilon > 0$ sufficiently small such that the open ball $\mathbb{B}_\varepsilon(x)$ is contained in U_0^c for all $x \in \Lambda_1$ and for every $x \in \Lambda_1$, since f is a local diffeomorphism, there exists an open set U_x such that $f\mid_{U_x} : U_x \to \mathbb{B}_\varepsilon(x)$ is a diffeomorphism. Note that the collection $\{\mathbb{B}_\varepsilon(x)\}_{x\in\Lambda_1}$ is an open cover of Λ_1. Since Λ_1 is compact, there is a finite subcover, let us say $\{\mathbb{B}_\varepsilon(x_i)\}_{i=1}^N$.

Fix $\lambda_0^{-1} < \lambda' < 1$, where λ_0 is the expansion constant of f and pick N' greater or equal to N, the cardinal of the finite subcover of Λ_1, such that for every $y \in \Lambda_1$, there is $i = i(y) \in \{1, \ldots, N'\}$ such that $\mathbb{B}_{\lambda'\varepsilon}(y) \Subset \mathbb{B}_\varepsilon(x_i)$, i.e. $\overline{\mathbb{B}_{\lambda'\varepsilon}(y)} \subset \mathbb{B}_\varepsilon(x_i)$.

Let us define $W=\bigcup_{i=1}^{N'}\mathbb{B}_\varepsilon(x_i)$ and $\widehat{W}=\bigcup_{i=1}^{N'}\overline{\mathbb{B}_\varepsilon(x_i)}$.

By remark (5) Λ_1 is invariant, then we have that for every x_i, there exists at least one $x_i^j\in\Lambda_1$ such that $f(x_i^j)=x_i$. Let us consider for every $1\le i\le N'$ all the possible pre-images by f of x_i that belongs to Λ_1, i.e. recall that f is a local diffeomorphism, hence for every point $x\in M$, the cardinal $\sharp\{f^{-1}(x)\}=N_f$ is constant, then for every $i\in\{1,\dots,N'\}$, there exist $K_i\subset\{1,\dots,N_f\}$ such that if $j\in K_i$ then $x_i^j\in\Lambda_1$ and $f(x_i^j)=x_i$. Therefore for every $i\in\{1,\dots,N'\}$ and for every $j\in K_i$, there exist open sets U_i^j such that $x_i^j\in U_i^j$ and $f\mid_{U_i^j}: U_i^j\to\mathbb{B}_\varepsilon(x_i)$ is a diffeomorphism. Given $i\in\{1,\dots,N'\}$, for every $j\in K_i$ consider the inverse branches, $\varphi_{i,j}:\mathbb{B}_\varepsilon(x_i)\to U_i^j$ such that

$$\begin{aligned}\varphi_{i,j}(x_i)=&\quad x_i^j,\\ f\circ\varphi_{i,j}(x)=&\quad x,\quad \forall\, x\in\mathbb{B}_\varepsilon(x_i).\end{aligned}$$

Now, consider $\Lambda^*=\bigcap_{n\ge0}f^{-n}(\widehat{W})$. Clearly, $\Lambda_1\subset\Lambda^*\subset U_0^c$ and Λ^* is an expanding set. In order to show that Λ^* is locally maximal, it is enough to show that $\Lambda^*\cap\partial\widehat{W}=\emptyset$, which is equivalent showing that $f^{-1}(\widehat{W})$ is contained in W. Just to make more clear what follows, let us rewrite $f^{-1}(\widehat{W})$ in terms of the inverse branches,

$$f^{-1}(\widehat{W})=f^{-1}(\bigcup_{i=1}^{N'}\overline{\mathbb{B}_\varepsilon(x_i)})=\bigcup_{i=1}^{N'}\bigcup_{j\in K_i}\overline{\varphi_{i,j}(\mathbb{B}_\varepsilon(x_i))}.$$

So, it is enough to show that

$$\overline{\varphi_{i,j}(\mathbb{B}_\varepsilon(x_i))}\subset\mathbb{B}_\varepsilon(x_{m_{i,j}}),$$

for some $x_{m_{i,j}}\in\{x_1,\dots,x_{N'}\}$. In fact,

$$\varphi_{i,j}(\mathbb{B}_\varepsilon(x_i))=U_i^j\subset\mathbb{B}_{\lambda_0^{-1}\varepsilon}(\varphi_{i,j}(x_i))\subset\mathbb{B}_{\lambda'\varepsilon}(\varphi_{i,j}(x_i))=\mathbb{B}_{\lambda'\varepsilon}(x_i^j)$$

then, there exists $m_{i,j}\in\{1,\dots,N'\}$ such that $\mathbb{B}_{\lambda'\varepsilon}(x_i^j)\Subset\mathbb{B}_\varepsilon(x_{m_{i,j}})$, and the assertion holds.

Since $\Lambda_1\subset\Lambda^*\subset U_0^c$ and Λ_1 intersects the image by f of every arc γ in U_0^c with diameter larger than δ_0, then follows that Λ^* also verifies the latter property. In particular, Λ^* intersects every arc γ in U_1^c with diameter larger than δ_0. ∎

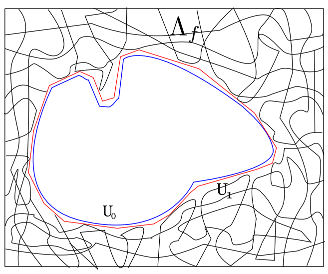


FIGURE 1. Λ_f *looks like a net which is an expanding set that "separates"*

Remark 6. *We want to highlight that for diffeomorphisms there exist examples of hyperbolic sets that are not contained in any locally maximal hyperbolic set, see for instance* [Cro02] *and* [Fis06]*. A similar construction seems feasible for endomorphisms. In our context, hypothesis (3) allows to overpass this problem and guarantees that* Λ_1 *is an invariant set. Moreover, we can consider a finite covering* $\{\mathbb{B}_\varepsilon(x_i)\}_{i=1}^{N'}$ *for* Λ_1*, with* $x_i \in \Lambda_1$*, in such a way that for every point* $y \in \Lambda_1$*, there is* x_i *such that* $\overline{\mathbb{B}_{\lambda'\varepsilon}(y)} \subset \mathbb{B}_\varepsilon(x_i)$*. Thus we conclude that* Λ^* *is contained in the interior of* $\widehat{W}$ *and therefore the expanding set* Λ_1 *is either locally maximal or is contained in a locally maximal expanding set.*

2.3. **Continuation of the Expanding Locally Maximal Set.** First, in proposition 1 we prove that $g\mid_{\Lambda_g}$ is conjugate to $f\mid_{\Lambda_f}$, where Λ_g is the maximal invariant set associated to g, for g sufficiently close to f. This is standard in hyperbolic theory using a shadowing's lemma argument. We provide the proof for completeness and to show how the conjugacy can be extended to a neighborhood. This is done in proposition 2. We want to remark that to construct the topological conjugacy between $g\mid_{\Lambda_g}$ and $f\mid_{\Lambda_f}$ is not necessary that Λ_f be locally maximal invariant, however, this property is essential in the proof of proposition 2.

Definition 2.2. *The sequence* $\{x_n\}_{n\in\mathbb{Z}}$ *is said to be a* $\delta-$pseudo orbit *for* f *if* $d(f(x_n), x_{n+1}) \leq \delta$ *for every* $n \in \mathbb{Z}$.

Definition 2.3. *We say that a* $\delta-$*pseudo orbit* $\{x_n\}_{n\in\mathbb{Z}}$ *for* f *is* $\varepsilon-$shadowed *by a full orbit* $\{y_n\}_{n\in\mathbb{Z}}$ *for* f *if* $d(y_n, x_n) \leq \varepsilon$ *for every* $n \in \mathbb{Z}$.

It follows that the *Shadowing Lemma* holds for C^1 endomorphisms.

Lemma 3. *Let* M *be a Riemannian manifold,* $U \subset M$ *open,* $f : U \to M$ *a* C^1 *expanding endomorphism, and* $\Lambda \subset U$ *a compact invariant expanding set for* f*. Then there exists a neighborhood* $\mathcal{U}(\Lambda) \supset \Lambda$ *such that whenever* $\eta > 0$ *there is an* $\varepsilon > 0$ *so that every* $\varepsilon-$*pseudo orbit for* f *in* $\mathcal{U}(\Lambda)$ *is* $\eta-$*shadowed by a full orbit of* f *in* Λ*. If* Λ *is locally maximal invariant set, then the shadowing full orbit is contained in* Λ.

Proof. For details, see for instance [Liu91].

Definition 2.4. *Let* $f : M \to M$ *and* $g : N \to N$ *be two maps and let* Λ_f *and* Λ_g *be invariant sets by* f *and* g *respectively. We say that* $f : \Lambda_f \to \Lambda_f$ *is* topologically conjugate *to* $g : \Lambda_g \to \Lambda_g$ *if there exists a homeomorphism (in the relative topology)* $h : \Lambda_f \to \Lambda_g$ *such that* $h \circ f = g \circ h$.

This a typical notion for hyperbolic sets, see [Shu87].

In order to fix some notation in what follows, we denote by Λ_f the expanding locally maximal set for $f : \Lambda_f$ is either Λ_1, in the case it is locally maximal, or it is Λ^* given in Lemma 2. We also denote by U the isolating block of Λ_f.

Proposition 1. *There exists* $\mathcal{V}_1(f)$ *an open neighborhood of* f *in* $E^1(\mathbb{T}^n)$ *such that if* $g \in \mathcal{V}_1(f)$*, then* g *is expanding on* $\Lambda_g = \bigcap_{n\geq 0} g^{-n}(U)$ *and there exists an homeomorphism* $h_g : \Lambda_g \to \Lambda_f$ *that conjugate* $f\mid_{\Lambda_f}$ *and* $g\mid_{\Lambda_g}$ *and* h_g *is close to the identity.*

Proof. In order to get the conjugacy we use the Shadowing Lemma for C^1 expanding endomorphisms, see lemma 3.

Since Λ_f is an expanding locally maximal set for f, there exists $\beta > 0$ such that f is expansive with constant β in Λ_f.

Fix $0 < \eta < \beta$. By the endomorphism version of the Shadowing Lemma, there exists $\varepsilon > 0$ such that any $\varepsilon-$pseudo orbit for f within ε of Λ_f is uniquely $\eta-$shadowed by a full orbit in Λ_f.

Take N such that

$$\bigcap_{j=0}^{N} f^{-j}(U) \subset \{q : \, d(q, \Lambda_f) < \varepsilon/2\}.$$

There exists a C^0 neighborhood $\mathcal{V}(f)$ of f such that for g in $\mathcal{V}(f)$

$$\bigcap_{j=0}^{N} g^{-j}(U) \subset \{q : \, d(q, \Lambda_f) < \varepsilon/2\}$$

and for any $x \in \bigcap_{j=0}^{N} g^{-j}(U)$, we may consider $\{x_n\}_{n\in\mathbb{Z}}$ a full orbit for g, where $x_0 = x$, getting that $\{x_n\}_n$ is an $\varepsilon-$pseudo orbit for f.

Let $\Lambda_g = \bigcap_{n\geq 0} g^{-n}(U)$. Taking an open subset $\mathcal{V}_1(f)$ of $\mathcal{V}(f)$ small enough in the C^1 topology, then for $g \in \mathcal{V}_1(f)$, Λ_g is an expanding locally maximal set for g. If g is close enough to f, then g is also expansive with constant β. Moreover, the Shadowing Lemma also holds for g.

Take $g \in \mathcal{V}_1(f)$. Given $x \in \Lambda_g$, consider $\{x_n\}_{n\in\mathbb{Z}}$ a full orbit for g, where $x_0 = x$. As $\{x_n\}_n$ is an $\varepsilon-$pseudo orbit for f, there exists a unique full orbit $\{y_n\}_{n\in\mathbb{Z}}$ for f with $y_0 = y \in \Lambda_f$ that $\eta-$shadows $\{x_n\}_{n\in\mathbb{Z}}$.

Let us define $h_g : \Lambda_g \to \Lambda_f$ by $h_g(x) = y$, where y is given by the Shadowing Lemma. By the uniqueness of the shadowing point, this map is well defined. The continuity of h_g follows from the shadowing lemma.

Moreover, $h_g \circ g = f \circ h_g$. In fact, consider the sequence $\{z_n\}_{n\in\mathbb{Z}}$ where $z_n = g(x_n) = x_{n+1}$. This $\varepsilon-$pseudo-orbit is $\eta-$shadowed by a unique full orbit $\{w_n\}_{n\in\mathbb{Z}}$ for f, with $w_0 = w \in \Lambda_f$. Then, for every $n \in \mathbb{Z}$,

$$\begin{aligned} d(w_n, z_n) &= d(f^n(w_0), x_{n+1}) = d(f^n(h_g(z_0)), g^n(g(x_0))) \\ &= d(f^n(h_g \circ g(x_0)), g^n(g(x_0))) \\ &= d(f^{n+1} \circ f^{-1} \circ h_g \circ g(x_0), g^{n+1}(x_0)) < \eta \end{aligned}$$

Observe that $f^{-1} \circ h_g \circ g(x_0) = w_{-1}$ is $\eta-$shadowing x_0. So, by uniqueness, we have that $f^{-1} \circ h_g \circ g(x_0) = y_0$; i.e. $h_g \circ g(x) = f \circ h_g(x)$.

Since we can apply the Shadowing Lemma for Λ_g using the same constants as in the construction of h_g, we define a map $h_f : \Lambda_f \to \Lambda_g$ such that $h_f \circ f = g \circ h_f$. In fact, if $\{y_n\}_{n\in\mathbb{Z}}$ is a full orbit for f with $y_0 \in \Lambda_f$, then it is an $\varepsilon-$pseudo orbit for g. Hence, this pseudo orbit is uniquely shadowed by a full orbit $\{x_n\}_{n\in\mathbb{Z}}$ for g, with $x_0 \in \Lambda_g$. Thus, $h_f(y_0) = x_0$ and $d(y_n, x_n) < \eta$ for every $n \in \mathbb{Z}$; moreover, h_f is continuous and satisfies $h_f \circ f = g \circ h_f$ just as h_g.

Next, let us verify that h_g is one to one. Let $p_1,\ p_2 \in \Lambda_g$ be two points such that $h_g(p_1) = h_g(p_2)$. Note that $d(f^n(h_g(p_1)), g^n(p_1)) < \eta$ and $d(f^n(h_g(p_2)), g^n(p_2)) < \eta$ by construction. Then $h_g(p_1)$ is $\eta-$shadowed by p_1 and p_2, which by uniqueness gives that $p_1 = p_2$.

Finally, for $y \in \Lambda_f$, consider a full orbit of $h_f(y)$ by g. Since $d(g^n(h_f(y)), f^n(y))$ is small for all n and some f full orbit of y shadows the g full orbit of $h_f(y)$, we have that $h_g(h_f(y)) = y$. Hence, h_g is onto and therefore is a homeomorphism. ∎

The next proposition is a version for expanding endomorphisms of a result already provided for the case of hyperbolic diffeomorphisms in [Rob76, Theorem 4.1]. The goal is to show that we can extend the conjugation between $f|_{\Lambda_f}$ and $g|_{\Lambda_g}$ to an open neighborhood U of Λ_f in such a way that still is a homeomorphism that conjugate $f|_U$ and $g|_U$, noting that the conjugacy is unique just in Λ_f. We are going to use this extension in next section for proving that the property of Λ_f disconnecting a "nice" class of arcs is robust.

Proposition 2. *The homeomorphism $h_f : \Lambda_f \to \Lambda_g$ in proposition (1) can be extended as a homeomorphism H to an open neighborhood of Λ_f such that $H \circ f = g \circ H$.*

Proof. This proof is inspired in the geometrical approach given by Palis in [Pal68] and also used to prove the Grobman-Hartman Theorem in [Shu87, pp.96].

Other alternative proof consist in using inverse limit space, in such a way that the expanding set Λ_f becomes a hyperbolic set for a diffeomorphism and so Theorem 4.1 in [Rob76] can be applied. Observe, that to have a well defined inverse limit and that the induced set associated to Λ_f verifies the hypothesis of the mentioned theorem in [Rob76] is needed that Λ_f is locally maximal invariant.

The goal is to choose an appropriate isolating neighborhood U of Λ_f and to construct an homeomorphism from U onto itself, using the inverse branches of f and g, first defined in a fundamental domain D_f for f (i.e. a set D_f such that for every $x \in U \setminus \Lambda_f$, there exists $n \in \mathbb{N}$ such that $f^n(x) \in D_f$) and then extended to U using inverse iterates. Observe that the isolating block of Λ_f is also an isolating block of Λ_g. Now we can take a fundamental domain for g, D_g, as it was done for f. Note that D_f is defined as $U \setminus f^{-1}(U)$ and since $f^{-1}(U)$ is properly contained in U, it follows that the same holds for g and therefore D_f and D_g are homeomorphic. Then it is taken an homeomorphism H between both fundamental domains D_f and D_g. This homeomorphism is saturated to $U \setminus \Lambda_f$ by backward iteration, i.e. if $x \in U \setminus \Lambda_f$, let n be such that $f^n(x) \in D_f$, take $H \circ f^n(x)$ and then $g^{-n} \circ H \circ f^n(x)$ where g^{-n} is taken carefully using the corresponding inverse branches.

Denote by N_f the cardinal of $\{w \in f^{-1}(x)\}$. Since f is a local diffeomorphism, N_f is constant. Let $K \subset \{1, \ldots, N_f\}$ be such that for every $i \in K$, there exist $U_i^f \subset U$ and $\varphi_i^f : U \to U_i^f$ inverse branch of f such that $\varphi_i^f(U) = U_i^f$ and $f(U_i^f) = f \circ \varphi_i^f(U) = U$. Also, for g as in proposition 1, for every $i \in K$, there exist $U_i^g \subset U$ and $\varphi_i^g : U \to U_i^g$ the inverse branch of g such that $\varphi_i^g(U) = U_i^g$ and $g(U_i^g) = g \circ \varphi_i^g(U) = U$.

To construct an homeomorphism H on U satisfying $H \circ f = g \circ H$ and $H|_{\Lambda_f} = h_f$ we can begin as follows. Suppose that the restriction $H : \partial U \to \partial U$ is any well-defined orientation preserving diffeomorphism. The restriction of H to ∂U_i^f is then defined as follows $H(x) = \varphi_i^g \circ H \circ f(x)$ if $x \in \partial U_i^f$ because H conjugate f and g. Now we extend H to a diffeomorphism which send $U \setminus \bigcup_{i \in K} U_i^f$ bounded by ∂U and ∂U_i^f onto $U \setminus \bigcup_{i \in K} U_i^g$ bounded by ∂U and ∂U_i^g. Since we may assume that the Hausdorff distance between U and Λ_f is small, see lemma 2, then the initial H is close to the identity. Let us say that $d(H(x), x) < \eta$, where $\eta > 0$ is given arbitrarily.

Given $i,j\in K$, denote $U^f_{j,i}=\varphi^f_j\circ\varphi^f_i(U)$ and $U^f_{2\ i}=U^f_i\setminus\bigcup_{j\in K}U^f_{j,i}$. If $x\in\partial U^f_{j,i}$ then $H(x)=\varphi^g_j\circ\varphi^g_i\circ H\circ f^2(x)\in\partial U^g_{j,i}$. If $x\in U^f_{2\ i}\setminus\Lambda_f$ then $H(x)=\varphi^g_i\circ H\circ f(x)\in U^g_{2\ i}$.

Doing this process inductively we have that: Given $i_1,\dots,i_n\in K$, denote $U^f_{i_n,\dots,i_1}=\varphi^f_{i_n}\circ\cdots\circ\varphi^f_{i_1}(U)$ and $U^f_{n\ (i_{n-1},\dots,i_1)}=U^f_{i_{n-1},\dots,i_1}\setminus\bigcup_{i_n\in K}U^f_{i_n,\dots,i_1}$. If $x\in\partial U^f_{i_n,\dots,i_1}$ then $H(x)=\varphi^g_{i_n}\circ\cdots\circ\varphi^g_{i_1}\circ H\circ f^n(x)$. If $x\in U^f_{n\ (i_{n-1},\dots,i_1)}\setminus\Lambda_f$ then $H(x)=\varphi^g_{i_{n-1}}\circ\cdots\circ\varphi^g_{i_1}\circ H\circ f^{n-1}(x)$. And $H(x)=h_f(x)$ if $x\in\Lambda_f$.

Let us prove that H is continuous.

Given $x\in\Lambda_f$, let $(x_n)_n$ be a sequence in $U\setminus\Lambda_f$ such that $x_n\to x$, when $n\to\infty$. Let us prove that $H(x_n)\to H(x)$, when $n\to\infty$.

First, consider $\{z_k\}_{k\in\mathbb{Z}}$ an $f-$full orbit in Λ_f such that $z_0=x$ and for every $n\in\mathbb{N}$, consider $\{z^n_k\}_{k\in\mathbb{Z}}$ a full orbit by f associated to each x_n using the corresponding inverse branches (for the backward iterates) given by the full orbit of x, where $z^n_0=x_n$. Since f is continuous, for every $k\in\mathbb{Z}$, we have that $z^n_k\to z_k$ when $n\to\infty$.

Note that for every $n\in\mathbb{N}$, there exists $k_n>0$ such that $z^n_{k_n}\in U\setminus\bigcup_{i\in K}U^f_i$. Furthermore, $z^n_k\in U$ for every $k\in[-k_n,k_n]$. Since $H\circ f=g\circ H$, we get that $H(x_n)\in\bigcap_{k=-k_n}^{k_n}g^k(U)$.

Hence, for η and ε as in proposition 1 and for every $n\in\mathbb{N}$, we have that $\{z^n_k\}_{k=-k_n}^{k_n}$ is a finite $\varepsilon-$pseudo orbit for g and it is $\eta-$shadowed by a $g-$orbit of $H(x_n)$ until k_n for forward iterates and $-k_n$ for backward iterates.

Observe that as m goes to infinity, the finite pseudo orbit $y^m_n=\{z^n_k\}_{k=-m}^m$ becomes longer. Consider now the sequence $\{y^m_n\}_n$. Then $y^m_n\to\{z_k\}_{k=-m}^m$ when $n\to\infty$. Hence, the sets of shadowing points of the finite pseudo orbits $y^{k_n}_n$ converge to the shadowing point of the infinite pseudo orbit $\{z_k\}_k$, then $H(x_n)\to h_f(x)=H(x)$ when $n\to\infty$. ∎

2.4. **The Locally Maximal Set "Separates".**

The main goal of this section is to show that the locally maximal set for f has a topological property that persists under perturbation, roughly speaking means that Λ_f and Λ_g disconnect small open sets. Using that we prove that Λ_f intersects "some nice" class of arcs in U_1^c and which also intersect Λ_g for all g nearby f. The first question that arise is: which arcs belong to this "nice" class? The second questions in the context of proving the Main Theorem is: why is this property enough? The third question is: why does the "nice class" of arc exist? All these questions are answered along the section, but to give some brief insight about the main ideas we want to make some comments:

1. Roughly speaking, these "nice arcs" are arcs that have the property that can be used to build "nice cylinders" (see definition 2.7) containing the initial arc and such that Λ_f "separates" (see definition 2.9) this cylinder in a "robust way" (see lemmas 4 and 5).
2. It is enough to consider these "nice" class of arcs to finish the proof of the Main Theorem. Suppose that the existence of this class of arcs is proved and suppose that given any open set, there is an iterate by g that contains a "nice" arc (see claim 2.1 and lemma 6). Then there is a point in this iterate which its forward orbits stay in the expanding region and arguing as in the

beginning of subsection 2.1 it is concluded the density of the pre-orbit of any point for the perturbed map.

(3) Therefore, to finish, we show in claim (2.1) that every large arc admits a "nice" arc. Later it is shown that any open set has an iterate, in the universal covering, with an arbitrary large arc (see lemma 6).

Let us define the concepts involved in this section.

Definition 2.5. *(**Cylinder**) Given a differentiable arc γ and $r > 0$, it is said that $C(\gamma, r)$ is a* cylinder *centered at γ with radius r if*

$$C(\gamma, r) := \bigcup_{x \in \gamma} ([T_x\gamma]^{\perp})_r,$$

where $([T_x\gamma]^{\perp})_r$ denotes the closed ball centered at x with radius r intersected with $[T_x\gamma]^{\perp}$ the orthogonal to the tangent to γ in x.

Definition 2.6. *(**Simply connected cylinder**) Given a differentiable arc γ and $r > 0$, it is said that a cylinder $C(\gamma, r)$ is* simply connected *if it is retractile to a point.*

Remark 7. *Fixed the radius, a cylinder as defined in 2.5 could be not retractile to a point. In this case, working in the universal covering space, consider the convex hull of its lift and then project it on the manifold. We call the resulting set as* simply connected cylinder *as well and is denoted in the same way as above.*

Definition 2.7. *(**Nice cylinder**) Given an arc γ and $r > 0$, it is said that a cylinder $C(\gamma, r)$ is a* nice cylinder *if it is simply connected cylinder and if x_A and x_B are the extremal points of γ then $A := ([T_{x_A}\gamma]^{\perp})_r \subset \partial C(\gamma, r)$ and $B := ([T_{x_B}\gamma]^{\perp})_r \subset \partial C(\gamma, r)$. In this case, we say that A and B are the* top and bottom sides *of the cylinder. (See figure 2).*

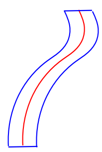


FIGURE 2. *Nice cylinder*

Remark 8. *In general, given any cylinder, as defined in 2.5, it has not necessarily top and bottom sides, and may not be simply connected.*

Hereafter, fix U_2 an open set such that $\overline{U_1} \subset U_2$, where U_1 is the same as in hypothesis (2) in the Main Theorem, and $\delta_0 < diam_{int}(U_2^c) < diam_{int}(U_0^c)$. Let $d_1 = d_H(U_2, U_1) > 0$, where d_H denotes the Hausdorff metric, and let $k \in \mathbb{N}$ such that $\delta_0' = \delta_0 + \frac{d_1}{3k} < diam_{int}(U_2^c)$.

Let us denote by $\widetilde{U}_0$ the lift of U_0, π the projection of $\mathbb{R}^n$ onto $\mathbb{T}^n$ and $\mathfrak{U}_0$ the convex hull of $\widetilde{U}_0 \cap [0,1]^n$. Consider $\mathrm{P}_i(\mathfrak{U}_0)$ the projection of $\mathfrak{U}_0$ in the $i-$th coordinate in the $n-$dimensional cube $[0,1]^n$. Since $diam(U_0) < 1$ and remark (1), for every $1 \leq i \leq n$, there exist $0 < k_i^- < k_i^+ < 1$ such that $k_i^- < \mathrm{P}_i(\mathfrak{U}_0) < k_i^+$. Note that $1 + k_i^- - k_i^+ > \delta_0'$ for every i, because $1 + k_i^- - k_i^+ > diam_{int}(U_0^c)$ by construction.

Let $R_i^m = \{x \in \mathbb{R}^n : k_i^- + m < x_i < k_i^+ + m\}$ with $m \in \mathbb{Z}$, $1 \leq i \leq n$ and x_i is the $i-$th coordinate of x. Thus, $\mathfrak{U}_0 \subset \bigcap_{m \in \mathbb{Z}, 1 \leq i \leq n} R_i^m$. Denote by $L_i^+ = \{x \in \mathbb{R}^n : x_i = k_i^+\}$ and $L_i^- = \{x \in \mathbb{R}^n : x_i = k_i^-\}$. Let $\tilde{f}$ be the lift of f.

The next claim answer the third question stated at the beginning of the section.

Claim 2.1. *Let $m > 2\sqrt{n}$ be fixed. Given any arc γ in $\mathbb{R}^n$ with $diam(\gamma) > m$, there exist an arc $\gamma' \subset \gamma$, $1 \leq i \leq n$ and $j \in \mathbb{Z}$ such that $\partial\gamma' \cap (L_i^+ + j) \neq \emptyset$, $\partial\gamma' \cap (L_i^- + j + 1) \neq \emptyset$ and $P_i^j(\gamma') \subset [k_i^+ + j, k_i^- + j + 1]$, where $P_i^j(\gamma')$ denotes the projection of γ' on the interval $[j, j+2]$ of the $i-$th coordinate. Moreover, γ' admits a nice cylinder $\gamma^* = \pi(\gamma')$ in U_2^c, with diameter of γ^* larger than δ_0 and also admitting a nice cylinder contained in U_1^c. (See figure 3)*

Proof. Take γ an arc with diameter larger than m, then the projection of γ in the $i-$th coordinate contains an interval of the kind formed by k_i^+ and $1 + k_i^-$ for some i (or formed by $k_i^+ + j$ and $k_i^- + j + 1$ for some $j \in \mathbb{Z}$). If it is not true, γ would be in a $n-$dimensional cube with sides smaller than $k_i^+ - k_i^- < 1$ and this cube has diameter smaller than $\sqrt{n}$, but this contradict the fact that $diam(\gamma) > m$. Hence, we may pick an arc γ' in γ such that $\partial\gamma' \cap (L_i^+ + j) \neq \emptyset$, $\partial\gamma' \cap (L_i^- + j + 1) \neq \emptyset$ and $P_i^j(\gamma') \subset [k_i^+ + j, k_i^- + j + 1]$ for some $1 \leq i \leq n$ and some $j \in \mathbb{Z}$. Therefore, diameter of γ' is greater than δ_0 and in consequence its projection in $\mathbb{T}^n$ also has diameter greater than δ_0.

Moreover, since the projection of γ' by P_i is in between $k_i^+ + j$ and $k_i^- + 1 + j$, we may construct a cylinder centered at γ' and radius $\frac{d_1}{2}$ such that this cylinder is "far" away from $\widetilde{U}_0$, so this cylinder could be simply connected or, if it is not simply connected cylinder, it has holes that are different from $\widetilde{U}_0$. In the case that the cylinder is not simply connected, we consider the convex hull of the cylinder, since the original cylinder is bounded by $L_i^+ + j$ and $L_i^- + j + 1$, then the convex hull stay in between these two hyperplanes and therefore it does not intersect $\widetilde{U}_0$. By abuse of notation, let us denote this set by $C(\gamma', \frac{d_1}{2})$, it is a simply connected cylinder. Observe that by construction, this cylinder will have top and bottom sides, thus $C(\gamma', \frac{d_1}{2})$ is a nice cylinder.

Take $\gamma^* = \pi(\gamma')$, note that γ' can be choose such that γ^* is contained in U_2^c and the diameter of γ^* is larger than δ_0, then projecting the nice cylinder for γ' in $\mathbb{T}^n$ we obtain a nice cylinder for γ^* which is denoted by $C(\gamma^*, \frac{d_1}{2})$. This nice cylinder has the property that every arc that goes from bottom to top side has diameter at least δ_0 and all this process can be made in such a way that the nice cylinder is in U_1^c. ∎

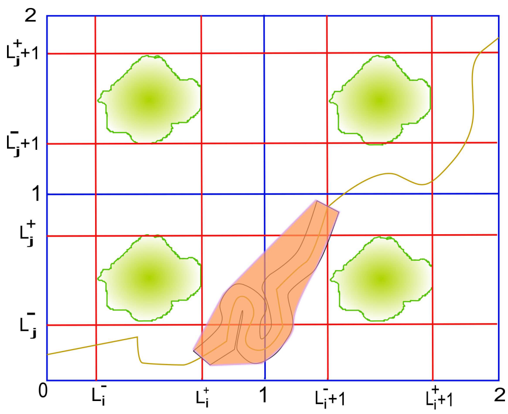


FIGURE 3. *Every arc admits a sub-arc with a nice cylinder*

Definition 2.8. *(**Lateral border**) Given a differentiable arc γ and $r > 0$. The* lateral border *S of the cylinder $C(\gamma, r)$ is $\partial C(\gamma, r)$ minus the top and bottom sides of the cylinder if they exist.*

Observe that nice cylinders have lateral borders.

Definition 2.9. *(**Separated horizontally**) We say that a nice cylinder $C(\gamma, r)$ is* separated horizontally *by a set Λ if there exists a connected component of Λ, let say Λ_c, such that:*

- *Λ_c intersects $C(\gamma, r)$ across the lateral border.*
- *$C(\gamma, r)$ minus Λ_c has at least two connected component.*

Now, we are going to prove that the locally maximal set for f, found in section 2.2, has the geometrical property of separating horizontally these nice cylinders as the one in claim 2.1.

Lemma 4. *Given any arc γ in U_2^c with diameter greater than δ_0 that admits a nice cylinder as in claim (2.1), Λ_f separates horizontally its nice cylinder.*

Proof. Let us denote by T the nice cylinder associated to γ as in the statement and let A and B denote the top and bottom sides of T respectively. Let $\varepsilon > 0$ be arbitrarily small.

Let T' be a bigger cylinder containing T joint together with two security regions, denote by S_A and S_B, and such that the distance between the lateral border of T and the lateral border of T' is small, for instance $d_H(T, T') = \frac{d_1}{6k}$, see figure (4). For security regions S_A and S_B we mean two strips of $\frac{d_1}{6k}$ thickness glued to the sides A and B of T, or in other words, S_A (respectively S_B) is the set of points in T^c such that the distance from these points to A (respectively B) is less or equal to $\frac{d_1}{6k}$. This set T' was constructed in such a way that its diameter is greater than δ_0'.

Since γ is in U_2^c and its diameter is greater than δ_0, we can assure that $T \cap \Lambda_f$ is non empty. Consider all the connected components of $T \cap \Lambda_f$. For every $x \in T \cap \Lambda_f$, we assign K_x the connected component of $T \cap \Lambda_f$ that contains x. Observe that we may define an equivalence relation: $x \sim x'$ if and only if $K_x = K_{x'}$. Then we pick one component from each class, or in other words we pick just the connected components that are two by two disjoints.

We claim that Λ_f separates T horizontally, i.e.; there exists one component K_x such that $K_x \cap \partial T \neq \emptyset$ and K_x separates T in more than one connected component.

Suppose it does not happen, i.e. none of the K_x separates T horizontally. Take U_x open set in T' such that $K_x \subset U_x$, $\partial U_x \cap \Lambda_f = \emptyset$, ∂U_x is connected and ∂U_x does not divides T horizontally. If there are many K_y accumulating in one K_x, then we could have a same open set U_x containing more than one connected component K_y.

Observe that the collection $\{U_x\}$ is an open cover of $T \cap \Lambda_f$. Since it is compact, there is a finite subcover $\{U_i\}_{i=1}^N$, i.e. $T \cap \Lambda_f \subset \mathcal{U} := \bigcup_{i=1}^N U_i$.

If the connected components of $\mathcal{U}$ does not separates horizontally T, it is easy to construct a curve going from A to B with diameter greater than δ_0 and empty intersection with the U_i's; hence, this curve does not intersects the set Λ_f. But this contradicts the fact that every curve in U_1^c with diameter larger than δ_0 intersects Λ_f. Then the connected components of $\mathcal{U}$ separate T horizontally, denote by C_j the connected components of T minus these connected components of $\mathcal{U}$ that separates T horizontally.

Observe that every C_j is path connected, since they are the complement of a finite union of open sets in a simply connected set T. There exist a finite quantity of C_j, let us say m. We can reorder these sets enumerating from the top side. If we denote by V_j each of the connected components of $T \cap \mathcal{U}$ that separate T horizontally, we have two cases, either C_j is in between two consecutive V_j and V_{j+1} (or V_{j-1} and V_j) or C_j just intersects one V_j on the border.

The idea is to build a curve from top to bottom of T connecting C_j with C_{j+1} in such a way that the diameter of the arc is greater than δ_0 but without intersecting Λ_f, which is a contradiction because it is again in U_1^c and has diameter greater than δ_0, then this curve must intersects Λ_f.

It is enough to show that we can pass from C_j to C_{j+1} without touching Λ_f. For this, we must observe that every V_j is a union of finitely many U_i, let us say $U_{i_1}, \ldots, U_{i_j}$. Pick a curve γ_j in C_j going from top to bottom, i.e. γ_j goes from ∂V_j to ∂V_{j+1} (or ∂V_{j-1} and ∂V_j) and γ_j does not intersects the interior of V_j and V_{j+1} (or V_{j-1} and V_j), then there exists $i_s \in \{i_1, \ldots, i_j\}$ such that $\gamma_j \cap \partial U_{i_s} \neq \emptyset$. After that continue this arc picking a curve following by the border of U_{i_s} until C_{j+1}, which has empty intersection with Λ_f by construction, if it is not possible to do in one step, pick another U_{i_k} and repeat the process. Note that this process finish in finitely many times. The resulting arc from joint together all this segment has diameter greater than δ_0 and with empty intersection with Λ_f as we wanted. ∎

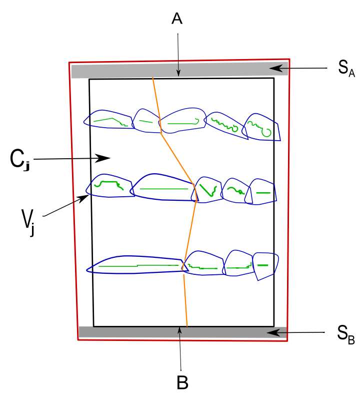


FIGURE 4. Λ_f *splits "horizontally" every nice cylinder in at least two connected component*

Remark 9. *In proposition 1, remembering that $d(h_g, id) < \eta$, we may fix $\eta < \min\{\frac{d_1}{6k}, \delta_0, \beta\}$. So for this η, there exists $\varepsilon_0 > 0$ given by the shadowing lemma, see lemma 3 and proposition 1, and ε_0 determine $\mathcal{V}_1(f)$ given in proposition 1.*

Lemma 5. *Given $g \in \mathcal{V}_1(f)$ and given an arc γ in U_2^c with diameter greater than δ_0' such that it admits a nice cylinder $C(\gamma, \frac{d_1}{2})$, then $\gamma \cap \Lambda_g$ is not empty.*

Proof. Let $g \in \mathcal{V}_1(f)$. Take an arc γ in U_2^c with diameter greater than δ_0' such that $C(\gamma, \frac{d_1}{2})$ is a nice cylinder.

By construction, we may assume that every arc taken in the nice cylinder that goes from top to bottom has diameter greater or equal to the diameter of γ. We take two security regions inside the cylinder, in the top and bottom sides of the cylinder respectively, with $\frac{d_1}{6k}$ of thickness each one, i.e. two strips glued to the top and bottom sides of the cylinder such that each one is the set of points in the cylinder within distance to top (respectively bottom) side less or equal to $\frac{d_1}{6k}$, see figure (5). Let us denote by C' the cylinder resulting of taking out these two security strips from the original cylinder $C(\gamma, \frac{d_1}{2})$, then the diameter of C' is still greater than δ_0.

Hence, the diameter of $\gamma' = \gamma \cap C'$ is greater than δ_0 and it is in U_1^c. Lemma 4 implies that Λ_f separates horizontally C', hence γ' intersects Λ_f, let us denote by x_f the point in the intersection.

Since $x_f \in \Lambda_f$, proposition 1 and remark (9), there exists $x_g \in \Lambda_g \cap \mathbb{B}_\eta(x_f)$. Note that Λ_f separates $\mathbb{B}_\eta(x_f)$ in at least two connected component. Because $f|_U$ and $g|_U$ are conjugate, follows that Λ_g separates $\mathbb{B}_\eta(x_f)$ in at least two connected component as well. Therefore, Λ_g must intersects γ. ∎

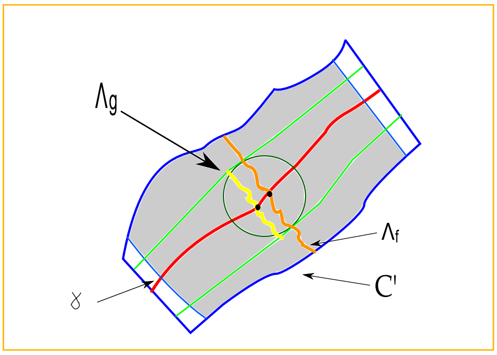


FIGURE 5. Λ_g *intersects* γ

2.5. **Getting Arcs of Large Diameter.** In this subsection we show that under the hypothesis of volume expanding, the diameter of the iterates of an open set grows on the covering.

Lemma 6. *For every $g \in \mathcal{V}_1(f)$ and given V an open path connected set in $\mathbb{T}^n$, there exists $m_0 = m_0(V, g) \in \mathbb{N}$ such that $diam(\tilde{g}^{m_0}(\widetilde{V})) > m$, where $\tilde{g}$ and $\widetilde{V}$ are the lift of g and V, respectively, and m was given in claim 2.1. In particular, it contains an arc with diameter greater than m.*

Proof. Let us suppose that the Lemma is false in the covering space. Let $g \in \mathcal{V}_1(f)$ and V be an open path connected set in $\mathbb{T}^n$. If there exists $k_0 > 0$ such that $d_k = diam(\tilde{g}^k(\widetilde{V})) < k_0$, then $vol(\tilde{g}^k(\widetilde{V})) \leq \left(\frac{d_k}{2}\right)^n$. But since g is volume expanding, there exists constant $\lambda > 1$ such that $vol(\tilde{g}^k(\widetilde{V})) > \lambda^k vol(\widetilde{V})$, for $k \geq 1$. Iterating by $\tilde{g}$, and since $\tilde{g}$ is a diffeomorphisms in the covering space, the volume increase and furthermore the diameter of its iterates grows in the covering space. Hence, there exists $m_0 \in \mathbb{N}$ such that $diam(\tilde{g}^{m_0}(\widetilde{V})) > m$. ∎

Remark 10. *For the case that V is an open connected set, observe that given a point in V there exists an open ball centered in this point and contained in V such that it is path connected. Then we may apply Lemma 6 to this ball and obtain a similar statement for V.*

2.6. **Getting Sets of Large Radius.** In this subsection, we show that open sets intersecting Λ_g, for g close enough to f, has large internal radius after large iterates.

Lemma 7. *There exist $\mathcal{V}_2(f)$ and $R > 0$ such that for every $g \in \mathcal{V}_2(f)$, if there is $x \in M$ such that $g^n(x) \notin U_0$ for every $n \geq 0$, then there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\mathbb{B}_R(g^N(x)) \subset g^N(\mathbb{B}_\varepsilon(x))$.*

Proof. We may pick U_3 an open subset contained in U_0 such that $m\{Df\mid_{U_3^c}\} > \lambda'$, with $1 < \lambda' < \lambda_0$. Take $\mathcal{V}_2(f)$ an open subset perhaps smaller than $\mathcal{V}_1(f)$ such that $m\{Dg\mid_{U_3^c}\} > \lambda'$ holds for every $g \in \mathcal{V}_2(f)$. Let us fix $R = d_H(U_0, U_3) > 0$.

Given $0 < \varepsilon < R$, take $N \in \mathbb{N}$ such that $(\lambda')^{-N}R < \varepsilon/2$. Then $\mathbb{B}_{(\lambda')^{-N}R}(x) \subset \mathbb{B}_\varepsilon(x)$.

Observe that $\mathbb{B}_R(g^n(x)) \cap U_3 = \emptyset$, for every $n \geq 0$. Also,

$$g^k(\mathbb{B}_{(\lambda')^{-N}R}(x)) = \mathbb{B}_{(\lambda')^{-N+k}R}(g^k(x)) \subset \mathbb{B}_R(g^k(x)),$$

for every $0 \leq k \leq N$. In particular, $g^k(\mathbb{B}_{(\lambda')^{-N}R}(x)) \cap U_3 = \emptyset$, for every $0 \leq k \leq N$. Then

$$g^N(\mathbb{B}_{(\lambda')^{-N}R}(x)) = \mathbb{B}_R(g^N(x)) \subset g^N(\mathbb{B}_\varepsilon(x)).$$

∎

Remark 11. *Let us note that lemma 7 holds for every point in Λ_g.*

2.7. **End of the Proof of Main Theorem.** Let $f \in E^1(\mathbb{T}^n)$ satisfying the hypotheses of the Main Theorem. Lemma 2 implies that we may assume the existence of Λ_f an expanding locally maximal set for f.

Fix $0 < \alpha < R$, arbitrarily small. Given $x \in \mathbb{T}^n$, since $\{w \in f^{-i}(x) : i \in \mathbb{N}\}$ is dense, there exists $n_0 \in \mathbb{N}$ such that

$$\{w \in f^{-i}(x) : 0 \leq i \leq n_0\} \text{ is } \alpha/2\text{-dense.}$$

Take a neighborhood $\mathcal{U}(f) \subset \mathcal{V}_2(f)$, where $\mathcal{V}_2(f)$ was given in lemma 7, such that for every $g \in \mathcal{U}(f)$ follows that

$$\{w \in g^{-i}(x) : 0 \leq i \leq n_0\} \text{ is } \alpha/2\text{-close to } \{w \in f^{-i}(x) : 0 \leq i \leq n_0\}.$$

Hence, $\{w \in g^{-i}(x) : 0 \leq i \leq n_0\}$ is α-dense.

Let V be an open connected set in $\mathbb{T}^n$. By lemma 6, there exists $m_0 \in \mathbb{N}$ such that $diam(\tilde{g}^{m_0}(\widetilde{V})) > m$. Then we may pick an arc γ in $\tilde{g}^{m_0}(\widetilde{V})$ with diameter larger than m and applying claim (2.1) follows that there exists a connected piece γ' of γ such that $\gamma^* = \pi(\gamma')$ is in U_2^c, diameter of γ^* is larger than δ_0' and it admits a nice cylinder $C(\gamma^*, \frac{d_1}{2})$. By lemma 5 follows that $\gamma^* \cap \Lambda_g$ is not empty, let y be a point in the intersection.

Hence, for this point y, there exists $\varepsilon_0 = \varepsilon_0(y) > 0$ such that $\mathbb{B}_{\varepsilon_0}(y) \subset g^{m_0}(V)$, by lemma 7 taking $0 < \varepsilon < \varepsilon_0$, we get that there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\mathbb{B}_R(g^N(y)) \subset g^N(\mathbb{B}_\varepsilon(y)) \subset g^{m_0+N}(V).$$

Hence, $\mathbb{B}_\alpha(g^N(y)) \subset g^{m_0+N}(V)$. Since the $\alpha-$density, we have that

$$\{w \in g^{-i}(x) : 0 \leq i \leq n_0\} \cap \mathbb{B}_\alpha(g^N(y)) \neq \emptyset.$$

Therefore, denoting by $p = m_0 + N$,

$$\{w \in g^{-i}(x) : 0 \leq i \leq n_0\} \cap g^p(V) \neq \emptyset.$$

Taking the $p-$th pre-image by g, we obtain that there is $i_0 \in \mathbb{N}$ such that

$$\{w \in g^{-i}(x) : 0 \leq i \leq i_0\} \cap V \neq \emptyset.$$

Thus, for every $g \in \mathcal{U}(f)$ follows that $\{w \in g^{-i}(x) : i \in \mathbb{N}\}$ is dense in $\mathbb{T}^n$ for every $x \in \mathbb{T}^n$. ∎

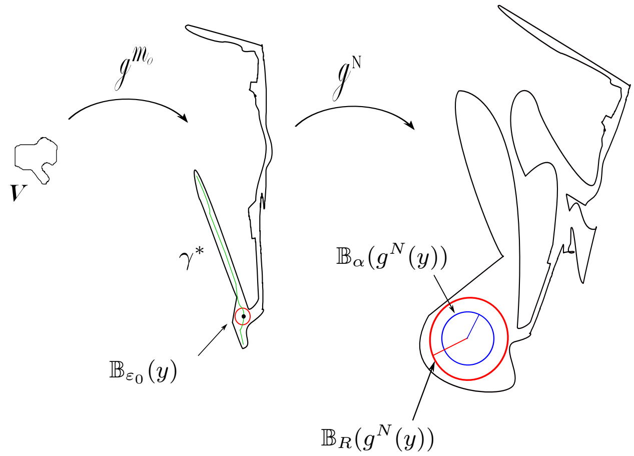


Figure 6. *Iterations by the perturbed map*

2.8. **The Main Theorem Revisited.** In this section, we state a general geometrical version of the Main Theorem. Observe that using hypotheses (2) and (3) of the Main Theorem, we showed in sections 2.2 and 2.4 the existence of a locally maximal expanding set for f which separates large nice cylinders, and in section 2.3 we proved that this geometrical property persist under perturbation, i.e. there is a set Λ_f locally maximal which intersects a nice class of arcs in U_0^c and this property also holds for the perturbed. The hypothesis of f being volume expanding guarantees that given any open set in the covering space, we are able to choose some iterates such that it contains an arc with diameter large enough to apply claim 2.1 and lemma 4. Hence, the Main Theorem may be enunciated as follows:

Main Theorem Revisited *Let $f \in E^1(\mathbb{T}^n)$ be volume expanding such that the pre-orbit of every point are dense. Suppose that there exist an open set U_0 with diam$(U_0) < 1$ and Λ_f a locally maximal expanding set for f in U_0^c such that every arc γ in U_0^c with diameter large enough intersect Λ_f. Then, the pre-orbit of every point are C^1 robustly dense.*

Observe that in the present version, it is already assumed the existences of an expanding locally maximal invariant set that intersects large enough arcs. The proof goes showing that the separation property is robust and this is done in the same way that is done in the Main Theorem.

Remark 12. *Note that the Main Theorem implies the Main Theorem Revisited, but we do not know if the reciprocal is true.*

3. Robust transitive endomorphisms with invariant splitting

Now, we consider the case that the endomorphisms exhibits a type of partially hyperbolic splitting. First we give the definition of partially hyperbolic endomorphisms which is slightly different than the one for diffeomorphisms due to the fact that any point has different pre-images which implies that the unstables subbundles are not unique (actually, they depend on the inverse branches).

Definition 3.1. ***(Unstable cone family)*** *Given $f : M \to M$ a local diffeomorphism, let V be an open subset of M such that $f|_V$ is a diffeomorphism onto its image. Denote by φ the inverse branches of f restricted to V; more precisely, $\varphi : f(V) \to V$ such that $f \circ \varphi(x) = x$ if $x \in f(V)$. A continuous cone field $\mathcal{C}^u = \{\mathcal{C}^u_x\}_x$ defined on V is called* unstable *if it is forward invariant:*

$$Df(x')\,\mathcal{C}^u_{x'} \subset \mathcal{C}^u_{f(x')}$$

for every $x' \in V \cap \varphi(V)$.

Remark 13. *Given a point x, there is not necessarily a unique unstable subbundle, i.e. for each inverse path $\{x_k\}_{k\geq 0}$, it means $x_0 = x$ and $f(x_{k+1}) = x_k$ for $k \geq 0$, there exists an unstable direction belonging to $\mathcal{C}^u$.*

Definition 3.2. ***(Complementary splitting)*** *We say that a splitting $\mathbb{E}^c_x + \mathcal{C}^u_x$ is* complementary *if the unstable cone $\mathcal{C}^u_x$ contains an invariant subspace whose dimension is equal to the dimension of the manifold minus the dimension of the central subbundle.*

Definition 3.3. ***(Partially hyperbolic endomorphism with expanding extremal direction)*** *It is said that an endomorphism f is* partially hyperbolic with expanding extremal direction *provided for every $x \in M$ there exists a complementary splitting $\mathbb{E}^c_x + \mathcal{C}^u_x$, where $\{\mathcal{C}^u_x\}_x$ is a family of unstable cones, and there exists $0 < \lambda < 1$ such that for every inverse branches φ of f follows that*

1. *$\|D\varphi(x)\,v\| < \lambda$, for all $v \in \mathcal{C}^u_x$.*
2. *$\|Df(x')\,|_{\mathbb{E}^c(x')}\|\|D\varphi(x)v\| < \lambda$, for all $v \in \mathcal{C}^u_x$, where $\varphi(x) = x'$, $f(x') = x$.*

3.1. Theorem 1: Splitting Version.

Now, we state a version of the Main Theorem for the case when the tangent bundle splits into two non-trivial subbundles, one with an expanding behavior and the other one with nonuniform behavior but dominated by the expanding one.

Theorem 1. *Let $f \in E^1(\mathbb{T}^n)$ be a locally diffeomorphism partially hyperbolic with expanding extremal direction satisfying the following properties:*

1. *$\{w \in f^{-k}(x) : k \in \mathbb{N}\}$ is dense for every $x \in \mathbb{T}^n$.*
2. *There exist $\delta_0 > 0$, $\lambda_0 > 1$ and $k_0 \in \mathbb{N}$ such that for every $x \in \mathbb{T}^n$, if γ is a disc tangent to the unstable cone $\mathcal{C}^u_x$ with internal diameter larger than δ_0, there exists a point $y \in \gamma$ such that $m\{Df^i\,|_{\mathbb{E}^c(f^k(y))}\} > \lambda_0^i$, for all $i > 0$, for all $k > k_0$.*

Then, for every g close enough to f, $\{w \in g^{-k}(x) : k \in \mathbb{N}\}$ is dense for every $x \in \mathbb{T}^n$.

3.2. **Proof of Theorem 1.** The proof of Theorem 1 is similar to the proof given in [PS06b], where it is proved that any partially hyperbolic diffeomorphism satisfying a hypothesis like the one stated in Theorem 1 and such that the strong stable foliation is minimal, then the strong stable foliation is robustly minimal. The key hypothesis in the statement of the main theorem in [PS06b] says that in any compact piece of the unstable foliation, there exists a point such that the central bundle has uniform expanding behavior along the forward orbit, and this is exactly what we have. The goal consists in proving that this property is robust under perturbation.

Given a local diffeomorphism f as in the statement of Theorem 1, we want to show that any small perturbation g preserve the property of density of the pre-orbit of any point. Our strategy is to prove that given any disc tangent to the unstable cones for g with large enough internal diameter has a point such that the central direction along the forward orbit by g is uniformly expanding.

Observe that given any open set, since we have a direction that is indeed expanding, the diameter along the unstable direction of the iterates growth. Then we are able to pick a disc inside this iterate such that the disc is tangent to the unstable cones with diameter large enough to apply the last property. Hence, there exists a point which its forward orbit is expanding in all direction, then there is some iterate such that it contains a ball of a fix radius ε.

Since g is close enough to f, we have that the pre-orbit by g are $\varepsilon-$dense. Therefore, given any open set, by the property of the unstable discs, there exists an iterate such that it intersects the pre-orbit by g of any point. Thus, we conclude the density of the pre-orbit of any point by the perturbation.

Moreover, the proof of Theorem 1 can also be performed in the spirit of Main Theorem. In fact, it is possible to show that

$$\bigcap_{l\geq 0} f^{-l}(\{x : m\{Df^n \mid_{\mathbb{E}^c(f^k(x))}\} > \lambda_0^n,\ n > 0,\ k > k_0\})$$

is an invariant expanding set such that separates unstable discs. This provides a geometrical interpretation. ∎

3.3. **Remarks About the Main Theorem and Theorem 1.** Observe that in the Main Theorem we asked for large arcs to contain points such that its forward iterations remain in the expanding region. The same is required in Theorem 1 but just for large unstable discs: there is a point there that its forward iterates remain in “an expanding region”for the central bundle.

The main difference in their proof arise from the fact that in the version with splitting, since we know that we have uniform expansion in one direction, any disc with internal diameter larger than δ_0 and tangent to this direction, growths up to length $\delta_1 > \delta_0$ in a bounded uniform time, independently of the disc. Note that we cannot guarantee that only assuming volume expansion.

Observe that in the Main Theorem is not assumed that f does not have any splitting. In fact, it could also be partially hyperbolic. However, knowing in advance that the endomorphism is partially hyperbolic then it is possible to get sufficient conditions for robust transitivity weaker than the one required by the Main Theorem.

4. C^1 Robust transitivity and volume expansion

Before showing the relation between C^1 robust transitivity and volume expansion (Theorem 2), let us recall some definitions that are involved in the statement

Definition 4.1. *The set $\Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U})$ is C^r* robustly transitive *if $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(\overline{U})$ is transitive for every endomorphism g C^r close enough to f, where U is an open set. It is said that a map f is C^r* robustly transitive *if there exists a C^r neighborhood $\mathcal{U}(f)$ such that every $g \in \mathcal{U}(f)$ is transitive.*

Definition 4.2. *We say that f restricted to an invariant set Λ has* no dominated splitting in a C^r robust way *if there exists a C^r open neighborhood $\mathcal{U}(f)$ of f such that for every $g \in \mathcal{U}(f)$ the tangent space $T\Lambda$ does not admit any dominated splitting.*

Theorem 2. *Let $f \in E^1(M)$ be a local diffeomorphism and U open set in M such that $\Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U})$ is C^1 robustly transitive set and it has no splitting in a C^1 robust way. Then f is volume expanding.*

Proof. The proof of this theorem is similar to the one of Theorem 4 in [BDP03, pp.361], nevertheless we include the main steps of the proof.

Let us consider $f \in E^1(M)$ a local diffeomorphism and denote by $\Lambda_f(U)$ (non-trivial) C^1 robustly transitive set for f (note that U could be the entire manifold). The idea of the proof is to assume that f is not volume expanding and show that for every $\mathcal{U}(f)$ C^1 neighborhood of f in $E^1(M)$, there exists $\psi \in \mathcal{U}(f)$ such that ψ has a sink and therefore ψ cannot be transitive.

Suppose that f is not volume expanding. Since f is onto, it cannot be uniform volume contracting in the entire manifold, so there are points in the manifold such that we have expansion, i.e. $1 \leq |det(Df^k(x))|$ for some $k \geq 0$, but it does not expand too much, i.e. $|det(Df^k(x))| \leq 1+\epsilon$, with ϵ small. Then there are sequences $x_n \in \Lambda_f(U)$, $k_n \in \mathbb{N}$ and $\tau_n > 1$, with $k_n \to \infty$ and $\tau_n \to 1^+$, such that

$$1 \leq |det(Df^{k_n}(x_n))| < \tau_n^{k_n}.$$

This is equivalent to say that

$$\frac{1}{k_n}\sum_{i=0}^{k_n-1} \log(|det(Df(f^i(x_n)))|) < \log(\tau_n).$$

We may take k_n such that $f^i(x_n) \neq f^j(x_n)$ for all $i \neq j$, $i, j \in \{0, \dots, k_n\}$. Consider for each n the Dirac measure δ_n supported in $\{x_n, f(x_n), \dots, f^{k_n}(x_n)\}$, i.e. $\delta_n = \frac{1}{k_n}\sum_{i=0}^{k_n-1} \delta_{f^i(x_n)}$. As the space of probabilities is compact with the weak star topology, there exists a subsequence of $\{\delta_n\}_n$ that converges to an invariant probability measure μ such that

$$\int \log|det(Df(x))| d\mu(x) \leq 0.$$

In fact, a classical argument proves that μ is invariant by f, since $f_*(\mu) - \mu$ is the weak star limit of $\frac{1}{k_{n_i}}(\delta_{f^{k_{n_i}}(x_{n_i})} - \delta_{x_{n_i}})$, which converge to zero. Observing that

$$\begin{aligned}\int \log|det(Df(x))| d\delta_n &= \tfrac{1}{k_n}\textstyle\sum_{i=0}^{k_n-1} \log(|det(Df^i(x_n))|)\\ &= \tfrac{1}{k_n}\log(|det(Df^{k_n}(x_n))|) \leq \log(\tau_n),\end{aligned}$$

and since $\tau_n \to 1^+$ we deduce that

$$\int \log |det(Df(x))| d\mu(x) \leq 0.$$

By the ergodic decomposition theorem, there is an ergodic and $f-$invariant measure ν such that

$$\int \log |det(Df(x))| d\nu(x) \leq 0.$$

Using the ergodic closing lemma for nonsingular endomorphisms (see [Cas09]), given $\varepsilon > 0$ there is g close to f and a $g-$periodic point y such that

$$\frac{1}{m_\varepsilon} \sum_{i=0}^{m_\varepsilon - 1} \log(|det(Dg(g^i(y)))|) < \varepsilon,$$

where m_ε is the period of y. Note that if $\varepsilon \to 0$, then $m_\varepsilon \to \infty$. So, taking $\varepsilon > 0$ arbitrarily small and m_ε big, using Franks' Lemma [Fra71] we get φ close to g such that $\varphi^{m_\varepsilon}(y) = y \in \Lambda_\varphi(U)$ and

$$\frac{1}{m_\varepsilon} \sum_{i=0}^{m_\varepsilon - 1} \log(|det(D\varphi(\varphi^i(y)))|) < 0,$$

this means that $|det(D\varphi^{m_\varepsilon}(y))| < \lambda < 1$. Observe that we are assuming the dimension of the manifold greater or equal to 2, so the fact that the modulus of the jacobian of φ be lower than 1 does not imply that all the eigenvalues have modulus smaller than 1.

Since $\Lambda_\varphi(U)$ is C^1 robustly transitive, after a perturbation, we may assume that the relative homoclinic class $H(y, \varphi, U)$ of y is the whole $\Lambda_\varphi(U)$ (see [BDP03] for definition). Now, consider the dense subset $\Sigma \subset \Lambda_\varphi(U)$ consisting of all the hyperbolic periodic points of $\Lambda_\varphi(U)$ homoclinically related to y.

If φ is close enough to f, then the tangent bundle does not admit a splitting as well. Using the idea of the proof of Lemma 6.1 in [BDP03, pp. 407] and, after that, Franks' Lemma, we obtain that there exists ψ a perturbation of φ and a point $p \in \Sigma$ such that all the eigenvalues of $D\psi^{m(p)}(p)$ have modulus strictly lower than 1, where $m(p)$ is the period of p. This means that the maximal invariant set in U of ψ contains a sink, but this is a contradiction since we choose ψ sufficiently close to f such that $\Lambda_\psi(U)$ is still transitive. ∎

Remark 14. *If $\Lambda_f(U)$ admits a splitting, then the extremal indecomposable subbundle is volume expanding. The proof is similar to the proof of theorem 2, restricting Df to the extremal subbundle.*

Remark 15. *Theorem 2 implies that volume expanding of the extremal bundle is a necessary condition for an endomorphism, which is local diffeomorphism, to be a robust transitive map. However, volume expanding is not a sufficient condition that guarantees robust transitivity for a local diffeomorphism. For instance, consider a product of an expanding endomorphism times an irrational rotation: this map is volume expanding and transitive but not robust transitive.*

Remark 16. *It is expected that if f is robustly transitive and has no invariant subbundles in a robust way, then f is a local diffeomorphism. This result depends on whether the Ergodic Closing Lemma holds even if there are critical points, since* *for maps with critical points already exists a version of Connecting Lemma, Closing Lemma and Franks' Lemma, which are the principal results involved in the proof of Theorem 2.*

5. Examples of Robust Transitive Endomorphisms

In this section we show that there exist examples of robust transitive endomorphisms verifying the hypotheses of our main results. The first two examples correspond to endomorphisms without any splitting where is applied the Main Theorem and the revisited version, and they can be considered as an endomorphisms version of the one constructed in [BV00]; those examples are *Derived from Expanding* endomorphisms. The last two ones correspond to partially hyperbolic endomorphisms, they can be considered as an endomorphisms version of the one constructed in [BD96] and [NP10], and they are not isotopic to expanding ones.

5.1. **Example 1: Applying Main Theorem.** Consider $\mathcal{E} : \mathbb{T}^n \to \mathbb{T}^n$ an expanding endomorphism, with $n \geq 2$. Let us consider a Markov partition of $\mathcal{E}$ and observe that its elements are given by $n-$dimensional closed rectangles. Note that taking a large $m > 1$, the topological degree of $\mathcal{E}^m$ is equal to the topological degree of $\mathcal{E}$ to the power $m-$th and the Markov partition can be chosen in such a way that the number of its elements is equal to the topological degree of $\mathcal{E}^m$, so without loss of generality we may assume that the initial map has many elements in the partition as we want. More precisely, if $N = \textit{topological degree}(\mathcal{E})$, we may assume that N is large and therefore the Markov partition has N elements. Denote by R_i the elements of the partition, with $1 \leq i \leq N$; R_i is closed, $int(R_i)$ is nonempty and $int(R_i) \cap int(R_j) = \emptyset$ if $i \neq j$.

Now, consider $\psi : \mathbb{T}^n \to \mathbb{T}^n$ a map isotopic to the identity and denote by $\widehat{R}_i = \psi(R_i)$ for every i. The idea of using this map is to deform the elements of the Markov partition and get a new partition which elements are not all of the same size (it could contain some very small elements and others very big).

Set U_0 an open set in $\mathbb{T}^n$ such that if $\widetilde{U}$ is the convex hull of the lift of U_0, then $\widetilde{U} \cap [0,1]^n$ is contained in the interior of $[0,1]^n$, i.e. $diam(U_0) < 1$. Note that there exists $\widehat{R}_i$ such that $\widehat{R}_i \cap U_0$ is nonempty. We also request that there are many $\widehat{R}_i$ contained in U_0^c, observe that this condition is feasible since the initial map has many elements in the partition.

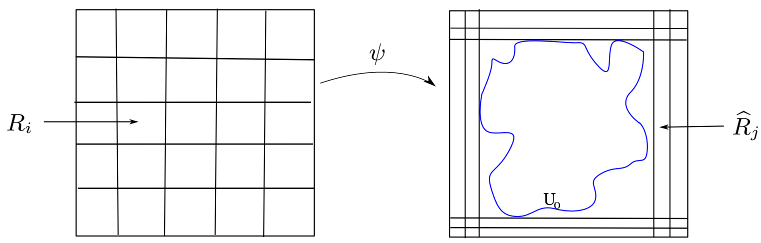


Figure 7. *Deforming the initial Markov Partition*

Define $f_0 : \mathbb{T}^n \to \mathbb{T}^n$ by $f_0 = \psi \circ \mathcal{E}$. We assume that there exist $p \in U_0$ and $q_i \in U_0^c$ fixed points for f_0, with $1 \leq i \leq n-1$. This is possible because we may start with an expanding map which has as many fixed points as we need.

Let us suppose that p and q_i are expanding for f_0 in all directions, it means that all the eigenvalues associated to these points has modulus greater than 1. Pick $\varepsilon > 0$ small enough such that $\mathbb{B}_\varepsilon(q_i) \cap U_0 = \emptyset$ and $\mathbb{B}_\varepsilon(q_i) \cap \mathbb{B}_\varepsilon(q_j) = \emptyset$ if $i \neq j$.

Let us denote the decomposition of the tangent space as follows

$$T_x(\mathbb{T}^n) = \mathbb{E}_1^u \prec \mathbb{E}_2^u \prec \cdots \prec \mathbb{E}_{n-1}^u \prec \mathbb{E}_n^u,$$

where $\prec$ denotes that $\mathbb{E}_i^u$ dominates the expanding behavior of $\mathbb{E}_{i-1}^u$.

Next we deform f_0 by a smooth isotopy supported in $U_0 \cup (\bigcup \mathbb{B}_\varepsilon(q_i))$ in such a way that:

(1) The continuation of p goes through a pitchfork bifurcation, appearing two periodic points r_1, r_2, such that both are repeller and p becomes a saddle point. But the new map f still expand volume in U_0.
(2) Two expanding eigenvalues of q_i become complex expanding eigenvalues. More precisely, we mix the two expanding subbundles of $T_{q_i}(\mathbb{T}^n)$ corresponding to $\mathbb{E}_i^u(q_i)$ and $\mathbb{E}_{i+1}^u(q_i)$, obtaining $T_{q_i}(\mathbb{T}^n) = \mathbb{E}_1^u \prec \mathbb{E}_2^u \prec \cdots \prec \mathbb{F}_i^u \prec \mathbb{E}_{n-1}^u \prec \mathbb{E}_n^u$, where $\mathbb{F}_i$ is two dimensional and correspond to the complex eigenvalues.
(3) Outside $U_0 \cup (\bigcup \mathbb{B}_\varepsilon(q_i))$, f coincides with f_0.
(4) f is expanding in U_0^c.
(5) There exists $\sigma > 1$ such that $|det(Df(x))| > \sigma$ for every $x \in \mathbb{T}^n$.

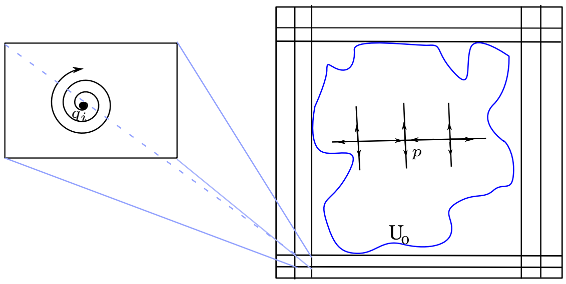


FIGURE 8. *f isotopic to f_0*

5.1.1. *Property of Large Arcs.*

Claim 5.1. *Every large arc in U_0^c has a point such that its forward orbits remain in U_0^c.*

Proof. Take d the maximum of the diameter of the elements of the partition contained in U_0^c. Note that every arc in U_0^c with diameter larger than d cannot be contained in the interior of any element of the partition, more precisely it has to intersect at least two elements of the partition. Hence, the image by f of this arc γ has diameter 1. So there exists a piece of $f(\gamma)$ in U_0^c intersecting at least one element of the partition across two parallel sides, let us call γ^1. Choose a pre-image of γ^1 in γ and call it γ_1.

Repeating the process for γ^1, we have that there is γ^2 a piece of $f(\gamma^1)$ verifying the same condition as γ^1. Then, choose γ_2 a pre-image of γ^2 by f^2 in γ.

Thus, we construct a sequence of nested arcs in γ. The intersection is non empty and a point in this intersection satisfy our claim. ∎

5.1.2. *Remarks and variation of Example 1.*

(1) q_i's are fixed points for f with complex expanding eigenvalues. Note that the existence of these points ensures that the tangent bundle does not admit any invariant subbundle. We could also start with an expanding map having, besides p, periodic points q_i with complex eigenvalues. In such a case, it is enough to make p goes through a pitchfork bifurcation.
(2) This example shows that U_0 can be as big as we desired while it verifies the hypothesis of having diameter less than 1.
(3) It can be constructed in any dimension.

5.2. **Example 2: Applying the Main Theorem Revisited.** Let us consider $\mathcal{E} : \mathbb{T}^n \to \mathbb{T}^n$ an expanding endomorphism, with $n \geq 2$. Assume that the initial map has many elements in the Markov partition, let us say N elements.

Denote by R_i the elements of the partition, with $1 \leq i \leq N$. Since $\mathcal{E}$ is expanding, R_i are closed, $int(R_i)$ are nonempty and $int(R_i) \cap int(R_j) = \emptyset$ if $i \neq j$. Choose finitely many of these elements, $\{R_{i_j}\}_{j=1}^{k}$, such that $R_{i_j} \cap R_{i_s} = \emptyset$ if $i_j \neq i_s$, i.e. they are two by two disjoints. Consider the pre-images of every R_{i_j}, let us say $\mathcal{E}^{-1}(R_{i_j}) = \{P^l_{i_j}\}_{l=1}^{N}$. Denote by $P^0_{i_j} = R_{i_j}$. Next, we keep $P^r_{i_j}$ such that $P^r_{i_j} \cap P^l_{i_s} = \emptyset$ whenever $0 \leq r \neq l \leq N$ and $i_j \neq i_s$. Finally, let us denote by $\{P_i\}_i$ the collection of these latter subsets, so they are two by two disjoints. See figure 9.

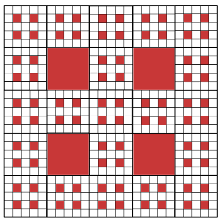


FIGURE 9. $\{P_i\}_i$ *collection*

Now, consider $\psi : \mathbb{T}^n \to \mathbb{T}^n$ a map isotopic to the identity and denote by $\widehat{P}_i = \psi(P_i)$ for every i. See figure 10.

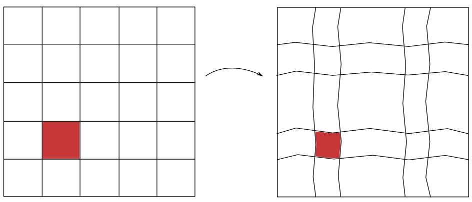
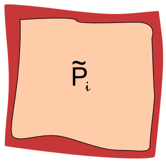


FIGURE 10. *Deforming the Markov partition*

Choose $\widetilde{P}_i$ an open connected subset such that its closure is contained in the interior of $\widehat{P}_i$. Let $\phi_i : \mathbb{T}^n \to \mathbb{T}^n$ be a map isotopic to the identity such that

- $\phi_i \mid_{\widetilde{P}_i}$ is not expanding.
- $\phi_i \mid_{\widehat{P}_i^c}$ is the identity.



Define $\phi : \mathbb{T}^n \to \mathbb{T}^n$ by

$$\phi(x) = \begin{cases} \phi_i(x), & \text{if } x \in \widehat{P}_i \\ \\ x, & \text{if } x \notin \bigcup_i \widehat{P}_i \end{cases}$$

Hence, ϕ is equal to the identity in $[\bigcup_i \widehat{P}_i]^c$, expands volume but is not expanding in $\bigcup_i \widehat{P}_i$.

Once we have defined all these maps, we consider the map $f = \phi \circ \psi \circ \mathcal{E}$ from $\mathbb{T}^n$ onto itself and denote by $U_0 = int(\bigcup_i \widehat{P}_i)$. Observe that f verifies that:

(i) f is a volume expanding endomorphism.
(ii) f is an expanding map in U_0^c.
(iii) $\Lambda_f = \bigcap_{n\geq 0} f^{-n}(U_0^c)$ is an expanding locally maximal set for f which has the property that separate large nice cylinders.

Since (i) and (ii) are immediate from the construction of f, we concentrate our interest in proving (iii).

5.2.1. Λ_f *Separates Large Nice Cylinders.* Note that by the construction of U_0, we have that the elements of the pre-orbit of U_0 are two by two disjoints. Let us consider $d_0 = \max\{diam(c.c. \bigcup_{k\geq 0} f^{-k}(U_0))\}$. Since the definition of U_0, $0 < d_0 < 1$. Observe that Λ_f looks like a Sierpinski set, see figure 11.

Claim 5.2. *If γ is an arc in U_0^c with diameter 1, then γ intersects Λ_f.*

Proof. Let γ be an arc in U_0^c such that $diam(\gamma) = 1$. Suppose that γ does not intersect Λ_f.

Remember that $\Lambda_f = \mathbb{T}^n \setminus \bigcup_{k\geq 0} f^{-k}(U_0)$, it means that if $x \in \Lambda_f$, then $f^k(x) \notin U_0$ for all $k \geq 0$. Therefore, γ is contained in one pre-image of U_0 or in a union of pre-images of U_0.

Observe that γ cannot be contained in just one pre-image of U_0, because if it is contained in $f^{-k}(U_0)$ for some $k \geq 0$, then $diam(\gamma) < diam(f^{-k}(U_0)) < d_0$, which is absurd because $d_0 < 1$.

Hence, γ should be contained in a union of pre-images of U_0, since γ is compact we can cover with a finite union of pre-images of U_0. But we know that the pre-images of U_0 are two by two disjoints, hence there exist points in γ that cannot be covered by the pre-images of U_0. In particular, γ intersects Λ_f. ∎

Remark 17. *We have already proved the existence of the invariant expanding locally maximal set Λ_f. Moreover, by claim (5.2) we get that this invariant set intersects every arc with large diameter. Then by the Main Theorem Revisited follows that this map is robustly transitive.*

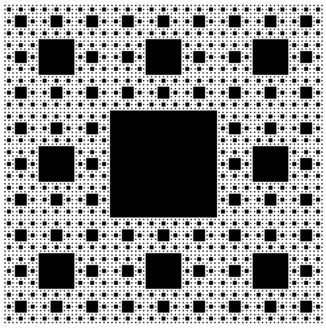


FIGURE 11. Λ_f *looks like a Sierpinski set*

5.2.2. *Remarks About Example 2.*

(1) We can apply our Main Theorem Revisited to this example, obtaining in particular that f is robustly transitive.
(2) The $\widehat{P}_i$'s can be as many and as big as we want.
(3) We can construct many examples starting with this initial map. In particular, we can construct examples without invariant subbundles, such as putting a fix point in the complement of the U_0 with complex eigenvalues and doing a derived from an expanding endomorphisms inside of some $\widehat{P}_i$.

5.3. **Example 3: Applying Theorem 1.** The idea of next example is to build an endomorphism in the 2-Torus which is a skew-product and contains a "blender" for endomorphisms. This example is more or less a standard adaptation for endomorphisms of examples obtained in [BD96] for diffeomorphisms. The main goal is to get "blenders" for endomorphisms and since we do not necessarily need to deal with stable foliation, the task is easier than the case of diffeomorphisms. For more information about blenders see [BD96].

First, let us identify the 2-Torus with $[0,1]^2$ and let us establish some notation before defining the map. Pick $0<a<b<3/4$ and $1/4<c<d<1$. Denote $J_1=[0,b]$, $J_2=[a,3/4]$, $J_3=[1/4,d]$ and $J_4=[c,1]$. Note that $J_1 \cap J_2 = [a,b]$ and $J_3 \cap J_4 = [c,d]$. This decomposition is associated to the horizontal fibers.

Next, fix $N > 3$ and pick $0 < a_1 < b_1 < a_2 < b_2 < a_3 < b_3 < a_4 < b_4 < 1$ such that $b_i - a_i = 1/N$. Let us denote by $I_i = [a_i, b_i]$ with $1 \leq i \leq 4$. Note that they are two by two disjoint and do not contain 0 or 1. We associate this decomposition to the vertical fibers.

Let us call $R_i = J_i \times I_i$ where $i = 1,2,3,4$.

Now, define $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$\Phi(x,y) = (\varphi_y(x), \mathcal{E}(y)),$$

where $\varphi_y, \mathcal{E} : S^1 \to S^1$ are defined as follows:

(1) $\mathcal{E}$ is an expanding endomorphism such that:
    - $\mathcal{E}(I_i) = [0,1]$ for every i.
    - There exists $a_i < c_i < b_i$ such that $\mathcal{E}(c_i) = c_i$.
(2) φ_y is defined by $\varphi_y(x) = f_i(x)$, if $y \in I_i$, where $f_i : S^1 \to S^1$ are diffeomorphisms defined as follows:
    - f_1 and f_2 satisfy the following properties:

- (i) f_1 has two fixed points, 0 and $a'\in(3/4,1)$, where 0 is a repeller and a' is an attractor for f_1.
- (ii) f_2 has two fixed points, $3/4$ and $a''\in(a',1)$, where $3/4$ is a repeller and a'' is an attractor for f_2.
- (iii) $f_1(J_1) = f_2(J_2) = [0,3/4]$.
- (iv) $|f_i'|_{J_i}| > 1$ for $i = 1,2$.

- f_3 and f_4 satisfy the following properties:
  - (i') f_3 has two fixed points, $c' \in (0,1/4)$ and $1/4$, where c' is an attractor and $1/4$ is a repeller for f_3.
  - (ii') f_4 has two fixed points, $c''\in(0,c')$ and 1, where c'' is an attractor and 1 is a repeller for f_4.
  - (iii') $f_3(J_3) = f_4(J_4) = [1/4,1]$.
  - (iv') $|f_i'|_{J_i}| > 1$ for $i = 3,4$.

(3) $|det(D\Phi)| = |\frac{\partial\varphi_y}{\partial x}\,\mathcal{E}'| > 1$.

(4) $\mathcal{E}' \gg \frac{\partial\varphi_y}{\partial y}$.

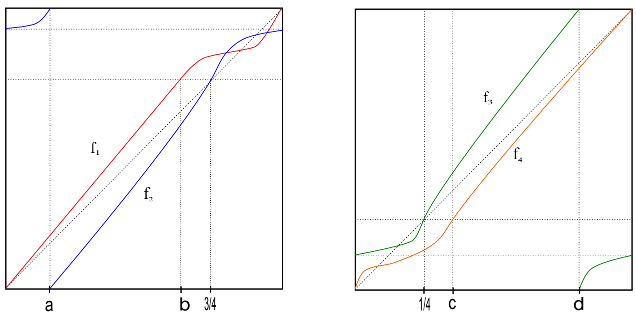


FIGURE 12. *Horizontal dynamics*

Hence, the horizontal fibers $F_i = S^1 \times \{c_i\}$ are invariant by Φ, see figure 13. Moreover, by condition (4), the image by Φ of every vertical fiber is almost a vertical fiber, in the sense that the tangent vector is close to a vertical one; more precisely, the unstable cones family are almost vertical.

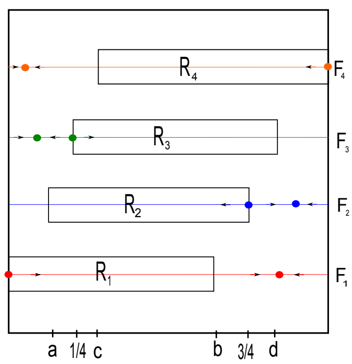


FIGURE 13. *This is how the dynamics Φ looks like*

Next, we consider $\Lambda_1^+ = \bigcap_{n\geq 0} \Phi^{-n}(R_1 \cup R_2)$ and $\Lambda_2^+ = \bigcap_{n\geq 0} \Phi^{-n}(R_3 \cup R_4)$. Let $\Lambda_1 = \bigcap_{n\in\mathbb{Z}} \Phi^{-n}(R_1 \cup R_2)$ and $\Lambda_2 = \bigcap_{n\in\mathbb{Z}} \Phi^{-n}(R_3 \cup R_4)$. Note that both sets, Λ_1 and Λ_2 are expanding locally maximal invariant sets and each one contains a blender.

5.3.1. Λ_1 *and* Λ_2 *Separate Large Vertical Segments.* Let us denote by $\ell_1^u(p)$ the vertical segment passing through p and length 1.

Claim 5.3. *For every* $p \in R_1 \cup R_2$, *follows that* $\ell_1^u(p) \cap \Lambda_1^+ \neq \emptyset$.

Proof. Let $p \in R_1 \cup R_2$, then $L_i = \ell_1^u(p) \cap R_i$ is non empty for some $i \in \{1, 2\}$. The image of $L_i = \ell_1^u(p) \cap R_i$ by Φ has length 1 and by property (4) of Φ follows that $\Phi(L_i)$ is almost vertical. Moreover, $L_i \cap F_i \neq \emptyset$ and $\Phi(L_i \cap F_i) \in F_i \subset R_i$. Then $\Phi(\ell_1^u(p)) \cap (R_1 \cup R_2) \neq \emptyset$. Call K_1^i the connected component $\Phi(\ell_1^u(p)) \cap R_i$. Note that $P_2(K_1^i) = I_i$, where P_2 is the projection in the second coordinate. Consider the pre-image of K_1^i by Φ in L_i and call it S_1^i.

Now, iterate K_1^i by Φ, doing a similar process we obtain K_2^i, the connected component $\Phi(K_1^i) \cap R_i$ such that $P_2(K_2^i) = I_i$. Again take a pre-image of K_2^i by Φ^2, giving a compact segment $S_2^i \subset S_1^i$. Repeating this process, we may construct a nested sequence of compact segment $\{S_k^i\}_k$ in each R_i. Thus, $\bigcap_k S_k^i$ is not empty and belong to $\ell_1^u(p) \cap \Lambda_1^+$. ∎

Claim 5.4. *For every* $p \in R_1 \cup R_2$, *follows that* $\ell_1^u(p) \cap \Lambda_1 \neq \emptyset$.

Proof. By claim (5.3), we know that there exists a point $z \in \ell_1^u(p) \cap \Lambda_1^+$, this means that $\Phi^n(z) \in R_1 \cup R_2$ for every $n \geq 0$.

Then, just remain to show that there exists a sequence $\{z_k\}_{k\geq 0} \subset R_1 \cup R_2$ such that $z_0 = z$ and $\Phi(z_k) = z_{k-1}$. The idea of the construction of such a sequence is to use now the property (2-*iii*) of overlapping in the horizontal dynamics.

Knowing that $\Phi(R_1) = f_1(J_1) \times [0, 1]$ and $\Phi(R_2) = f_2(J_2) \times [0, 1]$, since property (2-*iii*) we get that $\Phi(R_1) = \Phi(R_2) = [0, 3/4] \times [0, 1]$. Hence, $z_0 \in (R_1 \cup R_2) \cap \Phi(R_1)$ or $z_0 \in (R_1 \cup R_2) \cap \Phi(R_2)$, then there exists $z_1 \in R_1 \cup R_2$ such that $\Phi(z_1) = z_0$. Repeating this process we construct the require sequence. ∎

Claim 5.5. *For every* $p \in R_3 \cup R_4$, *follows that* $\ell_1^u(p) \cap \Lambda_2 \neq \emptyset$.

Proof. The proof is similar to claim (5.4) just making the necessary arrangement. ∎

Claim 5.6. *For every* $q \in \mathbb{T}^2$, *we have that either* $\ell_1^u(q) \cap \Lambda_1 \neq \emptyset$ *or* $\ell_1^u(q) \cap \Lambda_2 \neq \emptyset$.

Proof. Given any point $q \in \mathbb{T}^2$, note that $\ell_1^u(q) \cap R_i \neq \emptyset$ for some $1 \leq i \leq 4$. Hence, taking $p_i \in \ell_1^u(q) \cap R_i$ and noting that $\ell_1^u(p_i) = \ell_1^u(q)$, we may use claim (5.4) or (5.5) to conclude that either $\ell_1^u(q) \cap \Lambda_1 \neq \emptyset$ or $\ell_1^u(q) \cap \Lambda_2 \neq \emptyset$. ∎

5.3.2. *Remarks About Example 3.* This example was constructed in the 2-Torus with one dimensional central bundle, but we can construct it in any $\mathbb{T}^n$ and the dimension of the central bundle not need to be 1. Also, we can use more than 4 dynamics in the horizontal, that is more than four maps in the first variable. More precisely, we put 2 blenders in the dynamic, induced by these four maps, but we can consider as many blenders as we want.

5.4. **Example 4: Applying Theorem 1.** Let $\mathbb{B}_0$ be an open ball in $\mathbb{T}^m$ centered at 0 with radius $\alpha < 1$ and $\varphi_0 : \mathbb{T}^m \to \mathbb{T}^m$ be a differentiable map isotopic to the identity such that:

- $\varphi_0(0) = 0$
- There exist $0 < \lambda_0 < \lambda_1 < 1$ such that $\lambda_0 < m\{D\varphi_0\} < |D\varphi_0 |_{\mathbb{B}_0} | < \lambda_1$, i.e. φ_0 is a contraction in a disk.

Let us consider $\mathbb{D}_0$ the lift of $\mathbb{B}_0$ to $\mathbb{R}^m$ and $\widetilde{\varphi}_0$ the lift of φ_0. Note that $\widetilde{\varphi}_0(0) = 0$ and $\lambda_0 < m\{D\widetilde{\varphi}_0\} < |D\widetilde{\varphi}_0 |_{\mathbb{D}_0} | < \lambda_1$. By Proposition 2.3 [NP10], there exists $k \in \mathbb{N}$ such that for every small $\varepsilon > 0$, there exist $c_1, \dots, c_k \in \mathbb{B}_\varepsilon(0)$ and $\delta > 0$ such that $\mathbb{B}_\delta(0) \subset \overline{Orbit^+_{\mathcal{G}}(0)}$, where $\mathcal{G} = \mathcal{G}(\widetilde{\varphi}_0, \widetilde{\varphi}_0 + c_1, \dots, \widetilde{\varphi}_0 + c_k)$ and $Orbit^+_{\mathcal{G}}(0)$ is the set of points lying on some orbit of 0 under the iterated function system (IFS) $\mathcal{G}$; more precisely, if we denote by $\widetilde{\phi}_0 = \widetilde{\varphi}_0$ and $\widetilde{\phi}_i = \widetilde{\varphi}_0 + c_i$ for $i = 1, \dots, k$, then $Orbit^+_{\mathcal{G}}(0)$ is the set of sequence $\{\widetilde{\phi}_{\Sigma_l}(0)\}_{l=1}^{\infty}$ where $\Sigma_l = (\sigma_1, \dots, \sigma_l)$, $\widetilde{\phi}_{\Sigma_l} = \widetilde{\phi}_{\sigma_l} \circ \cdots \circ \widetilde{\phi}_{\sigma_1}$ and $\{\sigma_i\}_{i\in\mathbb{N}} \in \{0, \dots, k\}^{\mathbb{N}}$. (For more details about IFS see [NP10])

Now choose $p_1, \dots, p_r \in \mathbb{T}^m$ such that $\mathbb{T}^m \subset \bigcup_j \mathbb{B}_\delta(p_j)$.

If ϕ_i is the projection of $\widetilde{\phi}_i$ on $\mathbb{T}^m$, define for every j the IFS $\mathcal{G}_j = \mathcal{G}_j(\phi_0 + p_j, \phi_1 + p_j, \dots, \phi_k + p_j)$. Then $\mathbb{B}_\delta(p_j) \subset \overline{Orbit^+_{\mathcal{G}_j}(0)}$. Therefore, there exists an open set $\mathbb{D}_0 \subset \mathbb{B}_0$ such that $\bigcup \phi_i(\mathbb{D}_0) \supset \mathbb{D}_0$, i.e. the IFS has the covering property. Hence, $\bigcup_i \phi_i(\mathbb{B}_{\delta'}(p_j)) \supset \mathbb{B}_{\delta'}(p_j)$, with $0 < \delta' \leq \delta$. Moreover, $\mathcal{G}_j$ has also the overlapping property as in Example 3, in the previous section.

Define the skew-product $F : \mathbb{T}^m \times \mathbb{T}^n \to \mathbb{T}^m \times \mathbb{T}^n$ by

$$F(x, y) = (\psi_y(x), \mathcal{E}(y)),$$

where:

- $\mathcal{E} : \mathbb{T}^n \to \mathbb{T}^n$ is an expanding map with $(k + 1)r$ fixed points, let us denote the fixed points by $e^i_1, \dots, e^i_r$ with $0 \leq i \leq k$.
- For every $y \in \mathbb{T}^n$, $\psi_y : \mathbb{T}^m \to \mathbb{T}^m$ is a differentiable map isotopic to the identity such that $\psi_{e^i_j} = \phi_i + p_j$, with $0 \leq i \leq k$ and $1 \leq j \leq r$.

Hence, every fiber $\mathbb{T}^m \times \{e^i_j\}$ is invariant by F. Set $R^i_j = \mathbb{B}_{\delta'}(p_j) \times Q^i_j$, where Q^i_j is a small neighborhood of e^i_j in $\mathbb{T}^n$ such that $\mathcal{E}(Q^i_j) = \mathbb{T}^n$ and they are all disjoints for every i, j. Note that R^i_j are the analogous of R_i in the previous example.

Let $\Lambda_F := \bigcap_{n\in\mathbb{Z}} F^n(\bigcup_{i,j} R^i_j)$.

5.4.1. Λ_F *Separate Large Unstable Discs.*

Claim 5.7. *Λ_F verifies that for every $z \in \bigcup_{i,j} R^i_j$ follows that $\ell^u_1(z) \cap \Lambda_F \neq \emptyset$, where $\ell^u_1(z)$ is an unstable disc of internal diameter 1 passing through z.*

Proof. We may prove that there exists a point $z \in \bigcup_{i,j} R^i_j$ such that $F^n(z) \in \bigcup_{i,j} R^i_j$ for every $n \geq 0$ in a similar way as we proved claim (5.3) in previous example.

Moreover, for this z there exists $z_1 \in \bigcup_{i,j} R^i_j$ such that $F(z_1) = z$. In fact, the idea is more or less the same as in previous example, we must note that $F(R^i_j) = \psi_{e^i_j}(\mathbb{B}_{\delta'}(p_j)) \times \mathcal{E}(Q^i_j) = \phi_i(\mathbb{B}_{\delta'}(p_j)) \times \mathbb{T}^n$.

On the other hand, using the property of covering and overlapping follows that

$$\bigcup_{i,j} R^i_j = \bigcup_{i,j} \mathbb{B}_{\delta'}(p_j) \times Q^i_j \subset \bigcup_{i,j} \phi_i(\mathbb{B}_{\delta'}(p_j)) \times \mathcal{E}(Q^i_j) = F(\bigcup_{i,j} R^i_j).$$

Therefore, since $z \in \bigcup_{i,j} R^i_j$, there exists $z_1 \in \bigcup_{i,j} R^i_j$ such that $F(z_1) = z$. Inductively we can construct a sequence $\{z_k\}_{k\geq 0} \subset \bigcup_{i,j} R^i_j$ such that $z_0 = z$ and $F(z_k) = z_{k-1}$.

Thus, $z \in \ell^u_1(z) \cap \Lambda_F$. ∎

5.4.2. *Remarks About Example 4.* This example is a generalization of Example 3. The intention here is to show that we may apply Theorem 1 without taking into account the dimension of the central bundle and this could be as large as we want. Another observation is that the existence of blenders guarantee that our examples are robustly transitive and this example satisfies the property over the unstable discs with sufficiently large internal diameter intersecting the invariant expanding locally maximal set for the skew-product.

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† Departamento de Matemática, Facultad de Ciencias, Universidad de Los Andes, La Hechicera-Mérida, 5101, Venezuela
E-mail address: clizana@ula.ve

‡ IMPA, Estrada Dona Castorina,110, CEP 22460-320, Rio de Janeiro, Brazil
E-mail address: enrique@impa.br