# On the Quadratic Eigenvalue Complementarity Problem

Carmo Brás[*] , Alfredo N. Iusem[†] , Joaquim J. Júdice[‡]

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**Abstract**

We introduce several new results on the Quadratic Eigenvalue Complementarity Problem (QEiCP), focusing on the nonsymmetric case, i,e, without making symmetry assumptions on the matrices defining the problem. First we establish a new sufficient condition for existence of solutions of this problem, which is somewhat more manageable than previously existent ones. This condition works through the introduction of auxiliary variables which leads to the reduction of QEiCP to an Eigenvalue Complementarity Problem (EiCP) in higher dimension. Hence, this reduction suggests a new strategy for solving QEiCP, which is also analyzed in the paper. We also present an upper bound for the number of solutions of QEiCP and exhibit some examples of instances of QEiCP whose solution set has large cardinality, without attaining though the just mentioned upper bound. We also investigate the numerical solution of the QEiCP by solving a Variational Inequality Problem (VIP) on the $2n$-dimensional simplex, which is equivalent to a $2n$-dimensional EiCP. Some numerical experiments with a projection method for solving this VIP are reported, illustrating the value of this methodology in practice.

[*]Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Portugal, mb@fct.unl.pt. The work of this author was partially supported by CMA/FCT/UNL, under the project PEst-OE/MAT/UI0297/2011.

[†]Instituto de Matématica Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, RJ, CEP 22460-320, Brazil, iusp@impa.br. The work of this author was partially supported by CNPq grant no. 301280/86.

[‡]Instituto de Telecomunicações, Portugal, joaquim.judice@co.it.pt.

# 1 Introduction

Given matrices $B, C \in \mathbb{R}^{n\times n}$, the *Eigenvalue Complementarity Problem* (to be denoted EiCP$(B, C)$, see e.g. [17] and [18]), consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda Bx - Cx \tag{1}$$

$$w \geq 0, x \geq 0 \tag{2}$$

$$x^t w = 0 \tag{3}$$

$$e^t x = 1, \tag{4}$$

with $e = (1, 1, \dots, 1)^t \in \mathbb{R}^n$. The last constraint has been introduced, without loss of generality, in order to prevent the $x$ component of a solution to vanish. Usually, the matrix $B$ is assumed to be positive definite. In this paper, our basic assumption is the strict copositivity (SC) of $B$ (see Definition 1(i) in Section 2, and note that PD matrices are SC). The problem has many applications in engineering (see [15], [18]), and can be seen as a generalization of the well-known Generalized Eigenvalue Problem, denoted GEiP (see e.g. [8]). Indeed, GEiP consists of solving just (1) with $w = 0$, and a solution $(\lambda, x)$ of GEiP is just an eigenvalue and eigenvector of the matrix $B^{-1}C$ in the usual sense, when $B$ is nonsingular. If a triplet $(\lambda, x, w)$ solves EiCP, then the scalar $\lambda$ is called a *complementary eigenvalue* and $x$ is a *complementary eigenvector* associated to $\lambda$ for the pair $(B, C)$. The condition $x^t w = 0$ and the nonnegative requirements on $x$ and $w$ imply that either $x_i = 0$ or $w_i = 0$ for $1 \leq i \leq n$. These two variables are called complementary.

It is easy to prove that under strict copositivity of $B$, EiCP$(B, C)$ always has a solution, because it can be reformulated as the Variational Inequality Problem VIP$(\bar{F}, \Omega)$ with feasible set $\Omega = \{x \in \mathbb{R}^n \colon e^t x = 1, x \geq 0\}$ and operator $\bar{F} : \Omega \to \mathbb{R}^n$ given by

$$\bar{F}(x) = \frac{x^t C x}{x^t B x} Bx - Cx,$$

see [11]. Note that $\bar{F}$ is continuous in $\Omega$ as a consequence of the strict copositivity of $B$, and that $\Omega$ is convex and compact. It is well known that these two conditions ensure existence of solutions of VIP$(\bar{F}, \Omega)$ (see e.g. [4]). The reformulation of EiCP as a variational inequality problem is further developed in Section 3.

If the matrices $B$ and $C$ are both symmetric, then EiCP is called symmetric and reduces to the problem of finding a *Stationary Point* (SP) of the so-called Rayleigh Quotient function on the simplex $\Omega$ (see, e.g. [17, 18]), which is just a SP of the following Standard Quadratic Fractional Program

$$\begin{aligned} \text{Minimize} \quad & \frac{x^t C x}{x^t B x} \\ \text{subject to} \quad & e^t x = 1 \\ & x \geq 0. \end{aligned}$$

A number of techniques have been proposed for solving the EiCP and its extensions, see e.g. [1], [2], [6], [7], [9], [10], [11], [12], [14], [16] and [20]. As expected, the symmetric EiCP is easier to solve.

Recently an extension of the EiCP has been introduced in [19], where some applications are highlighted. It has been named *Quadratic Eigenvalue Complementarity Problem* (QEiCP), and it differs from EiCP through the existence of an additional quadratic term on $\lambda$. Its formal definition follows.

Given $A, B, C \in \mathbb{R}^{n\times n}$, QEiCP$(A, B, C)$ consists of finding $(\lambda, x, w) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that

$$w = \lambda^2 Ax + \lambda Bx + Cx, \tag{5}$$

$$w \geq 0, x \geq 0, \tag{6}$$

$$x^t w = 0, \tag{7}$$

$$e^t x = 1, \tag{8}$$

where, as before, $e = (1, 1, \dots, 1)^t \in \mathbb{R}^n$. As in the case of the EiCP, (8) has been introduced, without loss of generality, for preventing the $x$ component of a solution of the problem from vanishing. Note that when $A = 0$ QEiCP$(A, B, C)$ reduces to EiCP$(B, -C)$. The $\lambda$ component of a solution of QEiCP$(A, B, C)$ is called a *quadratic complementary eigenvalue* for $A, B, C$, and the $x$ component a *quadratic complementary eigenvector* for $A, B, C$ associated to $\lambda$.

The case of the symmetric QEiCP, i.e., when $A, B$ and $C$ are symmetric matrices, has been fully analyzed in [5], where each instance of QEiCP with $n \times n$ matrices is related to an instance of $EiCP$ with $2n \times 2n$ matrices. In this paper we remove such symmetry assumption, and focus on the general QEiCP. We also propose a connection between an $n$-dimensional QEiCP and a higher dimensional EiCP, but our connection procedure, developed in Section 2, differs from the one in [5].

We start by discussing the issue of existence of solutions. Contrary to the EiCP, the QEiCP may lack solutions, even under strict copositivity or positive definiteness of $A$. Indeed if we consider QEICP$(I, 0, I)$, then premultiplying (6) by $x$ and using (7), one gets $0 = (\lambda^2 + 1)\|x\|^2$, wich has no solution $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ because (8) implies that $x \neq 0$. In fact, this difference between EiCP and QEiCP in terms of existence of solutions mirrors the elementary fact that linear equations in one real variable always have solutions, while quadratic equations may fail to have them.

Thus, the issue of conditions on $(A, B, C)$ ensuring existence of solutions of QEiCP$(A, B, C)$ is a relevant one. The so-called co-regularity and co-hiperbolicity properties were introduced by A. Seeger in [19] as sufficient conditions for the existence of solutions of the QEiCP. In Section 2 we will present another sufficient condition, which neither implies nor is implied by Seeger's conditions, and discuss the relation between Seeger's conditions and ours.

Both Seeger's and our sufficiency proofs work through the reduction of QEiCP to two different variational inequality problems, and hence each condition suggest a strategy for solving QEiCP, which are discussed in Section 3.

An upper bound for the number of complementary eigenvalues for a pair $(B, C)$ has been established in [17] and [18]. In Section 4 we find a related upper bound for QEiCP, and exhibit an example with a large number of them (without attaining however the upper bound).

An enumerative method and a hybrid algorithm, combining the previous method and a semi-smooth approach, have been introduced in [6] and [7]. These algorithms are able to solve the QEiCP when the co-regularity anc co-hyperbolicity conditions are assumed to hold. In Section 5 we study the numerical solution of the QEiCP by solving its equivalent EiCP as a Variational Inequality Problem in the $2n$-dimensional simplex. We propose a projection algorithm discussed in [2] for solving this VIP. The numerical experiments reported in Section 5 indicate that the projection algorithm is able to solve the VIP for many test problems, but may fail to get a solution in some instances. A hybrid enumerative method in the spirit of [7] should be developed in the future for guaranteeing solution of the QEiCP under the sufficient condition presented in Section 2.

## 2 A sufficient condition for existence of solutions of QEiCP

In this section we will present a new sufficient condition for the existence of solutions of QEiCP and compare it with the one in [19]. The condition is based upon the study of the relation between an arbitrary $n$-dimensional QEiCP and two specific instances of EiCP with matrices in $\mathbb{R}^{2n\times 2n}$. A relation of this kind was also studied in [5], but the instance of EiCP chosen in this reference is different from the ones in this paper, which are tailored for addressing the existence issue.

For the sake of a simpler notation, we comit a slight notational abuse, and say that a pair $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n$ solves EiCP$(B, C)$ when the triplet $(\lambda, x, w)$, with $w = \lambda Bx - Cx$, is a solution of EiCP$(B, C)$ in the sense defined in Section 1. In the same fashion, we say that $(\lambda, x) \in \mathbb{R}\times\mathbb{R}^n$ solves QEiCP$(A, B, C)$ when the same occurs with the triplet $(\lambda, x, w)$, where $w = \lambda^2 Ax + \lambda Bx + Cx$.

Consider now QEiCP$(A, B, C)$ with $A, B, C \in \mathbb{R}^{n\times n}$ and define $D, G, H \in \mathbb{R}^{2n\times 2n}$ as

$$D = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \tag{9}$$

$$G = \begin{pmatrix} -B & -C \\ I & 0 \end{pmatrix}, \tag{10}$$

$$H = \begin{pmatrix} B & -C \\ I & 0 \end{pmatrix}. \tag{11}$$

Next we establish a relation between the solutions of QEiCP$(A, B, C)$ and those of EiCP$(D, G)$ and EiCP$(D, H)$. We emphasize that the following result holds without making any additional hypotheses on $A, B, C$.

**Proposition 1.** *a) Assume that $(\lambda, x)$ solves QEiCP$(A, B, C)$ and consider $D, G, H$ as in* (9)–(11).

*i) If $\lambda = 0$ then $(\lambda, z) = (0, z)$ solves both EiCP($D, G$) and EiCP($D, H$), where $z \in \mathbb{R}^{2n}$ is defined as $z = (0, x)$.*

*ii) If $\lambda > 0$ then $(\lambda, z)$ solves EiCP($D, G$), where $z \in \mathbb{R}^{2n}$ is defined as $z = (1+\lambda)^{-1}(\lambda x, x)$.*

*iii) If $\lambda < 0$ then the pair $(-\lambda, z)$ solves EiCP($D, H$), where $z \in \mathbb{R}^{2n}$ is defined as $z = (1-\lambda)^{-1}(-\lambda x, x)$.*

*b) Consider $D, G, H$ as in (9)–(11).*

*i) If $(\lambda, z)$ solves EiCP($D, G$) with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(\lambda, (1+\lambda)x)$ solves QEiCP($A, B, C$)*

*ii) If $(\lambda, z)$ solves EiCP($D, H$) with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$ and $\lambda \neq 0$, then $\lambda > 0$ and $(-\lambda, (1+\lambda)x)$ solves QEiCP($A, B, C$).*

*Proof.* a) For item (i), note that checking whether $(0, x)$ solves QEiCP($A, B, C$) reduces to verifying that $Cx \geq 0, x \geq 0, x^t Cx = 0$, and the same happens when verifying that $(0, (0, x))$ solves either EiCP($D, G$) or EiCP($D, H$). We deal now with item (ii). Note that checking that a pair $(\lambda, z)$ with $z = (u, v) \in \mathbb{R}^n \times \mathbb{R}^n$ solves EiCP($D, G$) is equivalent to verifying:

$$\lambda Au + Bu + Cv \geq 0, \tag{12}$$

$$\lambda v - u \geq 0, \tag{13}$$

$$u \geq 0, \quad v \geq 0, \tag{14}$$

$$u^t(\lambda Au + Bu + Cv) + v^t(\lambda u - v) = 0, \tag{15}$$

$$e^t u + e^t v = 1. \tag{16}$$

On the other hand, since $(\lambda, x)$ solves QEiCP($A, B, C$), we know that

$$\lambda^2 Ax + \lambda Bx + Cx \geq 0, \tag{17}$$

$$x \geq 0, \tag{18}$$

$$x^t(\lambda^2 Ax + \lambda Bx + Cx) = 0, \tag{19}$$

$$e^t x = 1. \tag{20}$$

If we take $u = \frac{\lambda}{1+\lambda}x, v = \frac{1}{1+\lambda}x$, then (13) follows immediately, and indeed with equality. Furthermore, (12) follows from (17), and (14) follows from (18) and positivity of $\lambda$. Also, the first term of the left hand side of (15) vanishes as a consequence of (19) and the second one because $\lambda v = u$. Regarding (16), note that $e^t u + e^t v = (1+\lambda)^{-1}(\lambda e^t x + e^t x) = e^t x = 1$, using (20) in the third equality. For item (iii), note that if $(\lambda, x)$ solves QEiCP($A, B, C$) then $(-\lambda, x)$ solves QEiCP($A, -B, C$). In such a case, since $-\lambda$ is positive, we can apply item (ii) to QEiCP($A, -B, C$), replacing $\lambda$ by $-\lambda$ and $B$ by $-B$. This gives the result, taking into account the definitions of $z$ and $H$.

b) Consider first item (i). We know that (12)–(16) hold with $(u,v) = (y,x)$, and we need to check that

$$(1+\lambda)(\lambda^2 Ax + \lambda Bx + Cx) \geq 0, \tag{21}$$

$$(1+\lambda)x \geq 0, \tag{22}$$

$$(1+\lambda)^2 \left[x^t(\lambda^2 Ax + \lambda Bx + Cx)\right] = 0, \tag{23}$$

$$(1+\lambda)e^t x = 1. \tag{24}$$

If $\lambda \geq 0$ then (22) follows immediately from (14). It is rather elementary to verify that if it holds that

$$y = \lambda x, \tag{25}$$

then (21) follows from (12), (23) follows from (19), and (24) follows from (20). Therefore $(\lambda, (1+\lambda)x)$ solves QEiCP$(A,B,C)$, provided $\lambda \geq 0$. We proceed to prove (25) component-wise, and at the same time we establish that $\lambda \geq 0$. Note that (13) and (14) imply that $x \geq 0, y \geq 0$ and

$$\lambda x - y \geq 0. \tag{26}$$

Taking into account (12), we conclude that the four factors in both terms of the left hand side of (15) are nonnegative, so that both terms vanish. Looking now at the second one component-wise, we have $x_i(\lambda x_i - y_i) = 0$. If $x_i > 0$ then we get $\lambda x_i = y_i$ as required. If $x_i = 0$, we get from (26) that $-y_i \geq 0$, and hence $y_i = 0$ by (14), so that $\lambda x_i = y_i$ holds trivially. We have shown that (25) holds, and hence $(\lambda, (1+\lambda)x)$ solves QEiCP$(A,B,C)$. Finally, positivity of $\lambda$ follows also from (25): since $(x,y) \geq 0$ by (14) and $(x,y) \neq 0$ by (16), $\lambda \leq 0$ entails a contradiction with (25).

For item (ii), we apply the same argument as in item (i) to EiCP$(D,H)$. Since $G$ and $H$ differ just by the sign of $B$, we conclude that $(\lambda, (1+\lambda)x)$ solves QEiCP$(A,-B,C)$. It now follows from the definition of QEiCP$(A,B,C)$ that $(-\lambda, (1+\lambda)x)$ solves it.

□

We comment that our sufficient condition requires only item (b) of Proposition 1; however, item (a) has some interesting consequences, see Remarks 1 and 2 below.

Now we rephrase the result of Proposition 1 just in terms of complementary eigenvalues.

**Corollary 1.** *Consider QEiCP$(A,B,C)$ with $A,B,C \in \mathbb{R}^{n\times n}$ and the matrices $D,G,H \in \mathbb{R}^{2n\times 2n}$ as defined in* (9)–(11)*. Then,*

*i) all quadratic complementary eigenvalues for $(A,B,C)$ are complementary eigenvalues for either $(D,G)$, $(D,H)$ or both,*

*ii) all nonzero complementary eigenvalues for $(D,G)$ are positive, and are quadratic complementary eigenvalues for $(A,B,C)$,* *iii) all nonzero complementary eigenvalues for $(D, H)$ are positive, and their additive inverses are quadratic complementary eigenvalues for $(A, B, C)$.*

*Proof.* Elementary from Proposition 1. $\square$

Corollary 1 signals a clear path for obtaining a sufficient condition for existence of solutions of QEiCP$(A, B, C)$: we must first find a sufficient condition for solvability of EiCP$(D, G)$ or EiCP$(D, H)$ (which in principle depends only on the matrix in the leading term in (1), namely $D$ in this case, and henceforth just on $A$, in terms of the data of the QEiCP), and then impose conditions ensuring that either 0 is a quadratic complementary eigenvalue for $(A, B, C)$, or that 0 is not a complementary eigenvalue of $(D, G), (D, H)$ (which, as mentioned in the proof of Proposition 1(a), depends only upon $C$).

We will present next some already known conditions which fit the recipe above, for which we need to recall the definitions of three classes of matrices (see e.g. [3]).

**Definition 1.** *i) A matrix $M \in \mathbb{R}^{n\times n}$ is said to be strictly copositive if $x^t Mx > 0$ for all $0 \neq x \in \mathbb{R}^n$, $x \geq 0$.*

*ii) The class $R_0 \subset \mathbb{R}^{n\times n}$ consists of those matrices $M \in \mathbb{R}^{n\times n}$ such that there exists no $x \in \mathbb{R}^n$ satisfying $x \geq 0, Mx \geq 0, x^t Mx = 0$.*

*iii) The class $S_0 \subset \mathbb{R}^{n\times n}$ consists of those matrices $M \in \mathbb{R}^{n\times n}$ such that there exists $0 \neq x \in \mathbb{R}^n$ satisfying $x \geq 0, Mx \geq 0$.*

**Proposition 2.** *i) If $M \in \mathbb{R}^{n\times n}$ is strictly copositive then EiCP$(M, C)$ has solutions for any $C \in \mathbb{R}^{n\times n}$.*

*ii) If $C \notin R_0$ then 0 is a quadratic complementary eigenvalue for $(A, B, C)$ for any $A, B \in \mathbb{R}^{n\times n}$.*

*iii) If $C \notin S_0$ then 0 is not a complementary eigenvalue for either $(D, G)$ or $(D, H)$ with $D, G, H$ as in (9)–(11).*

*Proof.* Item (i) has been proved in [11] (see also Section 3). Item (ii) is immediate from the definitions of QEiCP and $R_0$. For item (iii), assume that 0 is a complementary eigenvalue for $(D, G)$, with associated complementary eivenvector $0 \neq z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$. It follows immediately that $By + Cx \geq 0$, $-y \geq 0, x \geq 0, y \geq 0$, implying $y = 0$, and hence $Cx \geq 0$ and $x \neq 0$ because $z \neq 0$, so that we get a contradiction with the assumption that $C \notin S_0$. The same argument can be used for the case of $(D, H)$. $\square$

Now, all the pieces are in place for stating and proving our existence result for QEiCP.

**Theorem 1.** *Consider QEiCP$(A, B, C)$ and assume that either*

*i) $C \notin R_0$, or*

*ii)* $C \notin S_0$ *and* $A$ *is strictly copositive.*

*Then QEiCP($A, B, C$) has solutions. Additionally, under assumption (i)* $0$ *is a quadratic complementary eigenvalue for* $(A, B, C)$*, and under assumption (ii) there exist at least one positive and one negative quadratic complementary eigenvalue for* $(A, B, C)$*.*

*Proof.* If (i) holds then 0 is a quadratic complementary eigenvalue for $(A, B, C)$ by Proposition 2(ii). Assume now that (ii) holds. Strict copositivity of $A$ implies strict copositivity of $D$, so that both EiCP$(D, G)$ and EiCP$(D, H)$ have complementary eigenvalues by Proposition 2(i), which are nonzero by Proposition 2(iii), and hence positive by items (ii) and (iii) of Corollary 1. Hence there exist at least one positive and one negative quadratic complementary eigenvalue for $(A, B, C)$. $\square$

In the remainder of this section, we discuss the existence result given in Theorem 1. We start with a corollary, stating that the roles of $A$ and $C$ in item (ii) of Theorem 1 can be reversed.

**Corollary 2.** *Consider QEiCP($A, B, C$) and assume that* $A \notin S_0$ *and* $C$ *is strictly copositive. Then there exist at least one positive and one negative quadratic complementary eigenvalue for* $(A, B, C)$*.*

*Proof.* Apply Theorem 1(ii) to QEiCP$(C, B, A)$ and conclude that it has a solution $(\lambda, x)$ with $\lambda > 0$, so that

$$w = \lambda^2 Cx + \lambda Bx + Ax \geq 0, x \geq 0, w^t x = 0. \tag{27}$$

Let $\mu = \lambda^{-1}$. Divide the first inequality in (27) by $\lambda^2$, and get from (27) $\bar{w} = \mu^2 Ax + \mu Bx + Cx \geq 0, x \geq 0, \bar{w}^t x = 0$, so that $(\mu, x)$ solves QEiCP$(A, B, C)$ and $\mu > 0$. Proceeding in the same fashion with QEiCP$(C, -B, A)$, get a solution $(\bar{\lambda}, \bar{x})$ of this problem with $\bar{\lambda} > 0$, take $\bar{\mu} = \bar{\lambda}^{-1}$ and conclude that $(\bar{\mu}, \bar{x})$ solves QEiCP$(A, -B, C)$, so that $-\bar{\mu}$ is a negative quadratic complementary eigenvalue for $(A, B, C)$. $\square$

We continue with four remarks related to the result in Theorem 1.

**Remark 1.** When we move from QEiCP$(A, B, C)$ to EiCP$(D, G)$, we can settle the issue of existence of solutions for the former excepting in one “undeterminated” case: when we only know that 0 is a complementary eigenvalue for $(D, G)$. If EiCP$(D, G)$ has no solutions then the same happens to QEiCP$(A, B, C)$ by Corollary 1(i), if EiCP$(D, G)$ has a solution $(\lambda, x)$ with $\lambda \neq 0$ then $\lambda$ is a quadratic complementary eigenvalue for $(A, B, C)$ by Corollary 1(ii), but the fact that 0 is a complementary eigenvalue for $(D, G)$ entails no conclusion at all about the existence of solutions of QEiCP$(A, B, C)$. The same considerations hold for EiCP$(D, H)$.

**Remark 2.** Another consequence of Corollary 1 is the following: if a method for finding all complementary eigenvalues for an arbitrary instance EiCP is available, applying it to EiCP$(D, G)$ and EiCP$(D, H)$ will provide all quadratic complementary eigenvalues of QEiCP$(A, B, C)$; in fact all complementary eigenvalues of these two EiCP’s will result in quadratic complementary eigenvalues for QEiCP$(A, B, C)$ (with the possible exception of 0, which can be checked separately) by virtue of

Corollary 1(ii)–(iii), and no quadratic complementary eigenvalue will be missed, as a consequence of Corollary 1(i).

**Remark 3.** We mention that strict copositivity of $A$ by itself is not sufficient for existence of solutions of QEiCP$(A, B, C)$. Considering QEiCP$(I, 0, I)$, it is easy to show that it lacks solutions, while $I$ is strictly copositive. In this case $C \in S_0$ and 0 is a complementary eigenvalue for $(D, G)$, but $C \in R_0$ and hence 0 is not a quadratic complementary eigenvalue of QEiCP$(I, 0, I)$.

**Remark 4.** Showing that a matrix $C$ either belongs or does not belong to $S_0$ reduces to a linear program. Furthermore, any one of the following two conditions is obviously sufficient for ensuring that $C \notin S_0$:

i) $-C$ is strictly copositive (or even positive definite),

ii) $C$ has a fully negative row, i.e. there exists $i \in \{1, \dots, n\}$ such that $C_{ij} < 0$ for all $j \in \{1, \dots, n\}$.

Finally, we close the section with the comparison between our sufficient condition for existence of solutions of QEiCP$(A, B, C)$ and an already known sufficient condition, introduced by A. Seeger in [19] and stated in the next proposition.

**Proposition 3.** *If* $(A, B, C)$ *satisfy*

$$x^t A x \neq 0, \tag{28}$$

$$(x^t B x)^2 \geq (x^t A x)(x^t C x), \tag{29}$$

*for all* $0 \neq x \in \mathbb{R}^n, x \geq 0$*, then QEiCP*$(A, B, C)$ *has solutions.*

*Proof.* See [19]. □

In [19], matrices $A$ satisfying (28) are called *co-regular* and triplets $(A, B, C)$ satisfying (29) are said to be *co-hyperbolic*.

For the comparison between the assumptions of Theorem 1 and Proposition 3, we say that a triplet $(A, B, C)$ satisfies (P) when either $C \notin S_0$ and $A$ is strictly copositive, or $C \notin R_0$, and that it satisfies (P') when $A$ is co-regular and $(A, B, C)$ is co-hyperbolic.

We mention that if $A$ is strictly copositive and $C$ satisfies condition (i) in Remark 4, then (P') holds, because in such a case one has $x^t A x \geq 0$, $x^t C x \leq 0$ for all $x \in \mathbb{R}^n_+$, so that the right hand side in (29) is nonpositive, making this inequality valid.

On the other hand, it is easy to exhibit instances in which (P) holds but (P') does not. For instance, take $A$ positive definite, $B = 0$ and $C$ satisfying condition (ii) in Remark 4 and having a positive diagonal element (i.e., there exist $i, k \in \{1, \dots, n\}$ such that $C_{ij} < 0$ for all $j \in \{1, \dots, n\}$ and $C_{kk} > 0$). Clearly (P) holds but, taking $x$ equal to the $k$-th vector in the canonical basis of $\mathbb{R}^n$, i.e, $e^k$ with $e^k_j = \delta_{jk}$ (Kronecker's delta), one has $x^t B x = 0$, $(x^t A x)(x^t C x) = A_{kk} C_{kk} > 0$. Hence (P') fails.

There are also many instances of QEiCP for which (P') holds but not (P). Take for instance $A = C = I$, $B = 2I$. Validity of (P') is immediate, but (P) fails because $I \in R_0 \cup S_0$, as can be easily verified. Hence, (P) and (P') are independent of each other.

We comment now on some features of (P) and (P').

i) (P) depends only upon the matrices $A$ and $C$, while (P') also involves the matrix $B$.

ii) The copositivity of $A$ in (P) and the co-regularity of $A$ in (P') are in a certain sense comparable in terms of the difficulty of checking their validity. In fact, co-regularity of $A$ is equivalent to copositivity of either $A$ or $-A$, because it implies that the sign of $\phi(x) = x^t Ax$ cannot change within the nonnegative orthant. Both hold when $A$ is positive definite, a standard condition for the matrix in the leading term for EiCP (note that (P') also holds when $A$ is negative definite).

iii) On the other hand, there is a remarkable difference between (P) and (P') in terms of the remaining conditions, i.e., besides copositivity in (P) and co-regularity in (P'). The co-hyperbolicity condition given by (29) is definitely quite hard to check, excepting in very special cases (e.g. when $A$ and $-C$ are both copositive, as mentioned above). At the same time, determining whether a given matrix belongs to the class $S_0$ reduces to solving a linear programming problem, a task much easier that determining copositivity, for instance.

Other differences between (P) and (P'), related to the variational inequality problems induced by each of them, are discussed in the next section.

# 3 The reformulation of QEiCP as nonlinear complementarity or variational inequality problems

The fact that EiCP can be reformulated as a nonlinear complementarity or a variational inequality problem was already mentioned in Section 1, and was in fact recognized very early in the history of the subject, see e.g. [11]. We show in this section that the same reformulations work for QEiCP (see e.g. [5]). We start by giving a general overview of the issue, for the sake of self-containement, and then we discuss it from the perspective of properties (P) and (P').

We recall now the definition of the nonlinear complementarity problem. Let $\mathbb{R}^m_+$ be the nonnegative orthant of $\mathbb{R}^m$. Given $F : \mathbb{R}^m_+ \to \mathbb{R}^m$, the *nonlinear complementarity problem* NCP($F$) consists of finding $z \in \mathbb{R}^m$ such that

$$z \geq 0, \quad F(z) \geq 0, \quad F(z)^t z = 0. \tag{30}$$

We rewrite EiCP and QEiCP as instances of NCP. Consider first EiCP($B, C$) and assume that $B$ is strictly copositive. Condition (3) can be rewritten as $\lambda(x^t Bx) - (x^t Cx) = 0$, so that if $(\lambda, x)$

solves EiCP$(B, C)$ then $\lambda = (x^t C x)/(x^t B x)$. If we consider now $\bar{F} : \mathbb{R}^n \to \mathbb{R}^n$ as introduced in Section 1, namely

$$\bar{F}(x) = \frac{x^t C x}{x^t B x} Bx - Cx, \tag{31}$$

then it is immediate that (1)–(3) are precisely the conditions defining NCP$(\bar{F})$, namely (30). Since (4) has been introduced just for ensuring that complementary eigenvectors are nonzero, we can encapsulate the relation between EiCP$(B, C)$ and NCP$(\bar{F})$ in the following proposition.

**Proposition 4.** *If $(x^*, \lambda^*)$ solves EiCP$(B, C)$ then $x^*$ solves NCP$(\bar{F})$ with $\bar{F}$ as in* (31). *If $\bar{x}$ is a nonzero solution of NCP$(\bar{F})$ then $(\lambda^*, x^*)$ solves EiCP$(B, C)$ with $x^* = \|\bar{x}\|_1^{-1} \bar{x}$, $\lambda^* = (\bar{x}^t C \bar{x})/(\bar{x}^t B \bar{x})$.*

*Proof.* Elementary; see the paragraph just before the statement of the proposition. □

We continue with QEiCP$(A, B, C)$, observing that if $(\lambda, x)$ solves QEiCP$(A, B, C)$ then (7) can be rewritten as $\lambda^2(x^t A x) + \lambda(x^t B x) + (x^T C x) = 0$ so that $\lambda$ can be obtained from $x$ by solving this quadratic equation. Assume now that $A$ is co-regular and $(A, B, C)$ is co-hyperbolic, and define $\lambda_1, \lambda_2 : \mathbb{R}^n_+ \to \mathbb{R}$ and $F_1, F_2 : \mathbb{R}^n_+ \to \mathbb{R}^n$ as

$$\lambda_1(x) = \frac{-x^t B x + \sqrt{(x^t B x)^2 - 4(x^t A x)(x^t C x)}}{2x^t A x}, \tag{32}$$

$$\lambda_2(x) = \frac{-x^t B x - \sqrt{(x^t B x)^2 - 4(x^t A x)(x^t C x)}}{2x^t A x}, \tag{33}$$

$$F_i(x) = \lambda_i(x)^2 A x + \lambda_i(x) B x + C x, \quad (i = 1, 2). \tag{34}$$

As in the case of EiCP, we get the following connection between QEiCP$(A, B, C)$ and NCP$(F_1)$, NCP$(F_2)$:

**Proposition 5.** *If $(x^*, \lambda^*)$ solves QEiCP$(A, B, C)$ then $x^*$ solves either NCP$(F_1)$ or NCP$(F_2)$, with $F_i$ as in* (34) *$(i = 1, 2)$. If $\bar{x}$ is a nonzero solution of NCP$(F_i)$ $(i = 1, 2)$ then $(\lambda^*, x^*)$ solves QEiCP$(A, B, C)$ with $x^* = \|\bar{x}\|_1^{-1} \bar{x}$, $\lambda^* = \lambda_i(\bar{x})$ $(i = 1, 2)$, as defined in* (32), (33).

*Proof.* Elementary, and similar to the proof of Proposition 4. □

An undesirable feature of this reduction of EiCP or QEiCP to NCP is that only nonzero solutions of NCP give rise to solutions to EiCP or QEiCP, and this request does not appear in the definition of NCP. It would be desirable to add this request to NCP as an additional constraint, as in (4), (8). The attempt to add additional constraints to a nonlinear complementarity problem leads naturally to a variational inequality problem, where the feasible set can be any closed and convex subset of $\mathbb{R}^m$, rather than the nonnegative orthant, as is the case for NCP.

We recall that given a closed and convex $K \subset \mathbb{R}^m$ and $F : K \to \mathbb{R}^m$, the *variational inequality problem* VIP$(F, K)$ consists of finding $\bar{z} \in K$ such that

$$F(\bar{z})^t(z - \bar{z}) \geq 0 \;\; \forall \; z \in K. \tag{35}$$

For our purposes, we are interested in a particular subset of $\mathbb{R}^m$ as the feasible set for VIP, namely the set $\Omega$ introduced in Section 1, defined as $\Omega = \{z \in \mathbb{R}^m : z \geq 0, e^t z = 1\}$, with $e = (1, 1, \ldots, 1)^t \in \mathbb{R}^m$.

Recall that $F : \mathbb{R}^m_+ \to \mathbb{R}^m$ is positively homogeneous when $F(\alpha z) = \alpha F(z)$ for all $(z, \alpha) \in \mathbb{R}^m_+ \times \mathbb{R}_+$. We have the following relation between NCP$(F)$ and VIP$(F, \Omega)$.

**Proposition 6.** *i) If $F$ is positively homogeneous and $\bar{z}$ is a nonzero solution of NCP($F$), then $z^* = \|\bar{z}\|_1^{-1} \bar{z}$ solves VIP($F, \Omega$).*

*ii) If $z^*$ solves VIP($F, \Omega$) and $F(z^*)^t z^* = 0$ then $z^*$ solves NCP($F$).*

*Proof.* i) Note that $\bar{z} \geq 0$ by the definition of NCP, and the normalization in the definition of $z^*$ guarantees that $z^*$ belongs to $\Omega$. Also, for all $z \in \Omega$ it holds that

$$F(z^*)^t(z - z^*) = \|\bar{z}\|_1^{-1} F(\bar{z})^t z - \|\bar{z}\|_1^{-2} F(\bar{z})^t \bar{z} = \|\bar{z}\|_1^{-1} F(\bar{z})^t z \geq 0,$$

using the complementarity condition in the second equality and the facts that $F(\bar{z}) \geq 0$ and that $z$ belongs to $\Omega$ (so that $z \geq 0$), in the inequality. We have proved that (35) holds.

ii) $z^* \geq 0$ because $z \in \Omega$, and $F(z^*)^t z^* = 0$ by assumption. It remains to prove that $F(z^*) \geq 0$. By the definition of VIP, for all $y \in \Omega$ it holds that $0 = F(z^*)^t(z - z^*) = F(z^*)^t z$, using again the assumption that $F(z^*)^t z^* = 0$. Taking now as $z$ the elements of the canonical basis of $\mathbb{R}^m$, which belong to $\Omega$, we get $F(z^*)_i \geq 0$ $(1 \leq i \leq m)$, completing the proof.

□

We remark that it follows easily from (31)–(34) that the operators $\bar{F}$, $F_1$ and $F_2$ are positively homogeneous and satisfy $0 = \bar{F}(z)^t z, 0 = F_1(z)^t z = F_2(z)^t z$ for all $z \in \Omega$, which means that both for EiCP and for QEiCP the related NCP and VIP are basically equivalent.

As mentioned in Section 1, compactness and convexity of $K$ and continuity of $F$ on $K$ guarantee existence of solutions of VIP$(F, K)$ (see [4]). This classical result, together with the comments in the previous paragraph and Propositions 4–6, easily provide proofs for Propositions 2(i) and 3.

We have finished with the announced overview, and now we focus on two alternative approaches for solving QEiCP$(A, B, C)$ through variational inequalities, namely solving VIP$(F_1, \Omega)$ and VIP$(F_2, \Omega)$, with $m = n$ and $F_1, F_2$ as in (34), or solving VIP$(\bar{F}, \Omega)$ with $m = 2n$ and $\bar{F}$ related to EiCP$(D, G)$ or EiCP$(D, H)$, where $D, G, H$ are given by (9)–(11). Taking into account (31), we write next the formula of $\bar{F}$ for this second option in terms of the original data of QEiCP, with $z = (y, x) \in \mathbb{R}^n \times \mathbb{R}^n$. We have

$$\bar{F}(y, x) = \left( \frac{y^t(I - C)x - y^t B y}{y^t A y + \|x\|_2^2} \right) \begin{bmatrix} Ay \\ x \end{bmatrix} + \begin{bmatrix} By + Cx \\ -y \end{bmatrix}. \tag{36}$$

We denote as Approach 1 the one which deals with VIP$(F_i, \Omega)$ $(i = 1, 2)$ and as Approach 2 the one dealing with VIP$(\bar{F}, \Omega)$, with $\bar{F}$ as in (36).

We proceed to make a conceptual comparison between these two approaches. One obvious drawback of Approach 2 is that it deals with a VIP in dimension $2n$ while both VIP's in Approach 1 work in dimension $n$.

On the other hand, Approach 2 exhibits al least two attractive features, which might overcome the dimensionality issue. In the first place, the formula of $\bar{F}$ given in (36) is simpler than the formulae of $F_1$, $F_2$ given by (32)–(34). In fact, the presence of the square root in (32) and (33) is certainly undesirable from a numerical point of view, and could make the evaluation of the $F_i$'s at a given point costlier than an evaluation of $\bar{F}$, despite the dimensionality issue.

The second advantage of Approach 2 over Approach 1 refers to a robustness property. As we have mentioned, both approaches require similar assumptions on $A$ (co-regularity or strict copositivity) in order to ensure that the denominators in (32), (33) and (36) do not vanish, and when these assumptions are not valid both approaches might fail. The situation is however different when we look at the additional assumptions, in particular at the co-hyperbolicity of $(A, B, C)$ for the case of Approach 1, which, as already mentioned, is pretty hard to check. Assume that we have an instance of QEiCP with strictly copositive $A$, but such that neither the assumptions on $C$ in Property (P) nor the co-hyperbolicity in Property (P') have been checked. Suppose also that we solve the respective VIP's with some feasible method, i.e. one which approaches a solution $z$ of the problem through a sequence $\{z^k\} \subset \Omega$, and evaluates the operators $\bar{F}$, $F_1$ or $F_2$ at the $z^k$'s (and possibly at other feasible points too). Regardless of the assumptions on $C$, when using Approach 2 a solution $(\lambda, z)$ of EiCP$(D, G)$ or EiCP$(D, H)$ will be found, and if $\lambda \neq 0$ this will provide a solution of QEiCP$(A, B, C)$ as a result of Theorem 1. If $\lambda = 0$, it can be immediately checked whether $(0, z)$ solves QEiCP$(A, B, C)$, and only when this does not happen the procedure fails: it cannot be determined whether $(A, B, C)$ has or not some nonzero quadratic complementarity eigenvalue. The situation is rather worse when following Approach 1. If at some iteration the evaluation of $F_1$ or $F_2$ is required at a point $x$ where the co-hyperbolicity condition fails, i.e. such that $(x^T Bx)^2 < 4(x^t Ax)(x^t Cx)$, then the method just breaks when attempting to evaluate the square root in (32) or (33), and nothing is obtained in terms of solutions of QEiCP$(A, B, C)$. In this sense, Approach 2 looks more robust than Approach 1; the former always provides a pair $(\lambda, x)$, and a solution of QEiCP whenever $\lambda \neq 0$, while the latter is likely to stop at any iteration when the co-hyperbolicity condition is not known to hold.

# 4 A nonlinear programming formulation of the QEiCP

Let $A \in SC$ and $C \notin S_0$. If $D$ and $G$ are the matrices given by (9) and (10) respectively, then, by Proposition 1, EiCP $(D, G)$ has at least a solution $(\bar{\lambda}, \bar{z})$ such that $\bar{\lambda} > 0$ and $\bar{z} = (\bar{y}, \bar{x})$ satisfies $\bar{y} = \bar{\lambda}\bar{x}$. Furthermore $(\bar{\lambda}, (1 + \bar{\lambda})\bar{x})$ is a solution of QEiCP. Hence there is a vector $\bar{w} \geq 0$ such that

$(\bar{x}, \bar{y}, \bar{w}, \bar{\lambda})$ satisfies the following constraints

$$\lambda \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} - \begin{pmatrix} -B & -C \\ I & 0 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} w \\ 0 \end{pmatrix}$$
$$e^t y + e^t x = 1$$
$$y^t w = x^t w = 0$$
$$y = \lambda x$$
$$x, y, w, \lambda \geq 0$$

By introducing the vector $\bar{v}$ defined by $\bar{v} = \bar{\lambda}\bar{y}$, then $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ is a solution of the following system

$$Av + By + Cx = w \tag{37}$$

$$e^t y + e^t x = 1 \tag{38}$$

$$e^t v + e^t y = \lambda \tag{39}$$

$$w, x, y, v \geq 0 \tag{40}$$

$$w^t v = w^t y = w^t x = 0 \tag{41}$$

$$y = \lambda x \tag{42}$$

$$v = \lambda y \tag{43}$$

Let

$$K = \{(x, y, v, w, \lambda) : (x, y, v, w, \lambda) \text{ satisfies (37)-(40)}\} \tag{44}$$

and consider the nonlinear program

$$\begin{array}{ll} \text{NLP}: & \text{Minimize } f(x, y, v, w, \lambda) = \|y - \lambda x\|_2^2 + \|v - \lambda y\|_2^2 + (x + y + v)^t w \\ & \text{subject to } (x, y, v, w, \lambda) \in K. \end{array} \tag{45}$$

Then the following result holds:

**Proposition 7.** *Let $A \in SC$, $C \notin S_0$ and $K$ be the set given by* (44)*. Then the NLP* (45) *has a global minimum $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ such that $f(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda}) = 0$ and $(\bar{\lambda}, (1+\bar{\lambda})\bar{x})$ is a solution of QEiCP.*

Since computing a global minimum of NLP (45) is a difficult task, it is interesting to investigate when a stationary point of $f$ on $K$ provides a solution of QEiCP. The following result answers to this question.

**Proposition 8.** *A stationary point $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ of $f$ on $K$ is a global minimum of NLP* (45) *with $f(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda}) = 0$ if and only if the Lagrange multipliers associated to the equalities* (38) *and* (39) *are equal to zero.*

*Proof.* Let $u, \gamma_0$ and $\theta_0$ be the Lagrange multipliers associated to the equalities (37), (38) and (39) respectively and $\alpha, \beta, \gamma$ and $\theta$ be the Lagrange multipliers associated to the nonnegative constraints $w \geq 0$, $x \geq 0$, $y \geq 0$ and $v \geq 0$ respectively. Hence $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ satisfies the following KKT conditions:

$$x + y + v = u + \alpha \tag{46}$$

$$-2\lambda(y - \lambda x) + w = -C^t u + \beta + \gamma_0 e \tag{47}$$

$$2[(y - \lambda x) - \lambda(v - \lambda y)] + w = -B^t u + \gamma + \gamma_0 e + \theta_0 e \tag{48}$$

$$2(v - \lambda y) + w = -A^t u + \theta + \theta_0 e \tag{49}$$

$$-2x^t(y - \lambda x) - 2y^t(v - \lambda y) = -\theta_0 \tag{50}$$

$$x,\ y,\ v,\ w\ \geq 0 \tag{51}$$

$$\beta,\ \gamma,\ \theta,\ \alpha\ \geq 0 \tag{52}$$

$$\beta^t x = \gamma^t y = \theta^t v = \alpha^t w = 0 \tag{53}$$

Multiplying (46), (47), (48) and (49) by $w^t$, $x^t$, $y^t$ and $v^t$ respectively, using (53) and adding the resulting equalities term by term, we get

$$2[(y - \lambda x)^t(y - \lambda x) + (v - \lambda y)^t(v - \lambda y) + w^t(x + y + v)] = \gamma_0 + \theta_0 \lambda$$

i.e.,

$$2f(x, y, v, w, \lambda) = \gamma_0 + \theta_0 \lambda \tag{54}$$

Now, if $\gamma_0 = \theta_0 = 0$, then $f(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda}) = 0$ and $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ is a global minimum of NLP. Conversely, if $(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda})$ is a global minimum of NLP with $f(\bar{x}, \bar{y}, \bar{v}, \bar{w}, \bar{\lambda}) = 0$, then $\gamma_0 + \theta_0 \lambda = 0$. Furthermore $\theta_0 = 0$ by (50) and $\gamma_0 = 0$. □

Note: A Nonlinear Program similar to NLP (45) can be constructed associated to EiCP $(D, H)$.

# 5 Local algorithms for QEiCP

In Section 3, it is shown that a solution to the QEiCP can be found by solving the VIP$(\bar{F}, \Omega)$, where $\bar{F}$ is the mapping defined by (36) and

$$\Omega = \{z = (y, x) \in \mathbb{R}^{2n} : e^t y + e^t x = 1,\ y \geq 0, x \geq 0\}. \tag{55}$$

A popular technique for solving this VIP$(\bar{F}, \Omega)$ is to use the so-called Regularized Gap-Function $f_\alpha$ defined for a given $\alpha > 0$, by

$$f_\alpha(z) = -\min\{\bar{F}(z)^t(u - z) + \frac{\alpha}{2}\|u - z\|_2^2 : u \in \Omega\} \tag{56}$$

for each $z = (y, x) \in \Omega$. The following property holds:

**Proposition 9.** [2] *For each $\alpha > 0$, $\bar{z} = (\bar{y}, \bar{x}) \in \Omega$ is a global minimum of*

$$\begin{array}{ll} \text{Minimize} & f_\alpha(z) \\ \text{subject to} & z \in \Omega \end{array} \tag{57}$$

*with value $f_\alpha(\bar{z}) = 0$ if and only if $\bar{z}$ is a solution of VIP($\bar{F}, \Omega$).*

This property shows that solving VIP($\bar{F}, \Omega$) reduces to a global optimization problem of a continuously differentiable function on the simplex. Such a point is very difficult to compute in practice. A stationary point of $f_\alpha$ on $\Omega$ is much easier to find but there is no guarantee that it provides a solution of the VIP [4]. Furthermore the computation of such a point requires the gradient of $f_\alpha$, and the computation of this vector is quite involved. To alleviate the computational work we recommend the derivative-free projection algorithm (DFP) discussed in [2] for solving the VIP($\bar{F}, \Omega$). In order to briefly explain such a procedure, let $\bar{z} = (\bar{y}, \bar{x}) \in \Omega$ be a current point. Then a search direction is computed by

$$d = P_\Omega \left( \bar{z} - \frac{1}{\alpha} \bar{F}(\bar{z}) \right) - \bar{z} \tag{58}$$

where $P_\Omega(u)$ is the projection of $u \in \mathbb{R}^{2n}$ in the simplex that is, the unique global minimum and stationary point of

$$\begin{array}{ll} \text{Minimize} & \|u - v\|_2^2 \\ \text{subject to} & v \in \Omega. \end{array} \tag{59}$$

The computation of such a point can be done in polynomial-time by a number of algorithms [2]. Now, if $d = 0$ (i.e., $\|d\|_2 < \epsilon$ for some tolerance) then $\bar{z}$ is a solution of VIP. Otherwise $d$ satisfies $\bar{F}(\bar{z})^t d < 0$ [2] and we look for a stepsize $\delta > 0$ such that implies the reduction of the value of $f_\alpha$ according to the following Armijo type criterion:

$$f_\alpha(\bar{z} + \delta d) \leq f_\alpha(\bar{z}) + \delta \beta \bar{F}(\bar{z})^t d, \tag{60}$$

where $0 < \beta < 1$. As for the usual Armijo criterion, a number of trials of the form

$$\delta = \frac{1}{p^{1.4}}, \tag{61}$$

for $p = 1, 2, \ldots$ is done until the condition (60) is satisfied. Unfortunately there is no theoretical guarantee that such a procedure terminates with a stepsize $\delta > 0$ satisfying (60) after a finite number of trials. In this last case the algorithm terminates unsuccessfully. If a stepsize $\delta$ is computed then the point $\bar{z}$ is updated by $\tilde{z} = \bar{z} + \delta d$. It is easy to show that this new point $\tilde{z}$ belongs to $\Omega$ and a new iteration of the DFP method should be applied with such a point. The steps of the algorithm are presented below:

**DFP algorithm**

**Step 0.** Let $\bar{z} = (\bar{y}, \bar{x}) \in \Omega$ and $\epsilon$ a positive tolerance.

**Step 1.** Compute $d$ by (58).

**Step 2.** If $\|d\|_2 < \epsilon$, terminate with an approximate solution of VIP$(\bar{F}, \Omega)$.

**Step 3.** Try to compute a stepsize $\delta$ of the form (61) satisfying (60). If such a $\delta$ cannot be computed after a finite number of trials terminate the algorithm with a failure.

**Step 4.** Update $\bar{z} := \bar{z} + \delta d$ and go to Step 1 with the new point $\bar{z}$.

It is important to note that the verification of the criterion (60) only requires the computation of values of the regularized gap-function $f_\alpha$. Due to the definition (56) of $f_\alpha$, the computation of these values reduces to a Strictly Convex Quadratic Separable Program on the simplex, which can be solved in polynomial-time by a number of efficient algorithm [2]. Therefore the DFP method is very simple to implement. The main drawback of this approach is its unability for solving the VIP in general. The performance of the algorithm may be improved by a special choice of the initial point. However, such a choice is also a difficult task.

Another local approach for solving QEiCP consists of finding a stationary point of NLP (45) introduced in Section 4. Such a point can be computed by an active-set method, as that implemented in the well-known code MINOS [13] or any other efficient nonlinear programming algorithm.

In order to verify the efficiency and efficacy of these two local techniques for solving the QEiCP, we have performed some experiments on the solution of the QEiCP test problems discussed in [6] by MINOS and DFP.

For this latter algorithm three different starting points have been used in the experiments:

(INP1) the baricenter of the simplex, i.e.

$$\bar{x}_i = \bar{y}_i = \frac{1}{2n}, \text{ for all } i = 1, \ldots, n$$

(INP2) a vector of the canonical basis, i.e.

$$\bar{x}_i = 1, \, \bar{x}_j = 0, \, j \neq i$$

$$\bar{y}_j = 0, \, j = 1, \ldots, n$$

for some $i$ (we used $i = n$).

(INP3) the stationary point $(\bar{y}, \bar{x})$ computed by MINOS.

The numerical performance of these two algorithms is highlighted in Tables 1 and 2 below. In these tables, the following notations are used:

- $n$: order of the matrices $A$, $B$ and $C$.
- $f$: value of the objective function at the Stationary Point of NLP (45) computed by MINOS.
- $\lambda$: value of the variable $\lambda$ at the Stationary Point of NLP (45) computed by MINOS and the complementary eigenvalue when DFP terminates successfully.
- $\|d\|$: norm of the DFP direction at the termination of the DFP algorithm.
- It: number of iterations required by the algorithms to terminate.

| Problem | $n$ | $f$ | $\lambda$ | It |
|---|---|---|---|---|
| SeegerAdlyQ(3) | 3 | 1.6291E-13 | 0.2656 | 4 |
| Rand(0,1,5) | 5 | 4.3760E-17 | 0.8422 | 19 |
| Rand(0,1,10) | 10 | 1.5157E-03 | 0.9057 | 69 |
| Rand(0,1,20) | 20 | 1.1929E-05 | 1.0287 | 231 |
| Rand(0,1,30) | 30 | 9.5175E-04 | 0.8958 | 254 |
| Rand(0,1,40) | 40 | 1.6352E-03 | 0.9045 | 332 |
| Rand(0,1,50) | 50 | 5.2166E-04 | 0.9844 | 538 |
| Rand(0,10,5) | 5 | 6.3680E-03 | 0.8561 | 18 |
| Rand(0,10,10) | 10 | 1.8237E-03 | 0.8937 | 79 |
| Rand(0,10,20) | 20 | 1.6556E-03 | 1.0997 | 99 |
| Rand(0,10,30) | 30 | 1.0369E-02 | 0.8946 | 135 |
| Rand(0,10,40) | 40 | 1.8319E-03 | 0.9983 | 264 |
| Rand(0,10,50) | 50 | 5.6333E-03 | 0.7545 | 365 |
| Rand(0,100,5) | 5 | 8.0186E-17 | 1.1120 | 34 |
| Rand(0,100,10) | 10 | 3.6174E-03 | 0.8139 | 65 |
| Rand(0,100,20) | 20 | 1.7354E-02 | 1.0031 | 75 |
| Rand(0,100,30) | 30 | 7.3798E-03 | 1.0373 | 169 |
| Rand(0,100,40) | 40 | 2.8234E-03 | 0.9097 | 265 |
| Rand(0,100,50) | 50 | 4.1658E-03 | 0.9286 | 287 |

Table 1: Solution of QEiCP by a Stationary Point of NLP (45).

The following conclusions can be stated from these numerical results:

(i) MINOS usually finds a stationary point whose objective function value is quite small, i.e., close to a solution of QEiCP. In three cases the stationary point provides a solution to the QEiCP.

(ii) In general the DFP algorithm does not perform well for each one of the three choices of initial points.

(iii) The performance of the DFP algorithm seems to be quite influenced by scaling, as it is much more efficient when all the elements of the matrices A, B and C belong to the interval $[0, 1]$ (four out of six instances have been solved when the barycenter is chosen as the initial point).

| Problem | INP1 $\|d\|$ | INP1 It | INP1 $\lambda$ | INP2 $\|d\|$ | INP2 It | INP2 $\lambda$ | INP3 $\|d\|$ | INP3 It | INP3 $\lambda$ |
|---|---|---|---|---|---|---|---|---|---|
| SeegerAdlyQ(3) | 9.615E-17 | 2 | 0.0 | 0.0 | 1 | 0.0 | 3.310E-07 | 1 | 0.2656 |
| Rand(0,1,5) | 3.990E-02 | 8 | | 9.387E-07 | 28 | 0.8253 | 2.092E-09 | 1 | 0.8422 |
| Rand(0,1,10) | 9.229E-07 | 137 | 0.8107 | 9.631E-07 | 130 | 0.8107 | 1.884E-02 | 2 | |
| Rand(0,1,20) | 4.834E-07 | 68 | 1.1142 | 0.1231 | 20 | | 3.399E-03 | 2 | |
| Rand(0,1,30) | 9.830E-07 | 162 | 1.0546 | 0.1100 | 15 | | 3.072E-02 | 1 | |
| Rand(0,1,40) | 5.031E-02 | 13 | | 0.2197 | 4 | | 3.938E-02 | 2 | |
| Rand(0,1,50) | 9.930E-07 | 315 | 0.1153 | 0.1094 | 13 | | 2.275E-02 | 1 | |
| Rand(0,10,5) | 2.645E-04 | 16 | | 0.7071 | 2 | | 7.536E-02 | 1 | |
| Rand(0,10,10) | 0.7046 | 2 | | 0.6102 | 3 | | 0.5236 | 2 | |
| Rand(0,10,20) | 0.5419 | 2 | | 0.9696 | 3 | | 3.879E-02 | 1 | |
| Rand(0,10,30) | 0.5691 | 2 | | 0.4518 | 2 | | 9.243E-02 | 1 | |
| Rand(0,10,40) | 0.4267 | 2 | | 0.7071 | 3 | | 3.875E-02 | 1 | |
| Rand(0,10,50) | 0.5833 | 1 | | 0.7071 | 3 | | 6.277E-02 | 1 | |
| Rand(0,100,5) | 0.7071 | 2 | | 0.7071 | 2 | | 8.932E-09 | 1 | 1.1120 |
| Rand(0,100,10) | 0.7071 | 3 | | 0.7071 | 2 | | 5.536E-02 | 1 | |
| Rand(0,100,20) | 0.7071 | 2 | | 0.2487 | 6 | | 0.1180 | 1 | |
| Rand(0,100,30) | 0.6992 | 1 | | 0.7071 | 2 | | 2.301E-02 | 1 | |
| Rand(0,100,40) | 0.7655 | 1 | | 0.7071 | 3 | | 5.243E-02 | 1 | |
| Rand(0,100,50) | 0.7071 | 2 | | 0.7071 | 2 | | 2.628E-02 | 1 | |

Table 2: Solution of QEiCP by DFP algorithm.

(iv) When DFP algorithm starts with an initial point that is a Stationary Point of NLP (45), then it usually terminates with this point (It $= 1$).

(v) The matrix $C$ of the test problem SeegerAdlyQ(3) is $S_0$ and is not $R_0$. Therefore zero is a complementary eigenvalue of QEiCP. For this instance both the algorithms MINOS and DFP have been able to find a solution of the QEiCP. However, DFP computed the zero complementary eigenvalue while MINOS found a positive eigenvalue.

These results clearly indicate the need of designing a global optimization algorithm for computing a global minimum of NLP (45) or finding a solution of VIP by using the generalized gap-function or any other appropriate merit function associated to the VIP$(\bar{F}, \Omega)$ [4]. This algorithm should use the local techniques discussed in this paper in order to speed up the search. This will certainly an interesting topic of our future research.

## 6 On the number of quadratic complementary eigenvalues

In this section, we discuss the maximum number of quadratic complementary eigenvalues for QEiCP$(A, B, C)$. We obtain it through the introduction of an additional variable $y \in \mathbb{R}^n$, leading to a $2n$ dimensional problem, similar to the one used in Section 2. A related upper bound for the number of solutions of EiCP$(B, C)$ has been established in [16].

As mentioned in Section 1, given $B, C \in \mathbb{R}^{n\times n}$, a *generalized eigenvalue* is a complex number $\lambda$ such that there exists $0 \neq x \in \mathbb{C}^n$ satisfying $\lambda Bx - Cx = 0$. The number of generalized eigenvalues for a pair $(B, C)$ is bounded by $n$, because the equation above demands singularity of $\lambda B - C$, or equivalently $\det(\lambda B - C) = 0$. Hence, the announced bound follows from the fact that $\det(\lambda B - C)$ is a polynomial of degree at most $n$ as a function of $\lambda$, having therefore at most $n$ complex roots.

Based on the same property, it has been proved in Corollary 5.4 of [19] that the number $\theta_n$ of quadratic complementary eigenvalues of $(A, B, C)$, with $A, B, C \in \mathbb{R}^{n\times n}$, is bounded by $n2^n$.

Next we present an $n$-dimensional instance of QEiCP with $2^{n+1} - 2$ quadratic complementary eigenvalues. This means that $2^{n+1} - 2 \leq \theta_n \leq n2^n$. We comment that an example of an $n$-dimensional EiCP with $3(2^{n-1} - 1)$ complementary eigenvalues has been exhibited in [21]. This example could be used to generate an example of QEiCP with $3(2^n - 2)$ quadratic complementary eigenvalues, improving over our example. Since the adaptation is not trivial and the example in [21] is itself quite involved, we opted to present our much simpler example, for which the counting procedure is also rather elementary.

Consider $\mathbb{R}$ as a vector space over $\mathbb{Q}$ and take a set $\{r_1, r_2, \ldots, r_n\}$ of $n$ positive real numbers which are linearly independent over $\mathbb{Q}$. Define $r, e \in \mathbb{R}^n$ as $r = (r_1, r_2, \ldots, r_n)^t$, $e = (1, 1, \ldots, 1)^t$, and consider the matrices $A, B, C$ defined as $A = -I, B = 0, C = er^t$. Note that $C$ has rank 1. For each nonempty subset $J$ of $N$ define the vectors $x^J, w^J \in \mathbb{R}^n$, and the numbers $s^J, \bar{\lambda}^J, \widehat{\lambda}^J$ as

$$x_j^J = \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{if } j \notin J, \end{cases} \tag{62}$$

$$s^J = \sum_{j\in J} r_j, \qquad \bar{\lambda}^J = \sqrt{s^J}, \qquad \widehat{\lambda}^J = -\sqrt{s^J}, \tag{63}$$

$$w^J = \bar{\lambda}^2 Ax^J + Cx^J = \widehat{\lambda}^2 Ax^J + Cx^J. \tag{64}$$

We claim that all triplets in the set $\left\{(\bar{\lambda}^J, x^J, w^J)\right\}_{\emptyset\neq J\subset N} \cup \left\{(\widehat{\lambda}^J, x^J, w^J)\right\}_{\emptyset\neq J\subset N}$ are solutions of QEiCP$(A, B, C)$. We proceed to establish the claim.

It follows from (62), (63) that

$$\left(\bar{\lambda}^J\right)^2 Ax^J = \left(\widehat{\lambda}^J\right)^2 Ax^J = -s^J x^J,$$

$$Cx^J = er^t x^J = (r^t x^J)e = s^J e,$$

so that, in view of (64)

$$w^J = s^J(-x^J + e) = s^J(e - x^J). \tag{65}$$

Denoting $J^c = N \setminus J$, it is immediate that $e - x^J = x^{J^c}$, so that $e - x^J \geq 0$ and, taking into account (62), $(x^J)^t(e - x^J) = (x^J)^t x^{J^c} = 0$, so that we conclude, using (65), (64) and the fact that $B = 0$, that

$$\left(\bar{\lambda}^J\right)^2 Ax^J + \bar{\lambda}^J Bx^J + Cx^J = \left(\widehat{\lambda}^J\right)^2 Ax^J + \widehat{\lambda}^J Bx^J + Cx^J = w^J,$$

$$x^J \geq 0, \quad w^J \geq 0 \quad (x^J)^t w^J = 0,$$

which proves the claim.

No we claim that all the numbers in the set $\left\{\bar{\lambda}^J\right\}_{\emptyset \neq J \subset N} \cup \left\{\widehat{\lambda}^J\right\}_{\emptyset \neq J \subset N}$ are different. Since we consider only nonempty subsets of $N$, the $\bar{\lambda}^J$'s are positive, and the $\widehat{\lambda}^J$'s, being the additive inverses of the $\bar{\lambda}^J$'s, are negative, and so it suffices to show that the $\bar{\lambda}^J$'s are all different. We proceed to do so. Assume that $J, K \subset N$ are such that $\bar{\lambda}^J = \bar{\lambda}^K$. Then, by (63),

$$0 = \left(\bar{\lambda}^J\right)^2 - \left(\bar{\lambda}^K\right)^2 = s^J - s^K = \sum_{j \in J} r_j - \sum_{k \in K} r_k, \tag{66}$$

i.e., we have a linear combination of the $r_j$'s with rational coefficients (in fact, they are $0, 1$ or $-1$), which vanishes. By the linear independence of the $r_j$'s, all coefficients in the linear combination must vanish, and then it follows easily from (66) that $J = K$, so that the second claim holds. Since $N$ has $2^n - 1$ nonempty subsets, and we have two quadratic complementary eigenvalues of $(A, B, C)$ for each subset $J$ of $N$, namely $\bar{\lambda}^J$ and $\widehat{\lambda}^J$, we have proved that $(A, B, C)$ has at least $2^{n+1} - 2$ quadratic complementary eigenvalues.

We observe that the linear independence of the $r_i$'s over $\mathbb{Q}$ is not essential; it suffice to take the $r_i$'s so that all their "partial sums" are different. We could have taken, for instance, $r_i = 10^i$.

Observe that the bound for the number of quadratic complementary eigenvalues given by $\theta_n$ admits up to $2j$ eigenvalues for each subset $J$ of cardinality $j$, while our example has only 2, independently of the cardinality of $J$. In fact, since $\text{rank}(C) = 1$, the same holds for all its principal submatrices, and it is immediate that $C^J$ has one positive eigenvalue, namely $s^J$, with the associated positive eigenvector $e^J$ (meaning the $j$-th dimensional version of $e$), and also the eigenvalue 0, with $j - 1$ eigenvectors, which form a basis of the orthogonal complement of $e^J$.

It is not difficult to perturb the matrix $C$ so that each submatrix $C^J$ has $j$ different positive eigenvalues, and in such a way that no pair of eigenvalues of different principal submatrices of $C$ coincide, but it does not seem possible to do so while preserving at the same time nonnegativity of all the associated eigenvectors, as well as of all entries of $C$, and these two nonnegativity conditions seem to be necessary for the satisfaction of the complementarity constraints, in such a way that the eigenvalues of the principal submatrices of $C$ give rise to quadratic complementary eigenvalues of the triplet $(-I, 0, C)$. Possibly, in order to attain the upper bound given by $\theta_n$ (if it is at all attainable), an example must be constructed using a triplet $(A, B, C)$ with $(A, B) \neq (-I, 0)$.

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