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by

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Plateau's Problem for Parametric Double Integrals: I. Existence and Regularity in the Interior

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Abstract

We study Plateau's problem for two-dimensional parametric integrals

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, dudv,$$

the Lagrangian $F(x,z)$ of which is positive definite and at least semi-elliptic. It turns out that there always exists a conformally parametrized minimizer. Any such minimizer X is seen to be Hölder continuous in the parameter domain B and continuous up to its boundary. If F possesses a *perfect dominance function* G of class C^2 we can establish higher regularity of X in the interior. In fact, we prove $X \in H^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^n) \cap C^{1,\sigma}(B, \mathbb{R}^n)$ for some $\sigma > 0$. Finally we discuss the existence of perfect dominance functions.

1 Introduction and main results

Let $\mathbb{B}^n$ be the space of bivectors $\zeta = \xi \wedge \eta$ with $\xi, \eta \in \mathbb{R}^n$, $n \geq 2$. If $\xi = (\xi^1, \xi^2, \dots, \xi^n)$, $\eta = (\eta^1, \eta^2, \dots, \eta^n)$, then $\zeta = (\zeta^{jk})_{j<k}$, where $\zeta^{jk} := \xi^j \eta^k - \xi^k \eta^j$. For $n = 3$, $\zeta = \xi \wedge \eta$ is the usual vector product of ξ and η, except that in this case we take ζ as $(\zeta^{23}, \zeta^{31}, \zeta^{12})$, in agreement with the standard notation.

The norm

$$|\zeta| := \left(\sum_{j<k} |\zeta^{jk}|^2 \right)^{1/2}$$

of $\zeta = \xi \wedge \eta$ can be expressed by the Lagrangian identity

$$|\xi \wedge \eta|^2 = |\xi|^2 |\eta|^2 - (\xi \cdot \eta)^2, \tag{1.1}$$

where $\xi \cdot \eta$ denotes the Euclidean inner product $\xi \cdot \eta = \xi^1\eta^1 + \cdots + \xi^n\eta^n$. This implies

$$|\xi \wedge \eta| \leq \frac{1}{2}(|\xi|^2 + |\eta|^2), \tag{1.2}$$

and we have equality if and only if $|\xi|^2 = |\eta|^2$ and $\xi \cdot \eta = 0$. We can identify $\mathbb{B}^n$ with $\mathbb{R}^N$, $N := n(n-1)/2$. In the sequel we consider Lagrangians $F(x, z)$ which are positively homogeneous of degree 1 with respect to z, i.e. which satisfy

$$F(x, tz) = tF(x, z) \text{ for all } t > 0, \ (x, z) \in \mathbb{R}^n \times \mathbb{R}^N. \tag{H}$$

DEFINITION 1.1 *A* parametric Lagrangian $F(x, z)$ *is a function of class* $C^0(\mathbb{R}^n \times \mathbb{R}^N)$ *satisfying condition (H). Such a Lagrangian is said to be* positive definite *if there are two numbers* m_1 *and* m_2 *with* $0 < m_1 \leq m_2$ *such that*

$$m_1|z| \leq F(x, z) \leq m_2|z| \quad \textit{for all} \ (x, z) \in \mathbb{R}^n \times \mathbb{R}^N. \tag{D}$$

A special Lagrangian of this kind is the *area integrand* $A(z) := |z|$. For any $X \in H^{1,2}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^2$, the *area functional*

$$\mathcal{A}_\Omega(X) := \int_\Omega A(X_u \wedge X_v)\, dudv$$

is well-defined and real-valued, and the same is true for any *parametric variational integral*

$$\mathcal{F}_\Omega(X) := \int_\Omega F(X, X_u \wedge X_v)\, dudv$$

with a parametric, positive definite Lagrangian $F(x, z)$. Moreover, condition (D) is equivalent to $m_1 A \leq F \leq m_2 A$, and implies

$$m_1\mathcal{A}_\Omega(X) \leq \mathcal{F}_\Omega(X) \leq m_2\mathcal{A}_\Omega(X) \quad \text{for any} \quad X \in H^{1,2}(\Omega, \mathbb{R}^n).$$

DEFINITION 1.2 *A parametric Lagrangian* F *is said to be* semi-elliptic *on* $\Omega \times \mathbb{R}^N$, $\Omega \subset \mathbb{R}^n$, *if it is convex with respect to* z, *i.e. if*

$$F(x, t_1z_1 + t_2z_2) \leq t_1F(x, z_1) + t_2F(x, z_2) \tag{C}$$

for $t_1, t_2 \in [0, 1]$, $t_1 + t_2 = 1$, $x \in \Omega$, $z \in \mathbb{R}^N$. *We call* F elliptic *if for every* $R_0 > 0$ *there is some* $\lambda^*(R_0) > 0$ *such that* $F - \lambda^*(R_0)A$ *is semi-elliptic on* $\overline{B}_{R_0}(0) \times \mathbb{R}^N$.

If F is of class C^2 on $\mathbb{R}^n \times (\mathbb{R}^N - \{0\})$, condition (C) is equivalent to the assumption that $F_{zz}(x,z)$ is positive semi-definite for $z \neq 0$, and therefore ellipticity of F means that $F_{zz}(x,z) - \lambda^* A_{zz}(z)$ is positive semi-definite for all $(x,z) \in \overline{B}_{R_0}(0) \times (\mathbb{R}^N - \{0\})$ and some $\lambda^*(R_0) > 0$.

Since $|z|\zeta \cdot A_{zz}(z)\zeta = |\zeta|^2 - |z|^{-2}(z \cdot \zeta)^2 = |P_z^\perp \zeta|^2$ for any $z, \zeta \in \mathbb{R}^N$, $z \neq 0$, and $P_z^\perp \zeta := \zeta - |z|^{-2}(z \cdot \zeta)z$, ellipticity of F means

$$|z|\zeta \cdot F_{zz}(x,z)\zeta \geq \lambda^*(R_0)|P_z^\perp \zeta|^2 \tag{1.3}$$

for some $\lambda^*(R_0) > 0$ and any $x \in \overline{B}_{R_0}(0) \subset \mathbb{R}^n$, $z, \zeta \in \mathbb{R}^N$, $z \neq 0$. Notice that $|P_z^\perp \zeta|^2 = |z|^{-2}|\zeta \wedge z|^2$ by virtue of (1.1).

Definition 1.3 *If $F(x,z)$ is a parametric Lagrangian, we denote the function $f : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}$ defined by*

$$f(x,p) := F(x, p_1 \wedge p_2) \quad \text{for} \quad p = (p_1, p_2) \in \mathbb{R}^n \times \mathbb{R}^n \cong \mathbb{R}^{2n} \tag{1.4}$$

as associated Lagrangian of F.

Note that $a(p) := A(p_1 \wedge p_2) = |p_1 \wedge p_2|$ is the associated Lagrangian of A. In the terminology of J. Ball (cf. [3], p. 99), a *parametric Lagrangian is semi-elliptic if and only if its associated Lagrangian f is polyconvex.* Correspondingly, F is elliptic if and only if its associated Lagrangian f is *strictly polyconvex*, which means that $f - \lambda^* a$ is polyconvex for some $\lambda^* > 0$.

Now we formulate the *Plateau problem.* Let Γ be a closed, rectifiable curve in $\mathbb{R}^n$, and denote by B the open unit disk $B := \{(u,v) : u^2 + v^2 < 1\}$, which we will use as parameter domain of the competing surfaces $X : B \to \mathbb{R}^n$. By $\mathcal{C}(\Gamma)$ we denote the class of surfaces $X \in H^{1,2}(B, \mathbb{R}^n)$ whose trace $X|_{\partial B}$ on ∂B is a continuous, weakly monotonic mapping of ∂B onto Γ. Note that $\mathcal{C}(\Gamma)$ is nonempty. Set $\mathcal{F} := \mathcal{F}_B$, $\mathcal{A} := \mathcal{A}_B$, that is,

$$\mathcal{F}(X) := \int_B F(X, X_u \wedge X_v)\, du dv \quad \text{for} \quad X \in \mathcal{C}(\Gamma). \tag{1.5}$$

Then we call the minimum problem "$\mathcal{F} \to$ min in $\mathcal{C}(\Gamma)$" the *Plateau problem for the parametric variational integral* (1.5). In Section 2 we prove the following existence result.

Theorem 1.4 *If $F \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ satisfies* (H),(D), *and* (C)*, then there exists a minimizer X of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ which is conformally parametrized, i.e. $X \in \mathcal{C}(\Gamma)$ satisfies*

$$\mathcal{F}(X) = \inf\nolimits_{\mathcal{C}(\Gamma)} \mathcal{F}, \tag{1.6}$$

and

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \quad a.e. \quad on \;\; B. \tag{1.7}$$

In [7] we have proved this result assuming that F is elliptic. We should like to thank Stefan Müller who kindly pointed out to us that semi-ellipticity of F instead of ellipticity is sufficient for proving existence of a conformal minimizer. Concerning the preceding results of Sigalov, Cesari, Danskin (1951–1952), Morrey (1961, 1966) and Reshetnyak (1962), we refer to our paper [7], pp. 251–252.

The next result is essentially due to C.B. Morrey; we will give a proof at the end of Section 2.

THEOREM 1.5 *Suppose that* $F \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ *satisfies* (H),(D),(C). *Then every conformally parametrized minimizer* X *of* $\mathcal{F}$ *in* $\mathcal{C}(\Gamma)$ *satisfies*

$$X \in C^0(\overline{B}, \mathbb{R}^n) \cap C^{0,\gamma}(B, \mathbb{R}^n), \quad \gamma = m_1/m_2,$$

as well as the Morrey condition

$$\int_{B_r(w_0)} |\nabla X|^2 \, dudv \le \left(\frac{r}{R}\right)^{2\gamma} \int_{B_R(w_0)} |\nabla X|^2 \, dudv \tag{1.8}$$

for any $w_0 = (u_0, v_0) \in B$ *and* $0 < r \le R \le R_0 := 1 - |w_0|$.

REMARKS. 1. To prove (1.8) and continuity of X up to the boundary ∂B, one replaces X locally by a suitable harmonic vector and uses the minimum property of X, see Section 2. Notice, however, that $X \in C^0(\overline{B}, \mathbb{R}^n)$ can also be inferred from $X|_{\partial B} \in C^0(\partial B, \mathbb{R}^n)$ and (1.8) alone, without using the minimum property (1.6); see [6], Lemma 3. This observation may be useful for proving continuity up to the boundary for stationary surfaces of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$.

2. In [7] it was shown that minimizers of (1.5) are Hölder continuous up to the boundary under a chord-arc condition on the boundary curve Γ. In [9] we prove higher regularity up to the boundary for minimizers.

3. Set $\bar{\mathcal{C}}(\Gamma) := \mathcal{C}(\Gamma) \cap C^0(\overline{B}, \mathbb{R}^n)$. Then Theorem 1.5 states that every conformally parametrized minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ lies in $\bar{\mathcal{C}}(\Gamma)$. In particular, we have $\inf_{\mathcal{C}(\Gamma)} \mathcal{F} = \inf_{\bar{\mathcal{C}}(\Gamma)} \mathcal{F}$.

Following Morrey (see [12], pp. 390–394) we introduce the notion of a *dominance function* $G(x,p)$ of a parametric Lagrangian $F(x,z)$. The existence of suitable dominance functions turns out to be crucial for proving higher regularity of conformally parametrized minimizers of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$.

DEFINITION 1.6 (i) *Let $F(x,z)$ be a parametric Lagrangian with the associated Lagrangian $f(x,p) = F(x, p_1 \wedge p_2)$, $p = (p_1, p_2) \in \mathbb{R}^{2n}$. Then a function $G : \mathbb{R}^n \times \mathbb{R}^{2n} \to \mathbb{R}$ is said to be a* dominance function for F *if it is continuous and satisfies the following two conditions:*

(D1) $\quad f(x,p) \le G(x,p)$ *for any* $(x,p) \in \mathbb{R}^n \times \mathbb{R}^{2n}$,

(D2) $\quad f(x,p) = G(x,p)$ *if and only if* $|p_1|^2 = |p_2|^2$, $p_1 \cdot p_2 = 0$.

(ii) *A dominance function G of the parametric Lagrangian F is called* quadratic *if*

(D3) $\quad G(x,tp) = t^2 G(x,p)$ *for all* $t > 0$, $(x,p) \in \mathbb{R}^n \times \mathbb{R}^{2n}$,

and it is said to be positive definite *if there are two numbers μ_1, μ_2 with $0 < \mu_1 \le \mu_2$, such that*

(D4) $\quad \mu_1 |p|^2 \le G(x,p) \le \mu_2 |p|^2$ *for any* $(x,p) \in \mathbb{R}^n \times \mathbb{R}^{2n}$.

(iii) A function $G \in C^0(\mathbb{R}^n \times \mathbb{R}^{2n}) \cap C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ is called a perfect dominance function for the parametric Lagrangian F *if it satisfies* (D1)–(D4) *as well as the ellipticity condition*

(1.9) $\quad \pi \cdot G_{pp}(x,p)\pi \ge \lambda(R_0)|\pi|^2$ *for* $|x| \le R_0$ *and* $p, \pi \in \mathbb{R}^{2n}, p \ne 0$,

where $\lambda(R_0) > 0$ is a number depending only on the parameter $R_0 > 0$. Condition (1.9) means that

$$G_{p_\alpha^j p_\beta^k}(x,p)\pi_\alpha^j \pi_\beta^k \ge \lambda(R_0)\pi_\alpha^j \pi_\alpha^j, \quad \lambda(R_0) > 0.$$

Here and in the sequel we use the following *convention:* Greek indices $\alpha, \beta, \ldots$ run from 1 to 2, and Latin indices $j, k, \ldots$ from 1 to n. Correspondingly, repeated indices $\alpha, \beta, \ldots, j, k, \ldots$, are to be summed from 1 to 2, or n, respectively.

Let us consider some *examples.*

$\boxed{1}$ The area integrand $A(z) = |z|$ with the associated Lagrangian

(1.10) $$a(p) := |p_1 \wedge p_2| = \sqrt{|p_1|^2|p_2|^2 - (p_1 \cdot p_2)^2}$$

has the perfect dominance function

(1.11) $$D(p) := \frac{1}{2}|p|^2 = \frac{1}{2}|p_1|^2 + \frac{1}{2}|p_2|^2, \quad p = (p_1, p_2).$$

$\boxed{2}$ The function

$$G(x,p) := \frac{1}{2}|p|^2 + Q(x) \cdot (p_1 \wedge p_2)$$

with $Q \in C^0(\mathbb{R}^n, \mathbb{R}^N)$ is a dominance function for

$$F(x,z) := |z| + Q(x) \cdot z.$$

These are the Lagrangians appearing in capillarity theory.

$\boxed{3}$ Every Lagrangian $F(x,z)$ with the properties (H) and (D) possesses a dominance function $G(x,p)$ satisfying (D1)–(D4). For example, if $f(x,p) = F(x, p_1 \wedge p_2)$, $p = (p_1, p_2)$, we can take

$$\begin{aligned} (1.12) \qquad & G(x,p) := \\ & \{f^2(x,p) + \frac{1}{4}(m_1+m_2)^2[\frac{1}{4}(|p_1|^2 - |p_2|^2)^2 + (p_1 \cdot p_2)^2]\}^{1/2}, \end{aligned}$$

cf. [11], pp. 571–572. Another choice is

$$G(x,p) := g(x,p)D(p), \tag{1.13}$$

where

$$g(x,p) := \begin{cases} m_1 & \text{for } z = 0 \\ F(x, z/|z|) & \text{for } z \neq 0 \end{cases}, \quad z := p_1 \wedge p_2.$$

Denote by Π and Π_0, respectively, the following algebraic surfaces in $\mathbb{R}^{2n}$:

$$\Pi := \{(p_1,p_2) \in \mathbb{R}^{2n} : p_1 \wedge p_2 = 0\}, \tag{1.14}$$

$$\Pi_0 := \{(p_1,p_2) \in \mathbb{R}^{2n} : |p_1|^2 = |p_2|^2, \ p_1 \cdot p_2 = 0\}. \tag{1.15}$$

We note that $\Pi \cap \Pi_0 = \{0\}$.

The associated Lagrangian $f(x,p)$ of a parametric Lagrangian $F(x,z)$ can never be twice differentiable at $p = 0$ (except if $F(x,.) \equiv 0$.) Therefore, in general, the dominance functions $G(x,p)$ defined by (1.12) or (1.13) are of class C^2 on $\mathbb{R}^n \times (\mathbb{R}^{2n} - \Pi)$ if we assume that F is of class C^2 on $\mathbb{R}^n \times (\mathbb{R}^N - \{0\})$, but we do not have $G \in C^2(\mathbb{R}^n \times \mathbb{R}^{2n})$ except for special cases.

If $G \in C^2(\mathbb{R}^n \times \mathbb{R}^{2n})$ and if the variational integral

$$\mathcal{G}_\Omega(X) := \int_\Omega G(X, \nabla X)\, dudv \tag{1.16}$$

corresponding to G is *conformally invariant*, i.e., if $\mathcal{G}_\Omega(X) = \mathcal{G}_{\Omega^*}(X \circ \tau)$ for any biholomorphic map $\tau : \Omega^* \to \Omega$ and any simply connected domain Ω in $\mathbb{C} \cong \mathbb{R}^2$, then G has a particular form:

PROPOSITION 1.7 (M. Grüter, [5], §2) *Let G be continuous and $G(x,.)$ of class $C^2(\mathbb{R}^{2n})$ for any $x \in \mathbb{R}^n$. Then the variational integral $\mathcal{G}_\Omega$ is conformally invariant if and only if G is of the form*

$$G(x,p) = \frac{1}{2} g_{jk}(x) p^j_\alpha p^k_\alpha + b_{jk}(x) \det(p^j, p^k), \quad p^j = (p^j_\alpha)_{1\le\alpha\le 2}, \tag{1.17}$$

where $g_{jk} = g_{kj}$, $b_{jk} = -b_{kj}$. Moreover, if G satisfies (D4), *then (g_{jk}) is positive definite; in fact,*

$$2\mu_1|\xi|^2 \le g_{jk}(x)\xi^j\xi^k \le 2\mu_2|\xi|^2.$$

Note that example (1.13) satisfies (D1)–(D4) and that the corresponding integral $\mathcal{G}_\Omega$ is conformally invariant, whereas G is of the form (1.17) (with $g_{jk} = \delta_{jk}$ and $b_{jk} = 0$) if and only if $F(x,z) = \omega(x)|z|$.

The next result is the key to proving higher regularity of conformally parametrized solutions $X \in \mathcal{C}(\Gamma)$ to Plateau's problem "$\mathcal{F} \to \min$ in $\mathcal{C}(\Gamma)$" since it shows that any such surface is a solution of the minimum problem

$$\text{(P)} \qquad \mathcal{G} \to \min \quad \text{in } \mathcal{C}(\Gamma).$$

(A slightly weaker result was already established in [7], pp. 265–266.)

THEOREM 1.8 *Suppose that $F \in C^0(\mathbb{R}^n \times \mathbb{R}^N)$ satisfies* (H),(D),(C), *and let G be an arbitrary dominance function of F with the corresponding variational integral $\mathcal{G}(X) := \mathcal{G}_B(X)$ as defined in* (1.16). *Then we have:*

(i) $\inf_{\mathcal{C}(\Gamma)} \mathcal{F} = \inf_{\bar{\mathcal{C}}(\Gamma)} \mathcal{F} = \inf_{\mathcal{C}(\Gamma)} \mathcal{G} = \inf_{\bar{\mathcal{C}}(\Gamma)} \mathcal{G}$.

(ii) *Any minimizer of $\mathcal{G}$ in $\mathcal{C}(\Gamma)$ is a conformally parametrized minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$.*

(iii) *Conversely, any conformally parametrized minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ is a minimizer of $\mathcal{G}$ in $\mathcal{C}(\Gamma)$.*

PROOF: (i) By Theorem 1.4 there exists an $X \in \mathcal{C}(\Gamma)$ satisfying (1.6) and (1.7). Taking (D1) and (D2) into account it follows that

$$\inf_{\mathcal{C}(\Gamma)} \mathcal{G} \le \mathcal{G}(X) = \mathcal{F}(X) = \inf_{\mathcal{C}(\Gamma)} \mathcal{F} \le \inf_{\mathcal{C}(\Gamma)} \mathcal{G},$$

and so we have

$$\inf_{\mathcal{C}(\Gamma)} \mathcal{F} = \inf_{\mathcal{C}(\Gamma)} \mathcal{G} = \mathcal{G}(X).$$

Because of Theorem 1.5 we now obtain (i), and (iii) is proved by the same reasoning.

(ii) Let X be a minimizer of $\mathcal{G}$ in $\mathcal{C}(\Gamma)$. Then we have

$$\inf_{\mathcal{C}(\Gamma)} \mathcal{F} \leq \mathcal{F}(X) \leq \mathcal{G}(X) = \inf_{\mathcal{C}(\Gamma)} \mathcal{G} = \inf_{\mathcal{C}(\Gamma)} \mathcal{F},$$

and therefore $\mathcal{F}(X) = \inf_{\mathcal{C}(\Gamma)} \mathcal{F}$. □

Recall from example $\boxed{1}$ that $D(p) = |p|^2/2$ is a perfect dominance function of the area integrand $A(z) = |z|$, and let

$$\mathcal{D}(X) := \int_B D(\nabla X)\, dudv = \frac{1}{2}\int_B |\nabla X|^2\, dudv \tag{1.18}$$

be the corresponding variational integral. Then we have $\mathcal{A}(X) \leq \mathcal{D}(X)$, and the equality sign holds true if and only if the conformality relations (1.7) are satisfied. Theorem 1.8, in particular, implies the nontrivial result

$$\inf_{\mathcal{C}(\Gamma)} \mathcal{A} = \inf_{\bar{\mathcal{C}}(\Gamma)} \mathcal{A} = \inf_{\mathcal{C}(\Gamma)} \mathcal{D} = \inf_{\bar{\mathcal{C}}(\Gamma)} \mathcal{D}, \tag{1.19}$$

as we had already pointed out in [7].

To carry out the regularity investigation for solutions of problem (P) we first derive the *weak Euler equation* for X,

$$\delta\mathcal{G}(X,\phi) = 0 \ \text{ for all } \ \phi \in \overset{\circ}{H}{}^{1,2}(B,\mathbb{R}^n) \cap L^\infty(B,\mathbb{R}^n) \tag{1.20}$$

with

$$\delta\mathcal{G}(X,\phi) = \int_B [G_p(X,\nabla X)\cdot\nabla\phi + G_x(X,\nabla X)\cdot\phi]\, dudv, \tag{1.21}$$

which is much more pleasant to handle than the weak Euler equation for X with respect to $\mathcal{F}$,

$$\delta\mathcal{F}(X,\phi) = 0 \ \text{ for all } \ \phi \in \overset{\circ}{H}{}^{1,2}(B,\mathbb{R}^n) \cap L^\infty(B,\mathbb{R}^n) \tag{1.22}$$

with

$$\delta\mathcal{F}(X,\phi) = \int_B [f_p(X,\nabla X)\cdot\nabla\phi + f_x(X,\nabla X)\cdot\phi]\, dudv, \tag{1.23}$$

although both equations coincide. This fact as well as the equations (1.20)–(1.23) and some useful properties of Lagrangians $h(x,p)$ which are homogeneous of degree 2 in p are proved in Section 3.

In order to prove higher regularity of solutions of (P) we consider equation (1.20). We could proceed by Morrey's well-known method (cf. [12], §§1.10, 1.11) if $G(x,p)$ had the very special from

$$G(x,p) = G_{jk}^{\alpha\beta}(x) p_\alpha^j p_\beta^k$$

with sufficiently smooth coefficients $G_{jk}^{\alpha\beta}(x)$ such that

$$G_{jk}^{\alpha\beta}(x)\xi^j\xi^k\eta_\alpha\eta_\beta \geq \lambda \xi^j\xi^j\eta_\alpha\eta_\alpha, \quad \lambda > 0.$$

In general, $G_{pp}(x,p)$ will be singular at least at $p=0$, and this is the essential difficulty we have to overcome. In particular, rank-one convexity of $G(x,p)$ with respect to p will not imply Gårding's inequality

$$\int_B G_{pp}(X,\nabla X)\nabla\phi\nabla\phi \, dudv \geq \lambda_1 \mathcal{D}(\phi) - \lambda_0 \int_B |\phi|^2 \, dudv$$

for $\phi \in \mathring{H}^{1,2}(B,\mathbb{R}^n)$; in fact, $G_{pp}(X,\nabla X)$ is not defined at points $w \in B$ where $\nabla X(w) = 0$. For this reason we impose the additional assumption that F *possesses a perfect dominance function* G, and we prove in Section 4 the following regularity result:

THEOREM 1.9 *Suppose that F satisfies* (H),(D),(C), *and that F possesses a perfect dominance function G. Then any conformally parametrized minimizer X of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$ is of class $H^{2,2}_{\mathrm{loc}}(B,\mathbb{R}^n) \cap C^{1,\sigma}(B,\mathbb{R}^n)$ for some $\sigma > 0$.*

Presently it is not clear to us whether or not any elliptic, positive definite parametric Lagrangian of class $C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$ possesses a perfect dominance function. However, in [12], p. 391, Morrey has sketched the construction of a rank-one convex dominance function G^* of class C^2 on $\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\})$ for an elliptic positive definite parametric integrand $F^* \in C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$. This remarkable result enables us to exhibit an interesting class of parametric Lagrangians F having perfect dominance functions which is much larger than the trivial class of functions $F(x,z) = |z| + Q(x)\cdot z$ with $\sup_{\mathbb{R}^n}|Q| \ll 1$. In fact we have

THEOREM 1.10 *Let $F^* \in C^2(\mathbb{R}^n \times (\mathbb{R}^N - \{0\}))$ satisfy* (H),(D) and (1.3). *Then for*

(1.24) $$k > k_0(R_0) := \max\{2(m_2 - \lambda^*(R_0)), -m_1/2\}$$

the parametric Lagrangian F defined by

(1.25) $$F(x,z) := kA(z) + F^*(x,z)$$

possesses a perfect dominance function. In particular, any conformally parametrized minimizer of

$$\mathcal{F}(X) = \int_B F(X, X_u \wedge X_v)\, dudv,$$

where F is of the form (1.25), is of class $H^{2,2}_{\mathrm{loc}}(B,\mathbb{R}^n) \cap C^{1,\sigma}(B,\mathbb{R}^n)$.

If $\lambda^*(R_0) > m_2$ we can choose $k = 0$ in (1.24) to obtain a perfect dominance function for $\mathcal{F}^*(X) := \int_B F^*(X, X_u \wedge X_v)\, dudv$, which leads to

Corollary 1.11 *Let F^* be a Lagrangian as in Theorem 1.10 and let $X \in \mathcal{C}(\Gamma)$ be a conformally parametrized minimizer of $\mathcal{F}^*$ with $R_0 := \|X\|_{C^0(\overline{B},\mathbb{R}^n)}$. If $\lambda^*(R_0) > m_2$, then $X \in H^{2,2}_{\mathrm{loc}}(B,\mathbb{R}^n) \cap C^{1,\sigma}(B,\mathbb{R}^n)$.*

Proof of Theorem 1.10: Morrey's construction which is carried out in detail and investigated further in [8] yields a dominance function $G^* \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ for F^* which satisfies (D1)–(D4) and the estimate

$$\pi \cdot G^*_{pp}(x,p)\pi \geq -k_0|\pi|^2 \ \text{ for all } \ \pi \in \mathbb{R}^{2n},\ (x,p) \in \mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}).$$

Set $G(x,p) := k\mathcal{D}(p) + G^*(x,p)$. Then G is of class $C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ and satisfies (D1)–(D4). Furthermore, for $p \neq 0$ we have

$$\pi \cdot G_{pp}(x,p)\pi = k|\pi|^2 + \pi \cdot G^*_{pp}(x,p)\pi \geq (k - k_0)|\pi|^2,$$

hence G is a perfect dominance function for F since $k - k_0 > 0$. $\square$

It remains an open problem to show that the branch points $w_0 = (u_0, v_0)$ of a conformally parametrized minimizer of $\mathcal{F}$ are isolated in B or even in $\overline{B}$.

We finally mention that also Morrey had envisioned a regularity result for minimizers of specific parametric variational problems (cf. [11], p.570, [12], pp. 363–364). However, we do not see how his method indicated in [12], p.364, would lead to $X \in C^{1,\sigma}(B,\mathbb{R}^n)$. Yet we should like to mention that we were in many ways inspired by Morrey's approach in [12].

2 Existence of minimizers

We now establish the existence of conformally parametrized solutions X to the Plateau problem "$\mathcal{F} \to \min$ in $\mathcal{C}(\Gamma)$" as stated in Theorem 1.4.

Proof of Theorem 1.4: (i) For $\epsilon > 0$ we consider the functional $\mathcal{F}^\epsilon : H^{1,2}(B, \mathbb{R}^n) \to \mathbb{R}$ defined by $\mathcal{F}^\epsilon(X) := \mathcal{F}(X) + \epsilon \mathcal{D}(X)$. Introducing the nonparametric Lagrangian $f^\epsilon(x,p) := f(x,p) + \epsilon |p|^2/2$ we have

$$\mathcal{F}^\epsilon(X) = \int_B f^\epsilon(X, \nabla X)\, dudv.$$

Since $f^\epsilon(x,p)$ is polyconvex and therefore quasiconvex in p and satisfies

$$\frac{1}{2}\epsilon|p|^2 \le f^\epsilon(x,p) \le \frac{1}{2}(m_2+\epsilon)|p|^2,$$

the functional $\mathcal{F}^\epsilon$ is (sequentially) weakly lower semicontinuous on the space $H^{1,2}(B,\mathbb{R}^n)$, cf. [1], and satisfies

$$\epsilon\mathcal{D}(X) \le \mathcal{F}^\epsilon(X) \le (m_2+\epsilon)\mathcal{D}(X) \quad \text{for any} \quad X \in H^{1,2}(B,\mathbb{R}^n).$$

For $X \in \mathcal{C}(\Gamma)$ we have $X(\partial B) = \Gamma$, and a suitable Poincaré inequality yields $\|X\|^2_{L^2(B,\mathbb{R}^n)} \le c_0 \mathcal{D}(X)$ for any $X \in \mathcal{C}(\Gamma)$ and some constant $c_0 > 0$, whence $\|X\|^2_{H^{1,2}(B,\mathbb{R}^n)} \le \epsilon^{-1}(2+c_0)\mathcal{F}^\epsilon(X)$ for any $X \in \mathcal{C}(\Gamma)$. Now we choose a sequence of surfaces $X_j \in \mathcal{C}(\Gamma)$ with

$$\lim_{j\to\infty} \mathcal{F}^\epsilon(X_j) = \inf_{\mathcal{C}(\Gamma)} \mathcal{F}^\epsilon =: d(\epsilon).$$

Since $\mathcal{F}^\epsilon$ is conformally invariant, we can assume that the sequence $\{X_j\}$ satisfies a three-point condition, i.e., there are three different points $w_1, w_2, w_3 \in \partial B$ and three different points $P_1, P_2, P_3 \in \Gamma$ such that $X_j(w_k) = P_k$, $k = 1,2,3$, for any $j \in \mathbb{N}$. Passing to an appropriate subsequence again denoted by $\{X_j\}$, we obtain $X_j \rightharpoonup X^\epsilon$ in $H^{1,2}(B,\mathbb{R}^n)$ as $j \to \infty$, and $X_j|_{\partial B} \to X^\epsilon|_{\partial B}$ in $C^0(\partial B, \mathbb{R}^n)$ as $j \to \infty$ for some $X^\epsilon \in \mathcal{C}(\Gamma)$; cf. for instance [4], vol. I, Section 4.3. Then we obtain

$$X^\epsilon(w_k) = P_k, \quad k = 1,2,3, \tag{2.1}$$

and $d(\epsilon) \le \mathcal{F}^\epsilon(X^\epsilon) \le \liminf_{j\to\infty} \mathcal{F}^\epsilon(X_j) = d(\epsilon)$, whence $\mathcal{F}^\epsilon(X^\epsilon) = d(\epsilon) = \inf_{\mathcal{C}(\Gamma)} \mathcal{F}^\epsilon$. That is, X^ϵ minimizes $\mathcal{F}^\epsilon$ in $\mathcal{C}(\Gamma)$, and we in particular obtain $\partial\mathcal{F}^\epsilon(X^\epsilon, \eta) = 0$ for the inner variation of $\mathcal{F}^\epsilon$ at X^ϵ for every vector field $\eta \in C^1(\overline{B}, \mathbb{R}^2)$. Since $\mathcal{F}$ is parameter invariant, we have $\partial\mathcal{F}(X^\epsilon,\eta) = 0$, and thus it follows that $\partial\mathcal{D}(X^\epsilon,\eta) = 0$ for any $\eta \in C^1(\overline{B},\mathbb{R}^2)$. This implies the conformality relations

$$|X^\epsilon_u|^2 = |X^\epsilon_v|^2, \quad X^\epsilon_u \cdot X^\epsilon_v = 0 \ \text{a.e. in} \ B, \tag{2.2}$$

and so we also have $\mathcal{A}(X^\epsilon) = \mathcal{D}(X^\epsilon)$.

By assumption (D) we obtain $(m_1 + \epsilon)\mathcal{D}(X^\epsilon) \leq \mathcal{F}^\epsilon(X^\epsilon)$, and for any $Z \in \mathcal{C}(\Gamma)$ we have

$$\mathcal{F}^\epsilon(X^\epsilon) = d(\epsilon) \leq \mathcal{F}^\epsilon(Z) \leq m_2\mathcal{A}(Z) + \epsilon\mathcal{D}(Z) \leq (m_2 + \epsilon)\mathcal{D}(Z).$$

Since $(m_2 + \epsilon)(m_1 + \epsilon)^{-1} \leq m_2/m_1$ for any $\epsilon > 0$, we arrive at

$$\mathcal{D}(X^\epsilon) \leq \frac{m_2}{m_1}\mathcal{D}(Z) \text{ for any } Z \in \mathcal{C}(\Gamma).$$

There is a minimal surface $Y \in \mathcal{C}(\Gamma)$, and the isoperimetric inequality yields $4\pi\mathcal{D}(Y) \leq \mathcal{L}^2(\Gamma)$, where $\mathcal{L}(\Gamma)$ denotes the length of Γ; see [4], vol. I, Section 6.3. Thus we obtain

$$\|X^\epsilon\|_{H^{1,2}(B,\mathbb{R}^n)} \leq c(m_1, m_2, \Gamma) \tag{2.3}$$

for some constant c depending only on m_1, m_2, and $\mathcal{L}(\Gamma)$.

(ii) By (2.1) and (2.3) there is some sequence of numbers $\epsilon_j \to 0$ and a function $X \in \mathcal{C}(\Gamma)$, such that

$$\begin{gathered} X^{\epsilon_j} \rightharpoonup X \text{ in } H^{1,2}(B, \mathbb{R}^n) \text{ and} \\ X^{\epsilon_j}|_{\partial B} \to X|_{\partial B} \text{ in } C^0(\partial B, \mathbb{R}^n) \text{ as } j \to \infty. \end{gathered} \tag{2.4}$$

Therefore, $d(0) := \inf_{\mathcal{C}(\Gamma)} \mathcal{F}$ satisfies $d(0) \leq \mathcal{F}(X) \leq \liminf_{j\to\infty} \mathcal{F}(X^{\epsilon_j})$, since $\mathcal{F}$ is (sequentially) weakly lower semicontinuous on $H^{1,2}(B, \mathbb{R}^n)$ (because $f(x, p) := F(x, p_1 \wedge p_2)$ is polyconvex (and therefore quasiconvex) and satisfies $0 \leq f(x, p) \leq m_2|p_1 \wedge p_2| \leq m_2|p|^2/2$, see [1]). From $\mathcal{F}(Z) \leq \mathcal{F}^\epsilon(Z)$ for any $Z \in \mathcal{C}(\Gamma)$ we infer that $d : [0, \infty] \to \mathbb{R}$ is nondecreasing. Therefore, $\lim_{\epsilon\to+0} d(\epsilon)$ exists, and we have

$$d(0) \leq \lim_{\epsilon\to+0} d(\epsilon) = \lim_{\epsilon\to+0} \mathcal{F}^\epsilon(X^\epsilon),$$

and (2.3) implies $\lim_{\epsilon\to+0} \mathcal{F}^\epsilon(X^\epsilon) = \lim_{\epsilon\to+0} \mathcal{F}(X^\epsilon)$. On the other hand, we have $\mathcal{F}^\epsilon(X^\epsilon) \leq \mathcal{F}^\epsilon(Z)$ for any $Z \in \mathcal{C}(\Gamma)$, and therefore $\lim_{\epsilon\to+0} \mathcal{F}^\epsilon(X^\epsilon) \leq \lim_{\epsilon\to+0} \mathcal{F}^\epsilon(Z) = \mathcal{F}(Z)$, whence $\lim_{\epsilon\to+0} \mathcal{F}^\epsilon(X^\epsilon) \leq d(0)$. We conclude that

$$d(0) = \inf_{\mathcal{C}(\Gamma)} \mathcal{F} = \mathcal{F}(X) = \lim_{\epsilon\to+0} \mathcal{F}^\epsilon(X^\epsilon) = \lim_{\epsilon\to+0} \mathcal{F}(X^\epsilon),$$

and, consequently, X is a minimizer of $\mathcal{F}$ in $\mathcal{C}(\Gamma)$.

(iii) Now we prove that X is conformally parametrized. In fact, since X minimizes $\mathcal{F}$ in $\mathcal{C}(\Gamma)$, we have $\mathcal{F}(X) \leq \mathcal{F}(X^{\epsilon_j})$. Adding $\epsilon_j\mathcal{D}(X)$ to both sides, it follows that $\mathcal{F}^{\epsilon_j}(X) \le \mathcal{F}(X^{\epsilon_j}) + \epsilon_j \mathcal{D}(X)$. On the other hand, X^{ϵ_j} minimizes $\mathcal{F}^{\epsilon_j}$ in $\mathcal{C}(\Gamma)$, and so we also have $\mathcal{F}^{\epsilon_j}(X^{\epsilon_j}) \le \mathcal{F}^{\epsilon_j}(X)$. Consequently, $\mathcal{F}^{\epsilon_j}(X^{\epsilon_j}) \le \mathcal{F}(X^{\epsilon_j}) + \epsilon_j \mathcal{D}(X)$, and therefore, $\epsilon_j \mathcal{D}(X^{\epsilon_j}) \le \epsilon_j \mathcal{D}(X)$, which implies $\mathcal{D}(X^{\epsilon_j}) \le \mathcal{D}(X)$ for any $j \in \mathbb{N}$. Thus we obtain the inequality $\limsup_{j\to\infty} \mathcal{D}(X^{\epsilon_j}) \le \mathcal{D}(X)$. Because of (2.4) we also have $\mathcal{D}(X) \le \liminf_{j\to\infty} \mathcal{D}(X^{\epsilon_j})$, and so we arrive at $\lim_{j\to\infty} \mathcal{D}(X^{\epsilon_j}) = \mathcal{D}(X)$. On account of (2.4) it follows that $\lim_{j\to\infty} \|X^{\epsilon_j} - X\|_{H^{1,2}(B,\mathbb{R}^n)} = 0$, and we can infer the conformality relations

$$|X_u|^2 = |X_v|^2, \quad X_u \cdot X_v = 0 \text{ a.e. on } B$$

from (2.2). This completes the proof of Theorem 1.4. □

Proof of Theorem 1.5: Define $Z \in \mathcal{C}(\Gamma)$ by $Z(w) := X(w)$ for $w \in B - \Omega$, and $Z(w) := H(w)$ for $w \in \Omega$, where $\Omega := B_r(w_0)$, $H - X \in \mathring{H}^{1,2}(\Omega, \mathbb{R}^n)$, and $\Delta H = 0$ in Ω. Then $\mathcal{F}(X) \le \mathcal{F}(Z)$, and according to (D),

$$m_1 \mathcal{A}_\Omega(X) \le \mathcal{F}_\Omega(X) \le \mathcal{F}_\Omega(H) \le m_2 \mathcal{A}_\Omega(H).$$

By (1.2) and (1.7) we have

$$\mathcal{A}_\Omega(X) = \frac{1}{2}\int_\Omega |\nabla X|^2\, dudv, \quad \mathcal{A}_\Omega(H) \le \frac{1}{2}\int_\Omega |\nabla H|^2\, dudv,$$

and so we obtain

$$m_1 \int_\Omega |\nabla X|^2\, dudv \le m_2 \int_\Omega |\nabla H|^2\, dudv. \tag{2.5}$$

Setting $\varphi(r) := \int_{B_r(w_0)} |\nabla X|^2\, dudv$ we can apply (1.7) again to obtain

$$\varphi(r) = 2\int_0^r \int_0^{2\pi} \frac{1}{\rho} |X_\theta(\rho,\theta)|^2\, d\theta,$$

if $X(\rho,\theta)$ denotes the transform of X to polar coordinates ρ, θ about the pole $w_0 \in B$. Hence $\varphi'(r) = 2r^{-1}\int_0^{2\pi} |X_\theta(r,\theta)|^2\, d\theta$ for almost all $r \in (0,R)$, and a well-known inequality states that

$$\int_{B_r(w_0)} |\nabla H|^2\, dudv \le \int_0^{2\pi} |H_\theta(r,\theta)|^2\, d\theta.$$

Since $X(r,.) = H(r,.)$ is absolutely continuous for almost all $r \in (0,R)$ we infer from (2.5) that $\varphi(r) \le (2\gamma)^{-1} r\varphi'(r)$ a.e. on $(0,R)$, and therefore

$$\frac{d}{dr}[r^{-2\gamma}\varphi(r)] \ge 0 \quad \text{a.e.} \quad \text{on} \quad (0,R).$$

Thus $r^{-2\gamma}\varphi(r)$ is nondecreasing on $(0,R]$, and we obtain

$$\varphi(r) \le (r/R)^{2\gamma}\varphi(R) \;\; \text{for} \;\; 0 < r \le R,$$

which proves (1.8). Morrey's "Dirichlet growth theorem" (cf. [12], Theorem 3.5.2) implies that $X \in C^{0,\gamma}(B,\mathbb{R}^n)$. Since $X|_{\partial B}$ is continuous, a reasoning due to Morrey finally yields $X \in C^0(\overline{B},\mathbb{R}^n)$; see [12], §4.3, or the proof of Theorem 9.4.2. □

3 Estimates for homogeneous functions, and the weak Euler equation

LEMMA 3.1 *Suppose that $h \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ and that $h(x,p)$ is positively homogeneous in p of degree two. Then we have:*

(i) $h_x(x,p)$, $h_{xx}(x,p)$ are positively homogeneous in p of degree two, $h_p(x,p)$, $h_{px}(x,p)$ are positively homogeneous in p of degree one, and $h_{pp}(x,p)$ is positively homogeneous in $p \neq 0$ of degree zero. Consequently, h, h_x, h_{xx}, h_p, h_{px}, are continuous in $\mathbb{R}^n \times \mathbb{R}^{2n}$ with

$$(3.1) \qquad \begin{gathered} h(x,0) = 0, \quad h_x(x,0) = 0, \quad h_{xx}(x,0) = 0, \\ h_p(x,0) = 0, \quad h_{px}(x,0) = 0 \end{gathered}$$

for all $x \in \mathbb{R}^n$, and h_{pp} is bounded and continuous on $\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\})$.

(ii) There are constants $c_0(R_0), c_1(R_0), c_2(R_0) > 0$, such that for $|x| \le R_0$ and for $p \in \mathbb{R}^n \times \mathbb{R}^{2n}$ we have

$$(3.2) \qquad |h(x,p)| + |h_x(x,p)| + |h_{xx}(x,p)| \le c_0(R_0)|p|^2,$$

$$(3.3) \qquad |h_p(x,p)| + |h_{px}(x,p)| \le c_1(R_0)|p|,$$

and if $p \neq 0$, then

$$(3.4) \qquad |h_{pp}(x,p)| \le c_2(R_0).$$

PROOF: Assertion (i) is obvious. Since the functions $h(x,p)$, $h_x(x,p)$, … , $h_{pp}(x,p)$ are bounded on the compact set

$$\{(x,p) \in \mathbb{R}^n \times \mathbb{R}^{2n} : |x| \leq R_0,\ |p| = 1\},$$

we infer (3.2)–(3.4) from the corresponding homogeneity properties of $h(x,p)$, … , $h_{pp}(x,p)$. □

In the following we deduce a Lipschitz condition for h_p, which is beneficial for the derivation of the weak Euler equation. An analogous argument will be used in our regularity proof in Section 4.

LEMMA 3.2 *Let $h(x,p) \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ be positively homogeneous in p of degree two. Then we have for $(x,p),(x',p') \in \mathbb{R}^n \times \mathbb{R}^{2n}$ with $|x|,|x'| \leq R_0$,*

$$|h_p(x',p') - h_p(x,p)| \leq c_2(R_0)|p'-p| + c_1(R_0)|p||x'-x|. \tag{3.5}$$

PROOF: In order to verify (3.5) we use the estimate

$$|h_p(x,p) - h_p(x',p')| \leq |h_p(x,p) - h_p(x',p)| + |h_p(x',p) - h_p(x',p')|.$$

From (3.3) we infer

$$|h_p(x,p) - h_p(x',p)| \leq \sup_{|\xi| \leq R_0} |h_{px}(\xi,p)||x-x'| \leq c_1(R_0)|p||x-x'|.$$

Secondly, if $0 \notin [p,p'] := \{p + t\triangle p : 0 \leq t \leq 1\}$, where $\triangle p := p' - p$, we have

$$h_p(x',p') - h_p(x',p) = \int_0^1 \frac{d}{dt} h_p(x', p + t\triangle p)\, dt,$$

whence

$$|h_p(x',p') - h_p(x',p)| \leq \int_0^1 |h_{pp}(x', p + t\triangle p)|\, dt\, |\triangle p|,$$

and by (3.4) we obtain

$$|h_p(x',p') - h_p(x',p)| \leq c_2(R_0)|p'-p|, \tag{3.6}$$

if $0 \notin [p,p']$. If $0 \in [p,p']$ we choose p_ϵ and p'_ϵ with $0 < \epsilon \leq \epsilon_0$, such that $0 \notin [p_\epsilon, p'_\epsilon]$, $p_\epsilon \to p$ and $p'_\epsilon \to p'$ as $\epsilon \to 0$. Then we get

$$|h_p(x',p'_\epsilon) - h_p(x',p_\epsilon)| \leq c_2(R_0)|p'_\epsilon - p_\epsilon|,$$

and by continuity of $h_p(x',.)$ we again arrive at (3.6), which also proves (3.5). □

Now we derive the weak Euler equation for any bounded minimizer X of the variational integral associated with a homogeneous integrand.

PROPOSITION 3.3 *Let $h \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ be positively homogeneous of degree two, and let*

$$\mathcal{H}(X) := \int_B h(X, \nabla X)\, dudv$$

be the corresponding functional. Then for any surface $X \in H^{1,2}(B, \mathbb{R}^n) \cap L^\infty(B, \mathbb{R}^n)$ and any $\phi \in \overset{\circ}{H}{}^{1,2}(B, \mathbb{R}^n) \cap L^\infty(B, \mathbb{R}^n)$ we have

$$\lim_{\epsilon \to 0} \epsilon^{-1}[\mathcal{H}(X + \epsilon\phi) - \mathcal{H}(X)] = \delta\mathcal{H}(X, \phi),$$

where $\delta\mathcal{H}(X, \phi)$ is defined by

$$\delta\mathcal{H}(X, \phi) := \int_B [h_p(X, \nabla X) \cdot \nabla\phi + h_x(X, \nabla X) \cdot \phi]\, dudv.$$

If, in addition, X is a minimizer of $\mathcal{H}$ in $\mathcal{C}(\Gamma)$, then $\delta\mathcal{H}(X, \phi) = 0$.

As an immediate consequence we obtain

COROLLARY 3.4 *Let $G \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ satisfy* (D3). *Then any bounded minimizer X of $\mathcal{G}$ in $\mathcal{C}(\Gamma)$ satisfies (1.20).*

PROOF OF PROPOSITION 3.3: Let $0 < |\epsilon| \leq 1$, and set for some fixed $w \in B$

$$x(t) := X(w) + \epsilon\, t\phi(w), \qquad p(t) := \nabla X(w) + \epsilon\, t \nabla\phi(w) \qquad \text{for } \ t \in [0, 1].$$

Then we have $|x(t) - x(0)| \leq |\epsilon||\phi(w)|$, and $|p(t) - p(0)| \leq |\epsilon||\nabla\phi(w)|$ for all $t \in [0, 1]$, and

$$\begin{aligned} &\epsilon^{-1}[h(x(1), p(1)) - h(x(0), p(0))] \\ &\quad = \epsilon^{-1}[h(x(1), p(1)) - h(x(1), p(0))] + \epsilon^{-1}[h(x(1), p(0)) - h(x(0), p(0))] \\ &\quad = \int_0^1 h_p(x(1), p(t))\, dt \cdot \nabla\phi(w) + \int_0^1 h_x(x(t), p(0))\, dt \cdot \phi(w). \end{aligned}$$

Setting $R_0 := \sup_B |X| + \sup_B |\phi|$ we obtain

$$\begin{aligned} &\big|\epsilon^{-1}[h(x(1), p(1)) - h(x(0), p(0))] \\ &\quad -[h_p(x(0), p(0)) \cdot \nabla\phi(w) + h_x(x(0), p(0)) \cdot \phi(w)]\big| \\ &\quad\leq \int_0^1 |h_p(x(1), p(t)) - h_p(x(0), p(0))||\nabla\phi(w)|\, dt \\ &\qquad + \int_0^1 |h_x(x(t), p(0)) - h_x(x(0), p(0))||\phi(w)|\, dt \\ &\quad\leq |\epsilon|\big\{c_2(R_0)|\nabla\phi(w)|^2 + c_1(R_0)|\nabla X(w)||\nabla\phi(w)||\phi(w)| \\ &\qquad + c_0(R_0)|\phi(w)||\nabla X(w)|^2\big\} \end{aligned}$$

by virtue of Lemma 3.2 and the inequality

$$|h_x(x(t),p(0)) - h_x(x(0),p(0))| \le |\epsilon|\, c_0(R_0)|p(0)|^2$$

which immediately follows from Lemma 3.1.

Thus we arrive at

$$\begin{aligned}
&\left|\epsilon^{-1}[\mathcal{H}(X+\epsilon\phi) - \mathcal{H}(X)] - \delta\mathcal{H}(X,\phi)\right| \\
&\quad\le\ |\epsilon|\Big\{c_2(R_0)\int_B |\nabla\phi(w)|^2\, dudv + c_1(R_0)\int_B |\nabla X(w)||\nabla\phi(w)||\phi(w)|\, dudv \\
&\qquad\qquad + c_0(R_0)\int_B |\nabla X(w)|^2|\phi(w)|\, dudv\Big\}.
\end{aligned}$$

Letting ϵ tend to zero we obtain the first assertion.

If X is a minimizer of $\mathcal{H}$ in $\mathcal{C}(\Gamma)$, then $\mathcal{H}(X+\epsilon\phi) - \mathcal{H}(X) \ge 0$ because $X+\epsilon\phi \in \mathcal{C}(\Gamma)$ for all $\epsilon \in \mathbb{R}$, and therefore $\lim_{\epsilon\to 0}\epsilon^{-1}[\mathcal{H}(X+\epsilon\phi)-\mathcal{H}(X)] = 0$. This proves the second assertion. □

LEMMA 3.5 *Suppose that $G \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \Pi))$ is a dominance function of a parametric Lagrangian F with the associated Lagrangian $f \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \Pi))$. Then we have*

$$(3.7)\qquad G_x(x,p) = f_x(x,p), \quad G_p(x,p) = f_p(x,p), \quad G_{px}(x,p) = f_{px}(x,p)$$

for $(x,p) \in \mathbb{R}^n \times \Pi_0$, and

$$(3.8)\qquad G_{pp}(x,p) \ge f_{pp}(x,p) \ \text{ for } \ (x,p) \in \mathbb{R}^n \times (\Pi_0 - \{0\}).$$

In particular, if $G \in C^2(\mathbb{R}^n \times (\mathbb{R}^{2n} - \{0\}))$ and if $X \in H^{1,2}(B,\mathbb{R}^n) \cap L^\infty(B,\mathbb{R}^n)$ is conformally parametrized then

$$(3.9)\qquad \delta\mathcal{F}(X,\phi) = \delta\mathcal{G}(X,\phi) \ \text{ for all } \ \phi \in \mathring{H}^{1,2}(B,\mathbb{R}^n) \cap L^\infty(B,\mathbb{R}^n),$$

where $\delta\mathcal{G}(X,\phi)$ and $\delta\mathcal{F}(X,\phi)$ are given by (1.21) and (1.23), respectively.

PROOF: Since G is a dominance function of F we have

$$\begin{aligned}
&(3.10)\qquad G(x,p) - f(x,p) \ \ge\ 0 \ \text{ on } \ \mathbb{R}^n \times \mathbb{R}^{2n}, \ \text{ and} \\
&(3.11)\qquad G(x,p) - f(x,p) \ =\ 0 \ \text{ on } \ \mathbb{R}^n \times \Pi_0.
\end{aligned}$$

Property (3.11) together with (3.1) in Lemma 3.1 applied to $h := G$ and $h := f$, respectively, imply

$$(3.12)\qquad G_x(x,p) = f_x(x,p) \ \text{ on } \ \mathbb{R}^n \times \Pi_0,$$

and (3.10) together with (3.11) yield $G_p(x,p) - f_p(x,p) = 0$ for all $(x,p) \in \mathbb{R}^n \times \Pi_0$, which leads to $G_{px}(x,p) - f_{px}(x,p) = 0$ for all $(x,p) \in \mathbb{R}^n \times \Pi_0$. Finally we get $G_{pp}(x,p) - f_{pp}(x,p) \geq 0$ for $(x,p) \in \mathbb{R}^n \times (\Pi_0 - \{0\})$, again by (3.10) and (3.11), since $\Pi \cap \Pi_0 = \{0\}$. The last assertion follows from (3.7). □

4 The proof of higher regularity

An important tool for proving higher regularity of conformally parametrized solutions to Plateau's problem is the following result due to Morrey (see [12], Lemma 5.4.1).

PROPOSITION 4.1 *Let Ω be a domain in $\mathbb{R}^2$, define $R(\Omega) > 0$ by* $\operatorname{meas}\Omega = \pi R^2(\Omega)$, *and let q be a function of class $L^1(\Omega)$ such that there are numbers $M_0 > 0$ and $\beta > 0$ with*

$$\int_{\Omega_r(w_0)} |q(u,v)|\, dudv \leq M_0\, r^\beta \tag{4.1}$$

for all $w_0 \in \mathbb{R}^2$, $r > 0$, and $\Omega_r(w_0) := \Omega \cap B_r(w_0)$.

Then, for any $z \in \overset{\circ}{H}{}^{1,2}(\Omega)$ and any $k \in \mathbb{N}$, the functions qz^k are of class $L^1(\Omega)$, and for any $\nu \in (0,\beta)$ there is a number $M_k(\beta,\nu)$, independent of z, such that

$$\int_{\Omega_r(w_0)} |q||z|^k\, dudv \leq M_0 M_k \|\nabla z\|^k_{L^2(\Omega)} R(\Omega)^{\nu/2} r^{\beta-(\nu/2)} \tag{4.2}$$

for all $w_0 \in \mathbb{R}^2$, $r > 0$, $\nu \in (0,\beta)$, and $k \in \mathbb{N}$.

REMARK. Clearly, an analogous result holds for any $z \in \overset{\circ}{H}{}^{1,2}(\Omega,\mathbb{R}^n)$, but the constants M_k may now depend on n, too.

In the next result, let Ω be the disk $B_\rho(0)$, $\rho > 0$, and set

$$\Omega_r(w_0) := \Omega \cap B_r(w_0), \quad \mathcal{D}_\Omega(z) := \frac{1}{2}\int_\Omega |\nabla z|^2\, dudv.$$

PROPOSITION 4.2 *Suppose that there are constants $M > 0$, $\beta > 0$, and $r_0 > 0$, such that*

$$\int_{\Omega_r(w_0)} |\nabla z|^2\, dudv \leq M r^\beta \quad \textit{for} \quad w_0 \in \overline{\Omega}, r \in (0,r_0). \tag{4.3}$$

Then, for $M_0 := \max\{M, 2r_0^{-\beta}\mathcal{D}_\Omega(z)\}$ *we obtain*

$$\int_{\Omega_r(w_0)} |\nabla z|^2\, dudv \le M_0 r^\beta \quad \textit{for} \quad w_0 \in \mathbb{R}^2, r > 0. \tag{4.4}$$

PROOF: (i) Let $w_0 \in \overline{\Omega}$. If $r_0 \le r$ we have

$$\int_{\Omega_r(w_0)} |\nabla z|^2\, dudv \le 2\mathcal{D}_\Omega(z) \le 2\left(\frac{r}{r_0}\right)^\beta \mathcal{D}_\Omega(z) \le M_0 r^\beta,$$

and for $0 < r < r_0$, the inequality $\int_{\Omega_r(w_0)} |\nabla z|^2\, dudv \le M_0 r^\beta$ follows from (4.3).

(ii) Suppose now that $w_0 \notin \overline{\Omega} = \overline{B}_\rho(0)$, and set $w_0 =: \rho_0 e^{i\theta_0}$, $w_0^* =: \rho e^{i\theta_0}$, where $0 < \rho < \rho_0$. Then, for any $r > 0$, we have

$$\Omega_r(w_0) \;=\; \Omega \cap B_r(w_0) \;\subset\; \Omega \cap B_r(w_0^*) \;=\; \Omega_r(w_0^*),$$

whence

$$\int_{\Omega_r(w_0)} |\nabla z|^2\, dudv \le \int_{\Omega_r(w_0^*)} |\nabla z|^2\, dudv \le M r^\beta,$$

taking (i) into account. □

Thus we can always pass from a "restricted Morrey condition" (4.3) as proved in Theorem 1.5 to a global Morrey condition (4.4) for $q := |\nabla X|^2$, which was assumed in Proposition 4.1.

Now we turn to the

PROOF OF THEOREM 1.9: *Step 1:* $X \in H^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$.
To prove this we operate with the well-known technique of Lichtenstein and Nirenberg estimating the difference quotients $\triangle_h \nabla X$ in $L^2_{\mathrm{loc}}(B, \mathbb{R}^{2n})$. Let us first recall some definitions and some fundamental facts (cf. [13]). Pick some unit vector $e \in \mathbb{R}^2$ and some $h \in \mathbb{R}$, $h \ne 0$, and define the shifted function $Z_h(w)$ and the difference quotient $\triangle_h Z(w)$ by $Z_h(w) := Z(w + he)$ and $\triangle_h Z := h^{-1}[Z_h - Z]$, respectively. If Z is defined on B, and if $\Omega := B_\rho(0)$, $0 < \rho < 1$, $|h| < 1 - \rho$, then Z_h and $\triangle_h Z$ are defined on Ω. Let D_u and D_v be the partial derivatives with respect to u and v, i.e. $\nabla = (D_u, D_v)$. Then we have $(\triangle_h \nabla Z)(w) = (\nabla \triangle_h Z)(w)$, and $(\nabla Z)_h(w) = \nabla Z_h(w)$, for $w \in \Omega = B_\rho(0)$ and $|h| < 1 - \rho$. Furthermore, if $Z \in H^{1,2}(B_R)$, $B_R := B_R(w_0) \subset\subset B$, and $|h| < h_0 := 1 - |w_0| - R$, then

$$\|\triangle_h Z\|_{L^2(B_R)} \le \|D_u Z\|_{L^2(B_{R+|h|})}, \qquad \text{and} \quad \lim_{h\to 0} \|\triangle_h Z - D_u Z\|_{L^2(B_R)} = 0,$$

provided that $e = e_1 = (1,0)$, and if $e = e_2 = (0,1)$, then the corresponding relations hold with D_u replaced by D_v.

If $Z, Y \in L^2(B, \mathbb{R}^N)$ and if Z or Y has compact support in B, then for the L^2-inner product

$$(Z, Y) := \int Z \cdot Y \, dudv := \int_{\mathbb{R}^2} Z \cdot Y \, dudv$$

we have the following "rule of integration by parts":

$$(Z, \triangle_h Y) = -(\triangle_{-h} Z, Y) \ \text{ if } \ |h| \ll 1.$$

Moreover, we have the "product rule"

$$\triangle_h (Z \cdot Y) = (\triangle_h Z) \cdot Y + Z_h \cdot \triangle_h Y = (\triangle_h Z) \cdot Y_h + Z \cdot \triangle_h Y.$$

Now we choose some "friend" $\eta \in C_c^\infty(B_{2r}(w_0))$ on a disk $B_{2r}(w_0) \subset\subset B$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(w_0)$, and $|\nabla \eta| \le 2/r$ on the annulus $T_{2r} := B_{2r}(w_0) - \overline{B}_r(w_0)$, $\nabla \eta \equiv 0$ on $\mathbb{R}^2 - T_{2r}$.

Then

$$\phi := -\triangle_{-h}(\eta^2 \triangle_h X), \quad |h| \ll 1, \tag{4.5}$$

is an admissible test vector for the weak Euler equation

$$\int_B [G_p(X, \nabla X) \cdot \nabla \phi + G_x(X, \nabla X) \cdot \phi] \, dudv = 0,$$

and "integration by parts" yields

$$\begin{aligned} &\int [\triangle_h G_p(X, \nabla X)] \cdot \nabla[\eta^2 \triangle_h X] \, dudv \\ &\qquad = \ -\int [\triangle_h G_x(X, \nabla X)] \cdot (\eta^2 \triangle_h X) \, dudv. \end{aligned}$$

Since $\nabla[\eta^2 \triangle_h X] = \eta^2 \nabla \triangle_h X + 2\eta \nabla \eta \triangle_h X$, we obtain

$$\int [\triangle_h G_p(X, \nabla X)] \cdot \eta^2 \nabla \triangle_h X \, dudv = J_1 + J_2 \tag{4.6}$$

with

$$\begin{aligned} J_1 &:= -\int [\triangle_h G_p(X, \nabla X)] \cdot 2\eta \nabla \eta \triangle_h X \, dudv, \\ J_2 &:= -\int [\triangle_h G_x(X, \nabla X)] \cdot \eta^2 \triangle_h X \, dudv. \end{aligned} \tag{4.7}$$

Now we study the "dangerous" term $[\triangle_h G_p(X, \nabla X)] \cdot \eta^2 \nabla \triangle_h X$:

$$
\begin{aligned}
\triangle_h G_p(X, \nabla X) &= \frac{1}{h}[G_p(X_h, \nabla X_h) - G_p(X, \nabla X)] \\
&= \frac{1}{h}[G_p(X_h, \nabla X_h) - G_p(X, \nabla X_h)] \\
&\quad + \frac{1}{h}[G_p(X, \nabla X_h) - G_p(X, \nabla X)].
\end{aligned} \tag{4.8}
$$

We want to rewrite these expressions at some point $w \in B_{2r}(w_0)$. To this end we set

$$
\begin{gathered}
x(0) := X(w), \quad x(1) := X_h(w), \quad p(0) := \nabla X(w), \quad p(1) := \nabla X_h(w), \\
x(s) := sx(1) + (1-s)x(0), \quad p(s) := sp(1) + (1-s)p(0).
\end{gathered}
$$

Then $\dot{x}(s) = x(1) - x(0) = h\triangle_h X(w)$, $\quad \dot{p}(s) = p(1) - p(0) = h\nabla\triangle_h X(w)$, and we obtain

$$
\begin{aligned}
&\{[\triangle_h G_p(X, \nabla X)] \cdot \eta^2 \nabla \triangle_h X\}(w) \\
&= \int_0^1 G_{px}(x(s), \nabla X_h(w))\, ds\, \triangle_h X(w) \eta^2(w) \nabla \triangle_h X(w) \\
&\quad + \eta^2(w) h^{-1}[G_p(X(w), p(1)) - G_p(X(w), p(0))] \cdot \nabla \triangle_h X(w).
\end{aligned} \tag{4.9}
$$

If $p(s) \neq 0$ for $0 \leq s \leq 1$, the function $g(s) := G_p(X(w), p(s))$ is continuously differentiable on $[0,1]$, and we obtain

$$
\begin{aligned}
&h^{-1}[G_p(X(w), p(1)) - G_p(X(w), p(0))] \\
&= \frac{1}{h} \int_0^1 \frac{d}{ds} G_p(X(w), p(s))\, ds \\
&= \int_0^1 G_{pp}(X(w), p(s)) \nabla \triangle_h X(w)\, ds,
\end{aligned} \tag{4.10}
$$

and the ellipticity condition (1.9) implies

$$
\begin{aligned}
&h^{-1}[G_p(X(w), p(1)) - G_p(X(w), p(0)] \cdot \nabla \triangle_h X(w) \\
&\geq \lambda(R_0) |\nabla \triangle_h X(w)|^2,
\end{aligned} \tag{4.11}
$$

where $R_0 := \|X\|_{C^0(\overline{B}, \mathbb{R}^n)}$ is finite by virtue of Theorem 1.5.

If $p(0) = p(1)$, we have

$$
(\nabla \triangle_h X)(w) = (\triangle_h \nabla X)(w) = h^{-1}[\nabla X_h(w) - \nabla X(w)] = 0,
$$

and so (4.11) is trivially satisfied.

If $p(0) \neq p(1)$, but $p(s_0) = 0$ for some $s_0 \in [0,1]$, we choose some $\zeta \in \mathbb{R}^{2n}$ with $|\zeta| = 1$ and $\zeta \perp p(1) - p(0)$, and we form $p_\epsilon(s) := p(s) + \epsilon\zeta$ for $s \in [0,1]$, $\epsilon > 0$. Then $p_\epsilon(s) \neq 0$ for $0 \leq s \leq 1$ and $\epsilon > 0$, because, otherwise, we had $p_\epsilon(s_1) = 0$ for some $(s_1, \epsilon) \in [0,1] \times [0,\infty)$ whence $p(s_1) = -\epsilon\zeta$, and therefore $p(s_1) - p(s_0) = -\epsilon\zeta$. This implies

$$-\epsilon = -\epsilon|\zeta|^2 = \zeta \cdot [p(s_1) - p(s_0)] = (s_1 - s_0)\zeta \cdot [p(1) - p(0)] = 0,$$

a contradiction. Thus we can replace (4.10) by

$$h^{-1}[G_p(X(w), p_\epsilon(1)) - G_p(X(w), p_\epsilon(0))] = \int_0^1 G_{pp}(X(w), p_\epsilon(s)) \nabla \triangle_h X(w)\, ds,$$

since $\dot{p}_\epsilon(s) = \dot{p}(s) = h\nabla\triangle_h X(w)$, and by (1.9) we obtain

$$h^{-1}[G_p(X(w), p_\epsilon(1)) - G_p(X(w), p_\epsilon(0))] \cdot \nabla\triangle_h X(w) \geq \lambda(R_0)|\nabla\triangle_h X(s)|^2$$

for any $\epsilon > 0$. If we let ϵ tend to zero, $p_\epsilon(s)$ tends to $p(s)$, and since $G_p(x,p)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^{2n}$ (see Lemma 3.1), we obtain inequality (4.11), which is now established for any $w \in B_{2r}(w_0)$.

On account of Lemma 3.1 it follows that

$$|G_{px}(x(s), \nabla X_h(w))| \leq c_1(R_0)|\nabla X_h(w)|, \tag{4.12}$$

whence

$$\begin{aligned}
&\Big|\int_0^1 G_{px}(x(s), \nabla X_h(w))\, ds\, \triangle_h X(w)\eta^2(w)\nabla\triangle_h X(w)\Big| \\
&\quad\leq \; c_1(R_0)\eta^2(w)|\nabla X_h(w)||\triangle_h X(w)||\nabla\triangle_h X(w)| \quad \text{a.e. on } B_{2r}(w_0).
\end{aligned}$$

In conjunction with (4.9) and (4.11) this inequality implies

$$\begin{aligned}
&\int [\triangle_h G_p(X, \nabla X)] \cdot \eta^2 \nabla\triangle_h X\, dudv \\
&\quad\geq \; \lambda(R_0)\int \eta^2|\nabla\triangle_h X|^2\, dudv \\
&\qquad -c_1(R_0)\int \eta^2|\nabla X_h||\triangle_h X||\nabla\triangle_h X|\, dudv.
\end{aligned} \tag{4.13}$$

Moreover, we infer from Lemma 3.1 that

$$|G_{pp}(X(w), p)| \leq c_2(R_0) \;\text{ if }\; p \neq 0, \tag{4.14}$$

and by (4.10) we obtain

$$|h^{-1}[G_p(X(w),\nabla X_h(w)) - G_p(X(w),\nabla X(w))]| \le c_2(R_0)|\nabla \triangle_h X(w)|,$$

if $0 \notin [p(0),p(1)]$, and the same approximation argument as before implies that this estimate remains valid if $0 \in [p(0),p(1)]$.

On account of (4.12) and of

$$\begin{aligned} h^{-1}[G_p(X_h(w),\nabla X_h(w)) &- G_p(X(w),\nabla X_h(w))] \\ &= \int_0^1 G_{px}(x(s),\nabla X_h(w))\triangle_h X(w)\, ds, \end{aligned}$$

it follows that

$$\begin{aligned} |h^{-1}[G_p(X_h(w),\nabla X_h(w)) &- G_p(X(w),\nabla X_h(w))]| \\ \le\ & c_1(R_0)|\nabla X_h(w)||\triangle_h X(w)|. \end{aligned}$$

Thus, by (4.8), we arrive at

$$|\triangle_h G_p(X,\nabla X)|(w) \le c_1(R_0)|\nabla X_h(w)||\triangle_h X(w)| + c_2(R_0)|\nabla \triangle_h X(w)|$$

a.e. on $B_{2r}(w_0)$. Therefore we can estimate J_1 by

$$\begin{aligned} |J_1| \quad \le \quad & c_1(R_0)\int 2\eta|\nabla\eta||\nabla X_h||\triangle_h X|^2\, dudv \\ & + c_2(R_0)\int 2\eta|\nabla\eta||\nabla\triangle_h X||\triangle_h X|\, dudv. \end{aligned} \tag{4.15}$$

In order to estimate J_2 we write

$$\begin{aligned} &\triangle_h G_x(X,\nabla X) = h^{-1}[G_x(X_h,\nabla X_h) - G_x(X,\nabla X)] \\ &= h^{-1}[G_x(X_h,\nabla X_h) - G_x(X,\nabla X_h)] + h^{-1}[G_x(X,\nabla X_h) - G_x(X,\nabla X)] \\ &\qquad = \int_0^1 G_{xx}(x(s),\nabla X_h)\triangle_h X\, ds + \int_0^1 G_{xp}(X,p(s))\nabla\triangle_h X\, ds. \end{aligned}$$

By Lemma 3.1 we have

$$\begin{aligned} |G_{xx}(x(s),\nabla X_h(w))| \quad &\le \quad c_0(R_0)|\nabla X_h(w)|^2 \\ |G_{xp}(X(w),p(s))| \quad &\le \quad c_1(R_0)|p(s)| \le c_1(R_0)[|\nabla X(w)| + |\nabla X_h(w)|] \end{aligned}$$

a.e. on $B_{2r}(w_0)$, and thus it follows that

$$\begin{aligned} |J_2| \quad \le \quad & c_0(R_0)\int \eta^2|\nabla X_h|^2|\triangle_h X|^2\, dudv \\ & + c_1(R_0)\int \eta^2|\nabla\triangle_h X|[|\nabla X| + |\nabla X_h|]|\triangle_h X|\, dudv. \end{aligned} \tag{4.16}$$

Now we recall that $|\nabla\eta| \le 2/r$. Then we infer from (4.6), (4.13), (4.15), (4.16), that

$$\begin{aligned} &\lambda(R_0)\int \eta^2|\nabla\triangle_h X|^2\,dudv \\ &\le\ 2c_1(R_0)\int \eta^2|\nabla X_h||\triangle_h X||\nabla\triangle_h X|\,dudv \\ &\quad +\frac{4}{r}c_2(R_0)\int \eta|\triangle_h X||\nabla\triangle_h X|\,dudv+\frac{4}{r}c_1(R_0)\int \eta|\nabla X_h||\triangle_h X|^2\,dudv \\ &\quad +c_0(R_0)\int \eta^2|\nabla X_h|^2|\triangle_h X|^2\,dudv+c_1(R_0)\int \eta^2|\nabla X||\triangle_h X||\nabla\triangle_h X|\,dudv. \end{aligned}$$

Using the estimate $2ab \le \epsilon a^2+\epsilon^{-1}b^2$ for $\epsilon>0$, we obtain

$$\begin{aligned} &\lambda(R_0)\int \eta^2|\nabla\triangle_h X|^2\,dudv \\ &\quad\le\ \epsilon\int \eta^2|\nabla\triangle_h X|^2\,dudv \\ &\qquad +c^*(R_0,\epsilon)\Big[\int \eta^2|\nabla X|^2|\triangle_h X|^2\,dudv+\int \eta^2|\nabla X_h|^2|\triangle_h X|^2\,dudv \\ &\qquad\qquad +r^{-2}\int_{T_{2r}}|\triangle_h X|^2\,dudv\Big] \end{aligned}$$

for some constant $c^*(R_0,\epsilon)$ depending only on R_0 and ϵ. By choosing $\epsilon := \lambda(R_0)/2$ we can absorb the term $\epsilon\int\eta^2|\nabla\triangle_h X|^2\,dudv$ by the left-hand side of this inequality. Multiplying the result by $2/\lambda(R_0)$ and setting

$$J' := \int \eta^2|\nabla X|^2|\triangle_h X|^2\,dudv,\quad J'' := \int \eta^2|\nabla X_h|^2|\triangle_h X|^2\,dudv, \tag{4.17}$$

it follows on account of

$$\int_{T_{2r}}|\triangle_h X|^2\,dudv \le 2\mathcal{D}(X)\ \text{ for }\ |h|\ll 1 \tag{4.18}$$

that

$$\int \eta^2|\nabla\triangle_h X|^2\,dudv \le c(R_0)[J'+J''+r^{-2}\mathcal{D}(X)]\ \text{ for }\ |h|\ll 1, \tag{4.19}$$

where the constant $c(R_0)$ depends only on R_0.

In order to estimate J' and J'' we apply Proposition 4.1 to $q := |\nabla X|^2$, or to $q := |\nabla X_h|^2$, and to $\Omega := B_{\rho_0}(0)$ with $0<\rho_0<1$. For this purpose we note that, by Theorem 1.5,

$$\int_{B_\rho(\zeta_0)}|\nabla X|^2\,dudv \le \Big(\frac{\rho}{R}\Big)^{2\gamma}\int_{B_R(\zeta_0)}|\nabla X|^2\,dudv$$

if $\zeta_0 \in \Omega$, and $0 < \rho \leq R \leq 1 - \rho_0$, and therefore also

$$\int_{B_\rho(\zeta_0)} |\nabla X_h|^2 \, dudv \leq \left(\frac{\rho}{R}\right)^{2\gamma} \int_{B_R(\zeta_0)} |\nabla X_h|^2 \, dudv$$

if $\zeta_0 \in \Omega$, $\quad 2|h| < 1 - \rho_0$, and $0 < \rho \leq R \leq 2^{-1}(1 - \rho_0)$. Then we obtain for $R := 2^{-1}(1 - \rho_0)$, $\quad \zeta_0 \in \Omega$, $\quad |h| < R$, $\quad M := 2R^{-2\gamma}\mathcal{D}(X)$, $\quad 0 < \rho < R$, that

$$\int_{B_\rho(\zeta_0)} |\nabla X|^2 \, dudv \leq M\rho^{2\gamma}, \quad \int_{B_\rho(\zeta_0)} |\nabla X_h|^2 \, dudv \leq M\rho^{2\gamma},$$

and all the more so for $\Omega_\rho(\zeta_0) := \Omega \cap B_\rho(\zeta_0)$,

$$\int_{\Omega_\rho(\zeta_0)} |\nabla X|^2 \, dudv \leq M\rho^{2\gamma}, \quad \int_{\Omega_\rho(\zeta_0)} |\nabla X_h|^2 \, dudv \leq M\rho^{2\gamma}.$$

By Proposition 4.2 it follows that

$$\int_{\Omega_\rho(\zeta_0)} |\nabla X|^2 \, dudv \leq M_0\rho^{2\gamma}, \quad \int_{\Omega_\rho(\zeta_0)} |\nabla X_h|^2 \, dudv \leq M_0\rho^{2\gamma}$$

for all $\zeta_0 \in \mathbb{R}^2$ and all $\rho > 0$, $|h| \leq 2^{-1}(1 - \rho_0)$, if we set

$$M_0 := 2R^{-2\gamma}\mathcal{D}(X) = 2M.$$

Thus the assumptions of Proposition 4.1 are fulfilled for $\Omega = B_{\rho_0}(0)$, $R(\Omega) = \rho_0$, $q = |\nabla X|^2$ or $|\nabla X_h|^2$, $\beta = 2\gamma$ and $z := \eta \triangle_h X$, we get

$$\max\{\int_{\Omega_\rho(\zeta_0)} |\nabla X|^2 |\eta \triangle_h X|^2 \, dudv, \int_{\Omega_\rho(\zeta_0)} |\nabla X_h|^2 |\eta \triangle_h X|^2 \, dudv\}$$
$$\leq M_0 M_2 \, \rho_0^{\gamma/2} \rho^{2\gamma - \gamma/2} \int_\Omega |\nabla(\eta \triangle_h X)|^2 \, dudv, \quad M_2 = M_2(2\gamma, \gamma).$$

for all $\zeta_0 \in \mathbb{R}^2$ and all $\rho > 0$.

Suppose now that $B_{2r}(w_0) \subset B_{\rho_0}(0) = \Omega$, $\rho_0 \in (0, 1)$, and choose $\zeta_0 = w_0$, $\rho = r$. Then we obtain for $\gamma^* := 2\gamma - \gamma/2$

$$J' + J'' \leq c^* r^{\gamma^*} \int |\nabla(\eta \triangle_h X)|^2 \, dudv$$

for some number $c^* > 0$ independent of r and h provided that $|h| < \frac{1}{2}(1 - \rho_0)$. Since $\nabla(\eta \triangle_h X) = \eta \nabla \triangle_h X + (\nabla \eta) \triangle_h X$, it follows that

$$|\nabla(\eta \triangle_h X)|^2 \leq 2\eta^2 |\nabla \triangle_h X|^2 + 8r^{-2} |\triangle_h X|^2,$$

and thus we arrive at

$$J' + J'' < 4c^* r^{\gamma^*} \int \eta^2 |\nabla \triangle_h X|^2 \, dudv + 16 c^* r^{-2+\gamma^*} \int_{T_{2r}} |\triangle_h X|^2 \, dudv.$$

On account of (4.18) we finally see that

$$J' + J'' \le 4c^* r^{\gamma^*} \int \eta^2 |\nabla \triangle_h X|^2 \, dudv + 32 c^* r^{-2+\gamma^*} \mathcal{D}(X), \tag{4.20}$$

if $|h| \ll 1$. Let us choose r so small that $c^* r^{\gamma^*} \le [8c(R_0)]^{-1}$ and insert the estimate (4.20) of $J' + J''$ into (4.19). Then we obtain

$$\int \eta^2 |\nabla \triangle_h X|^2 \, dudv \le c(r) \mathcal{D}(X) \quad \text{for} \quad |h| \ll 1, \tag{4.21}$$

where the constant $c(r)$ depends on r, ρ_0, R_0 and the other constants related to G, but not on h. Hence

$$\int_{B_r(w_0)} |\nabla \triangle_h X|^2 \, dudv \le c(r) \mathcal{D}(X) \quad \text{for} \quad |h| \ll 1.$$

If we set $e = e_1$ or e_2, respectively, and let h tend to zero, we infer that the weak derivatives $\nabla D_u X$ and $\nabla D_v X$ exist and are of class $L^2(B_r(w_0), \mathbb{R}^{2n})$; in fact, we obtain

$$\int_{B_r(w_0)} |\nabla^2 X|^2 \, dudv \le 2c(r) \mathcal{D}(X). \tag{4.22}$$

A covering argument leads to $X \in H^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^n)$. This concludes the first part of our regularity proof.

Step 2: $X \in C^{1,\sigma}(B, \mathbb{R}^n)$ for some $\sigma \in (0,1)$.

To prove this result, we insert instead of (4.5) a slightly modified test vector into the weak Euler equation

$$\int_B [G_p(X, \nabla X) \cdot \nabla \phi + G_x(X, \nabla X) \cdot \phi] \, dudv = 0,$$

namely $\phi := -\eta^2 \triangle_{-h} \triangle_h X$, where the "friend" η has the same properties as in Step 1. We obtain

$$\begin{aligned} &\int G_p(X, \nabla X) \cdot \nabla [-\eta^2 \triangle_{-h} \triangle_h X] \, dudv \\ &\qquad = \int G_x(X, \nabla X) \cdot \eta^2 \triangle_{-h} \triangle_h X] \, dudv. \end{aligned} \tag{4.23}$$

Let $\xi \in \mathbb{R}^n$ be an arbitrary constant vector which will be fixed later on. We can write $\triangle_{-h}\triangle_h X = \triangle_{-h}(\triangle_h X - \xi)$ and

$$\triangle_{-h}[\eta^2(\triangle_h X - \xi)] = \eta^2 \triangle_{-h}(\triangle_h X - \xi) + (\triangle_{-h}\eta^2)(\triangle_h X - \xi)_{-h};$$

therefore

$$\nabla[-\eta^2\triangle_{-h}\triangle_h X] = -\triangle_{-h}\nabla[\eta^2(\triangle_h X - \xi)] + \nabla[(\triangle_{-h}\eta^2)(\triangle_h X - \xi)_{-h}].$$

Inserting this expression into (4.23) it follows that $I_1 = I_2 + I_3$, where we have set

$$\begin{aligned} I_1 &:= \int [\triangle_h G_p(X, \nabla X)] \cdot \{\nabla[\eta^2(\triangle_h X - \xi)]\}\, dudv, \\ I_2 &:= -\int G_p(X, \nabla X) \cdot \nabla[(\triangle_{-h}\eta^2)(\triangle_h X - \xi)_{-h}]\, dudv, \\ I_3 &:= \int G_x(X, \nabla X) \cdot \eta^2 \triangle_{-h}\triangle_h X\, dudv. \end{aligned}$$

We decompose I_1 into $I_1 = I_1' + I_1''$, where

$$\begin{aligned} I_1' &:= \int [\triangle_h G_p(X, \nabla X)] \cdot \eta^2 \nabla\triangle_h X\, dudv, \\ I_1'' &:= \int [\triangle_h G_p(X, \nabla X)] \cdot 2\eta\nabla\eta(\triangle_h X - \xi)\, dudv. \end{aligned}$$

By (4.13) we have

$$I_1' \geq \lambda(R_0)\int \eta^2|\nabla\triangle_h X|^2\, dudv - c_1(R_0)\int \eta^2|\nabla X_h||\triangle_h X||\nabla\triangle_h X|\, dudv,$$

and analogously to (4.15) it follows that

$$\begin{aligned} |I_1''| \leq\ & c_1(R_0)\int 2\eta|\nabla\eta||\nabla X_h||\triangle_h X||\triangle_h X - \xi|\, dudv \\ & + c_2(R_0)\int 2\eta|\nabla\eta||\nabla\triangle_h X||\triangle_h X - \xi|\, dudv. \end{aligned}$$

From $I_1' = -I_1'' + I_2 + I_3$ we then infer

$$\lambda(R_0)\int \eta^2|\nabla\triangle_h X|^2\, dudv \leq I_2 + I_3 + c_1(R_0)[I_4 + I_5] + c_2(R_0)I_6, \tag{4.24}$$

where we have set

$$\begin{aligned} I_4 &:= \int \eta^2 |\nabla X_h||\triangle_h X||\nabla \triangle_h X|\, dudv, \\ I_5 &:= \int 2\eta |\nabla \eta||\nabla X_h||\triangle_h X||\triangle_h X - \xi|\, dudv, \\ I_6 &:= \int 2\eta |\nabla \eta||\nabla \triangle_h X||\triangle_h X - \xi|\, dudv. \end{aligned}$$

Now we estimate I_2. To this end we introduce $\triangle^1_{-k}$ and $\triangle^2_{-k}$ as the difference-quotient operators $\triangle_{-k}$ with respect to e_1 and e_2, i.e., we "symbolically" have

$$D_u = D_1 = \lim_{k\to 0} \triangle^1_{-k}, \quad D_v = D_2 = \lim_{k\to 0} \triangle^2_{-k}.$$

Then we have $I_2 = \lim_{k\to 0} I_2^k$, where

$$\begin{aligned} I_2^k &:= \int [\triangle_k^1 G_{p^1}(X, \nabla X)] \cdot [(\triangle_{-h}\eta^2)(\triangle_h X - \xi)_{-h}]\, dudv \\ &\quad + \int [\triangle_k^2 G_{p^2}(X, \nabla X)] \cdot [(\triangle_{-h}\eta^2)(\triangle_h X - \xi)_{-h}]\, dudv. \end{aligned}$$

Similarly to (4.15) we obtain

$$\begin{aligned} |I_2^k| &\le c_1(R_0) \sum_{j=1}^2 \int |\nabla X_k||\triangle_k^j X||\triangle_{-h}\eta^2||(\triangle_h X - \xi)_{-h}|\, dudv \\ &\quad + c_2(R_0) \sum_{j=1}^2 \int |\nabla \triangle_k^j X||\triangle_{-h}\eta^2||(\triangle_h X - \xi)_{-h}|\, dudv. \end{aligned}$$

Because of Step 1 we have $\nabla X \in H^{1,2}_{\mathrm{loc}}(B, \mathbb{R}^n) \cap L^s_{\mathrm{loc}}(B, \mathbb{R}^n)$ for any $s \in [1, \infty)$, and therefore $|\nabla \triangle_k^j X| \to |\nabla D_j X|$ and $|\nabla X_k||\triangle_k^j X| \to |\nabla X||D_j X|$ in $L^2_{\mathrm{loc}}(B)$ as $k \to \infty$. This leads to

$$\begin{aligned} |I_2| &\le c_1(R_0) \int |\nabla X|^2 |\triangle_{-h}\eta^2||(\triangle_h X - \xi)_{-h}|\, dudv \\ &\quad + c_2(R_0) \int |\nabla^2 X||\triangle_{-h}\eta^2||(\triangle_h X - \xi)_{-h}|\, dudv. \end{aligned}$$

Since $|G_x(X, \nabla X)| \le c_0(R_0)|\nabla X|^2$, we furthermore obtain

$$|I_3| \le c_0(R_0) \int \eta^2 |\nabla X|^2 |\triangle_{-h}\triangle_h X|\, dudv.$$

Inequality (4.24) then implies

$$\begin{aligned}
&\lambda(R_0)\int \eta^2|\nabla\triangle_h X|^2\,dudv\\
&\le\ c(R_0)\Big[\int |\nabla X|^2|\triangle_{-h}\eta^2||(\triangle_h X-\xi)_{-h}|\,dudv\\
&\quad+\int |\nabla^2 X||\triangle_{-h}\eta^2||(\triangle_h X-\xi)_{-h}|\,dudv\\
&\quad+\int \eta^2|\nabla X|^2|\triangle_{-h}\triangle_h X|\,dudv\\
&\quad+\int \eta^2|\nabla X_h||\triangle_h X||\nabla\triangle_h X|\,dudv\\
&\quad+\int 2\eta|\nabla\eta||\nabla X_h||\triangle_h X||\triangle_h X-\xi|\,dudv\\
&\quad+\int 2\eta|\nabla\eta||\nabla\triangle_h X||\triangle_h X-\xi|\,dudv\Big]
\end{aligned}$$

for some constant $c(R_0)$ depending on R_0, but independent of h.

We apply this inequality to $\triangle_h^1$ with the constant $\xi_1\in\mathbb{R}^n$ and to $\triangle_h^2$ with the constant $\xi_2\in\mathbb{R}^n$, add the resulting inequalities and let h tend to zero. Then it follows that, for some new constant $c^*(R_0)$,

$$\begin{aligned}
&\lambda(R_0)\int \eta^2|\nabla^2 X|^2\,dudv\\
&\le\ c^*(R_0)[\int \eta^2|\nabla X|^2|\nabla^2 X|\,dudv\\
&\quad+\int \eta|\nabla\eta||\nabla^2 X||\nabla X-C|\,dudv\\
&\quad+\int \eta|\nabla\eta||\nabla X|^2|\nabla X-C|\,dudv],
\end{aligned}$$

where $C=(\xi_1,\xi_2)$ denotes an arbitrary vector of $\mathbb{R}^{2n}$. Since $|\nabla\eta|\le 2/r$ and $ab\le\epsilon a^2+(4\epsilon)^{-1}b^2$ for any $\epsilon>0$, we infer that

$$\begin{aligned}
&\lambda(R_0)\int \eta^2|\nabla^2 X|^2\,dudv\\
&\le\ c^*(R_0)\Big[\epsilon\int \eta^2|\nabla^2 X|^2\,dudv+\frac{1}{4\epsilon}\int \eta^2|\nabla X|^4\,dudv\\
&\quad+\epsilon\int \eta^2|\nabla^2 X|^2\,dudv+\frac{4r^{-2}}{4\epsilon}\int_{T_{2r}}|\nabla X-C|^2\,dudv\\
&\quad+\int \eta^2|\nabla X|^4\,dudv+4r^{-2}\int_{T_{2r}}|\nabla X-C|^2\,dudv\Big].
\end{aligned}$$

Choosing $\epsilon > 0$ so small that $2\epsilon c^*(R_0) < \lambda(R_0)/2$, we can absorb all terms on the right-hand side which involve $\int \eta^2 |\nabla^2 X|^2 \, dudv$ by the left-hand side, thus obtaining

$$
\begin{aligned}
(4.25) \qquad \int_{B_r(w_0)} |\nabla^2 X|^2 \, dudv &\le \int \eta^2 |\nabla^2 X|^2 \, dudv \\
&\le c^{**}(R_0)\Big[\int_{B_{2r}(w_0)} |\nabla X|^4 \, dudv + r^{-2} \int_{T_{2r}} |\nabla X - C|^2 \, dudv\Big],
\end{aligned}
$$

provided that $B_{2r}(w_0) \subset\subset B$. Now we choose C as the mean value of ∇X on $T_{2r} = B_{2r}(w_0) - \overline{B}_r(w_0)$, i.e. $C := \fint_{T_{2r}} \nabla X \, dudv$. By Poincaré's inequality we obtain

$$
(4.26) \qquad \int_{T_{2r}} |\nabla X - C|^2 \, dudv \le K_P \, r^2 \int_{T_{2r}} |\nabla^2 X|^2 \, dudv
$$

for some number $K_P \ge 1$ independent of $r \in (0, 1 - |w_0|)$. Since $X \in H^{2,2}_{\mathrm{loc}}(B, \mathbb{R}^n)$, we have $X \in H^{1,s}_{\mathrm{loc}}(B, \mathbb{R}^n)$ for any $s \ge 2$, and by Hölder's inequality we get

$$
\int_{B_{2r}(w_0)} |\nabla X|^4 \, dudv \le \left(\int_{B_{2r}(w_0)} |\nabla X|^{4/\delta} \, dudv \right)^{\delta} (\,\mathrm{meas}\, B_{2r}(w_0))^{1-\delta}.
$$

Choose $\delta \in (0, 1/2)$, $0 < \rho_0 < R < 1$, and set $r_0 := 2^{-1}(R - \rho_0)$, and

$$
K(R, \delta) := (4\pi)^{1-\delta} \|\nabla X\|^4_{L^{4/\delta}(B_R(0), \mathbb{R}^{2n})}.
$$

Then we have

$$
(4.27) \qquad \int_{B_{2r}(w_0)} |\nabla X|^4 \, dudv \le K(R, \delta) r^{2-2\delta}
$$

for $w \in B_{\rho_0}(0)$ and $0 < r \le r_0$. Denote by $\kappa(R_0)$ the number

$$
\kappa(R_0) := c^{**}(R_0) K_P,
$$

which only depends on R_0. Then (4.25)–(4.27) imply

$$
(4.28) \qquad \int_{B_r(w_0)} |\nabla^2 X|^2 \, dudv \le \kappa \Big[\int_{T_{2r}} |\nabla^2 X|^2 \, dudv + K(R, \delta) r^{2-2\delta} \Big]
$$

for $w_0 \in B_{\rho_0}(0)$, $0 < r < r_0$. Now we "fill the hole" in $T_{2r} = B_{2r}(w_0) - \overline{B}_r(w_0)$ by adding $\kappa \int_{B_r(w_0)} |\nabla^2 X|^2 \, dudv$ to both sides of (4.28). Multiplying both sides of the resulting inequality by $(1+\kappa)^{-1}$, we arrive at

$$\int_{B_r(w_0)} |\nabla^2 X|^2 \, dudv \le \theta_0 \Big[\int_{B_{2r}(w_0)} |\nabla^2 X|^2 \, dudv + K(R,\delta) r^{2-2\delta} \Big] \tag{4.29}$$

for $w_0 \in B_{\rho_0}(0)$ and $0 < r < r_0$, where δ is some fixed number with $\delta \in (0, 1/2)$, and $\theta_0 = \theta_0(R_0)$ denotes the constant $\theta_0 := \kappa(\kappa+1)^{-1} \in (0,1)$. We choose some number τ with $2\delta < \tau < 1$, and set

$$\begin{aligned} \theta &:= \max\{\theta_0, 2^{-2+2\tau}\} \quad \in (0,1), \\ \sigma &:= \frac{-\log\theta}{2\log 2} > 0, \ \text{ i.e. } \ \theta = 2^{-2\sigma}, \\ r^* &:= [\theta^{-1} K(R,\delta)^{-1}(2^\tau - 1)]^{1/(\tau - 2\delta)}. \end{aligned}$$

Then

$$\omega(r) := \int_{B_r(w_0)} |\nabla^2 X|^2 \, dudv + r^{2-\tau}$$

is nondecreasing for $r \in (0, r_0]$ and satisfies

$$\omega(r) \le \theta \omega(2r) \ \text{ for } \ 0 < r \le r_1 := \min\{r_0, r^*\}.$$

For any $r \in (0, r_1)$ there is a $j \in \mathbb{N}$ such that $2^{-j} r_1 \le r < 2^{-j+1} r_1$, whence $\theta^j = 2^{-2j\sigma} \le (r/r_1)^{2\sigma}$, and

$$\begin{aligned} \omega(r) &\le \omega(2^{-j+1} r_1) \le \theta\omega(2^{-j+2} r_1) \le \theta^2 \omega(2^{-j+3} r_1) \\ &\le \ldots \le \theta^j \omega(2r_1) \le \omega(2r_1)(r/r_1)^{2\sigma}. \end{aligned}$$

This implies

$$\int_{B_r(w_0)} |\nabla^2 X|^2 \, dudv \le [(2r_1)^{2-\tau} + \int_{B_{2r}(w_0)} |\nabla^2 X|^2 \, dudv] \left(\frac{r}{r_1}\right)^{2\sigma}$$

for $w_0 \in B_{\rho_0}(0)$ and $0 < r < r_1$. If we set

$$M^*(\rho_0, R) := r_1^{-2\sigma} [(2r_0)^{2-\tau} + \int_{B_R(0)} |\nabla^2 X|^2 \, dudv],$$

it follows that

$$\int_{B_r(w_0)} |\nabla^2 X|^2 \, dudv \le M^*(\rho_0, R) r^{2\sigma} \tag{4.30}$$

for all $w_0 \in B_{\rho_0}(0)$ and all $r \in (0, r_1)$, where $r_1 = r_1(\rho_0, R)$. Morrey's "Dirichlet growth theorem" then implies $\nabla X \in C^{0,\sigma}(\overline{B}_{\rho_0}(0), \mathbb{R}^{2n})$, and thus we have proved that $X \in C^{1,\sigma}(B, \mathbb{R}^n)$ for some $\sigma > 0$. □

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