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by

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A SHORT PROOF OF THE SELF-IMPROVING REGULARITY OF QUASIREGULAR MAPPINGS

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Abstract. We provide a short proof of a theorem, due to Iwaniec and Martin [9] and Iwaniec [8], on the self-improving integrability of quasiregular mappings.

1. Introduction

Let $\Omega \subset \mathbf{R}^n$ be an open set and K a constant greater than 1. Then a mapping $f \in W^{1,n}_{loc}(\Omega, \mathbf{R}^n)$ is said to be K-quasiregular if

$$|Df(x)|^n \le K J(x,f) \quad \text{a.e. } x \in \Omega \tag{1.1}$$

where $|Df(x)|$ is the operator norm of the matrix $Df(x)$, the differential of f at the point x, and $J(x,f)$ is the Jacobian of f. The theory of quasiregular mappings is a central topic in modern analysis with important connections to a variety of topics as elliptic partial differential equations, complex dynamics, differential geometry and calculus of variations [13] [10].

A remarkable feature of quasiregular mappings is the self-improving regularity. In 1957 [2], Bojarski proved that for planar quasiregular mappings, there exists an exponent $p(2,K) > 2$ such that quasiregular mappings *a priori* in $W^{1,2}$ belong to $W^{1,p}$ for every $p < p(2,K)$. In 1973, Gehring [6] extended the result to n-dimensional quasiconformal mappings (homeomorphic quasiregular mappings) and proved the celebrated Gehring's Lemma. A bit later, Elcrat and Meyers [4] proved that Gehring's ideas can be further exploited to treat quasiregular mappings and partial differential systems.

The higher integrability result admits dual version. In two outstanding papers, Iwaniec and Martin [9] (for even dimensions) and Iwaniec [8] (for all dimensions) proved the following theorem.

Theorem 1.1. *There exists two numbers $q(n,K) < n < p(n,K)$ such that for every $q, p \in (q(n,K), p(n,K))$ every mapping $f \in W^{1,q}_{loc}(\Omega, \mathbf{R}^n)$*

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such that (1.1) holds belongs to $W^{1,p}_{loc}(\Omega, \mathbf{R}^n)$*. Moreover, for each* $\varphi \in C_0^\infty(\Omega)$ *we have the Caccioppoli type inequality*

$$\|\varphi Df\|_q \leq C_q(n, K)\|f \otimes \nabla\varphi\|_q,$$

where $\otimes$ *denotes the tensor product.*

The proofs in [8] have been simplified and are nicely presented in [10]. The key point is to prove the Caccioppoli inequality for a range of p smaller than n. Actually, in the case $q \leq n$, the constant $C_q(n, K)$ in the Caccioppoli inequality can be chosen independent of q. Then applying the Poincaré inequality, one infers that $|Df|^q$ satisfies a weak reverse Hölder's inequality. Then Gehring's lemma can be applied to verify the $L^{q+\delta}$ integrability of $|Df|$ with some $\delta = \delta(n, K) > 0$. The exponent will eventually exceed n by iterating the process, and the theorem is proved. The detailed argument is in [10, chapter 14].

We remark here that the Caccioppoli inequality can be employed studying removable sets for bounded quasiregular mappings, see [10, Theorem 17.3.1].

The proof in [9] and [8] for the Cacciopoli inequality is very deep. A whole theory about non-linear commutators, non-linear Hodge decompositions and other topics is developed. The even dimensional situation is almost as complicated.

In this note, we give a short proof of Theorem 1.1. The proof is self-contained and surprisingly simple. Our approach is inspired by the paper of Lewis [12]. We construct a Lipschitz continuous test-function by using a point-wise inequality for the Sobolev functions in terms of the maximal functions. More precisely, it is well known that for any mapping $g \in C_0^\infty(\Omega, \mathbf{R}^n)$,

$$\int_\Omega J(x, g)\, dx = 0. \tag{1.2}$$

An easy approximation argument shows that (1.2) is still true if $g \in W_0^{1,n}(\Omega, \mathbf{R}^n)$. Of course, (1.2) is generally not true if we only assume that $g \in W_0^{1,q}(\Omega, \mathbf{R}^n)$ for some $q < n$. In order to use the equality (1.2), which is the essential point of the argument, we modify one of the coordinate function of g, say g_1, by truncating it in terms of the maximal function of $|Dg|$ and construct a new function, which is Lipschitz continuous in Ω. Then the equality (1.2) is true for the modified mapping.

Several variations of this idea have been used to treat different problems, see [3], [14], [11]. The method seems to be very effective when dealing with problems involving exponents below the natural ones. This technique will be further exploited to settle the problem of self-improving regularity in the class of mappings of finite distortion [5].

The building block is the following well-known point-wise estimates for the Sobolev function, whose proofs rely on an argument due to Hedberg [7].

Lemma 1.2 (Point-wise inequalities for the Sobolev functions)**.** *Let* $u \in W^{1,p}(\mathbf{R}^n), 1 < q < \infty$, *and let* x *and* y *be Lebesgue points of* u *such that* $x \in B_0 = B(x_0, r)$. *Then*

$$|u(x) - u_{B_0}| \le crM(|\nabla u|\chi_{2B_0}))(x_0) \tag{1.3}$$

$$|u(x) - u(y)| \le c|x-y|(M(|\nabla u|)(x) + M(|\nabla u|)(y)), \tag{1.4}$$

where $c = c(n) > 0$, χ_E *is the characteristic function of the set* E, v_{B_0} *is the average of* v *over* $B_0 = B(x_0, r)$ *and* Mh *is the Hardy-Littlewood maximal function of* h.

Naturally one asks what the exact values of $q(n,K)$ and $p(n,K)$ are. The answer is known in the planar case only. In 1993, Astala [1] proved the Gehring-Reich conjecture: the exponents $q(2,K)$ and $p(2,K)$ are $\frac{2K}{K+1}$ and $\frac{2K}{K-1}$, respectively. For the higher dimensional cases, there are no good estimates for these thresholds. Unfortunately, our proof does not improve the existing results in this respect.

2. The proof

As explained in the introduction, it is clear that our task is to prove the Caccioppoli inequality with a uniform constant for exponents smaller than n. That is, we need to prove that there is $q(n,K) < n$ such that for any $q(n,K) < q \le n$ if $f \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$ satisfies (1.1) then

$$\int_\Omega |\varphi Df|^q \, dx \le C(n,K) \int_\Omega |f \otimes \nabla\varphi|^q \, dx, \tag{2.1}$$

for all test-functions $\varphi \in C_0^\infty(\Omega)$. Clearly we may assume that $\varphi \in C_0^\infty(B_0)$, $B_0 = B(x_0, r) \Subset \Omega$, and $\varphi \ge 0$.

We will approximate the first component f_1 of f suitably. Let $u = f_1\varphi$ and extend it to be zero in $\mathbf{R}^n \setminus B_0$. Then $u \in W^{1,q}(\mathbf{R}^n)$. Denote for $\lambda > 0$,

$$F_\lambda = \{x \in B(x_0, r) : M(g)(x) \le \lambda \text{ and } x \text{ is a Lebesgue point of } u\},$$

where $g = |\varphi Df| + |f \otimes \nabla\varphi|$ in B_0 and $g = 0$ in $\mathbf{R}^n \setminus B_0$. It is easy to show that u is $c\lambda$-Lipschitz continuous on the set $F_\lambda \cup (\mathbf{R}^n \setminus B_0)$ for $c = c(n) \ge 1$. Indeed, suppose that $x, y \in F_\lambda$. Since $|\nabla u| \le c(n)g$, then it follows from (1.4) that

$$\begin{aligned} |u(x) - u(y)| &\le c|x-y|(M(|\nabla u|)(x) + M(|\nabla u|)(y)) \\ &\le c|x-y|(M(g)(x) + M(g)(y)) \\ &\le c\lambda|x-y|. \end{aligned} \tag{2.2}$$

If $x \in F_\lambda$ and $y \in \mathbf{R}^n \setminus B_0$, set $\rho = 2\,\mathrm{dist}(x, \mathbf{R}^n \setminus B(x_0, r))$. Since $|\{x \in B(x,\rho) : u(x) = 0\}| \geq |B(x,\rho) \cap (\mathbf{R}^n \setminus B_0)| \geq c(n)|B(x,\rho)|$, the Poincaré inequality yields

$$|u_{B(x,\rho)}| \leq c(n)\rho|\nabla u|_{B(x,\rho)} \leq c\rho Mg(x) \leq c\lambda|x-y|$$

Thus by (1.3),

$$\begin{aligned} |u(x) - u(y)| = |u(x)| &\leq |u(x) - u_{B(x,\rho)}| + |u_{B(x,\rho)}| \\ &\leq c\rho M(|\nabla u|)(x) + c\lambda|x-y| \\ &\leq c\rho Mg(x) + c\lambda|x-y| \leq c\lambda|x-y|. \end{aligned} \tag{2.3}$$

If $x, y \in \mathbf{R}^n \setminus B_0$, then the claim is clear. Since all the other cases follow by symmetry, it follows that $u|_{F_\lambda \cup (\mathbf{R}^n \setminus B_0)}$ is Lipschitz continuous with the constant $c\lambda$. We extend $u|_{F_\lambda \cup (\mathbf{R}^n \setminus B_0)}$ to a Lipschitz continuous function u_λ in $\mathbf{R}^n$ with the same constant by the classical McShane extension theorem. Then we consider the mapping $f_\lambda = (u_\lambda, \varphi f_2, \varphi f_3, ..\varphi f_n)$. Since $f \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$, an easy approximation argument shows that if $q \geq n-1$,

$$\int_\Omega J(x, f_\lambda)\,dx = 0,$$

and hence,

$$\int_{F_\lambda} J(x, \varphi f)\,dx \leq -\int_{\Omega \setminus F_\lambda} J(x, f_\lambda)\,dx \tag{2.4}$$

Now $|f_i \nabla\varphi| \leq C(n)|f \otimes \nabla\varphi|$ and $|\nabla(\varphi f_i)| \leq c(n)g$. Putting these estimates together with (2.4) and expressing the Jacobian as a wedge product of differential forms, we obtain that

$$\int_{F_\lambda} \varphi^n J(x,f) \leq c(n)(\lambda\int_{\Omega\setminus F_\lambda} g^{n-1} + \int_{F_\lambda} |f \otimes \nabla\varphi| g^{n-1}).$$

This inequality holds for all $\lambda > 0$. Then, we multiply it by $\lambda^{-1-\epsilon}$ for some $\epsilon > 0$, which will be determined later. We integrate with respect to λ over $(0,\infty)$, and finally change the order of integration to obtain that

$$\begin{aligned} \int_\Omega \varphi^n J(x,f) \int_{Mg(x)}^\infty \lambda^{-1-\epsilon}\,d\lambda\,dx \leq c(n) &\left(\int_\Omega g^{n-1} \int_0^{Mg(x)} \lambda^{-\epsilon}\,d\lambda\,dx \right. \\ &\left. + \int_\Omega |f \otimes \nabla\varphi| g^{n-1} \int_{Mg(x)}^\infty \lambda^{-1-\epsilon}\,d\lambda\,dx \right). \end{aligned} \tag{2.5}$$

We use the quasiregularity of f which so far have not been used. It follows that

$$\int_\Omega \varphi^n |Df|^n M(g)^{-\epsilon}\, dx \le c(n)K \int_\Omega |f \otimes \nabla\varphi| g^{n-1} M(g)^{-\epsilon}\, dx + \frac{\epsilon c(n)K}{1-\epsilon} \int_\Omega g^{n-1} M(g)^{1-\epsilon}\, dx \tag{2.6}$$

From now on we only use Hölder's inequality and the Hardy-Littlewood maximal theorem to conclude the proof by choosing suitable small ϵ. On one hand we have that

$$\begin{aligned} \int_\Omega (|Df||\varphi|)^{n-\epsilon}\, dx &\le (\int_\Omega |Df|^n |\varphi|^n M(g)^{-\epsilon}\, dx)^{\frac{n-\epsilon}{n}} (\int_\Omega M(g)^{n-\epsilon}\, dx)^{\frac{\epsilon}{n}} \\ &\le c(n)(\int_\Omega |Df|^n |\varphi|^n M(g)^{-\epsilon}\, dx)^{\frac{n-\epsilon}{n}} (\int_\Omega g^{n-\epsilon}\, dx)^{\frac{\epsilon}{n}}, \end{aligned} \tag{2.7}$$

where we have used the Hardy-Littlewood maximal theorem

$$\int_\Omega M(g)^{n-\epsilon}\, dx \le c(n) \int_\Omega g^{n-\epsilon}\, dx, \tag{2.8}$$

where the constant $c(n) > 0$ can be chosen such that it does not depend on ϵ if $0 \le \epsilon \le 1$. Now we observe that we may assume that

$$\int_\Omega g^{n-\epsilon}\, dx \le 2^{n-\epsilon} \int_\Omega (|Df||\varphi|)^{n-\epsilon}\, dx. \tag{2.9}$$

Otherwise the Caccioppoli inequality (2.1) holds with the constant $c(n, K) = 1$. Thus it follows from (2.7) that

$$\int_\Omega g^{n-\epsilon}\, dx \le c(n) \int_\Omega |Df|^n |\varphi|^n M(g)^{-\epsilon}\, dx. \tag{2.10}$$

On the other hand, we estimate the right hand side of (2.6) by Hölder's inequality and (2.8). By (2.10) and (2.6),

$$\begin{aligned} \int_\Omega g^{n-\epsilon}\, dx &\le c(n) \int_\Omega |Df|^n |\varphi|^n M(g)^{-\epsilon}\, dx \\ &\le c(n)K(\int_\Omega |f \otimes \nabla\varphi|^{n-\epsilon}\, dx)^{\frac{1}{n-\epsilon}} (\int_\Omega g^{n-\epsilon}\, dx)^{\frac{n-\epsilon-1}{n-\epsilon}} \\ &+ c(n)K\frac{\epsilon}{1-\epsilon} \int_\Omega g^{n-\epsilon}\, dx \end{aligned} \tag{2.11}$$

Then we conclude the proof by choosing $0 < \epsilon < 1$ small enough such that $c(n)K\epsilon/(1-\epsilon) \le 1/2$.

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