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Regularity and blow-up analysis for
$J$-holomorphic maps

by

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# Regularity and blow-up analysis for $J$-holomorphic maps

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## Abstract

*If $u \in H^1(M, N)$ is a weakly J-holomorphic map from a compact without boundary almost hermitian manifold $(M, j, g)$ into another compact without boundary almost hermitian manifold $(N, J, h)$. Then it is smooth near any point $x$ where $Du$ has vanishing Morrey norm $\mathcal{M}^{2,2m-2}$, with $2m = dim(M)$. Hence $H^{2m-2}$ measure of the singular set for a stationary J-holomorphic map is zero. Blow-up analysis and the energy quantization theorem are established for stationary J-holomorphic. Connections between stationary J-holomorphic maps and stationary harmonic maps are given for either almost Kähler manifolds $M$ and $N$ or symmetric $\nabla^h J$.*

§1 Introduction and statements of results

Let $(M, j, g)$ (respectively $(N, J, h)$) be a smooth hermitian almost complex manifold with dimension $2m$ (respectively $2n$). Assume further that $(N, h)$ is compact without boundary and isometrically embedded into some euclidean $R^k$ via the Nash's embedding theorem. Denote the Sobolev space $H^1(M, N) = \{v \in H^1(M, R^k) \mid v(x) \in N \text{ for a. e. } x \in M\}$.

**Definition 1.1**. A map $u \in H^1(M, N)$ is said to be weakly $(j, J)$-holomorphic map (or $J$-holomorphic map for abbreviation) if $du$ preserves the almost complex structures in the sense:

$$du(j(x)(X)) = J(u)(du(X)), \text{ for a. e. } x \in M, \forall X \in T_x M \qquad (1.1)$$

Note that $J$-holomorphic maps are higher dimensional natural extensions

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of pseudo-holomorphic curves, which have been very important subjects and have had so many important applications in four-dimensional geometric topology, since the pioneering works by Gromov [G]. Moreover, the compactnees for pseudo-holomorphic curves was also a very interesting problem from the analytic point of views (cf [Y] [PW]). In a very recent work, Riviere-Tian [RT] made the study for $j$-holomorphic maps from almost complex 4-manifold $(M^4, j)$ into algebraic varieties $N \subset CP^n$, in connections with C. Taubes' works on Seiberg-Witten and Gromov invariants for symplectic 4-manifolds. In particular, it was proven by [RT] that any locally approximable $j$-holomorphic map is smooth away from isolated points.

In this paper we are interested in regularity for weakly $J$-holomorphic maps. A point $x \in M$ is a *regular* point for $u$ if there is a $r > 0$ such that $u \in C^\infty(B_r(x), N)$, here $B_r(x)$ denotes the geodesic ball with radius $r$, centered at $x$. It is clear that the set of regular points of $u$ is an open subset of $M$, whose complement is called *singular* set of $u$. Observe that if $x \in M$ is a regular point, then

$$\lim_{r \downarrow 0} \max \{ s^{2-m} \int_{B_s(y)} |Du|^2 | B_s(y) \subset B_r(x) \} = 0 \qquad (1.2)$$

By the regularity theory of minimizing harmonic maps by Schoen-Uhlenbeck [SU] and for stationary harmonic maps by Hélein [H], Evans [E], Bethuel [B], we know that the smallness condition (1.2) is also sufficient for smoothness. Although we know (see remark (8.16) of Eells-Lemaire [EL] page 51) that $J$-holomorphic maps are not necessarily harmonic maps, our first result confirms that the same regularity criterion holds for weakly $J$-holomorphic maps. We first recall that the condition (1.2) can be expressed in terms of the Morrey space and recall the following definition (see, e.g. Giaquinta [G]).

**Definition 1.2**. For $p \in (1, 2]$ and an open set $\Omega \subset M$, the Morrey space

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$\mathcal{M}^{p,2m-p}$ is defined by

$$\mathcal{M}^{p,2m-p}(\Omega) = \{f \in L^2(\Omega) :$$

$$\|f\|_{\mathcal{M}^{p,2m-p}(\Omega)} \equiv (\sup_{B_r(x) \subset \Omega} r^{p-2m} \int_{B_r(x)} |f|^p)^{\frac{1}{p}} < \infty\}$$

Now we state our first theorem

**Theorem A**. *There exists an $\epsilon_0 > 0$ such that if $u \in H^1(M, N)$ is a weakly J-holomorphic map and satisfies, for $B_r(x) \subset M$,*

$$\|Du\|_{\mathcal{M}^{2,2m-2}(B_r(x))} \leq \epsilon_0 \tag{1.3}$$

*then $u \in C^\infty(B_{\frac{r}{2}}(x), N)$.*

Our idea to prove theorem A follows from the two new observations: (1) Under the assumption that $u(B_r(x))$ is contained in a coordinate chart $U$ of $N$, we can use the local coordinate frame on $U$ to express $J$ as $SO(2n)$-valued function so that the eqn. (1.1) can imply $\Delta u^\alpha = f^\alpha$, with $f^\alpha$ having a jacobian structure, hence the ideas for proving the regularity theorem for stationary harmonic maps into spheres (see, [H1] [E] or [CWY]) is applicable to yield that $u$ is Hölder continuous in $B_{\frac{r}{2}}(x)$; (2) In general, we can modify the enlargement idea, due to Hélein [H], to isometrically embed $(N, h)$ into a higher dimensional manifold $(\tilde{N}, \tilde{h})$ which admits a global smooth orthonormal frame $\{e_\alpha\}_{\alpha=1}^l$ ($l = \dim(\tilde{N})$), then we can derive from the eqn. (1.1) that

$$\operatorname{div}(\langle Du, e_\alpha(u) \rangle) = f_\alpha(x) g_\alpha(x), \quad 1 \leq \alpha \leq l$$

with $f_\alpha$ having jacobian structure and $g_\alpha \in H^1 \cap L^\infty$. Hence we can adopt Bethuel's idea ([B]) for regularity of stationary harmonic maps into general target manifolds, see §2 below for details. In this way, we find that the proof of regularity properties of J-holomorphic maps is very much related to that of stationary harmonic maps.

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For $m = 1$, i.e. $M$ a Riemannian surface. Observe that $\mathcal{M}^{2,0} = L^2(M)$ so that the absolute continuity of $\int |Du|^2$ that the condition (1.3) is satisfied for any $x \in M$, and sufficiently small $r > 0$. Hence, as a byproduct of theorem A, we find a new proof of the interior regularity theorem of weakly pseudo-holomorphic curves by Ye [Y] on his proof of Gromov's compactness theorem for pseudo-holomorphic curves (see also Wolfson [W] or Parker-Wolfson [PW]). More precisely,

**Theorem B** ([Y]). *Assume that $M$ is a compact Riemannian surface without boundary. If $u \in H^1(M, N)$ is a weakly J-holomorphic curve. Then $u \in C^\infty(M, N)$.*

For $m \geq 2$, observe that $\mathcal{M}^{2,2m-2}$ is a scaling-invariant subspace of $L^2(M)$ whose elements behave like $L^{2m}(M)$ from the point of view of scalings. In §3 below, we introduce a class of $J$-holomorphic maps $u \in H^1(M, N)$, called stationary $J$-holomorphic maps, which have vanishing first variations with respect to the domain variations, i.e. for any smooth vector field $X$ on $M$ with compact support,

$$\frac{d}{dt}|_{t=0} \int_M |Du(F_t(x))|^2 = 0 \tag{1.4}$$

where $F_t : M \rightarrow M$ is a parameter family of diffeomorphisms generated by $X$. It follows from Proposition 3.2 that any stationary $J$-holomorphic map satisfies the energy monotonicity inequality: there is a $C_0 = C_0(M, j, g) > 0$ such that

$$e^{C_0 r} r^{2-2m} \int_{B_r(x)} |Du|^2(x) + 2 \int_{B_R(x) \backslash B_r(x)} e^{C_0|y-x|} |y - x|^{2-2m} |\frac{\partial u}{\partial |y - x|}|^2$$

$$\leq e^{C_0 R} R^{2-2m} \int_{B_R(x)} |Du|^2(x) \tag{1.5}$$

for any $x \in M$ and $0 < r \leq R \leq R_0 = R_0(M, j, g)$. A direct consequence of (1.5) is: for any $x \in M$ and $0 < r \leq R_0$,

$$\|Du\|_{\mathcal{M}^{2,2m-2}(B_r(x))} \leq C_0 r^{2-2m} \int_{B_r(x)} |Du|^2 \tag{1.6}$$

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Hence theorem A yields the partial regularity for stationary $J$-holomorphic maps, which is an analogy to the partial regularity for stationary harmonic maps. More precisely,

**Theorem C**. *Suppose that $u \in H^1(M, N)$ is a stationary $J$-holomorphic map. Define*

$$\Sigma = \{x \in M \mid \lim_{r \to 0} r^{2-2m} \int_{B_r(x)} |Du|^2 > 0\}$$

*Then $H^{2m-2}(\Sigma) = 0$ and $u \in C^\infty(M \setminus \Sigma, N)$.*

Based on both the energy monotonicity inequality (1.5) and the small energy regularity theorem A, we find that the blow-up techniques for stationary harmonic maps developed by Lin [L] can be modified to study the convergence issues for sequences of stationary $J$-holomorphic maps.

From now on, we call a nonconstant smooth $J$-holomorphic map $\omega$ : $(S^2, j_0) \to (N, J)$ as a pseudo-holomorphic $S^2$, here $j_0$ is the standard complex structure on $S^2$. We prove

**Theorem D**. *Let $\{u_k\} \subset H^1(M, N)$ be stationary $J$-holomorphic maps which converges weakly to $u \in H^1(M, N)$. Then $u$ is a weakly $J$-holomorphic map and there exists a closed $(2m-2)$-rectifiable set $\Sigma \subset M$, with $H^{2m-2}(\Sigma) < \infty$, such that*

*(i) $u_k \to u$ in $H^1_{loc} \cap C^1_{loc}(M \setminus \Sigma, N)$. In particular, $u \in C^\infty(M \setminus \Sigma, N)$.*

*(ii)*

$$\frac{1}{2}|Du_k|^2 \, dx \to \frac{1}{2}|Du|^2 \, dx + \nu$$

*as convergence of Radon measures, for some nonnegative Radon measure $\nu$ on $M$. Moreover, $\nu = \theta H^{2m-2} \mathbf{L} \Sigma$ for some nonnegative $H^{2m-2}$-measurable function*

$$\theta(x) = \lim_{r \to 0} \frac{\nu(B_r(x))}{|B_r(x)|} \geq \epsilon_0^2, \text{ for } H^{2m-2} a. \ e. \ x \in \Sigma$$

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*(iii) For $H^{2m-2}$ a. e. $x \in \Sigma$, $T_x\Sigma \subset R^{2m}$ is $j_x$-holomorphic $(2m-2)$-plane, i. e. $j_x(T_x\Sigma) = T_x\Sigma$.*

*(iv) $u_k \nrightarrow u$ in $H^1(M, N) \Leftrightarrow H^{2m-2}(\Sigma) > 0$. Moreover, there exists at least one pseudo-holomorphic $S^2$ in $(N, J)$.*

The main difference between our proof of theorem D and §2 of [L] is that we need to verify that the concentration set $\Sigma$ is $j$-holomorphic $(2m-2)$-rectifiable set. Once we achieve this, then both the conformality and the removablity of isolated singularity for pseudo-holomorphic curves (cf. [Y] [PW]) guarantee that the restriction of a bubble on $(T_x\Sigma)^\perp$ can be lifted to be a pseudo-holomorphic $S^2$.

It is a very important problem to quantify the density function $\theta$ for the defect measure $\nu$ in the content of blow-up analysis for stationary harmonic maps. In [LR], Lin-Riviere were able to quantify, under that assumption that $N$ is a standard sphere, $\theta$ as the finite sum of energies of harmonic $S^2$. Likewise, to quantify $\theta$ in the content of stationary $J$-holomorphic maps is also a problem of great importance, such as Gromov's compactness for pseudo-holomorphic curves (cf. [Y] [PW]). For a map $v : S^2 \to N$, let $E(v, S^2) = \frac{1}{2} \int_{S^2} |Dv|^2$ denote its Dirichlet energy. In this aspect, we prove

**Theorem E**. *Under the same assumptions and notations as in theorem D. We have, for $H^{2m-2}$ a. e. $x \in \Sigma$, there is $1 \leq l_x \leq [\frac{\nu(M)}{\epsilon_0^2}]$ such that*

$$\theta(x) = \sum_{i=1}^{l_x} E(\omega_i, S^2) \tag{1.7}$$

*for some pseudo-holomorphic $S^2$'s, $\{\omega_i\}_{i=1}^{l_x}$.*

The ideas to prove theorem E are based on the observations that on $(T_x\Sigma)^\perp$ the blow-up sequences are both approximated $(j_0, J)$-holomorphic maps and approximated conformal maps, with perturbation errors uniformly small in $L^2$, see §3 below. Therefore, the mixtures of ideas from the proof of

theorem A with ideas from Sacks-Uhlenbeck [SaU] and Lin-Riviere [LR] can yield the conclusion.

Note that the obstruction to strong convergence in $H^1$ for stationary $J$-holomorphic maps are pseudo-holomorphic $S^2$'s. Therefore, if $N$ supports no pseudo-holomorphic $S^2$'s, then we can apply the Federer's dimension reduction argument (cf [F]) to prove

**Theorem F**. *Assume that $(N, J, h)$ doesn't support any pseudo-holomorphic $S^2$. If $u \in H^1(M, N)$ is stationary $J$-holomorphic. Then $dim_H(sing(u)) \leq 2m - 6$. In particular, $sing(u)$ is discrete for $m = 3$.*

The paper is written as follows. In §2, we prove theorem A and C. In §3, we prove theorem D, E and F. In §4, we discuss the relationship between $J$-holomorphic maps and harmonic maps in the case either $(M, j, g)$ and $(N, J, h)$ are almost Kähler manifolds or $\nabla^h J$ is symmetric.

§2 Regularity for $J$-holomorphic maps and proof of theorem A

In this section, we prove theorem A. The proof is divided into two cases: (1) $u(B_r(x))$ is in a coordinate chart of $N$; (2) no restriction on $u(B_r(x))$. It follows from the higher order regularity theory that it suffices to prove $u$ is Hölder continuous under the smallness assumption (1.3) (see, e.g., [Y]). It follows from the Morrey decay Lemma [M] that the key step to prove the Hölder continuity is the following self-improving Lemma.

**Lemma 2.1** *There exist $\epsilon_0 > 0$, $\theta_0 \in (0, \frac{1}{4})$, and $C_0 > 0$ depending only on $(M, j, g)$ such that if $u \in H^1(M, N)$ is a weakly $J$-holomorphic map and satisfies, for $B_r(x) \subset M$, $\|Du\|_{\mathcal{M}^{2,2m-2}(B_r(x))} \leq \epsilon_0$, then*

$$\|Du\|_{\mathcal{M}^{1,2m-1}(B_{\theta_0 r}(x))} \leq \frac{1}{2}\|Du\|_{\mathcal{M}^{1,2m-1}(B_r(x))} + C_0 r \qquad (2.1)$$

**Proof**. We proceed it by two cases.

**Case 1**. *There exists a coordinate chart $U \subset N$ such that $u(y) \in U$ for a.e. $y \in B_r(x)$.*

Note that this is the type of conditions appeared in Giaquinta-Giusti [GG]]. For simplicity, we assume $x = 0 \in M$. Since $\mathcal{M}^{2,2m-2}(B_r(0))$ is non-decreasing with respect to $r$, we may assume that $r > 0$ is chosen to be sufficiently small so that there is a normal coordinate system $(x_1, \cdots, x_{2m})$ on $B_r(0)$. On $U$, let $(y_1, y_2, \cdots, y_{2n-1}, y_{2n})$ denote the coordinate system and $(\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_{2n}})$ denote the coordinate frame field. Using these coordinate systems, the almost complex structure $J$ can be written as

$$J(y) = \sum_{\alpha,\beta=1}^{2n} J_{\alpha\beta}(y)\frac{\partial}{\partial y_\alpha} \otimes dy_\beta$$

for some skew-symmetric $(J_{\alpha\beta}(y)) \in C^\infty(U, SO(2n))$, and $j$ can be written as

$$j(x) = \sum_{i,k=1}^{2m} j_{ik}(x)\frac{\partial}{\partial x_i} \otimes dx_k$$

for some skew-symmetric $(j_{ik}(x)) \in C^\infty(B_r(0), SO(2m))$. Moreover,

$$\frac{\partial}{\partial x_i} = \sum_{k=1}^{2m} j_{ki}(x)(j(x)(\frac{\partial}{\partial x_k}))$$

Now the $J$-holomorphic map eqn. (1.1) becomes: $\forall 1 \leq \alpha \leq 2n$,

$$\frac{\partial u^\alpha}{\partial x_i} = \sum_{k=1}^{2n} j_{ki}(x)(du(j(x)(\frac{\partial}{\partial x_k})))^\alpha$$

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$$= \sum_{k=1}^{2m} j_{ki}(x)(J(u)(\frac{\partial u}{\partial x_k}))^{\alpha}$$

$$= \sum_{k=1}^{2m} \sum_{\beta=1}^{2n} j_{ki}(x) J_{\alpha\beta}(u) \frac{\partial u^{\beta}}{\partial x_k} \tag{2.2}$$

Let $\Delta_g = -\sum_{i,l=1}^{2m} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_l}(\sqrt{g} g^{li} \frac{\partial}{\partial x_i})$ denote the Laplace-Beltrami operator on $(M, g)$, here $(g^{li}) = g^{-1}$. Then we have

**Claim 1**. *In $B_r(0)$, for $1 \le \alpha \le 2n$,*

$$-\Delta_g u^{\alpha} = \frac{1}{\sqrt{g}} \sum_{\beta=1}^{2n} \sum_{1 \le k < l \le 2m} [\frac{\partial}{\partial x_k}(\sqrt{g}(\sum_{i=1}^{2m} g^{li} j_{ik}(x)) J_{\alpha\beta}(u)) \frac{\partial u^{\beta}}{\partial x_l}$$

$$- \frac{\partial}{\partial x_l}(\sqrt{g}(\sum_{i=1}^{2m} g^{li} j_{ik}(x)) J_{\alpha\beta}(u)) \frac{\partial u^{\beta}}{\partial x_k}] \tag{2.3}$$

*Proof of Claim 1.* By taking one more derivative of the eqn. (2.2), we have

$$-\Delta_g u^{\alpha} = \frac{1}{\sqrt{g}} \sum_{\beta=1}^{2n} \sum_{k,l=1}^{2m} \frac{\partial}{\partial x_l}(\sqrt{g}(\sum_{i=1}^{2m} g^{li} j_{ki}(x)) J_{\alpha\beta}(u) \frac{\partial u^{\beta}}{\partial x_k})$$

$$= \sum_{\beta=1}^{2n} \sum_{k,l=1}^{2m} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_l}(\sqrt{g}(\sum_{i=1}^{2m} g^{li} j_{ki}(x)) J_{\alpha\beta}(u)) \frac{\partial u^{\beta}}{\partial x_k}$$

$$+ \sum_{\beta=1}^{2n} \sum_{k,l=1}^{2m} (\sum_{i=1}^{2m} g^{li} j_{ki}(x)) J_{\alpha\beta}(u) \frac{\partial^2 u^{\beta}}{\partial x_l \partial x_k}$$

$$= I + II$$

Now we need

**Claim 2**. *$g^{-1}j$ is skew-symmetric, i.e. $\sum_{i=1}^{2m}(g^{li} j_{ik}(x)) = -\sum_{i=1}^{2m}(g^{ki} j_{il}(x))$ for any $1 \le l, k \le 2m$.*

*Proof of Claim 2.* Since $g$ is hermitian with respect to $j$, we have

$$g_{kl} = g(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}) = g(j(x)(\frac{\partial}{\partial x_k}), j(x)(\frac{\partial}{\partial x_l}))$$

$$= j_{ik}(x) j_{pl}(x) g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_p}) = -j_{ki}(x) g_{ip} j_{pl}(x)$$

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This is equivalent to $g = -jgj$. Since $j^2 = -I_{2m}$, we have $gj = jg$ and $g^{-1}j = jg^{-1}$. Therefore

$$(g^{-1}j)^t = j^t(g^{-1})^t = -jg^{-1} = -g^{-1}j.$$

Now it is easy to see that $II = 0$ and

$$I = -\frac{1}{\sqrt{g}}\sum_{\beta=1}^{2n}\sum_{1\le k<l\le 2m}[\frac{\partial}{\partial x_l}(\sqrt{g}(\sum_{i=1}^{2m}g^{li}j_{ik}(x))J_{\alpha\beta}(u))\frac{\partial u^\beta}{\partial x_k}$$
$$-\frac{\partial}{\partial x_k}(\sqrt{g}(\sum_{i=1}^{2m}g^{li}j_{ik}(x))J_{\alpha\beta}(u))\frac{\partial u^\beta}{\partial x_l}]$$

Hence we obtain the eqn. (2.3).

Since each term in the summation of the right hand side of the eqn. (2.3) is of the jacobian structure $\{\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_l} - \frac{\partial f}{\partial x_l}\frac{\partial g}{\partial x_k}\}$ for some $f, g \in H^1(B_r(0))$, which belongs to the Hardy space $\mathcal{H}^1(B_r(0))$ by the theorem of [CLMS]. Moreover, it follows from the Poincaré inequality that $Du \in \mathcal{M}^{2,m-2}(B_r(0))$ implies that $u \in \mathrm{BMO}(B_r(0))$, and for any $1 < p \le 2$,

$$[u]_{\mathrm{BMO}(B_r(0))} \equiv \sup\{\inf_{c\in R^k} s^{-2m}\int_{B_s(x)}|u-c| \mid B_s(x) \subset B_r(0)\}$$
$$\le C\|Du\|_{\mathcal{M}^{p,2m-p}(B_r(0))} \le C\|Du\|_{\mathcal{M}^{2,2m-2}(B_r(0))} \quad (2.4)$$

Now we can apply the duality theorem between $\mathcal{H}^1$ and BMO (see Fefferman-Stein [FS]) to prove (2.1) as follows. For $y \in B_{\theta_0 r}(0)$, $0 < s \le \theta_0 r$, let $\Lambda = (2\theta_0)^{-1}$ with $\theta_0$ to be chosen later, and $v \in H^1(B_{\Lambda s}(y))$ be such that, for any $1 \le \alpha \le 2n$,

$$\Delta v^\alpha = 0, \quad \text{in } B_{\Lambda s}(y) \quad\quad (2.5)$$
$$v^\alpha = u^\alpha, \text{ on } \partial B_{\Lambda s}(y)$$

Let $w = u - v$. Multiplying both sides of the eqn. (2.3) by $w^\alpha$, integrating it over $B_{\Lambda s}(y)$, and summing over $\alpha$, we obtain

$$\int_{B_{\Lambda s}(y)}|Dw|^2\,dx$$

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$$
= \sum_{\alpha,\beta,i,k<l} \int_{B_{\Lambda s}(y)} \left[ \frac{\partial}{\partial x_l}(\sqrt{g}g^{li}j_{ik}(x)J_{\alpha\beta}(u))\frac{\partial w^\alpha}{\partial x_k} \right.
$$
$$
\left. - \frac{\partial}{\partial x_k}(\sqrt{g}g^{li}j_{ik}(x)J_{\alpha\beta}(u))\frac{\partial w^\alpha}{\partial x_l} \right]u^\beta
$$
$$
\leq C \sum_{\alpha,\beta,i,k<l} [u^\beta]_{\mathrm{BMO}(B_r(0))} \cdot
$$
$$
\|\frac{\partial}{\partial x_l}(\sqrt{g}g^{li}j_{ik}(x)J_{\alpha\beta}(u))\frac{\partial w^\alpha}{\partial x_k} - \frac{\partial}{\partial x_k}(\sqrt{g}g^{li}j_{ik}(x)J_{\alpha\beta}(u))\frac{\partial w^\alpha}{\partial x_l}\|_{\mathcal{H}^1(R^{2m})}
$$
$$
\leq C[u]_{\mathrm{BMO}(B_r(0))}\|Dw\|_{L^2(B_{\Lambda s}(y))}\|D(\sqrt{g}g^{-1}j(x)J(u))\|_{L^2(B_{\Lambda s}(y))}
$$

Direct calculations show that there is $C_1 > 0$, depending only on $g, j, J$, such that

$$
\|D(\sqrt{g}g^{-1}j(x)J(u))\|_{L^2(B_{\Lambda s}(y))} \leq C_1(\Lambda s)^m + \|Du\|_{L^2(B_{\Lambda s}(y))}
$$

Therefore, it follows from the standard decay estimate for harmonic functions that we have

$$
s^{1-2m}\int_{B_s(y)}|Du| \leq s^{1-2m}\int_{B_s(y)}|Dv| + s^{1-2m}\int_{B_s(y)}|Dw|
$$
$$
\leq C\Lambda^{-1}[v]_{B_{\Lambda s}(y)} + (s^{2-2m}\int_{B_{\Lambda s}(y)}|Dw|^2)^{\frac{1}{2}}
$$
$$
\leq C(\Lambda^{-1} + \Lambda^m s + \Lambda^{m-1}\epsilon_0)[u]_{\mathrm{BMO}(B_r(0))}
$$
$$
\leq C(\theta_0 + \theta_0^{-m}s + \theta_0^{1-m}\epsilon_0)[u]_{\mathrm{BMO}(B_r(0))}
$$

Therefore, by first choosing sufficiently small $\theta_0 > 0$ and then choosing much smaller $\epsilon_0 > 0$, we have

$$
\|Du\|_{\mathcal{M}^{1,2m-1}(B_{\theta_0 r}(0))} \leq \sup\{s^{1-2m}\int_{B_s(y)}|Du| \mid y \in B_{\theta_0 r}, s \leq \theta_0 r\}
$$
$$
\leq \frac{1}{2}\|Du\|_{\mathcal{M}^{1,2m-1}(B_r(0))} + Cr
$$

This yields (2.1) and the proof of Case 1 is complete. ∎

**Case 2**. *Modification of Hélein construction for global orthonormal frames*

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In order to push forward the ideas from the Case 1, we need to find an alternative to replace the local coordinate frame on the target manifold $(N, h)$. For this purpose, we modify the construction of global frames by Hélein [H] to our setting. First recall from [H] that there always exists a compact without boundary Riemanian manifold $(\tilde{N}, \tilde{h})$, with dimension $l > 2n$, and a totally geodesic isometric embedding $\Phi : (N, h) \to (\tilde{N}, \tilde{h})$ such that there is a tubular neighborhood $U, \subset \tilde{N}$, of $\Phi(N)$ with the property that the tangent bundle $T\tilde{N}$ restricted to $U$ is trivial. Therefore, we may assume that there is an orthonormal frame field $\{e_\alpha\}_{\alpha=1}^l$ which spans $T\tilde{N}|_U$. Now observe that the isometry $\Phi : (N, h) \to (\Phi(N), \tilde{h}|_{\Phi(N)})$ naturally pushes the almost complex structure $J$ on $N$ forward to give an almost complex structure $\tilde{J}$ on $\Phi(N)$. In fact, $\tilde{J}$ can be defined as follows: for any $y = \Phi(x) \in \Phi(N)$ and $\tilde{v} = (d\Phi)_\#(v) \in T_y\Phi(N)$, $\tilde{J}_y(\tilde{v}) \equiv (d\Phi)_\#(J_x v)$. Then

$$\tilde{J}_y^2(\tilde{v}) = \tilde{J}_y((d\Phi)_\#(J_x v)) = (d\Phi)_\#(J_x^2(v)) = -(d\Phi)_\#(v) = -\tilde{v}$$

Hence $\tilde{J}$ is an almost complex structure on $\Phi(N)$. Moreover, the metric $\tilde{h}|_{\Phi(M)}$ is hermitian with respect to $\tilde{J}$. In fact, for any $y = \Phi(x) \in \Phi(N)$ and $\tilde{X} = (d\Phi)_\#(X), \tilde{Y} = (d\Phi)_\#(Y) \in T_y\Phi(N)$, we have

$$\tilde{h}(y)(\tilde{X}, \tilde{Y}) = h(x)(X, Y) = h(x)(J_x X, J_x Y)$$
$$= \tilde{h}(y)((d\Phi)_\#(J_x X), (d\Phi)_\#(J_x Y)) = \tilde{h}(\tilde{J}_y(\tilde{X}), \tilde{J}_y(\tilde{Y}))$$

Now, it is easy to see that $\tilde{u} = \Phi(u) \in H^1(M, \Phi(N))$ is a weakly $\tilde{J}$-holomorphic map. In fact, for a. e. $x \in M$ and $X \in T_x M$, we have

$$d\tilde{u}(j_x(X)) = (d\Phi)_\#(du(j_x(X))) = (d\Phi)_\#(J(u)(du(X))$$
$$= \tilde{J}(\tilde{u})(d(\Phi(u))(X)) = \tilde{J}(\tilde{u})(d\tilde{u}(X)) \tag{2.6}$$

Now we extend $\tilde{J}$ from $T(\Phi(N))$ to $T\tilde{N}|_{\Phi(N)}$ as follows. Let $T\tilde{N}|_{\Phi(N)} = T(\Phi(N)) + (T(\Phi(N)))^\perp$, here $(T(\Phi(N)))^\perp \subset T\tilde{N}|_{\Phi(N)}$ denotes the normal

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bundle of $\Phi(N)$. Hence any vector field $v \in T\tilde{N}|_{\Phi(N)}$ is uniquely written as $v = v_1 + v_2$, with $v_1 \in T(\Phi(N))$ and $v_2 \in (T(\Phi(N)))^{\perp}$. Now we extend $\tilde{J}$ by setting

$$\tilde{J}(v) = \tilde{J}(v_1)$$

It is clear that $\tilde{J} : T\tilde{N}|_{\Phi(N)} \to T\tilde{N}|_{\Phi(N)}$ is a smooth linear transform whose restriction to $T(\Phi(N))$ is the almost complex transform pushed forward from $J$.

It suffices to prove (2.1) for $\tilde{u}$, since $\Phi$ is an isometry map. Now we can use the orthonormal frame field $\{e_\alpha\}_{\alpha=1}^{l}$ and its dual cotangent frame field $\{e_\alpha^*\}_{\alpha=1}^{l}$ to express $\tilde{J}$ as

$$\tilde{J}(y) = \sum_{\alpha,\beta=1}^{l} J_{\alpha\beta}(y)e_\alpha(y) \otimes e_\beta^*(y)$$

for some matrix-valued function $(J_{\alpha\beta}) \in C^\infty(M, R^{l \times l})$. As in the case 1, we assume that $x = 0$ and $B_r(0) \subset M$ is a geodesic ball so that the eqn. (2.6), see also the eqn. (2.2), yields

$$
\begin{aligned}
\frac{\partial \tilde{u}}{\partial x_i} &= \sum_{k=1}^{2m} j_{ki}(x) d\tilde{u}(j(x)(\frac{\partial}{\partial x_k})) \\
&= \sum_{k=1}^{2m} j_{ki}(x)(\tilde{J}(\tilde{u})(\frac{\partial \tilde{u}}{\partial x_k}) \\
&= \sum_{k=1}^{2m} \sum_{\beta=1}^{l} j_{ki}(x)\langle \frac{\partial \tilde{u}}{\partial x_k}, e_\beta(\tilde{u})\rangle \tilde{J}(\tilde{u})(e_\beta(\tilde{u})) \\
&= \sum_{k=1}^{2m} \sum_{\beta=1}^{l} j_{ki}(x)\langle \frac{\partial \tilde{u}}{\partial x_k}, e_\beta(\tilde{u})\rangle J_{\alpha\beta}(\tilde{u})e_\alpha(\tilde{u})
\end{aligned}
$$

Therefore, in $B_r(0)$, we have, for $1 \leq \alpha \leq l$,

$$\langle \frac{\partial \tilde{u}}{\partial x_i}, e_\alpha(\tilde{u})\rangle = \sum_{k=1}^{2m} \sum_{\beta=1}^{l} j_{ki}(x) J_{\alpha\beta}(\tilde{u})\langle \frac{\partial \tilde{u}}{\partial x_k}, e_\beta(\tilde{u})\rangle \tag{2.7}$$

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For $1 \leq \alpha \leq l$, define $Y_\alpha = (\langle \frac{\partial \tilde{u}}{\partial x_1}, e_\alpha(\tilde{u}) \rangle, \cdots, \langle \frac{\partial \tilde{u}}{\partial x_{2m}}, e_\alpha(\tilde{u}) \rangle) : B_r(0) \to R^{2m}$.
Then direct calculations, combined with the skew symmetricity of $gj$, imply

$$-\text{div}_g Y_\alpha = \frac{1}{\sqrt{g}} \sum_{k,i=1}^{2m} \frac{\partial}{\partial x_k}(\sqrt{g} g^{ki} \langle \frac{\partial \tilde{u}}{\partial x_i}, e_\alpha(\tilde{u}) \rangle)$$

$$= \sum_{\beta,i} \sum_{p<k} g^{ki} j_{pi}(x) J_{\alpha\beta}(\tilde{u})(\langle \frac{\partial \tilde{u}}{\partial x_p}, \frac{\partial(e_\beta(\tilde{u}))}{\partial x_k} \rangle - \langle \frac{\partial \tilde{u}}{\partial x_k}, \frac{\partial(e_\beta(\tilde{u}))}{\partial x_p} \rangle)$$

$$+ \frac{1}{\sqrt{g}} \sum_{\beta,i} \sum_{p<k} [\langle e_\beta(\tilde{u}), \frac{\partial}{\partial x_k}(\sqrt{g} g^{ki} j_{pi}(x) J_{\alpha\beta}(\tilde{u})) \frac{\partial \tilde{u}}{\partial x_p} \rangle$$

$$- \langle e_\beta(\tilde{u}), \frac{\partial}{\partial x_p}(\sqrt{g} g^{ki} j_{pi}(x) J_{\alpha\beta}(\tilde{u})) \frac{\partial \tilde{u}}{\partial x_k} \rangle] \qquad (2.8)$$

Observe that each term in the summation of the right hand side of the eqn. (2.8) is of the form $(\frac{\partial f}{\partial x_k} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_k})h$ for some $f, g \in H^1(B_r(0))$ and $h \in L^\infty(B_r(0)) \cap H^1(B_r(0))$. Therefore, the eqn. (2.8) is similar to the harmonic map equation into general target manifolds, written under the optimal gauge frame (see, e.g., [B]). We can then modify the argument of [B] to prove (2.1). To make the paper short, we only sketch a slightly simpler proof as follows. Since we can handle the effect of $(g, j)$ in eqn. (2.8) the same way as in Case 1. For simplicity, we assume that $g$ is the euclidean metric and $j$ is the standard almost complex structure on $R^{2m}$. Therefore, the eqn. (2.8) reduces to

$$-\text{div} Y_\alpha = \sum_\beta \sum_{i<k} J_{\alpha\beta}(\tilde{u})(\langle \frac{\partial \tilde{u}}{\partial x_i}, \frac{\partial(e_\beta(\tilde{u}))}{\partial x_k} \rangle - \langle \frac{\partial \tilde{u}}{\partial x_k}, \frac{\partial(e_\beta(\tilde{u}))}{\partial x_i} \rangle)$$

$$+ \sum_\beta \sum_{i<k} \langle e_\beta(\tilde{u}), \frac{\partial(J_{\alpha\beta}(\tilde{u}))}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_k} - \frac{\partial(J_{\alpha\beta}(\tilde{u}))}{\partial x_k} \frac{\partial \tilde{u}}{\partial x_i} \rangle \qquad (2.9)$$

For $y \in B_{\theta_0 r}(0)$ and $0 < s \leq \theta_0 r$, let $\Lambda = (2\theta_0)^{-1}$ and $\eta \in C_0^1(B_{2\Lambda s}(y))$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{\Lambda s}(y)$, and $|D\eta| \leq \frac{C}{\Lambda s}$. Denote $\tilde{u}_{y,2\Lambda s} = \frac{1}{|B_{2\Lambda s}(y)|} \int_{B_{2\Lambda s}(y)} \tilde{u}$. For $1 \leq \alpha \leq l$, consider $\langle D(\eta^2(\tilde{u} - \tilde{u}_{y,2\Lambda s})), e_\alpha(\tilde{u}) \rangle$. Then it follows from the standard Hodge decomposition theorem (cf. [MT]) that there exist $F_\alpha \in H_0^1(B_{2\Lambda s}(y))$ and $G_\alpha \in L^2(B_{2\Lambda s}(y), R^{2m})$ such that

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$\operatorname{div}(G_\alpha) = 0$ and

$$\langle D(\eta^2(\tilde{u} - \tilde{u}_{y,2\Lambda s})), e_\alpha(\tilde{u})\rangle = DF_\alpha + G_\alpha, \quad \text{in } B_{2\Lambda s}(y) \tag{2.10}$$

$$\|DF_\alpha\|_{L^p(B_{2\Lambda s}(y))} + \|G_\alpha\|_{L^p(B_{2\Lambda s}(y))} \le C_p\|D\tilde{u}\|_{L^p(B_{2\Lambda s}(y))} \tag{2.11}$$

for any $1 < p \le 2$. Since $\operatorname{div}(G_\alpha) = 0$ and $F_\alpha|_{\partial B_{2\Lambda s}(y)} = 0$, we have $\int_{B_{2\Lambda s}(y)} G_\alpha \cdot DF_\alpha = 0$. Therefore, $G_\alpha$ can be estimated as follows

$$\begin{aligned}
\int_{B_{2\Lambda s}(y)} |G_\alpha|^2 &= \int_{B_{2\Lambda s}(y)} G_\alpha \langle D(\eta^2(\tilde{u} - \tilde{u}_{y,2\Lambda s})), e_\alpha(\tilde{u})\rangle \\
&= -\int_{B_{2\Lambda s}(y)} G_\alpha \cdot D(e_\alpha(\tilde{u}))\eta^2(\tilde{u} - \tilde{u}_{y,2\Lambda s}) \\
&\le C\|G_\alpha \cdot D(e_\alpha(\tilde{u}))\|_{\mathcal{H}^1}\|\eta^2(\tilde{u} - \tilde{u}_{y,2\Lambda s})\|_{\text{BMO}} \\
&\le C\|G_\alpha\|_{L^2(B_{2\Lambda s}(y))}\|D(e_\alpha(\tilde{u}))\|_{L^2(B_{2\Lambda s}(y))}[\tilde{u}]_{\text{BMO}(B_r(0))} \\
&\le C\|G_\alpha\|_{L^2(B_{2\Lambda s}(y))}\|D\tilde{u}\|_{L^2(B_{2\Lambda s}(y))}[\tilde{u}]_{\text{BMO}(B_r(0))}
\end{aligned}$$

This yields

$$\|G_\alpha\|_{L^2(B_{2\Lambda s}(y))} \le Cs^{m-1}\|D\tilde{u}\|_{\mathcal{M}^{2,2m-2}(B_r(0))}[\tilde{u}]_{\text{BMO}(B_r(0))} \tag{2.12}$$

To estimate $F_\alpha$, we define three auxillary functions $F_\alpha^1 \in H^1(B_{2\Lambda s}(y))$, $F_\alpha^2 \in H_0^1(B_{2\Lambda s}(y))$, and $F_\alpha^3 \in H_0^1(B_{2\Lambda s}(y))$ as follows.

$$\Delta F_\alpha^1 = 0, \quad \text{in } B_{2\Lambda s}(y), \quad F_\alpha^1|_{\partial B_{2\lambda s}(y)} = F_\alpha|_{\partial B_{2\Lambda s}(y)} \tag{2.13}$$

$$\Delta F_\alpha^2 = \sum_\beta \sum_{i<k} J_{\alpha\beta}(\tilde{u})(\langle \frac{\partial \tilde{u}}{\partial x_i}, \frac{\partial(e_\beta(\tilde{u}))}{\partial x_k}\rangle - \langle \frac{\partial \tilde{u}}{\partial x_k}, \frac{\partial(e_\beta(\tilde{u}))}{\partial x_i}\rangle) \tag{2.14}$$

$$\Delta F_\alpha^3 = \sum_\beta \sum_{i<k} \langle e_\beta(\tilde{u}), \frac{\partial(J_{\alpha\beta}(\tilde{u}))}{\partial x_i}\frac{\partial \tilde{u}}{\partial x_k} - \frac{\partial(J_{\alpha\beta}(\tilde{u}))}{\partial x_k}\frac{\partial \tilde{u}}{\partial x_i}\rangle \tag{2.15}$$

It follows from (2.9), (2.13)-(2.15) that $F_\alpha = \sum_{i=1}^3 F_\alpha^i$. It follows from the estimate for harmonic functions, we have

$$s^{1-2m}\int_{B_s(y)} |DF_\alpha^1| \le C\Lambda^{-1}(\Lambda s)^{1-2m}\int_{B_{2\Lambda s}(y)} |DF_\alpha^1| \tag{2.16}$$

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To estimate $F_\alpha^2$, we first recall the dual characterization for the $L^p$ norm of gradient of functions in $W_0^{1,p}(B_{2\Lambda s}(y))$. For $p \in (1,2]$, denote $p' = \frac{p}{p-1}$. Then we have, for any $f \in W_0^{1,p}(B_{2\Lambda s}(y))$,

$$\|Df\|_{L^p(B_{2\Lambda s}(y))} \leq C \sup\{\int_{B_{2\Lambda s}(y)} Df \cdot Dg :$$
$$g \in W_0^{1,p'}(B_{2\Lambda s}(y)), \quad \|Dg\|_{L^{p'}(B_{2\Lambda s}(y))} \leq 1\}$$

Now let $p \in (1, \frac{2m}{2m-1})$ so that $p' > 2m$. Then for any $g \in W_0^{1,p'}(B_{2\Lambda s}(y))$ with $\|Dg\|_{L^{p'}(B_{2\Lambda s}(y))} \leq 1$, we have

$$\int_{B_{2\Lambda s}(y)} DF_\alpha^2 \cdot Dg = \int_{B_{2\Lambda s}(y)} \Delta F_\alpha^2 g$$
$$= \sum_{\beta, i<k} \int_{B_{2\Lambda s}(y)} J_{\alpha\beta}(\tilde{u})g(\langle \frac{\partial \tilde{u}}{\partial x_i}, \frac{\partial (e_\beta(\tilde{u}))}{\partial x_k}\rangle - \langle \frac{\partial \tilde{u}}{\partial x_k}, \frac{\partial (e_\beta(\tilde{u}))}{\partial x_i}\rangle)$$
$$\leq C \sum_{\beta, i<k} \|\langle \frac{\partial (J_{\alpha\beta}(\tilde{u})g)}{\partial x_i}, \frac{\partial (e_\beta(\tilde{u}))}{\partial x_k}\rangle - \langle \frac{\partial (J_{\alpha\beta}(\tilde{u})g)}{\partial x_k}, \frac{\partial (e_\beta(\tilde{u}))}{\partial x_i}\rangle\|_{\mathcal{H}^1}$$
$$\cdot \|\tilde{u}\|_{\mathrm{BMO}(B_r(0))}$$
$$\leq C \sum_\beta \|D\tilde{u}\|_{L^2(B_{2\Lambda s}(y))} \|D(J_{\alpha\beta}(\tilde{u})g)\|_{L^2(B_{2\Lambda s}(y))}[\tilde{u}]_{\mathrm{BMO}(B_r(0))}$$

By the Sobolev inequality and the Hölder inequality, we have

$$\|g\|_{L^\infty(B_{2\Lambda s}(y))} \leq C(\Lambda s)^{1-\frac{2m}{p'}}, \quad \|Dg\|_{L^2(B_{2\Lambda s}(y))} \leq C(\Lambda s)^{m-\frac{2m}{p'}}$$

Hence

$$\|D(J_{\alpha\beta}(\tilde{u})g)\|_{L^2(B_{2\Lambda s}(y))}$$
$$\leq C(\|Dg\|_{L^2(B_{2\Lambda s}(y))} + \|g\|_{L^\infty(B_{2\Lambda s}(y))})\|D\tilde{u}\|_{L^2(B_{2\Lambda s}(y))}$$
$$\leq C(1 + \|D\tilde{u}\|_{\mathcal{M}^{2,2m-2}(B_{2\Lambda s}(y))})(\Lambda s)^{m-\frac{2m}{p'}}$$

Therefore, by taking super over all such $g$'s, we obtain

$$(\Lambda s)^{1-2m} \int_{B_{2\Lambda s}(y)} |DF_\alpha^2| \leq (\Lambda s)^{\frac{p-2m}{p}} \|DF_\alpha^2\|_{L^p(B_{2\Lambda s}(y))}$$
$$\leq C\|D\tilde{u}\|_{\mathcal{M}^{2,2m-2}(B_r(0))}[\tilde{u}]_{\mathrm{BMO}(B_r(0))}$$
$$\leq C\epsilon_0 [\tilde{u}]_{\mathrm{BMO}(B_r(0))} \qquad (2.17)$$

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Similarly, one can prove

$$(\Lambda s)^{1-2m} \int_{B_{2\Lambda s}(y)} |DF_\alpha^3| \leq C\epsilon_0 [\tilde{u}]_{\mathrm{BMO}(B_r(0))} \tag{2.18}$$

Putting (2.16), (2.17) and (2.18) together, we get

$$s^{1-2m} \int_{B_s(y)} |DF_\alpha| \leq C\Lambda^{-1}(\Lambda s)^{1-2m} \int_{B_{2\Lambda s}(y)} |D\tilde{u}| \\ + C\Lambda^{2m-1}\epsilon_0 [\tilde{u}]_{\mathrm{BMO}(B_r(0))}$$

This, combined with the estimate (2.12) for $G_\alpha$, implies

$$s^{1-2m} \int_{B_s(y)} |D\tilde{u}| \leq C\Lambda^{-1}(\Lambda s)^{1-2m} \int_{B_{2\Lambda s}(y)} |D\tilde{u}| \\ + C\Lambda^{2m-1}\epsilon_0 [\tilde{u}]_{\mathrm{BMO}(B_r(0))}$$

Since

$$\max\{(\Lambda s)^{1-2m} \int_{B_{2\Lambda s}(y)} |D\tilde{u}|, \quad [\tilde{u}]_{\mathrm{BMO}(B_r(0))}\} \leq \|D\tilde{u}\|_{\mathcal{M}^{1,2m-1}(B_r(0))}$$

we obtain, by taking over all $y \in B_{\theta_0 r}(0)$ and $0 < s \leq \theta_0 r$,

$$\|D\tilde{u}\|_{\mathcal{M}^{1,2m-1}(B_{\theta_0 r}(0))} \leq C(\theta_0 + \theta_0^{1-2m}\epsilon_0)\|D\tilde{u}\|_{\mathcal{M}^{1,2m-1}(B_r(0))}$$

This, with the help of suitable choices of $\theta_0$ and $\epsilon_0$, yields

$$\|D\tilde{u}\|_{\mathcal{M}^{1,2m-1}(B_{\theta_0 r}(0))} \leq \frac{1}{2}\|D\tilde{u}\|_{\mathcal{M}^{1,2m-1}(B_r(0))}$$

Therefore (2.1) is proved and the proof of Lemma 2.1 is complete. ∎

**Completion of proof of theorem A.**

Since $\|Du\|_{\mathcal{M}^{2,2m-2}(B_r(x))} \leq \epsilon_0$, it follows that for any $y \in B_{\frac{r}{2}}(x)$ and $s \in (0, \frac{r}{2})$, one has $\|Du\|_{\mathcal{M}^{2,2m-2}(B_s(y))} \leq \epsilon_0$. Hence (2.1) implies, for any $y \in B_{\frac{r}{2}}(x)$

$$\|Du\|_{\mathcal{M}^{1,2m-1}(B_{\theta_0^k r}(y))} \leq 2^{-k}\|Du\|_{\mathcal{M}^{1,2m-1}(B_{\frac{r}{2}}(y))} + \frac{C_0 r}{1-\theta_0} \tag{2.18}$$

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This yields that there is $\alpha_0 \in (0, 1)$ such that

$$s^{1-2m} \int_{B_s(y)} |Du| \leq C_0 s^{\alpha_0}, \ \forall y \in B_{\frac{r}{2}}(x), \ 0 < s \leq \frac{r}{4}$$

Hence the Morrey decay Lemma implies that $u \in C^{\alpha_0}(B_{\frac{r}{2}}(x), N)$. One can apply the higher order regularity (see, e.g., [Y]) to conclude that $u \in C^{\infty}(B_{\frac{r}{2}}(x), N)$. ∎

§3. Blow-up analysis for $J$-holomorphic maps and proof of theorem D, E, F

In this section, we will prove theorem D, E, and F. As mentioned in §1, the two key ingredienst such as the small energy regularity theorem A and the energy monotonicity inequality (1.5) make it possible to adapt the ideas from [L] to our settings.

**Definition 3.1**. A weakly $J$-holomorphic map $u \in H^1(M, N)$ is said to be stationary if, in addition to (1.1), it is a critical point of the Dirichlet energy with respect to domain variations: for any smooth vector filed $X$ with compact support,

$$\frac{d}{dt}|_{t=0} \int_M |D(u(F_t(x)))|^2 = 0 \tag{3.1}$$

where $F_t$ is one parameter family of diffeomorphisms of $M$ generated by $X$.

It is readily seen (see, e.g., Price [P]) that (3.1) is equivalent to the first variational formula:

$$\int_M |Du|^2 \text{div}_g(X) - 2\langle du(\nabla_{e_i}^g X), du(e_i) \rangle = 0 \tag{3.2}$$

for any smooth vector field $X$ with compact support, here $\nabla^g$ denotes the Levi-Civita connection on $M$ and $\text{div}_g$ denotes the divergence and $\{e_i\}_{i=1}^{2m}$ is an orthonormal frame field with respect to $g$. Therefore, one has the following energy monotonicity inequality for stationary $J$-holomorphic maps (see, e.g. Price [P]).

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**Proposition 3.2.** *For $m \geq 2$. Let $u \in H^1(M, N)$ be a stationary $J$-holomorphic map. Then there exist $R_0 > 0$, $C_0 > 0$ depending only on $(M, g)$ such that*

$$e^{C_0 R} R^{2-2m} \int_{B_R(x)} |Du|^2 - e^{C_0 r} r^{2-2m} \int_{B_r(x)} |Du|^2$$

$$\geq 2 \int_{B_R(x) \setminus B_r(x)} e^{C_0 |y-x|} |y-x|^{2-2m} \Big| \frac{\partial u}{\partial |y-x|} \Big|^2 \tag{3.3}$$

*for any $x \in M$, $0 < r \leq R \leq R_0$.*

**Proof.** Since $M$ is compact, the injectivity radius $R_0$ is positive. For any $x_0 \in M$, there is a normal coordinate system on the geodesic ball $B_{R_0}(x_0)$ such that $x(x_0) = 0$, and

$$\max\{|g(x) - g_0|, \ |dg|(x)\} \leq C_0 |x|, \ \forall x \in B_{R_0}$$

where $g_0$ is the euclidean metric on $B_{R_0}$. For any $Y \in C_0^\infty(B_{R_0}, R^{2m})$, (3.2) implies

$$\Big| \int_{B_{R_0}} |Du|^2 \mathrm{div}(Y) - 2 \sum_{i,j} \langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \rangle Y_i^j \Big|$$

$$\leq C_0 \int_{B_{R_0}} |x| |Y(x)| |Du|^2(x) \tag{3.4}$$

here div denotes the divergence with respect to $g_0$. Now, by choosing $Y(x) = \eta(|x|)x$ for suitable cut-off function $\eta \in C_0^1([0, R_0])$, one obtains (3.3) from (3.4). ∎

Now we start to prove theorem D, E, and F.

**Proof of theorem D.**

Without loss of generality, we may assume that there is $u \in H^1(M, N)$ such that $u_k \to u$ weakly in $H^1(M, N)$ and

$$\frac{1}{2} |Du_k|^2(x) \, dx \to \mu \equiv \frac{1}{2} |Du|^2(x) \, dx + \nu \tag{3.5}$$

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as convergence of Radon measures for some nonnegative Radon measure $\nu \geq 0$. Let $\epsilon_0 > 0$ be given by theorem A. Define the concentration set $\Sigma \subset M$ by

$$\Sigma = \cap_{r>0}\{x \in M \mid \liminf_{k\to\infty} e^{ar^2} r^{2-2m} \int_{B_r(x)} |Du_k|^2 \geq \epsilon_0^2\} \qquad (3.6)$$

Then the monotonicity inequality (3.3) implies that $\Sigma$ is closed, and the Vitali's covering lemma, combined with the fact that $M$ is compact without boundary, implies

$$H^{2m-2}(\Sigma) \leq C \sup_k \int_M |Du_k|^2 < \infty \qquad (3.7)$$

It follows from both theorem A and corollary 3.3 that we may assume that, for any $l \geq 1$, $u_k$ is bounded in $C^l_{\text{loc}}(M\setminus\Sigma, N)$ and $u_k \to u$ in $C^l_{\text{loc}}(M\setminus\Sigma, N)$ as well. It is clear that (3.3) implies that for any $x \in M$, $e^{ar^2} r^{2-2m} \mu(B_r(x))$ is monotonically nondecreasing with respect to $r$, for $0 < r \leq r_0$. Hence

$$\Theta^{2m-2}(\mu, x) = \lim_{r\to0} r^{2-2m} \mu(B_r(x))$$

exists for all $x \in M$ and is upper semicontinuous. Moreover, there is a $C_0 > 0$ depending on $\sup_k \int_M |Du_k|^2$ such that

$$\Sigma = \{x \in M \mid \epsilon_0^2 \leq \Theta^{2m-2}(\mu, x) \leq C_0\}$$

Therefore, $\mu\mathbf{L}\Sigma$ is absolutely continuous with respect to $H^{2m-2}$ and

$$\mu\mathbf{L}\Sigma = \Theta^{2m-2}(\mu, \cdot)H^{2m-2}\mathbf{L}\Sigma \qquad (3.8)$$

since it follows from Federer-Ziemmer [FZ] that

$$\Theta^{2m-2}(|Du|^2\, dx, y) \equiv \lim_{r\to0} r^{2-2m} \int_{B_r(x)} |Du|^2 = 0, \text{ for } H^{2m-2} \text{ a. e. } y \in \Sigma$$

we obtain, for $H^{2m-2}$ a. e. $x \in \Sigma$,

$$\Theta^{2m-2}(\nu, x) = \Theta^{2m-2}(\mu, x) \geq \epsilon_0^2, \quad \text{and } \nu = \Theta^{2m-2}(\nu, \cdot)H^{2m-2}\mathbf{L}\Sigma$$

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where $\Theta^{2m-2}(\nu, x) = \lim_{r \to 0} r^{2-2m} \nu(B_r(x))$. Note that $u_k$ doesn't converge to $u$ in $H^1(M, N)$ if and only if

$$0 < \nu(M) = \int_\Sigma \Theta^{2m-2}(\nu, x) \, dH^{2m-2} \le \epsilon_0^2 H^{2m-2}(\Sigma) \Leftrightarrow H^{2m-2}(\Sigma) > 0$$

Now we want to prove that if $H^{2m-2}(\Sigma) > 0$ then $\Sigma$ is $(2m-2)$-rectifiable and we can blow-up $u_k$ near $\Sigma$ to get a pseudo holomorphic $S^2$. Since $\mu$ has positive and finite $\Theta^{2m-2}$-density everywhere on $\Sigma$, one can apply either the abstract rectifiablity theorem of D. Priess [Pd] or follow the elegant direct proof of Lin [L] to conclude the $(2m-2)$-rectifiablity of $\Sigma$. Here, we would like to present a third proof. It is based on the *generalized varifold approach* and the extended version of Allard's rectifiablity theorem on varifolds with controlled first variations [A]. This approach was outlined by Lin [L1] and Lin-Wang [LW] in a related context. For details, one may refer to [LW]. For any $x_0 \in M$, consider the geodesic ball $B_{R_0}(x_0)$. Recall that $V_{2m-2}^*(B_{R_0}(x_0))$, the space of generalized $(2m-2)$-varifolds, consists of all nonnegative Radon measures $V$ on $B_{R_0}(x_0) \times A_{2m-2}$, here

$$A_{2m-2} = \{A \in \mathcal{S}^{2m} \mid \text{tr}(A) = 2m - 2, \ -(2m-2)I_{2m} \le A \le I_{2m}\}$$

where $\mathcal{S}^{2m}$ denotes the space of symmetric $2m \times 2m$ matrices and $I_{2m}$ denotes the identity matrix of order $2m$. For $V \in V_{2m-2}^*(B_{R_0}(x_0))$, $\|V\| = \pi_\#(V)$ is its weight, where $\pi : B_{R_0}(x_0) \times A_{2m-2} \to B_{R_0}(x_0)$ is the first component projection map, and its *first variation* is defined by

$$\delta V(X) = \int_{B_{R_0}(x_0)} DX : A \, dV(x, A), \ \forall X \in C_0^1(B_{R_0}(x_0), R^{2m}) \qquad (3.9)$$

where : denotes the scalar product on $R^{2m \times 2m}$. For a subset $G \subset B_{R_0}(x_0)$,

$$\|\delta V\|(G) = \sup\{|\delta V(X)| : X \in C_0^1(B_{R_0}(x_0), R^{2m}), \ \text{spt}(X) \subset G\} < \infty$$

If $\|\delta V\| << \|V\|$, then the Resiz representation theorem implies that there is a *generalized mean curvature* $H \in L_{\|V\|}^1(B_{R_0}(x_0), R^{2m})$ such that

$$\delta V(X) = \int_{B_{R_0}(x_0)} \langle H, X \rangle \, d\|V\|, \ \forall X \in C_0^1(B_{R_0}(x_0), R^{2m}) \qquad (3.10)$$

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Now, for the above sequence $\{u_k\}$, we associate a sequence of generalized varifolds $V_{u_k} \in V_{2m-2}^*(B_{R_0}(x_0))$ as follows. For $x \in B_{R_0}(x_0)$, define $A(u_k)(x)$ by

$$A(u_k)(x) = I_{2m} - 2\frac{Du_k \otimes Du_k}{|Du_k|^2}(x), \text{ if } |Du_k|(x) \neq 0$$
$$= I_{2m-2}, \qquad\qquad \text{ if } |Du_k|(x) = 0$$

and $V_{u_k}(x, A) = \delta_{A(u_k)(x)}(A)\frac{1}{2}|Du_k|^2(x)\,dx$ for $(x, A) \in B_{R_0}(x_0) \times A_{2m-2}$, here $\delta_{A(u_k)(x)}$ denotes the delta mass centered at $A(u_k)(x)$. Then we have $\|V_{u_k}\| = \frac{1}{2}|Du_k|^2(x)\,dx$. Moreover, (3.4) implies

$$|\delta V_{u_k}(X)| \leq C_0 \int_{B_{R_0}(x_0)} |x - x_0||X(x)|\,d\|V_{u_k}\| \tag{3.11}$$

for any $X \in C_0^1(B_{R_0}(x_0), R^{2m})$. Therefore the generalized mean curvature $H_k = \frac{\delta V_{u_k}}{\|V_{u_k}\|} \in L_{\|V_{u_k}\|}^\infty(B_{R_0}(x_0), R^{2m})$ and

$$|H_k|(x) \leq C_0|x - x_0|, \ \forall x \in B_{R_0}(x_0) \tag{3.12}$$

Now, we can assume that there is a $V \in V_{2m-2}^*(B_{R_0}(x_0))$ such that $V_{u_k} \to V$ and

$$\|V_{u_k}\| \to \|V\| = \mu \equiv \frac{1}{2}|Du|^2(x)\,dx + \nu$$

It is clear that (3.12) implies that $H = \frac{\delta V}{\|V\|} \in L_{\|V\|}^\infty(B_{R_0}(x_0), R^{2m})$ and

$$|H|(x) \leq C_0|x - x_0|, \ \forall x \in B_{R_0}(x_0) \tag{3.13}$$

Now $V$ is a generalized $(2m-2)$-varifold with bounded first variation. We can slightly modify the proof of theorem 4.9 of [LW] to obtain

**Claim.** $V\mathbf{L}\{x \in B_{R_0}(x_0) \,|0 < \Theta^{2m-2}(\|V\|, x) < \infty\} = V\mathbf{L}\Sigma$ *is a $(2m-2)$-rectifiable varifold.*

In fact, since the Resiz representation theorem implies that $V = V_x\|V\|$ for some measurable function $V_x$ with values in the space of probability measures on $A_{2m-2}$, we have for $H^{2m-2}$ a. e. $x \in \Sigma$,

$$\Theta^{*,2m-2}(\Sigma, x) = \limsup_{r \to 0} r^{2-2m} H^{2m-2}(\Sigma \cap B_r(x)) \geq 2^{-2m-2}$$

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$\Theta^{2m-2}(\|V\|, \cdot), V_x$ are $H^{2m-2}$ approximately continuous at $x$

$$\lim_{r \to 0} r^{2-2m} \int_{B_r(x)} |H| \, d\|V\| = |H(x)|\Theta^{2m-2}(\|V\|, x) < \infty$$

Now for any $r_i \to 0$, we can find a subsequence $r_i' \to 0$ such that the rescalings of $V$, $\mathcal{D}_{x,r_i'}(V)$, satisfy

$$\mathcal{D}_{x,r_i'}(V) \to V_x H^{2m-2}\mathbf{L}T$$

for a $(2m-2)$ plane $T \subset R^{2m}$, according to the geometric Lemma 2.4 of Lin [L] which is applicable to our setting due to the fact that only the energy monotonicity inequality (3.3) is required. Moreover,

$$
\begin{aligned}
\|\delta(V_x H^{2m-2}\mathbf{L}T)\| &= \lim_{i \to \infty} \|\delta(\mathcal{D}_{x,r_i'}(V))\| \\
&= \lim_{i \to \infty} (r_i')^{3-2m}\mathcal{D}_{x,r_i'}\|\delta V\| \\
&= \lim_{i \to \infty} \frac{r_i' \int_{B_{r_i'}(x)} |H| \, d\|V\|}{r_i'^{2m-2}} = 0
\end{aligned}
$$

Hence the constancy theorem for varifolds (cf. Simon [S]) implies that $V_x = \delta_T$ and $T$ is unique, i.e. independent of the choices of $r_i'$. Therefore, $V\mathbf{L}\Sigma$ is $(2m-2)$-rectifiable and $\Sigma$ is a $(2m-2)$-rectifiable set.

We now assume $H^{2m-2}(\Sigma) > 0$ and need to extract a pseudo-holomorphic $S^2$ by suitably rescaling $u_k$ near points of $\Sigma$. The idea is very close to that of Lin [L] on bubbling of harmonic $S^2$ for stationary harmonic maps, but with the difference that we need to consider the rescalings of both $j$, $g$ at the mean time, and show that the bubbling plane is a $j_0$-holomorphic plane. Here, we again only sketch it.

First, pick up a generic point $x_0 \in \Sigma$ such that $\Theta^{2m-2}(|Du|^2 \, dx, x_0) = 0$, the tangent plane $T_{x_0}\Sigma$ exists, and $\Theta^{2m-2}(\nu, \cdot)$ is $H^{2m-2}$ approximately continuous at $x_0$. From now on, we identify $T_{x_0}\Sigma = \{(0,0)\} \times R^{2m-2}$ and $(T_{x_0}\Sigma)^{\perp} = R^2 \times \{(0, \cdots, 0)\}$. Let $B_{R_0}(x_0)$ be the geodesic ball centered at

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$x_0$, for any $r_i \downarrow 0$ and any $x \in B_{R_0 r_i^{-1}}$, define $D_{x_0, r_i}(x) = x_0 + r_i x$ and $\tilde{u}_i(x) = u_{k_i}(x_0 + r_i x)$, $g_i = g \circ D_{x_0, r_i}$, $j_i = (D_{x_0, r_i})^* j$ (i.e. for any $Y \in T_x B_{R_0 r_i^{-1}}$, $j_i(x)(Y) = d(D_{x_0, r_i})^{-1} \circ j \circ dD_{x_0, r_i}(Y)$). Then, $\tilde{u}_i : (B_{R_0 r_i^{-1}}, j_i, g_i) \to (N, J, h)$ is a $J$-holomorphic map. It is clear that we may assume that $g_i \to g_{x_0}$ and $j_i \to j_{x_0}$ in $C^2$ norm. Moreover, by the Cauchy diagonal process, we may assume that there is $k_i \to \infty$ such that $\tilde{u}_i \to$ constant weakly in $H^1$. The small energy regularity theorem A implies $\tilde{u}_i \to$ constant in $C^1_{\text{loc}}(B_2 \setminus \{(0,0)\} \times R^{2m-2}, N)$. Moreover,

$$|D\tilde{u}_i|^2 \, dx \to \nu_* \equiv \Theta^{2m-2}(\nu, x_0) H^{2m-2} \mathbf{L}(\{(0,0)\} \times R^{2m-2}) \qquad (3.14)$$

and the geometric Lemma 2.4 of [L] implies

$$\lim_{i \to \infty} \sum_{l=3}^{2m} \int_{B_2} |\frac{\partial \tilde{u}_i}{\partial x_l}|^2 = 0 \qquad (3.15)$$

Hence, the Fubini's theorem, the weak $L^1$-estimate for the Hardy-Littlewood maximal function, and the small energy regularity theorem A imply that there is $x_2^i \in B_{\frac{1}{4}}^{2m-2}$ such that $\tilde{u}_i$ is smooth near $(0, 0, x_2^i)$ and

$$\lim_{i \to \infty} \max_{0 < r \leq \frac{5}{4}} r^{2-2m} \int_{B_r^{2m-2}(x_2^i)} dx_2 \int_{B_1^2} \sum_{l=3}^{2m} |\frac{\partial \tilde{u}_i}{\partial x_l}|^2 dx_1 = 0 \qquad (3.16)$$

Now we can find $\delta_i > 0$ and $x_1^i \in B_{\frac{1}{2}}^2$ such that

$$\max_{x_1 \in B_{\frac{1}{2}}^2} \delta_i^{2-2m} \int_{B_{\delta_i}^2(x_1) \times B_{\delta_i}^{2m-2}(x_2^i)} |D\tilde{u}_i|^2 = \frac{\epsilon_0^2}{C(m)} \qquad (3.17)$$

is achieved at $x_1^i$. Here $C(m) > 0$ is to be chosen later. It is easy to see that $\delta_i \to 0$ and $x_1^i \to 0$. Now, let $v_i(y) = \tilde{u}_i((x_1^i, x_2^i) + \delta_i y)$, $\tilde{j}_i = (D_{0, \delta_i})^*(j_i)$, and $\tilde{g}_i = g_i \circ D_{0, \delta_i}$. It is clear that $v_i : (B_{R_i}, \tilde{j}_i, \tilde{g}_i) \to (N, J, h)$ is $J$-holomorphic, here $R_i = \frac{R_0}{r_i \delta_i}$. Moreover,

$$\lim_{i \to \infty} \sup_{0 < R < R_i} R^{2-2m} \int_{B_R^{2m-2}} \int_{B_{R_i}^2} \sum_{l=3}^{2m} |\frac{\partial v_i}{\partial x_l}|^2 = 0 \qquad (3.18)$$

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$$\sup_{i} \sup_{0<R<R_i} R^{2-2m} \int_{B_R} |Dv_i|^2 \leq C_0 \qquad (3.19)$$

and

$$\int_{B_1^2 \times B_1^{2m-2}} |Dv_i|^2 = \frac{\epsilon_0^2}{C(m)}$$

$$\geq \max\{\int_{B_1^2(y) \times B_1^{2m-2}} |Dv_i|^2 \; |y \in B_{R_i}^2\} \qquad (3.20)$$

For $a \in B_{R_i-1}^2 \times B_1^{2m-2}$, $\eta_1 \in C_0^\infty(B_1^2)$, and $\eta_2 \in C_0^\infty(B_1^{2m-2})$, we define

$$F_i(a) = \int_{B_1^2 \times B_1^{2m-2}} |Dv_i|^2(y+a)\eta_1(y_1)\eta_2(y_2)\, dy_1\, dy_2$$

Then the stationarity identities (3.2) and (3.4) imply, for $3 \leq k \leq 2m$,

$$\left|\frac{\partial F_i(a)}{\partial a_k} - 2\sum_{l=1}^{2m}\int_{B_2^{2m}} \frac{\partial v_i}{\partial y_l}\frac{\partial v_i}{\partial y_k}\frac{\partial(\eta_1\eta_2)}{\partial y_k}\right| \leq C\delta_i \int_{B_2^{2m}} |Dv_i|^2 \to 0 \qquad (3.21)$$

This implies, by choosing $C(m)$ sufficiently large, that

$$\int_{B_2^2 \times B_2^{2m-2}} |Dv_i|^2(y_1+b, y_2)\, dy_1 dy_2 \leq \epsilon_0^2 \qquad (3.22)$$

for all $b \in B_{R_i-1}^2$. Therefore, theorem A yields

$$\|v_i\|_{C^2(B_{R_i-1}^2 \times B_{\frac{7}{4}}^{2m-2})} \leq C(\epsilon_0)$$

and we can assume that there is a map $v \in C^1(B_{\frac{3}{2}}^{2m-2} \times R^2, N)$ such that

$$v_i \to v, \text{ in } C_{\text{loc}}^1(R^2 \times B_{\frac{3}{2}}^{2m-2}, N)$$

It is easy to see that $v : (R^{2m}, j_0, g_0) \to (N, J, h)$ is a $J$-holomorphic map,

$$\frac{\partial v}{\partial y_k} = 0, \quad \forall 3 \leq k \leq 2m$$

so that $v(y) = v(y_1, y_2) : R^2 \to N$, and (3.19)-(3.20) imply

$$\epsilon_0^2 \leq \int_{R^2} |Dv|^2 < \infty \qquad (3.23)$$

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Now we need to show: $R^2 = (T_{x_0}\Sigma)^{\perp}$ is $j_{x_0}$-holomorphic plane (i.e. $j_0(R^2) = R^2$). Once this is proven. Then we know that $v : (R^2, j_{x_0}, g_{x_0}) \to (N, J, h)$ is a pseudo-holomorphic map, which can be lifted to become a pseudo-holomorphic map from $S^2$ to $N$ by either the removable singularity theorem by [Y] [PW] or our corollary B. Suppose not, then $j_{x_0}(R^2) \cap R^2 = \{0\}$. For simplicity, we assume that $j_{x_0} = j_0$ is the standard complex structure and $g_{x_0} = g_0$ is the euclidean metric on $R^{2m}$. Hence $\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\}$ is an orthonormal basis of $R^2$. Moreover, there are $\lambda_1, \lambda_2 \in (-1, 1)$ and $e_1^{\perp}, e_2^{\perp} \in T_{x_0}\Sigma = \{(0, 0)\} \times R^{2m-2}$ such that

$$j_0(\frac{\partial}{\partial y_1}) = \lambda_1 \frac{\partial}{\partial y_2} + e_1^{\perp}$$
$$j_0(\frac{\partial}{\partial y_2}) = \lambda_2 \frac{\partial}{\partial y_1} + e_2^{\perp}$$

Note that $\langle Dv, e_1^{\perp} \rangle = \langle Dv, e_2^{\perp} \rangle = 0$. Therefore,

$$\int_{R^2} |\frac{\partial v}{\partial y_1}|^2 + |\frac{\partial v}{\partial y_2}|^2 = \int_{R^2} |J(v)(\frac{\partial v}{\partial y_1})|^2 + |J(v)(\frac{\partial v}{\partial y_2})|^2$$
$$= \int_{R^2} |dv(j_0(\frac{\partial}{\partial y_1}))|^2 + |dv(j_0(\frac{\partial}{\partial y_2}))|^2$$
$$= \lambda_1^2 \int_{R^2} |\frac{\partial v}{\partial y_2}|^2 + \lambda_2^2 \int_{R^2} |\frac{\partial v}{\partial y_1}|^2$$
$$\leq \max\{\lambda_1^2, \lambda_2^2\} \int_{R^2} |\frac{\partial v}{\partial y_1}|^2 + |\frac{\partial v}{\partial y_2}|^2 \qquad (3.24)$$

where we have used the fact that $J$ is an isometry and $v$ is $J$-holomorphic in the first two identities. Hence

$$(1 - \max\{\lambda_1^2, \lambda_2^2\}) \int_{R^2} |Dv|^2 \leq 0$$

This yields $\int_{R^2} |Dv|^2 = 0$, since $1 - \max\{\lambda_1^2, \lambda_2^2\} > 0$. We get a contradiction. Note that this argument also implies that, for $H^{2m-2}$ a. e. $x_0 \in \Sigma$, $T_{x_0}\Sigma$ is $j_{x_0}$-holomorphic, i.e. $j_{x_0}(T_{x_0}\Sigma) = T_{x_0}\Sigma$. The proof of theorem is complete. ∎

**Proof of theorem E.**

Suppose that $x_0 \in \Sigma$ satisfies: (1) $\Theta^{2m-2}(|Du|^2 \, dx, x_0) = 0$; (2) $T_{x_0}\Sigma = \{(0,0)\} \times R^{2m-2}$; (3) $\Theta^{2m-2}(\nu, \cdot)$ is $H^{2m-2}$-approximately continuous at $x_0$. Since (1)-(3) holds for $H^{2m-2}$ a.e. in $\Sigma$, it suffices to prove (3.24) holds at such a $x_0$. Recall from the process to obtain the first bubble in the proof of theorem 3.4 that we can choose $r_i \downarrow 0$ such that $\tilde{u}_i(\cdot) = u_{k_i}(D_{x_0,r_0}(\cdot))$ : $(B_{R_0 r_i^{-1}}, j_i \equiv (D_{x_0,r_i})^* j, g_i \equiv g \circ D_{x_0,r_i}) \to (N, J, h)$ is $J$-holomorphic. Moreover, it follows from (3.15) that there is a $x_2^i \in B_{\frac{1}{4}}^{2m-2}$ such that (3.16) holds. For $w_i(y) = \tilde{u}_i((0, x_2^i) + y)$, $a \in B_1^2 \times B_1^{2m-2}$, $\eta_1 \in C_0^\infty(B_1^2)$, and $\eta_2 \in C_0^\infty(B_1^{2m-2})$, consider

$$G_i(a) = \int_{B_1^2 \times B_1^{2m-2}} |Dw_i|^2 (y + a) \eta_1(y_1) \eta_2(y_2) \tag{3.25}$$

Then, similar to (3.21), we have, for $3 \leq k \leq 2m$,

$$\left| \frac{\partial G_i}{\partial a_k} - 2 \sum_{l=1}^{2m} \int_{B_2^{2m}} \frac{\partial w_i}{\partial y_l} \frac{\partial w_i}{\partial y_k} \frac{\partial(\eta_1 \eta_2)}{\partial y_k} \right| \leq C r_i \int_{B_2^{2m}} |Dw_i|^2 \to 0 \tag{3.26}$$

so that we can apply the Allard's strong constancy Lemma ([A1]) as in [L] or Lin-Riviere [LR] or [LW] to conclude that

$$\lim_{i \to \infty} \| E(w_i(\cdot, y_2), B_1^2) - \Theta^{2m-2}(\nu, x_0) \|_{L^1(B_1^{2m-2})} = 0 \tag{3.27}$$

where $E(w_i(\cdot, y_2), B_1^2) = \int_{B_1^2} |Dw_i|^2(y_1, y_2) \, dy_1$ Now, we apply the Fubini's theorem and the weak $L^1$-estimate for the Hardy-Littlewood maximal function again to obtain a $y_2^i \in B_{\frac{1}{4}}^{2m-2}$, which may be different from $x_2^i$, such that

$$\lim_{i \to \infty} |E(w_i(\cdot, y_2^i), B_1^2) - \Theta^{2m-2}(\nu, x_0)| = 0 \tag{3.28}$$

and (3.16) holds with $\tilde{u}_i$, $x_2^i$ replaced by $w_i$ and $y_2^i$. Now we can repeat the process for the first bubble as many times as possible to extract all bubbles, $\{\omega_l\}_{l=1}^{l_{x_0}}$, for some $1 \leq l_{x_0} \leq [\frac{\Theta^{2m-2}(\nu, x_0)}{\epsilon_1^2}]$, such that

$$\lim_{i \to \infty} E(w_i(\cdot, y_2^i), B_1^2) \geq \sum_{l=1}^{l_{x_0}} E(\omega_l, S^2) \tag{3.29}$$

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where $\epsilon_1$ depends only on $N$ (cf. [Y] or [PW]) and is given by

$$\epsilon_1^2 = \inf\{E(\omega, S^2) | \omega \in C^\infty(S^2, N) \text{ is nonconstant pseudo-holomorphic map}\}$$

Therefore, (3.24) is proven if we show that (3.29) is an equality.

First, by an induction argument on $l_{x_0}$, it suffices to prove (3.29) for $l_{x_0} = 1$ (cf. [DT] ($m = 1$) and [LR] [LW] ($m \geq 2$) for details). Let $\omega_1$ be the only bubble, it follows from theorem 3.4 that there exist $\delta_i \to 0$ and $y_1^i \in B_1^2 \to 0$ such that $v_i(\cdot) = w_i((y_1^i, 0) + \delta_i \cdot) = \tilde{u}_i((y_1^i, y_2^i) + \delta_i \cdot)$ converges to $\omega_1$ in $H^1 \cap C^2(R^2 \times B_2^{2m-2}, N)$ locally. For simplicity, we assume that $(y_1^i, y_2^i) = (0, 0)$. As in [DT] [LR] or [LW], $l_{x_0} = 1$ implies that, for any sufficiently small $\epsilon > 0$ and sufficiently large $R > 0$,

$$r^{2-2m} \int_{(B_{2r}^2 \setminus B_r^2) \times B_r^{2m-2}} |D\tilde{u}_i|^2 \leq \epsilon^2, \quad \forall R\delta_i \leq r \leq 1 \qquad (3.30)$$

Therefore, theorem A yields

$$|y_1||D\tilde{u}_i|(y_1, 0) \leq C\epsilon, \quad \forall 2R\delta_i \leq |y_1| \leq \frac{1}{2} \qquad (3.31)$$

or

$$|y_1||Dv_i|(y_1, 0) \leq C\epsilon, \quad \forall 2R \leq |y_1| \leq \frac{1}{2\delta_i} \qquad (3.32)$$

Observe that, for $l_{x_0} = 1$, (3.29) is an equality is equivalent to

$$\int_{B_{\frac{\delta}{2\delta_i}}^2 \setminus B_{2R}^2} |Dv_i|^2(y_1, 0) \, dy_1 = o(i^{-1}, \delta, R^{-1}) \qquad (3.33)$$

here $\lim_{R \to \infty, \delta \to 0} \lim_{i \to \infty} o(i^{-1}, \delta, R^{-1}) = 0$. Since $v_i : (B_{R_0(r_i\delta_i)^{-1}}, \tilde{j}_i, \tilde{g}_i) \to (N, J, h)$ is $J$-holomorphic, $\tilde{j}_i \equiv (D_{x_0, r_i\delta_i})^* j \to j_{x_0}$ and $\tilde{g}_i \equiv g \circ D_{x_0, r_i\delta_i} \to g_{x_0}$ in $C^2$ norm, and $R^2 \times \{(0, \cdots, 0)\} = (T_{x_0}\Sigma)^\perp$ is $j_{x_0}$-holomorphic by theorem 3.4, we have two propositions for $\tilde{v}_i(y) = v_i(y, 0) : R^2 \to N$:

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**Proposition 3.3**. For $y = (y_1, y_2) \in R^2$,

$$\frac{\partial \tilde{v}_i}{\partial y_2} = J(\tilde{v}_i)(\frac{\partial \tilde{v}_i}{\partial y_1}) + d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_1})) \tag{3.34}$$

$$\frac{\partial \tilde{v}_i}{\partial y_1} = -J(\tilde{v}_i)(\frac{\partial \tilde{v}_i}{\partial y_2}) - d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_2})) \tag{3.35}$$

*Proof.* Since (3.35) can be obtained by the same way as (3.34), it suffices to indicate the proof of (3.34). We assume that $j_{x_0}(\frac{\partial}{\partial y_1}) = \frac{\partial}{\partial y_2}$. Hence we have

$$\frac{\partial \tilde{v}_i}{\partial y_2} = d\tilde{v}_i(j_{x_0}(\frac{\partial}{\partial y_1}))$$

$$= d\tilde{v}_i(\tilde{j}_i(\frac{\partial}{\partial y_1})) + d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_1}))$$

$$= J(\tilde{v}_i)(d\tilde{v}_i(\frac{\partial}{\partial y_1})) + d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_1}))$$

$$= J(\tilde{v}_i)(\frac{\partial \tilde{v}_i}{\partial y_1}) + d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_1}))$$

This implies (3.34).

**Proposition 3.4** (almost conformality). *For $y = (y_1, y_2) \in R^2$,*

$$|\langle \frac{\partial \tilde{v}_i}{\partial y_1}, \frac{\partial \tilde{v}_i}{\partial y_2} \rangle| \leq C \|j_{x_0} - \tilde{j}_i\|_{C^0} |D\tilde{v}_i|^2 \tag{3.36}$$

$$|\langle \frac{\partial \tilde{v}_i}{\partial y_1}, \frac{\partial \tilde{v}_i}{\partial y_1} \rangle - \langle \frac{\partial \tilde{v}_i}{\partial y_2}, \frac{\partial \tilde{v}_i}{\partial y_2} \rangle| \leq C \|j_{x_0} - \tilde{j}_i\|_{C^0} |D\tilde{v}_i|^2 \tag{3.37}$$

*Proof.* For simplicity, we only verify (3.37). In fact, (3.35) implies

$$\langle \frac{\partial \tilde{v}_i}{\partial y_1}, \frac{\partial \tilde{v}_i}{\partial y_1} \rangle - \langle \frac{\partial \tilde{v}_i}{\partial y_2}, \frac{\partial \tilde{v}_i}{\partial y_2} \rangle$$

$$= 2\langle J(\tilde{v}_i)(\frac{\partial \tilde{v}_i}{\partial y_2}), d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_2})) \rangle + |d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_2}))|^2$$

$$\leq C \|j_{x_0} - \tilde{j}_i\|_{C^0} |D\tilde{v}_i|^2$$

This yields (3.37).

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Our idea to prove (3.33) is as follows. We first prove there is no angular energy concentrated in the neck region, which can be done by modifying the argument of Sacks-Uhlenbeck [SaU] in their proof of removable isolated singularity theorem, and then use (3.36) and (3.37) to control the radial energy by the angular energy in the neck region.

For this purpose, we assume that there is a global orthonormal frame field $\{e_\alpha\}_{\alpha=1}^{2n}$ of $TN$ (in general, we can follow the modified Hélein's construction of global frame as in Case 2 of proof for theorem A in §2 to ensure such an existence of a global frame). As before, we can write $J = \sum_{\alpha,\beta} J_{\alpha\beta} e_\alpha \otimes e_\beta^*$ so that (3.34) and (3.35) become:

$$\langle \frac{\partial \tilde{v}_i}{\partial y_2}, e_\alpha(\tilde{v}_i)\rangle = \sum_\beta J_{\alpha\beta}(\tilde{v}_i)\langle \frac{\partial \tilde{v}_i}{\partial y_1}, e_\beta(\tilde{v}_i)\rangle$$

$$+ \langle d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_1})), e_\alpha(\tilde{v}_i)\rangle \qquad (3.38)$$

$$\langle \frac{\partial \tilde{v}_i}{\partial y_1}, e_\alpha(\tilde{v}_i)\rangle = -\sum_\beta J_{\alpha\beta}(\tilde{v}_i)\langle \frac{\partial \tilde{v}_i}{\partial y_2}, e_\beta(\tilde{v}_i)\rangle$$

$$- \langle d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_2})), e_\alpha(\tilde{v}_i)\rangle \qquad (3.39)$$

Denote $F_\alpha = (\langle \frac{\partial \tilde{v}_i}{\partial y_1}, e_\alpha(\tilde{v}_i)\rangle, \langle \frac{\partial \tilde{v}_i}{\partial y_2}, e_\alpha(\tilde{v}_i)\rangle)$ and

$$G_\alpha = (-\langle d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_2})), e_\alpha(\tilde{v}_i)\rangle, \langle d\tilde{v}_i((j_{x_0} - \tilde{j}_i)(\frac{\partial}{\partial y_1})), e_\alpha(\tilde{v}_i)\rangle)$$

Then, by taking one more derivative of (3.38) and (3.39), we have, on $B_{\frac{2\delta}{\delta_i}}^2 \subset R^2$,

$$\text{div}(F_\alpha) = \sum_\beta J_{\alpha\beta}(\tilde{v}_i)(\langle \frac{\partial \tilde{v}_i}{\partial y_1}, \frac{\partial e_\beta(\tilde{v}_i)}{\partial y_2}\rangle - \langle \frac{\partial \tilde{v}_i}{\partial y_2}, \frac{\partial e_\beta(\tilde{v}_i)}{\partial y_1}\rangle)$$

$$+ \sum_\beta \langle \frac{\partial \tilde{v}_i}{\partial y_1}\frac{\partial J_{\alpha\beta}(\tilde{v}_i)}{\partial y_2} - \frac{\partial \tilde{v}_i}{\partial y_2}\frac{\partial J_{\alpha\beta}(\tilde{v}_i)}{\partial y_1}, e_\beta(\tilde{v}_i)\rangle$$

$$+ \text{div}(G_\alpha) \qquad (3.40)$$

Now we extend $\tilde{v}_i(y, 0)$ from $B_{\frac{2\delta}{\delta_i}}^2 \setminus B_R^2$ to the whole $R^2$, still denoted as itself, such that its $H^1(R^2)$-norm is bounded by $H^1(B_{\frac{2\delta}{\delta_i}}^2 \setminus B_R^2)$-norm. Similar to

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the Hodge decomposition in §2, we know that there are $H_\alpha \in H^1(R^2, R)$ and $I_\alpha \in L^2(R^2, R^2)$ such that $\text{div}_{R^2}(I_\alpha) = 0$ and

$$F_\alpha = \bar{D}H_\alpha + I_\alpha, \quad \|\bar{D}H_\alpha\|_{L^2(R^2)} + \|I_\alpha\|_{L^2(R^2)} \leq C\|\bar{D}\tilde{v}_i\|_{L^2(B^2_{\frac{2\delta}{\delta_i}} \setminus B^2_R)} \quad (3.41)$$

Here we use $\bar{D} = (\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2})$ and $\bar{\Delta}$ to denote the gradient and the Laplacian on $R^2$, and $D$ to denote the gradient in $R^{2m}$. It follows from (3.16) that

$$\int_{B^2_{\frac{2\delta}{\partial \delta_i}} \setminus B^2_R} |D\tilde{v}_i|^2 = \int_{B^2_{\frac{2\delta}{\partial \delta_i}} \setminus B^2_R} |\bar{D}\tilde{v}_i|^2 + o(i^{-1})$$

Hence, on $B^2_{\frac{2\delta}{\delta_i}} \setminus B^2_R$, we have

$$\begin{aligned}
\bar{\Delta}H_\alpha = &\sum_\beta J_{\alpha\beta}(\tilde{v}_i)((\langle \frac{\partial \tilde{v}_i}{\partial y_1}, \frac{\partial e_\beta(\tilde{v}_i)}{\partial y_2}\rangle - \langle \frac{\partial \tilde{v}_i}{\partial y_2}, \frac{\partial e_\beta(\tilde{v}_i)}{\partial y_1}\rangle)) \\
&+ \sum_\beta \langle \frac{\partial \tilde{v}_i}{\partial y_1}\frac{\partial J_{\alpha\beta}(\tilde{v}_i)}{\partial y_2} - \frac{\partial \tilde{v}_i}{\partial y_2}\frac{\partial J_{\alpha\beta}(\tilde{v}_i)}{\partial y_1}, e_\beta(\tilde{v}_i)\rangle \\
&+ \text{div}(G_\alpha) \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (3.42)
\end{aligned}$$

Since $\text{div}_{R^2}(I_\alpha) = 0$, there is a $u_\alpha \in H^1(R^2)$ such that $I_\alpha = (\frac{\partial u_\alpha}{\partial y_2}, -\frac{\partial u_\alpha}{\partial y_1})$. Therefore, direct calculations imply

$$\bar{\Delta}u_\alpha = \langle \frac{\partial \tilde{v}_i}{\partial y_1}, \frac{\partial e_\alpha(\tilde{v}_i)}{\partial y_2}\rangle - \langle \frac{\partial \tilde{v}_i}{\partial y_2}, \frac{\partial e_\alpha(\tilde{v}_i)}{\partial y_1}\rangle \quad\quad (3.43)$$

To estimate $H_\alpha$, we proceed as follows. For $R, \delta > 0$ fixed, denote $P_i = B^2_{\frac{2\delta}{\delta_i}} \setminus B^2_R$ and $P_i^k = B^2_{e^{k+1}R} \setminus B^2_{e^k R}$ for $0 \leq k \leq k_i = [\ln(\frac{\delta}{R\delta_i})]$. Note that $P_i = \cup_{k=0}^{k_i} P_i^k$. Define the piecewise radial harmonic function $\Psi_i$ on $P_i$ as follows. For $0 \leq k \leq k_i$, define

$$\bar{\Delta}\Psi_i = 0, \quad\quad\quad\quad\quad\quad\quad \text{in } P_i^k \quad\quad (3.44)$$

$$\Psi_i = \frac{1}{|\partial B^2_{e^k R}|}\int_{\partial B^2_{e^k R}} H_\alpha, \quad\quad \text{on } \partial B^2_{e^k R}$$

$$\Psi_i = \frac{1}{|\partial B^2_{e^{k+1}R}|}\int_{\partial B^2_{e^{k+1}R}} H_\alpha, \quad\quad \text{on } \partial B^2_{e^{k+1}R}$$

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Observe that the first two terms of the right hand side of eqn. (3.42) is less than $C|D\tilde{v}_i|^2$. Now, multiplying both eqn. (3.42) and (3.44) by $H_\alpha - \Psi_i$ subtracting each other and then integrating over $P_i^k$, for $0 \le k \le k_i$, we obtain

$$\int_{P_i^k} |\bar{D}(H_\alpha - \Psi_i)|^2$$
$$\le \int_{\partial P_i^k} \langle \frac{\partial(H_\alpha - \Psi_i)}{\partial \nu}, H_\alpha - \Psi_i \rangle + C \int_{P_i^k} |D\tilde{v}_i|^2 |H_\alpha - \Psi_i|$$
$$+ \int_{P_i^k} \langle G_\alpha, \bar{D}(H_\alpha - \Psi_i) \rangle + \int_{\partial P_i^k} \langle G_\alpha, \nu \rangle (H_\alpha - \Psi_i) \tag{3.45}$$

where $\nu$ denotes the unit outward normal of $\partial P_i^k$.

Since $|G_\alpha| \le C\|j_{x_0} - \tilde{j}_i\|_{C^0} |D\tilde{v}_i|$, we have

$$| \int_{P_i^k} \langle G_\alpha, \bar{D}(H_\alpha - \Psi_i) \rangle |$$
$$\le C\|j_{x_0} - \tilde{j}_i\|_{C^0} \int_{P_i^k} |D\tilde{v}_i| |\bar{D}(H_\alpha - \Psi_i)|$$
$$\le \frac{1}{4} \int_{P_i^k} |\bar{D}(H_\alpha - \Psi_i)|^2 + C\|j_{x_0} - \tilde{j}_i\|_{C^0} \int_{P_i^k} |D\tilde{v}_i|^2 \tag{3.46}$$

Observe that
$$\int_{\partial P_i^k} \langle \frac{\partial \Psi_i}{\partial \nu}, H_\alpha - \Psi_i \rangle = 0$$

and (3.32) implies

$$\max_{P_i^k} |H_\alpha - \Psi_i| \le \operatorname{osc}_{P_i^k} |\tilde{v}_i| \le C\epsilon \tag{3.47}$$

Therefore, we have

$$\int_{P_i^k} |\bar{D}(H_\alpha - \Psi_i)|^2$$
$$\le C(\epsilon + \|j_{x_0} - \tilde{j}_i\|_{C^0}) \int_{P_i^k} |D\tilde{v}_i|^2$$
$$+ \int_{\partial P_i^k} \langle \frac{\partial H_\alpha}{\partial \nu}, H_\alpha - \Psi_i \rangle + \int_{\partial P_i^k} \langle G_\alpha, \nu \rangle (H_\alpha - \Psi_i) \tag{3.48}$$

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Hence, taking summations over $0 \le k \le k_i$, we obtain

$$\int_{P_i} |\bar{D}(H_\alpha - \Psi_i)|^2$$

$$\le C(\epsilon + \|j_{x_0} - \tilde{j}_i\|_{C^0}) \int_{P_i} |D\tilde{v}_i|^2$$

$$+ (\int_{\partial B^2_{\frac{\delta}{\delta_i}}} - \int_{\partial B^2_R})(\langle \frac{\partial H_\alpha}{\partial |y|}, H_\alpha - \Psi_i \rangle + \langle G_\alpha, \frac{y}{|y|} \rangle (H_\alpha - \Psi_i))$$

$$= I_i + II_i$$

It is clear that by choosing sufficiently small $\epsilon > 0$ and sufficiently large $i \gg 1$ we have

$$|I_i| \le 2\epsilon \int_{P_i} |D\tilde{v}_i|^2$$

For $II_i$, we know

$$|II_i| \le C(\int_{\partial B^2_{\frac{\delta}{\delta_i}}} + \int_{\partial B^2_R})(|\bar{D}H_\alpha| + \|j_{x_0} - \tilde{j}_i\|_{C^0}|D\tilde{v}_i|)|H_\alpha - \Psi_i|$$

Hence, using the Poincaré inequality on $\partial B^2_{\frac{\delta}{\delta_i}}$ and $\partial B^2_R$ and the Fubini's theorem, we get

$$|II_i| \le C\int_{B^2_{\frac{2\delta}{\delta_i}} \setminus B^2_{\frac{\delta}{\delta_i}}} |D\tilde{v}_i|^2 + C\int_{B^2_{2R} \setminus B^2_R} |D\tilde{v}_i|^2$$

which converges to zero as $i \to \infty, \delta \to 0, R \to \infty$. Therefore, we have

$$\int_{P_i} |\bar{D}(H_\alpha - \Psi_i)|^2 \le 2\epsilon \int_{P_i} |D\tilde{v}_i|^2 + o(i^{-1}, \delta, R^{-1}) \qquad (3.49)$$

Now we want to estimate the $\|I_\alpha\|_{L^2(P_i)}$. This step can be done in the same way by [LR] as follows. First, the eqn. (3.43) implies that

$$\|I_\alpha\|_{L^{2,1}(R^2)} \le \|Du_\alpha\|_{L^{2,1}(R^2)}$$

$$\le C\|D^2 u_\alpha\|_{L^1(R^2)}$$

$$\le C\|\langle \frac{\partial \tilde{v}_i}{\partial y_1}, \frac{\partial e_\alpha(\tilde{v}_i)}{\partial y_2} \rangle - \langle \frac{\partial \tilde{v}_i}{\partial y_2}, \frac{\partial e_\alpha(\tilde{v}_i)}{\partial y_1} \rangle\|_{\mathcal{H}^1(R^2)}$$

$$\le C\|\bar{D}\tilde{v}_i\|^2_{L^2(P_i)} \qquad (3.50)$$

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On the other hand, it follows from (3.32) that we can conclude that

$$\|I_\alpha\|_{L^{2,\infty}(P_i)} \leq \|\bar{D}u_\alpha\|_{L^{2,\infty}(P_i)}$$
$$\leq C \max_{y \in P_i} |y||\bar{D}\tilde{v}_i|(y) \leq C\epsilon \qquad (3.51)$$

Hence, by (3.50) (3.51) and the interpolation between $L^{2,1}$ and $L^{2,\infty}$, we have

$$\|I_\alpha\|^2_{L^2(P_i)} \leq C\|I_\alpha\|_{L^{2,1}(R^2)}\|I_\alpha\|_{L^{2,\infty}(P_i)} \leq C\epsilon \int_{P_i} |\bar{D}\tilde{v}_i|^2 \qquad (3.52)$$

Therefore we have

$$\int_{P_i} r^{-2}|\frac{\partial \tilde{v}_i}{\partial \theta}|^2 \leq \sum_\alpha \int_{P_i} |D(H_\alpha - \Psi_i)|^2 + |I_\alpha|^2$$
$$\leq 2\epsilon \int_{P_i} |D\tilde{v}_i|^2 + o(i^{-1}, \delta, R^{-1}) \qquad (3.53)$$

where we have used the polar coordinate $y = (r, \theta)$. Finally, we apply the almost conformality identities (3.36) and (3.37) to get

$$\int_{P_i} |\frac{\partial \tilde{v}_i}{\partial r}|^2 \leq \int_{P_i} r^{-2}|\frac{\partial \tilde{v}_i}{\partial \theta}|^2 + C\|j_{x_0} - \tilde{j}_i\|_{C^0} \int_{P_i} |D\tilde{v}_i|^2$$
$$\leq 2\epsilon \int_{P_i} |D\tilde{v}_i|^2 + o(i^{-1}, \delta, R^{-1}) + C\|j_{x_0} - \tilde{j}_i\|_{C^0} \int_{P_i} |D\tilde{v}_i|^2 \qquad (3.54)$$

Combining (3.53) with (3.54) together, we obtain

$$\int_{P_i} |\bar{D}\tilde{v}_i|^2 \leq C\epsilon + o(i^{-1}, \delta, R^{-1}) \qquad (3.55)$$

Since $\epsilon$ is as small as we want, (3.55) yields (3.33). The proof of theorem 3.6 is complete. ∎

**Proof of theorem F**.

We can follow the Federer's dimension reduction argument (cf. [F] [SU]) to obtain the result. For simplicity, we only indicate that (1): for $m = 2$, $u$ is

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smooth; (2) for $m = 3$ sing$(u)$ consists of isolated points. First, let $x_0 \in M$ be a singular point for $u$, then

$$\Theta^{2m-2}(u, x_0) \equiv \lim_{r \downarrow 0} r^{2-2m} \int_{B_r(x_0)} |Du|^2 \geq \epsilon_1^2 > 0 \qquad (3.56)$$

Then, for any $r_i \downarrow 0$, consider $u_i(x) = u(x_0 + r_i x) : B_2^{2m} \to N$. Then it is easy to see that $u_i : (B_2^{2m}, j_i, g_i) \to (N, J, h)$ are stationary $J$-holomorphic maps, here $j_i = (D_{x_0, r_i})^*(j)$ and $g_i = g \circ D_{x_0, r_i}$. Moreover, we can assume

$$\sup_{i \geq 1} \{ E(u_i, B_2^{2m}) = (2r_i)^{2-2m} \int_{B_{2r_i}(x_0)} |Du|^2 \} \leq 2\Theta^{2m-2}(u, x_0) < \infty \quad (3.57)$$

Since $j_i$ and $g_i$ are uniformly nice in $C^3(B_2^{2m})$, we can assume that

$$\max\{ \|j_i - j_{x_0}\|_{C^2(B_2^{2m})}, \quad \|g_i - g_{x_0}\|_{C^2(B_2^{2m})} \} \to 0$$

We can then assume that there is $\phi \in H^1(B_2^{2m}, N)$, which is $J$-holomorphic with respect to $(B_2^{2m}, j_{x_0}, g_{x_0})$, such that $u_i \to \phi$ weakly in $H^1(B_2^{2m}, N)$. Since $(N, J, h)$ is assumed to support no pseudo-holomorphic $S^2$'s, it follows from theorem 3.4 that $u_i \to \phi$ strongly in $H^1(B_2^{2m}, N)$. Hence $\phi$ is a stationary $J$-holomorphic map satisfying

$$\frac{\partial \phi}{\partial r}(x) = 0, \text{ for a. e. } x \in B_2^{2m} \qquad (3.58)$$

and

$$\int_{S^{2m-1}} |D\phi|^2(\theta) \, dH^{2m-1}(\theta) = \Theta^{2m-2}(u, x_0) \geq \epsilon_1^2 > 0 \qquad (3.59)$$

Since

$$d\phi(j_{x_0}(\frac{\partial}{\partial r})) = J(\phi)(\frac{\partial \phi}{\partial r})$$

(3.58) implies

$$d\phi(j_{x_0}(\frac{\partial}{\partial r})) = 0, \text{ for a. e. } x \in B_2^{2m}$$

We may assume that $j_{x_0}, g_{x_0}$ are standard on $R^{2m}$. Then the integral curves for $j_{x_0}(\frac{\partial}{\partial r})$ are fibers of the Hopf fibration:

$$H(z_1, \cdots, z_m) = [z_1, \cdots, z_m] : C^m \equiv R^{2m} \to CP^{m-1}$$

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Therefore, there exists a $\tilde{\phi} : CP^{m-1} \equiv S^{2m-1}/S^1$ (cf. [MS]) such that for a. e. $z = (z_1, \cdots, z_m) \in C^m \setminus \{0\}$,

$$\phi(z) = \tilde{\phi}(H(\frac{z}{|z|}))$$

Moreover, the $J$-holomorphicity of $\phi$ implies that

$$\tilde{\phi} : (CP^{m-1}, \bar{j}_0, \bar{g}_0) \to (N, J, h)$$

is a $J$-holomorphic map, here $\bar{j}_0$ is the standard complex structure and $\bar{g}_0$ is the Fubini-Study metric on $CP^{m-1}$; the stationarity of $\phi$ also implies that $\tilde{\phi} : (CP^{m-1}, \bar{g}_0) \to (N, h)$ is stationary.

For $m = 2$. Suppose that $sing(u) \neq \emptyset$. Then the above argument implies that $\tilde{\phi} : CP^1 \equiv S^2 \to N$ is a $J$-holomorphic map which, by (3.59), satisfies $\epsilon_1^2 \leq \int_{S^2} |D\tilde{\phi}|^2 < \infty$, which contradicts with the assumption that $(N, J, h)$ doesn't support pseudo-holomorphic $S^2$'s. Hence $u \in C^\infty(M^4, N)$.

Now for $m = 3$. Suppose that $sing(u)$ is not isolated. Then there exist $\{x_i\}_{i=1}^\infty, x_0 \in sing(u)$ such that $x_i \to x_0$. Consider $v_i(x) = u(x_0 + \lambda_i x) : B_2^{2m} \to N$, with $\lambda_i = 2|x_i - x_0| \to 0$. Then one can show that there is a stationary $J$-holomorphic map $\phi : R^6 \to N$ such that $v_i \to \phi$ strongly in $H^1(B_2^6, N)$. Moreover, $\phi$ is homogeneous of degree zero and there is a $z_1$, with $|z_1| = \frac{1}{2}$, such that $\{0, z_1\} \subset sing(\phi)$. Now, if we blow-up $\phi$ near $z_1$, then we obtain another stationary $J$-holomorphic map $\psi : R^6 \to N$, which is independent of $z_1$-direction and hence independent of $j_0(z_1)$-direction as well. Hence the singular set of $\psi$ contains a two dimensional plane, which is impossible by theorem C. ∎

**Remark 3.5**. It is a very interesting question to ask: *whether a weak limit map $u \in H^1(M, N)$ from a sequence of stationary $J$-holomorphic maps is a stationary $J$-holomorphic map*. It follows from theorem D that any such a map $u$ is weakly $J$-holomorphic which is smooth away from a closed set with

finite $H^{2m-2}$-measure. The answer is positive, under the assumption that both $(M, j, g)$ and $(N, J, h)$ are almost Kähler manifolds, and $\Pi_2(N) = 0$, see the remark 4.4 of §4 .

§4 *J-holomorphic maps as harmonic maps*

In this section, we examine a few cases where $J$-holomorphic maps become harmonic maps, i.e. critical points of the Dirichlet energy with respect to the variations in $N$. One may compare it with the well-known fact that any holomorphic map between Kähler manifolds is harmonic map (cf. [EL] page 51).

**Proposition 4.1**. *For any domain $\Omega \subset M$, we have*

$$E(u, \Omega) = \frac{1}{4} \int_\Omega |du - J \circ du \circ j|^2 - E_J(u, \Omega), \qquad (4.1)$$

*where $E(u, \Omega) = \frac{1}{2} \int_\Omega |Du|^2$ and $E_J(u, \Omega) = \frac{1}{2} \int_\Omega \langle J \circ du, du \circ j \rangle$.*

**Proof**. Note that

$$\langle J \circ du \circ j, J \circ du \circ j \rangle = \langle du \circ j, du \circ j \rangle = \langle du, du \rangle$$

and

$$\langle du, J \circ du \circ j \rangle = \langle J \circ du, J^2 \circ du \circ j \rangle = -\langle J \circ du, du \circ j \rangle$$

we have

$$\int_\Omega |du - J \circ du \circ j|^2$$
$$= \int_\Omega (|du|^2 + \langle J \circ du \circ j, J \circ du \circ j \rangle - 2\langle du, J \circ du \circ j \rangle)$$
$$= 2 \int_\Omega |du|^2 + \langle J \circ du, du \circ j \rangle$$

This clearly gives (4.1). ∎

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From now on, we further assume that $(N, J, h)$ (respectively $(M, j, g)$) is an almost Kähler manifold, i.e. $\omega^N(\cdot, \cdot) = h(J \cdot, \cdot)$ (respectively $\omega^M(\cdot, \cdot) = g(j \cdot, \cdot)$) is an almost Kähler form. Then we have the following proposition (see, corollary (8.15) of [EL] page 51 for the case that both $j$ and $J$ are integrable).

**Proposition 4.2**. *Assume that $(M, j, g)$ and $(N, J, h)$ are almost Kähler manifolds. Then any weakly $J$-holomorphic map $u \in H^1(M, N)$ is a weakly harmonic map.*

**Proof**. Let $\Pi_N$ be the nearest point projection of a neighborhood of $N$ in $R^l$ onto $N$. Then $u$ is a weakly harmonic map iff

$$\frac{d}{dt}\Big|_{t=0} \int_M |D(\Pi_N(u + t\phi))|^2 = 0, \ \forall \phi \in C_0^1(M, R^l), \qquad (4.2)$$

Direct calculations give, for $t \geq 0$,

$$\langle J \circ d(\Pi_N(u + t\phi)), d(\Pi_N(u + t\phi)) \circ j \rangle = \langle \omega^M, (\Pi_N(u + t\phi))^* \omega^N \rangle$$

Since $(\Pi_N(u + t\phi))^* \omega^N = (u + t\phi)^* \Pi_N^* \omega^N$, we claim:

$$\frac{d}{dt}\Big|_{t=0} \int_M \langle \omega^M, (\Pi_N(u + t\phi))^* \omega^N \rangle = 0 \qquad (4.3)$$

Suppose that (4.3) were proven. Denote $u_t = \Pi_N(u + t\phi)$. Since $u$ is weakly $J$-holomorphic, we have

$$\int_M |du_t - J \circ du_t \circ j|^2 \geq \int_M |du - J \circ du \circ j|^2$$

hence

$$\frac{d}{dt}\Big|_{t=0} \int_M |du_t - J \circ du_t \circ j|^2 = 0$$

This, combined with (4.1) and (4.3), implies (4.2). Now we return to the proof of (4.3). Since for a.e. $x \in M$ $d(\Pi_N^* \omega^N) = 0$, we have (cf. also the

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proof of [RT] Proposition II.1)

$$\frac{d}{dt}|_{t=0}(\Pi_N(u+t\phi))^*\omega^N$$

$$= \frac{d}{dt}|_{t=0}(u+t\phi)^*\Pi_N^*\omega^N$$

$$= d(u^*(i_{\phi(x)}\Pi_N^*\omega^N)) + u^*i_{\phi(x)}d(\Pi_N^*\omega^N)$$

$$= d(u^*(i_{\phi(x)}\Pi_N^*\omega^N))$$

where $i_{\phi(x)}$ denotes the interior product by $\phi(x)$. Therefore

$$\frac{d}{dt}|_{t=0}\int_M \langle \omega^M, u_t^*\omega^N \rangle$$

$$= \int_M \langle \omega^M, \frac{d}{dt}|_{t=0}u_t^*\omega^N \rangle$$

$$= \int_M \langle \omega^M, d(u^*(i_{\phi(x)}\Pi_N^*\omega^N))) \rangle$$

$$= \frac{1}{(2m-1)!}\int_M d(u^*(i_{\phi(x)}\Pi_N^*\omega^N)(\omega^M)^{m-1})$$

$$= 0$$

Here we have used both the Stokes' theorem and $\partial M = \emptyset$. This yields (4.3). Hence the proof of (4.2) is complete. ∎

**Proposition 4.3**. *Assume that $(M, j, g)$ and $(N, J, h)$ are almost Kähler manifolds. Then any weakly $J$-holomorphic map $u \in H^1(M, N)$ satisfying $d(u^*\omega^N) = 0$, in the sense of distributions, is stationary. In particular, it is a stationary harmonic map.*

**Proof**. For any smooth vector field $X$ on $M$ with compact support, let $F_t$ be one parameter family of diffeomorphisms of $M$ generated by $X$. Denote $u_t(x) = u(F_t(x))$, we need to show that (3.1) holds. It follows from (4.1) and the fact $u$ is weakly $J$-holomorphic that it suffices to prove

$$\int_M \langle \omega^M, u_t^*\omega^N \rangle = \int_M \langle \omega^M, u^*\omega^N \rangle \tag{4.4}$$

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We can assume that the support of $X$ is contained in a geodesic ball $B \subset M$, i.e. $\mathrm{spt}(X) \subset B$. Since $d\omega^M = 0$ and $B$ is a ball, we can assume that there is a smooth one form $\phi$ on $B$ such that $\omega^M = d\phi$ holds on $B$. Since $u_t^* \omega^N = F_t^* u^* \omega^N$ and $d(u^* \omega^N) = 0$, we have $d(u_t^* \omega^N) = F_t^* d(u^* \omega^N) = 0$. Hence

$$
\begin{aligned}
\int_M \langle \omega^M, u_t^* \omega^N \rangle &= \int_B \langle d\phi, u_t^* \omega^N \rangle \\
&= \frac{1}{(2m-1)!} \int_B d(\phi \wedge (\omega^M)^{m-2}) \wedge u_t^* \omega^N \\
&= \frac{1}{(2m-1)!} \int_B d(\phi \wedge (\omega^M)^{m-2} \wedge u_t^* \omega^N) \\
&= 0
\end{aligned}
$$

here we have used the Stokes's theorem in the last step. This gives (4.4). Therefore, (3.1) is proved. The fact that $u$ is also a weakly $J$-holomorphic map follows from the previous proposition. $\blacksquare$

**Remark 4.4.** (i) We would like to remark that [RT] introduced *local approximablity* to denote maps $u \in H^1(M, N)$ which can be locally approximated by smooth maps in $H^1$-norm, which is equivalent to $d(u^* \omega^N) = 0$ weakly.

(ii) Under the same conditions as those in Proposition 4.3. If, in additions, $\Pi_2(N) = 0$. Then any weakly $J$-holomorphic map $u \in H^1(M, N)$ is a stationary harmonic map.

**Proof of (ii).** According to Proposition 4.3, it suffices to prove

$$
d(u^* \omega^N) = 0
$$

Since $\Pi_2(N) = 0$, it follows from the density theorems, due to Bethuel [B1] and Hang-Lin [HL], that, for any geodesic ball $B \subset M$, there exist $\{u_k\} \subset C^\infty(B, N) \cap H^1(B, N)$ such that

$$
\lim_{k \to \infty} \|u_k - u\|_{H^1(B,N)} = 0
$$

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Hence, on $B$,

$$d(u^* \omega^N) = \lim_{k \to \infty} d(u_k^* \omega^N) = \lim_{k \to \infty} u_k^*(d\omega^N) = 0$$

Therefore, the conclusion of the remark 4.4 follows from the Proposition 4.3.

■

**Proposition 4.5.** *Assume that $(M, j, g)$ and $(N, J, h)$ are almost hermitian manifolds. If the covariant derivative of $J$ satisfies the symmetry condition:*

$$(\nabla_X^h J)(Y) = (\nabla_Y^h J)(X), \ \forall X, Y \in TN \tag{4.5}$$

*here $\nabla^h$ denotes the Levi-Civita connection on $N$ with respect to $h$. Then any $J$-holomorphic map $u \in H^1(M, N)$ is a weakly harmonic map. In particular, any stationary $J$-holomorphic map $u \in H^1(M, N)$ is a stationary harmonic map.*

**Proof.** For simplicity, we assume that $u \in C^\infty(M, N)$ and verify that $u$ satisfies the harmonic map equation: $\tau(u)(p) = 0$ for any $p \in M$ (without the smoothness assumption, one can express our calculations in the form of distributions by integration of parts). By choosing a normal coordinate $(x_1, \cdots, x_{2m})$ centered at $p$ such that $p = 0 \equiv (0, \cdots, 0)$, $g_{ij}(0) = \delta_{ij}$, $\frac{\partial g_{ij}}{\partial x_k}(0) = 0$, and $j(0) = j_0$ which is the standard complex structure on $R^{2m}$ such that, at $p$,

$$j(0)(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{m+i}}, \ \ j(0)(\frac{\partial}{\partial x_{m+i}}) = -\frac{\partial}{\partial x_i}, \ \ \forall 1 \le i \le m$$

Let $\nabla$ denote the Levi-Civita connection on $u^* TN$. Then, at $p$,

$$\tau(u)(p) = \sum_{i,j} g^{ij} \nabla_{\frac{\partial}{\partial x_i}} (\frac{\partial u}{\partial x_j})|_{x=0} = \sum_{i=1}^{2m} \nabla_{\frac{\partial}{\partial x_i}} (\frac{\partial u}{\partial x_i})|_{x=0}$$

$$= -\sum_{i=1}^{m} [\nabla_{\frac{\partial}{\partial x_i}} (J(u)(du(j(x)(\frac{\partial}{\partial x_i})))) + \nabla_{\frac{\partial}{\partial x_{m+i}}} (J(u)(du(j(x)(\frac{\partial}{\partial x_{m+i}}))))]|_{x=0}$$

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$$= -\sum_{i=1}^{m}[(\nabla_{\frac{\partial}{\partial x_i}} J(u))(du(j(x)(\frac{\partial}{\partial x_i}))) + (\nabla_{\frac{\partial}{\partial x_{m+i}}} J(u))(du(j(x)(\frac{\partial}{\partial x_{m+i}}))) ]|_{x=0}$$

$$- \sum_{i=1}^{m} J(u)((\nabla_{\frac{\partial}{\partial x_i}} du)(j(x)(\frac{\partial}{\partial x_i})) + (\nabla_{\frac{\partial}{\partial x_{m+i}}} du)(j(x)(\frac{\partial}{\partial x_{m+i}})))|_{x=0}$$

$$- \sum_{i=1}^{m} J(u)(du(\nabla_{\frac{\partial}{\partial x_i}}(j(x)(\frac{\partial}{\partial x_i}))) + du(\nabla_{\frac{\partial}{\partial x_{m+i}}}(j(x)(\frac{\partial}{\partial x_{m+i}}))))|_{x=0}$$

$$= -(I + II + III)$$

It is not difficult to see that, at $x = 0$,

$$III = \sum_{i=1}^{m} J(u)(du(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_{m+i}} - \nabla_{\frac{\partial}{\partial x_{m+i}}}\frac{\partial}{\partial x_i}))$$

$$= \sum_{i=1}^{m} J(u)(du([\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{m+i}}])) = 0$$

$$II = \sum_{i=1}^{m} J(u)(\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial u}{\partial x_{m+i}} - \nabla_{\frac{\partial}{\partial x_{m+i}}}\frac{\partial u}{\partial x_i}) = 0$$

$$I = \sum_{i=1}^{m}(\nabla_{\frac{\partial}{\partial x_i}} J(u))(\frac{\partial u}{\partial x_{m+i}}) - (\nabla_{\frac{\partial}{\partial x_{m+i}}} J(u))(\frac{\partial u}{\partial x_i}) = 0$$

Here we have used (4.5) in the last step. This completes the verification. ∎

**Remark 4.6**. Since the Nijenhuis tensor $N_J$ can be expressed in the form (cf. [MS])

$$N_J(X, Y) = (\nabla_{JY}^{h} J)X - (\nabla_{JX}^{h} J)Y + (\nabla_{Y}^{h} J)JX - (\nabla_{X}^{h} J)JY$$

(4.5) implies that $N_J \equiv 0$. Hence the well-known theorem of Newlander-Nirenberg implies that $J$ is integrable so that $(N, J, h)$ is a hermitian manifold.

<div align="center">

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