Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

On moving Ginzburg-Landau filament vortices

by

Changyou Wang

Preprint no.: 57 2002

On moving Ginzburg-Landau filament vortices

Changyou Wang
Department of Mathematics
University of Kentucky
Lexington, KY 40506

Abstract

In this note, we establish a quantization property for the heat equation of Ginzburg-Landau functional in R^4 which models moving filament vortices. It asserts that if the energy is sufficiently small on a parabolic ball in $R^4 \times R_+$ then there is no filament vortices in the parabolic ball of $\frac{1}{2}$ radius. This extends a recent result of Lin-Rivière [LR3] in R^3 but the problem is open for R^n with $n \geq 5$.

§1 **Introduction**

For $n \geq 2$ and $\epsilon > 0$, the heat equation for the Ginzburg-Landau functional on R^n is:

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} - \Delta u_\epsilon &= \frac{1}{\epsilon^2}(1 - |u_\epsilon|^2)u_\epsilon, \quad (x,t) \in R^n \times R_+ \\ u_\epsilon(x,0) &= g_\epsilon(x), \qquad\qquad\qquad x \in R^n \end{aligned} \tag{1.1}$$

Here $g_\epsilon : R^n \to R^2$ are given maps. Notice that (1.1) is the negative gradient flow for the Ginzburg-Landau functional

$$E_\epsilon(v) = \int_{R^n} \frac{1}{2}|Dv|^2 + \frac{1}{4\epsilon^2}(1 - |v|^2)^2. \tag{1.2}$$

Asympotic behaviors for minimizers of E_ϵ in dimension two was first studied by Bethuel-Brezis-Hélein [BBH] (see also Struwe [S1] and recent important works by Pacard-Rivière [PR] on steady solutions to (1.1)). Moreover, such static theories were also developed by Rivière [R1] [R2] and Lin-Rivière

[LR1] in high dimensions in connection with codimension two area minimizing currents, escipally a crucial quantization property for steady solutions to the equation (1.1) was proved by Lin-Rivière [LR2] for $n = 3$ and late by Bethuel-Brezis-Orlandi [BBO] for all $n \geq 3$. The asympototic for the equation (1.1) in dimension two was initiated by Lin [L1][L2] and also studied by Jerrard-Soner [JS]. Notice that in the context of understanding the limiting behavior for sequence of solutions u_ϵ to either static or parabolic versions of the equation (1.1), one encounters the main difficulty from the possibilities that u_ϵ may vanish on sets, called Ginzburg-Landau vortex, where the equation (1.1) degenerates and $\frac{u_\epsilon}{|u_\epsilon|}$ have nontrivial topological obstructions as well. On the other hand, it is well-known that existences of vortices requires the Ginzburg-Landau energy at least of the order $\log \frac{1}{\epsilon}$. In other words, g_ϵ above is assumed to have $E_\epsilon(g_\epsilon) = O(\log(\frac{1}{\epsilon}))$, this, combined with the energy inequality for (1.1), implies

$$E_\epsilon(u_\epsilon(\cdot, t)) \leq O(\log \frac{1}{\epsilon}). \tag{1.3}$$

From the analytic point of view, the size estimate for the bad set, which leads to vortices at the limit, $B_\epsilon = \{(x, t) \in R^n \times R_+ : |u_\epsilon|(x, t) \leq \frac{1}{2}\}$ plays a critical role in obtain $W^{1,p}$ compactness for suitable $p \in (1, 2)$ (see, e.g. [BBH] and [PR]). To obtain sharp size estimates of B_ϵ, one often needs to obtain the so-called η-compactness property for u_ϵ which rough says that if $E_\epsilon(u_\epsilon)$ is of order $\eta \log \frac{1}{\epsilon}$ for sufficiently small $\eta > 0$ then there is no interior bad points for u_ϵ, which was established for (i): minimizers of E_ϵ by Rivière [R1] [R2] for $n = 3$ and by Lin-Rivière [LR1] for $n \geq 3$; (ii) critical points of E_ϵ by Lin-Rivière [LR2] for $n = 3$ and by Bethuel-Brezis-Orlandi [BBO] for all $n \geq 3$. Moreover, such η-compactness property was also proved for solutions to the equation (1.1) very recently by Lin-Rivière [LR3] in the case $n = 3$. It was believed that their result still holds for R^n with $n \geq 4$. In this note, we confirm such a belief in the case that $n = 4$. More precisely, we prove

Theorem A. *For $n = 4$ and $\epsilon > 0$. Let $u_\epsilon : R^4 \times R_+ \to R^2$ be solutions to the equation (1.1) satisfying $|u_\epsilon| \le 1$ and $|Du_\epsilon| \le \frac{C_0}{\epsilon}$. Then there exist $\epsilon_0 > 0$ and $\eta > 0$ depending only on C_0 such that if for $(x_0, t_0) \in R^4 \times R_+$, $0 < \rho < \sqrt{t_0}$, and $\epsilon \le \epsilon_0$*

$$\frac{1}{\rho^4}\int_{t_0-\rho^2}^{t_0}\int_{R^4}(\frac{1}{2}|Du_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2})e^{\frac{|x-x_0|^2}{4(t-t_0)}} \le \eta \log\frac{\rho}{\epsilon}, \tag{1.4}$$

then

$$|u_\epsilon(x_0, t_0)| \ge \frac{1}{2}.$$

We would like to remark that the idea developed by Lin-Riviere [LR2] [LR3] was to interpolate between the Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$ on generic two dimensional slices which therefore worked very well in R^3, but it seems unclear how to extend them to R^n with $n \ge 4$. On the other hand, there is the interpolation technique between L^1 and L^∞ developed by Bethuel-Brezis-Orlandi [BBO] avaiable for the statics case in R^n for all $n \ge 3$, where they made very clever uses of the energy monotonicity formular for static solutions to the equation (1.1). Our method starts with the observation that there exists an energy monotonicity inequality for all time slice $R^n \times \{t\}$ when $n = 4$, which enables us to adapt the main ideas from [BBO] and some of those ideas from [LR3]. Notice that one can always view solutions to the equation (1.1) in $R^3 \times R_+$ as solutions to the equation (1.1) in $R^4 \times R_+$ which are independent of the fourth spatial variable. Hence, our proof also gives a somewhat different proof of a main theorem of [LR3].

The paper is organized as follows. In §2, we derive the needed elliptic type energy inequality in $R^4 \times \{t\}$; In §3, we recall the parabolic type energy monotonicity inequalities established by Struwe [S2] and Lin-Rivière [LR3] and extract a good time slice; In §4, we illustrate the main estimate by performing an intrinsic Hodge decomposition of a suitable quantity on good time slices and prove theorem A.

§2 **Euclidean monotonicity at time slice for** $n = 4$

This section is devoted to a slice monotonicity inequality (2.1) for $u_\epsilon : R^n \times R_+ \to R^2$ satisfying (1.1) when $n = 4$. For $n \geq 4$, $x \in R^n$, $r > 0$, and $t > 0$, we denote

$$E_\epsilon(x, r) = \int_{B_r(x)} (\frac{1}{2}|Du_\epsilon|^2 + \frac{n(1-|u_\epsilon|^2)^2}{2(n-2)\epsilon^2})(y)\,dy$$

as the Ginzburg-Landau energy of u_ϵ over $B_r(x)$ at time t. Then we have the following differential inequality.

Lemma 2.1. *For $n \geq 4$. For $\epsilon > 0$, let $u_\epsilon : R^n \times R_+ \to R^2$ be a solution to (1.1). Then, for any $(x, t) \in R^n \times R_+$ and $r > 0$, one has*

$$\begin{aligned}
&\frac{d}{dr}(r^{2-n}E_\epsilon(x, r) + \frac{r^{3-n}}{3-n}\int_{B_r(x)} |\frac{\partial u_\epsilon}{\partial t}||\frac{\partial u_\epsilon}{\partial r}|) \\
&\geq r^{2-n}\int_{\partial B_r(x)} |\frac{\partial u_\epsilon}{\partial r}|^2 + \frac{(1-|u_\epsilon|^2)^2}{2(n-2)\epsilon^2} + \frac{r^{3-n}}{3-n}\int_{\partial B_r(x)} |\frac{\partial u_\epsilon}{\partial t}||\frac{\partial u_\epsilon}{\partial r}| \quad (2.1)
\end{aligned}$$

Proof. For simplicity, we assume that $x = 0$ and denote u for u_ϵ. Multiplying (1.1) by $x \cdot Du$, integrating over B_r and using integration by parts, we obtain

$$\begin{aligned}
\int_{B_r} u_t x \cdot Du &= \int_{B_r} \Delta u x \cdot Du - \frac{1}{4\epsilon^2} x \cdot D(1-|u|^2)^2 \\
&= \int_{B_r} D \cdot (Du x \cdot Du) - Du \cdot D(x \cdot Du) - x \cdot D\frac{(1-|u|^2)^2}{4\epsilon^2} \\
&= r\int_{\partial B_r} |\frac{\partial u}{\partial r}|^2 - \int_{B_r} |Du|^2 \\
&- \int_{B_r} x \cdot D(\frac{1}{2}|Du|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2})
\end{aligned}$$

$$= r\int_{\partial B_r}(|\frac{\partial u}{\partial r}|^2 - \frac{1}{2}|Du|^2 - \frac{(1-|u|^2)^2}{4\epsilon^2})$$
$$+ (n-2)\int_{B_r}(\frac{1}{2}|Du|^2 + \frac{n(1-|u|^2)^2}{4(n-2)\epsilon^2})$$

This yields

$$(n-2)E_\epsilon(0,r) = \int_{B_r} u_t x\cdot Du + r\int_{\partial B_r}(\frac{1}{2}|Du|^2 - |\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2})$$

Therefore

$$\frac{d}{dr}(r^{2-n}E_\epsilon(0,r)) = (2-n)r^{1-n}E_\epsilon(0,r) + r^{2-n}\int_{\partial B_r}(\frac{1}{2}|Du|^2 + \frac{n(1-|u|^2)^2}{4(n-2)\epsilon^2})$$
$$= -r^{1-n}\int_{B_r} u_t x\cdot Du + r^{2-n}\int_{\partial B_r}(|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{2(n-2)\epsilon^2})$$

Observe that

$$-r^{1-n}\int_{B_r} u_t x\cdot Du \geq -r^{2-n}\int_{B_r}|u_t||\frac{\partial u}{\partial r}|$$
$$= -\frac{d}{dr}(\frac{r^{3-n}}{3-n}\int_{B_r}|u_t||\frac{\partial u}{\partial r}|) + \frac{r^{3-n}}{3-n}\int_{\partial B_r}|u_t||\frac{\partial u}{\partial r}|$$

Hence

$$\frac{d}{dr}(r^{2-n}E_\epsilon(0,r) + \frac{r^{3-n}}{3-n}\int_{B_r}|u_t||\frac{\partial u}{\partial r}|)$$
$$\geq r^{2-n}\int_{\partial B_r}(|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{2(n-2)\epsilon^2}) + \frac{r^{3-n}}{3-n}\int_{\partial B_r}|\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}|$$

This completes the proof of (2.1). ■

Now we state the consequence of Lemma 2.1 for $n = 4$, namely the following slice energy monotonicty inequality.

Proposition 2.2. *For $\epsilon > 0$, let $u_\epsilon : R^4 \times R_+ \to R^2$ be a solution to (1.1). Then, for any $(x,t) \in R^4 \times R_+$ and $0 \leq r \leq R < \infty$, it holds:*

$$r^{-2}E_\epsilon(x,r) + \int_r^R \frac{dr}{r^2}\int_{\partial B_r(x)}(\frac{1}{2}|\frac{\partial u_\epsilon}{\partial r}|^2 + (4\epsilon^2)^{-1}(1-|u_\epsilon|^2)^2$$
$$\leq 2R^{-2}E_\epsilon(x,R) + 2\int_{B_R(x)}|\frac{\partial u_\epsilon}{\partial t}|^2. \tag{2.2}$$

In particular,

$$\int_{B_R(x)} |y-x|^{-2}\frac{(1-|u_\epsilon(y)|^2)^2}{\epsilon^2} \le 8R^{-2}E_\epsilon(x,R) + 8\int_{B_R(x)} |\frac{\partial u_\epsilon}{\partial t}|^2. \quad (2.3)$$

Proof. It is clear that (2.2), with r tending to zero, gives (2.3). Therefore, it suffices to prove (2.2). First notice that, integrating (2.1) with $n=4$ from r to R, we have

$$\begin{aligned}
&R^{-2}E_\epsilon(x,R)\\
&\ge r^{-2}E_\epsilon(x,r) + R^{-1}\int_{B_R(x)}|\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| - r^{-1}\int_{B_r(x)}|\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}|\\
&+\int_r^R s^{-2}\int_{\partial B_s(x)}(|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}) - \int_r^R s^{-1}\int_{\partial B_s(x)}|\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}|
\end{aligned}$$

Now, we need to use the fact $n=4$ for the following estimates.

$$\begin{aligned}
r^{-1}\int_{B_r(x)}|\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| &\le \frac{1}{2}r^{-2}\int_{B_r(x)}|\frac{\partial u}{\partial r}|^2 + \frac{1}{2}\int_{B_r(x)}|\frac{\partial u}{\partial t}|^2\\
&\le \frac{1}{2}r^{-2}\int_{B_r(x)}|\frac{\partial u}{\partial r}|^2 + \frac{1}{2}\int_{B_R(x)}|\frac{\partial u}{\partial t}|^2
\end{aligned}$$

Applying the Young inequality again, we also have, for $r \le s \le R$,

$$s^{-1}\int_{\partial B_s(x)}|\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| \le \frac{1}{2}s^{-2}\int_{\partial B_s(x)}|\frac{\partial u}{\partial r}|^2 + \frac{1}{2}\int_{\partial B_s(x)}|\frac{\partial u}{\partial t}|^2$$

so that

$$\int_r^R s^{-1}\int_{\partial B_s(x)}|\frac{\partial u}{\partial t}||\frac{\partial u}{\partial r}| \le \frac{1}{2}\int_r^R s^{-2}\int_{\partial B_s(x)}|\frac{\partial u}{\partial r}|^2 + \int_{B_R(x)}|\frac{\partial u}{\partial t}|^2.$$

Putting these inequality together, we obtain

$$\begin{aligned}
R^{-2}E_\epsilon(x,R) \ge{}& \frac{1}{2}r^{-2}E_\epsilon(x,r) - \int_{B_R(x)}|\frac{\partial u}{\partial t}|^2\\
&+\int_r^R s^{-2}\int_{\partial B_s(x)}(\frac{1}{2}|\frac{\partial u}{\partial r}|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2})
\end{aligned}$$

This implies (2.2). ■

§3 Parabolic monotonicity and extracting a good time

In this section, we gather two of the necessary parabolic energy monotonicty inequality which was first proved by Struwe [S2] in the context of heat flow for harmonic maps, and slightly variance of which was established by Lin-Rivière [LR3] and then extract a good time slice. The formular below are valid for all $n \geq 2$.

Lemma 3.1 (Energy monotonicity) *Let $u_\epsilon : R^n \to R_+ \to R^2$ be solutions to the equation (1.1) and $(x_0, t_0) \in R^n \times R_+$. Then for any $0 < \rho \leq \sqrt{t_0}$*

$$\frac{d}{d\rho}[\frac{1}{\rho^n}\int_{t_0-\rho^2}^{t_0}\int_{R^n}(\frac{1}{2}|Du_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2})e^{\frac{|x-x_0|^2}{4(t-t_0)}}]$$
$$= \frac{1}{\rho^{n+1}}\int_{t_0-\rho^2}^{t_0}\int_{R^n}[\frac{1}{2(t_0-t)}|(x-x_0)\cdot Du_\epsilon + 2(t-t_0)\frac{\partial u_\epsilon}{\partial t}|^2$$
$$\frac{(1-|u_\epsilon|^2)^2}{2\epsilon^2}]e^{\frac{|x-x_0|^2}{4(t-t_0)}} \tag{3.1}$$

Proof. It follows exactly same lines of the proof of Lemma 2.1 of [LR3]. We omit it here. ■

We also need the following identity which indicates how the energy decays along the spatial infinity.

Lemma 3.2. *Under the same notations as Lemma 3.1. For any $t_0 > 0$ and $0 < \rho \leq \sqrt{t_0}$. Then the following holds:*

$$\int_{t_0-\rho^2}^{t_0}\int_{R^n}[(1+\frac{|x|^2}{4(t_0-t)})(\frac{1}{2}|Du_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2})e^{\frac{|x|^2}{4(t-t_0)}}$$
$$\frac{1}{4(t_0-t)}|x\cdot Du_\epsilon + 2(t-t_0)\frac{\partial u_\epsilon}{\partial t}|^2 e^{\frac{|x|^2}{4(t-t_0)}}]$$
$$\leq \rho^2\int_{R^n}\int_{R^n\times\{t_0-\rho^2\}}[\frac{1}{2}|Du_\epsilon|^2 + \frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2}]e^{\frac{-|x|^2}{4\rho^2}}$$
$$+\int_{t_0-\rho^2}^{t_0}\frac{x}{4(t_0-t)}\cdot Du_\epsilon\cdot[x\cdot Du_\epsilon + 2(t-t_0)\frac{\partial u_\epsilon}{\partial t}]e^{\frac{|x|^2}{4(t-t_0)}} \tag{3.2}$$

Proof. It again follows from the same argument as that of Lemma 2.2 of [LR3]. ■

Now we describe the extraction of a good time slice as follows. We follow closely from §2.2 of [LR3] and the reader may refer to [LR3] for more details. For simplicity, we assume that $(x_0,t_0)=(0,0)$ and the equation (1.1) holds in $R^4\times R_-$. Assume that (1.4) holds for some $\rho>0$. Then, by integration of (3.1) from ϵ to ρ and the Fubini's theorem, there exists a $\rho_1=\rho_\epsilon\in(\epsilon,\rho)$ such that :

$$\frac{1}{\rho_1^4}\int_{-\rho_1^2}^{0}\int_{R^4} j_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}\le\eta \tag{3.3}$$

Here

$$j_\epsilon(u_\epsilon)\equiv\frac{1}{2|t|}|x\cdot Du_\epsilon+2t\frac{\partial u_\epsilon}{\partial t}|^2+\frac{(1-|u_\epsilon|^2)^2}{2\epsilon^2} \tag{3.4}$$

so that

$$\frac{1}{\rho_1^2}\inf_{\rho\in(\frac{\rho_1}{2},\rho_1)}\int_{R^4\times\{-\rho^2\}} j_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}}\le 2\eta \tag{3.5}$$

Denote

$$E=\frac{1}{\rho_1^4}\int_{-\rho_1^2}^{0}\int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}} \tag{3.6}$$

where

$$e_\epsilon(u_\epsilon)\equiv(\frac{1}{2}|Du_\epsilon|^2+\frac{(1-|u_\epsilon|^2)^2}{4\epsilon^2})$$

Then (3.1) implies

$$\begin{aligned}
E &\le \inf_{\frac{\rho_1}{2}\le\rho\le\rho_1}\frac{1}{\rho^4}\int_{-\rho^2}^{0}\int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}+\int_{\frac{\rho_1}{2}}^{\rho_1}\rho^{-5}\int_{R^4} j_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}\\
&\le \inf_{\frac{\rho_1}{2}\le\rho\le\rho_1}\frac{1}{\rho^4}\int_{-\rho^2}^{0}\int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}+\frac{4}{\rho_1^4}\int_{-\rho_1^2}^{0}\int_{R^4} j_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}\\
&\le \inf_{\frac{\rho_1}{2}\le\rho\le\rho_1}\frac{1}{\rho^4}\int_{R^4} e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}+4\eta
\end{aligned}$$

As in [LR3], we may assume

$$E>>C\eta \tag{3.7}$$

so that

$$\inf_{\frac{\rho_1}{2}\le\rho\le\rho_1}\frac{1}{\rho^4}\int_{R^4}e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}}\le E\le 2\inf_{\frac{\rho_1}{2}\le\rho\le\rho_1}\frac{1}{\rho^4}\int_{R^4}e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho^2}} \tag{3.8}$$

Therefore, there exists a $\rho_0\in[\frac{\rho_1}{2},\rho_1]$ such that

$$\max\{\frac{1}{\rho_0^4}\int_{-\rho_0^2}^0\int_{R^4}j_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}},\frac{1}{\rho_0^2}\int_{R^4\times\{-\rho_0^2\}}j_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}}\}\le C\eta \tag{3.9}$$

$$\frac{1}{\rho_0^4}\int_{-\rho_0^2}^0\int_{R^4}e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}\le E\le\frac{C}{\rho_0^4}\int_{-\rho_0^2}^0\int_{R^4}e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}} \tag{3.10}$$

$$\frac{1}{\rho_0^2}\int_{R^4\times\{-\rho_0^2\}}e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}}\le E\le\frac{C}{\rho_0^2}\int_{R^4\times\{-\rho_0^2\}}e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \tag{3.11}$$

These inequalities, combined with Lemma 3.2, also yield

$$\frac{1}{\rho_0^2}\int_{R^4\times\{-\rho_0^2\}}\frac{|x|^2}{|t|}e_\epsilon(u_\epsilon)e^{\frac{|x|^2}{4t}}\le CE \tag{3.12}$$

Observe that (3.9) and (3.11) also imply

$$\int_{R^4\times\{-\rho_0^2\}}|\frac{\partial u_\epsilon}{\partial t}|^2e^{-\frac{|x|^2}{4\rho_0^2}}\le CE \tag{3.13}$$

In particular, for any $\lambda>>1$ to be chosen later, one has

$$\int_{B_{4\lambda\rho_0}\times\{-\rho_0^2\}}|\frac{\partial u_\epsilon}{\partial t}|^2\le Ce^{4\lambda^2}E \tag{3.14}$$

Hence, applying the monotonicity inequality (2.3) for u_ϵ at $t=-\rho_0^2$, we obtain the following key inequality:

$$\int_{B_{2\lambda\rho_0}(x)\times\{-\rho_0^2\}}|y-x|^{-2}\frac{(1-|u_\epsilon|^2)^2}{\epsilon^2}\le Ce^{4\lambda^2}E,\forall x\in B_{2\lambda\rho_0} \tag{3.15}$$

On the other hand, (3.9) also yields:

$$\frac{1}{\rho_0^2}\int_{B_{4\lambda\rho_0}\times\{-\rho_0^2\}}\frac{(1-|u_\epsilon|^2)^2}{\epsilon^2}\le Ce^{4\lambda^2}\eta \tag{3.16}$$

Notice that (3.12) implies that

$$\frac{1}{\rho_0^2}\int_{(R^4\setminus B_{\frac{\lambda\rho_0}{2}})\times\{-\rho_0^2\}} e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \le \frac{C}{\lambda^2}E. \tag{3.17}$$

This, combined with suitable choice of $\lambda >> 1$ according to the Fubini's theorem, gives

$$\frac{1}{\rho_0}\int_{\partial B_{\lambda\rho_0}} e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \le \frac{C}{\lambda^3}E \tag{3.18}$$

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}\times\{-\rho_0^2\}} e_\epsilon(u_\epsilon)e^{-\frac{|x|^2}{4\rho_0^2}} \ge \frac{E}{3}. \tag{3.19}$$

Together with the inequalities (3.9)–(3.18), we can proceed on the estimate of E by estimating the left hand side of (3.19) in §4 below.

§4 An intrinsic Hodge decomposition to estimate $u_\epsilon \times du_\epsilon$

This section is devoted to the proof of theorem A. The main techinical part is to obtain L^2-estimate of $u_\epsilon \times du_\epsilon$ on $B_{\lambda\rho_0}\times\{-\rho_0^2\}$. To do it, we need an intrinsic Hodge decompostion of $u_\epsilon \times du_\epsilon$ at $t = -\rho_0^2$. We adapt ideas from both [BBO] and [LR3] for this purpose.

From now on, we work on $t = -\rho_0^2$ and denote u as u_ϵ.

First, we define $H : B_{\lambda\rho_0} \to R^2$ by the auxillary Neumann problem:

$$\frac{\partial}{\partial x_i}(e^{-\frac{|x|^2}{4\rho_0^2}}\frac{\partial H}{\partial x_i}) = \frac{\partial}{\partial x_i}(e^{-\frac{|x|^2}{4\rho_0^2}}u\times\frac{\partial u}{\partial x_i}), \text{ in } B_{\lambda\rho_0} \tag{4.1}$$

$$\frac{\partial H}{\partial r} = u\times\frac{\partial u}{\partial r}, \qquad \text{on } \partial B_{\lambda\rho_0} \tag{4.2}$$

Observe that

$$\begin{aligned}
|\frac{\partial}{\partial x_i}(e^{-\frac{|x|^2}{4\rho_0^2}}u\times\frac{\partial u}{\partial x_i})| &= e^{-\frac{|x|^2}{4\rho_0^2}}|\frac{(-2\rho_0^2\frac{\partial u}{\partial t}+x\cdot Du)}{2\rho_0^2}\times u| \\
&\le e^{-\frac{|x|^2}{4\rho_0^2}}\frac{|-2\rho_0^2\frac{\partial u}{\partial t}+x\cdot Du|}{2\rho_0^2} \\
&\le 2\rho_0^{-1}e^{-\frac{|x|^2}{4\rho_0^2}}(j_\epsilon(u_\epsilon))^{\frac{1}{2}}
\end{aligned}$$

so that we can establish the following estimate for DH.

Lemma 4.1 *Under the same notations as above. The following holds: there exists a* $C_\lambda > 0$ *such that*

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |DH|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \leq C_\lambda \rho_0^{-2}\int_{B_{\lambda\rho_0}} j_\epsilon(u_\epsilon) e^{-\frac{|x|^2}{4\rho_0^2}} + \frac{C\lambda}{\rho_0}\int_{\partial B_{\lambda\rho_0}} |\frac{\partial u}{\partial r}|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \tag{4.3}$$

In particular, one has

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |DH|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \leq C_\lambda \eta + \frac{CE}{\lambda^2}. \tag{4.4}$$

Proof. First, notice that (4.4) is the consequence of (4.3) and the inequalities (3.9) and (3.18). Secondly, the proof of (4.3) can be obtained by copying lines of arguments of Lemma 2.4 of [LR3](page 836-839). We omit it here. ■

Observe that (4.1) and (4.2) can be rewritten into:

$$\frac{\partial}{\partial x_i}(e^{-\frac{|x|^2}{4\rho_0^2}}(\frac{\partial H}{\partial x_i} - u \times \frac{\partial u}{\partial x_i})\mathcal{I}_{B_{\lambda\rho_0}}) = 0 \tag{4.5}$$

in the sense of distributions on R^4, here $\mathcal{I}_{B_{\lambda\rho_0}}$ denotes the characteristic function of the ball $B_{\lambda\rho_0}$.

Define $\delta \in C^\infty(R_+, R_+)$ by $\delta(r) = r^2$ for $0 \leq r \leq 2\lambda\rho_0$, $\delta(r) = (4\lambda\rho_0)^2$ for $r \geq 4\lambda\rho_0$ and $(2\lambda\rho_0)^2 \leq \delta(r) \leq (4\lambda\rho_0)^2$ for $r \in [2\lambda\rho_0, 4\lambda\rho_0]$. Let $g_{ij}(x) = e^{-\frac{\delta(|x|)}{4\rho_0^2}}\delta_{ij}$ be the new conformal metric on R^4, which is readily seen to be bilipschitzly equivalent to the standard metric on R^4. Denote d_g^* as the adjoint of d with respect to g and $\Delta_g \equiv d_g^* d + dd_g^*$ as the Laplace-Beltrami operator with respect to g. Notice that (4.5) is equivalent to

$$d_g^*((dH - u \times du)\mathcal{I}_{B_{\lambda\rho_0}}) = 0, \text{ in } R^4 \tag{4.6}$$

Therefore, by the classical Hodge decompostion theory (see, e.g., Iwaniec-Martin [IW]), there exists a 2-form $\alpha \in H_g^1(R^4, \Lambda^2(R^4))$ such that

$$d_g^*\alpha = (dH - u \times du)\mathcal{I}_{B_{\lambda\rho_0}}, \quad d\alpha = 0 \tag{4.7}$$

$$\|D\alpha\|_{L_g^2(R^4)} \le C(\|Du\|_{L_g^2(B_{\lambda\rho_0})} + \|DH\|_{L_g^2(B_{\lambda\rho_0})}) \tag{4.8}$$

Here H_g^1 (or L_g^1 respectively) denotes H^1 (or L^2 respectively) with respect to g. Notice that

$$\|Df\|^2_{L_g^2(R^4)} = \int_{R^4} |Df|^2(x) e^{-\frac{\delta(|x|)}{4\rho_0^2}}$$

In order to estimate $D\alpha$ in L_g^2, we modify the approach of [BBO] as follows. Let $\beta \in (0, \frac{1}{2}$ be determined later, and $f : R_+ \to [1, \frac{1}{1-\beta}]$ be a smooth function such that $f(t) = \frac{1}{t}$ for $t \ge 1-\beta$, $f(t) = 1$ for $t \le 1 - 2\beta$, and $|f'| \le 4$. Define on R^4 the function a such that $a(x) = f^2(|u|(x))$ on $B_{\lambda\rho_0}$ and $a(x) = 1$ elsewhere, so that $0 \le a - 1 \le 4\beta$ holds on R^4. Observe that $f^2(|u|^2)u \times du = f(|u|)u \times d(f(|u|)u)$. Therefore, (4.7) implies

$$\begin{aligned} d(ad_g^*\alpha) &= \mathcal{I}_{B_{\lambda\rho_0}} d(f(|u|)u) \times d(f(|u|u) \\ &\quad + f(|u|)u \times du \wedge d|x| \sigma^g_{\partial B_{\lambda\rho_0}} - d(\mathcal{I}_{B_{\lambda\rho_0}} adH) \\ &= \omega_1 + \omega_2 + \omega_3 \end{aligned} \tag{4.9}$$

where $\sigma^g_{\partial B_{\lambda\rho_0}}$ denotes the surface measure of $\partial B_{\lambda\rho_0}$ with respect to the metric g. Observe that if $|u| \ge 1-\beta$ then $d(f(|u|u) \times d(f(|u|)u) = d(\frac{u}{|u|}) \times d(\frac{u}{|u|}) = 0$, otherwise we have $1 \le \beta^{-2}(1-|u|^2)^2$ so that

$$|\omega_1|(x) \le C\epsilon^{-2} \le C\beta^{-2}\frac{(1-|u(x)|^2)^2}{\epsilon^2}, \forall x \in B_{\lambda\rho_0} \tag{4.10}$$

Using the fact that $d\alpha = 0$, we get

$$\Delta_g\alpha = dd_g^*\alpha = d(ad_g^*\alpha) + d((1-a)d_g^*\alpha) = \omega_1 + \omega_2 + \omega_3 + d((1-a)d_g^*\alpha) \tag{4.11}$$

Denote $G(x, y) = G(|x - y|)$ as the fundamental solution of Δ_g on R^4. Then it follows from the bilipschitz equivalence between g and the euclidean metric on R^4 that there exists a $C > 0$ such that

$$Ce^{-4\lambda^2}|x-y|^{-2} \le G(x,y) \le Ce^{4\lambda^2}|x-y|^{-2}, |D_y G(x,y)| \le Ce^{4\lambda^2}|x-y|^{-3} \tag{4.12}$$

Let $\alpha_i = G * \omega_i$ for $1 \le i \le 3$. Then $\alpha_4 = \alpha - \sum_{i=1}^3 \alpha_i$ solves

$$\Delta_g \alpha_4 = d((1-a)d_g^* \alpha) \tag{4.13}$$

Direct calculations, using $|a - 1| \le 4\beta$ and smallness of β, yield

$$\|D\alpha_4\|_{L_g^2(R^4)}^2 \le C\beta \sum_{i=1}^3 \|D\alpha_i\|_{L_g^2(R^4)}^2 \tag{4.14}$$

The main difficulty comes from estimates of $D\alpha_1$ which can be done as follows, due to the monotonicity inequality (3.15) and (3.16). Indeed, by the maximum principle, we have $\|\alpha_1\|_{L^\infty(R^4)} = \|\alpha_1\|_{L^\infty(B_{\lambda\rho_0})}$ and, by (4.10), (4.12), and (3.15),

$$\begin{aligned}\|\alpha_1\|_{L^\infty(B_{\lambda\rho_0})} &\le \sup_{x \in B_{\lambda\rho_0}} \int_{B_{\lambda\rho_0}} G(x-y)|\omega_1|(y) \\ &\le C_\lambda \beta^{-2} \sup_{x \in B_{\lambda\rho_0}} \int_{B_{\lambda\rho_0}} |x-y|^{-2} \frac{(1-|u(y)|^2)^2}{\epsilon^2} \\ &\le C_\lambda \beta^{-2} E \end{aligned} \tag{4.15}$$

This, combined with (3.16), implies

$$\|D\alpha_1\|_{L_g^2(R^4)}^2 \le \|\omega_1\|_{L^1(R^4)} \|\alpha_1\|_{L^\infty(R^4)} \le C_\lambda \beta^{-2} \rho_0^2 \eta E \tag{4.16}$$

For α_3, using integration by parts and (4.4), we have

$$\|D\alpha_3\|_{L_g^2(R^4)}^2 \le C\|DH\|_{L_g^2(B_{\lambda\rho_0})}^2 \le C_\lambda \eta \rho_0^2 + \frac{C\rho_0^2 E}{\lambda^2}. \tag{4.17}$$

For α_2, we can modify the Lemma A1 of appendix in [BBO] to conclude that

$$\|D\alpha_2\|^2_{L^2_g(R^4)} \le C\lambda\rho_0 \|Du\|^2_{L^2_g(\partial B_{\lambda\rho_0})} \tag{4.18}$$

this, combined with (3.18), gives

$$\|D\alpha_2\|^2_{L^2_g(R^4)} \le \frac{C\rho_0^2}{\lambda^2} E \tag{4.19}$$

Putting these estimates for α_i for $1 \le i \le 4$ and Lemma 4.1 together, we then obtain

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |u\times du|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \le C_\lambda \eta + \frac{CE}{\lambda^2} + C_\lambda \beta^{-2}\eta E \tag{4.20}$$

This, combined with the fact that $4|u|^2|du|^2 = 4|u\times du|^2 + |D|u|^2|^2$ and the following estimate (see (2.67) of [LR3] page 845)

$$\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |D|u|^2|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \le C\eta^{\frac{1}{4}}E + C\eta^{\frac{1}{2}} \tag{4.21}$$

implies

$$\begin{aligned}
&\frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |Du|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \\
&= \frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} (1-|u|^2)|Du|^2 e^{-\frac{|x|^2}{4\rho_0^2}} + \frac{1}{\rho_0^2}\int_{B_{\lambda\rho_0}} |u|^2|Du|^2 e^{-\frac{|x|^2}{4\rho_0^2}} \\
&\le \frac{C}{\rho_0^2}\int_{B_{\lambda\rho_0}} \frac{(1-|u|^2)}{\epsilon}|Du| e^{-\frac{|x|^2}{4\rho_0^2}} \\
&+ \frac{4}{\rho_0^2}\int_{B_{\lambda\rho_0}} (|u\times du|^2 + |D|u|^2|^2) e^{-\frac{|x|^2}{4\rho_0^2}} \\
&\le \frac{C_\lambda}{\rho^2}\int_{B_{\lambda\rho_0}} \frac{(1-|u|^2)^2}{\epsilon^2} e^{-\frac{|x|^2}{4\rho_0^2}} + (C_\lambda\eta + C\eta^{\frac{1}{2}}) \\
&+ (\lambda^{-1} + C\lambda^{-2} + C_\lambda\beta^{-2}\eta + C\eta^{\frac{1}{4}})E \\
&\le (\lambda^{-1} + C\lambda^{-2} + C_\lambda\beta^{-2}\eta + C\eta^{\frac{1}{4}})E + (C_\lambda\eta + C\eta^{\frac{1}{2}})
\end{aligned} \tag{4.22}$$

Therefore, for any given $\delta > 0$, we can first choose a sufficiently large $\lambda > 1$ and a sufficiently small β and then choose much smaller η so that

$$E \leq C\delta \tag{4.23}$$

so that, using the monotonocity inequality (3.1) again,

$$\frac{1}{\epsilon^6}\int_{-\epsilon^2}^{0}\int_{B_\epsilon(0)} \frac{(1-|u_\epsilon|^2)^2}{\epsilon^2} e^{\frac{|x|^2}{4t}} \leq \delta. \tag{4.24}$$

This, combined with the fact that $|Du_\epsilon| \leq C\epsilon^{-1}$, yields $|u_\epsilon(0,0)| \geq \frac{1}{2}$. Therefore, the proof of theorem A is complete. ■

References

[BBH] F. Bethuel, H. Brezis, F. Hélein, Ginzburg-Landau vortices. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhuser Boston, Inc., Boston, MA, 1994.

[BBO] F. Bethuel, H. Brezis, G. Orlandi, *Small energy solutions to the Ginzburg-Landau equation.* C. R. Acad. Sci. Paris Sr. I Math. 331 (2000), no. 10, 763–770.

[IM] T. Iwaniec, G. Martin, *Quasiregular mappings in even dimensions.* Acta Math. 170 (1993), no. 1, 29–81.

[JS] R. Jerrard, H. Soner, *Dynamics of Ginzburg-Landau vortices.* Arch. Rational Mech. Anal. 142 (1998), no. 2, 99–125.

[L1] F. H. Lin, *Some dynamical properties of Ginzburg-Landau vortices.* Comm. Pure Appl. Math. 49 (1996), no. 4, 323–359.

[L2] F. H. Lin, *Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds.* Comm. Pure Appl. Math. 51 (1998), no. 4, 385–441.

[LR1] F. H. Lin, T. Rivière, *Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents.* J. Eur. Math. Soc. (JEMS) 1 (1999), no. 3, 237–311.

[LR2] F. H. Lin, T. Rivière, *A quantization property for static Ginzburg-Landau vortices.* Comm. Pure Appl. Math. 54 (2001), no. 2, 206–228.

[LR3] F. H. Lin, T. Rivière, *A quantization property for moving line vortices.* Comm. Pure Appl. Math. 54 (2001), no. 7, 826–850.

[PR] F. Pacard, T. Rivière, Linear and nonlinear aspects of vortices. The Ginzburg-Landau model. Progress in Nonlinear Differential Equations and their Applications, 39. Birkhuser Boston, Inc., Boston, MA, 2000. x+342 pp. ISBN: 0-8176-4133-5

[R1] T. Rivière, *Lignes de tourbillon dans le modle ablien de Higgs. (French) [Vortex lines in the Higgs abelian model]* C. R. Acad. Sci. Paris Sr. I Math. 321 (1995), no. 1, 73–76.

[R2] T. Rivière, *Line vortices in the* U(1)*-Higgs model.* ESAIM Contrle Optim. Calc. Var. 1 (1995/96), 77–167 (electronic).

[S1] M. Struwe, *On the asymptotic behavior of minimizers of the Ginzburg-Landau model in* 2 *dimensions.* Differential Integral Equations 7 (1994), no. 5-6, 1613–1624.

[S2] M. Struwe, *On the evolution of harmonic maps in higher dimensions.* J. Differential Geom. 28 (1988), no. 3, 485–502.