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discrete Perona-Malik energies**

by

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Abstract. We prove that a suitable rescaling of biased Perona-Malik energies, defined in the discrete setting, Γ -converges to an anisotropic version of the Mumford-Shah functional. Numerical results are discussed.

Key words: anisotropic diffusion, image segmentation, Γ -convergence.

1 Introduction

Many authors tackled in the last years the problem of image segmentation, among them Mumford and Shah (see [13]) suggested a variational approach based on the minimization of the energy

$$\int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^1(S_u) + \int_{\Omega} |u - g|^2 dx, \quad (1.1)$$

while Perona and Malik (see [16]) proposed an evolution equation of the form

$$\frac{\partial u}{\partial t} = \operatorname{div} \left(\frac{\nabla u}{1 + |\nabla u|^2} \right) \quad u(x, 0) = g(x), \quad (1.2)$$

where $g : \Omega \rightarrow [0, 1]$ is the grey level function of the original image, u represents the segmentation and S_u is considered as the set of contours. Considering (1.2) as the gradient flow of the functional

$$\int_{\Omega} \log(1 + |\nabla u|^2) dx,$$

it turns out that the simultaneous smoothing and edge detection effects of the equation depend on the particular structure of the function $\log(1 + t^2)$: the quadratic behavior near the origin

is responsible for the denoising process while the concave and sublinear behavior at infinity is responsible for the edge detection. Moreover this kind of *convex-concave* potentials has been used also for the variational approximation (in the sense of Γ -convergence) of the Mumford-Shah functional both in the continuous setting (see [12],[5]) and in the discrete one, which we are going to deal with. Considering Gobbino's paper [12], Chambolle proposed in [7] a functional of the form

$$F_\varepsilon(u) = \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbb{Z}^2} \sum_{\substack{\xi \in \mathbb{Z}^2 \\ x + \varepsilon \xi \in \Omega}} \frac{1}{\varepsilon} f \left(\frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon |\xi|} \right) \frac{\rho(\xi)}{|\xi|} + \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbb{Z}^2} |u(x) - g(x)|^2, \quad (1.3)$$

where the function $f : [0, +\infty) \rightarrow [0, +\infty)$ is non-decreasing, continuous and satisfies

$$f'(0) = 1 \quad \text{and} \quad \lim_{t \rightarrow +\infty} f(t) = 1,$$

while the convolution term $\rho : \mathbb{Z}^2 \rightarrow [0, +\infty)$ is even and satisfies

$$\rho(0) = 0, \quad \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) < +\infty, \quad \rho(\xi) > 0 \quad \text{if } |\xi| = 1, \quad \text{and} \quad \rho(\xi) = \rho(\xi^\perp). \quad (1.4)$$

Chambolle proved that the Γ -limit is the anisotropic Mumford-Shah functional given by

$$c_\rho \int_{\Omega} |\nabla u|^2 dx + \int_{S_u} \phi(\nu) d\mathcal{H}^1, \quad (1.5)$$

where

$$c_\rho := \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \quad \text{and} \quad \phi(\nu) := \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) |\nu \cdot \hat{\xi}|$$

($\hat{\xi}$ stands for $\frac{\xi}{|\xi|}$). In our main theorem we prove that (1.5) is the Γ -limit also of the following *discrete Perona-Malik energies*

$$\begin{aligned} F_\varepsilon(u) &= \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbb{Z}^2} \sum_{\substack{\xi \in \mathbb{Z}^2 \\ x + \varepsilon \xi \in \Omega}} \frac{1}{a_\varepsilon} \log \left(1 + a_\varepsilon \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2 |\xi|} \right) \frac{\rho(\xi)}{|\xi|} \\ &\quad + \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbb{Z}^2} |u(x) - g(x)|^2, \end{aligned} \quad (1.6)$$

where $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$ and ρ satisfies (1.4).

In the framework of anisotropic diffusion equations the functional (1.6) can be seen as a discretization of

$$G_\varepsilon(u) = \frac{1}{a_\varepsilon} \int_{\Omega} \log(1 + a_\varepsilon |\nabla u|^2) dx + \int_{\Omega} |u - g|^2 dx,$$

whose gradient flow is

$$\frac{\partial u}{\partial t} = \operatorname{div} \left(\frac{\nabla u}{1 + a_\varepsilon |\nabla u|^2} \right) + 2(u - g). \quad (1.7)$$

The last equation resembles a variant of the Perona-Malik equation, known as *biased anisotropic diffusion* which was proposed by Nordström in [15].

Concerning the numerical results, which are discussed in detail in the final section, the main advantage of (1.6) is that the solutions obtained by a gradient descent algorithm, which can be interpreted as a discretization of (1.7), are comparable with the ones obtained by a Graduated Non Convexity technique.

2 Notations and statement of the main result

Given a vector $\tau \in \mathbb{R}^2$ let

$$\mathbb{Z}_\tau^2 = \{x \in \mathbb{R}^2 : x = m\tau + n\tau^\perp \text{ for } (m, n) \in \mathbb{Z}^2\}, \quad (2.1)$$

$$C_\tau = \{x \in \mathbb{Z}^2 : x = s\tau + r\tau^\perp \text{ for } (s, r) \in [0, 1) \times [0, 1)\}. \quad (2.2)$$

For every open subset $A \subseteq \mathbb{R}^2$ and $y \in \mathbb{R}^2$ we denote

$$l^1((y + \mathbb{Z}_\tau^2) \cap A) := \left\{ v : (y + \mathbb{Z}_\tau^2) \cap A \rightarrow \mathbb{R} \text{ such that } \sum_{x \in (y + \mathbb{Z}_\tau^2) \cap A} |v(x)| < +\infty \right\};$$

in the following every function $v \in l^1((y + \mathbb{Z}_\tau^2) \cap A)$ will be identified with the function $\tilde{v} \in L^1(A)$ which takes the constant value $v(x)$ in the square $x + C_\tau$ if $x \in (y + \mathbb{Z}_\tau^2) \cap A$, and zero otherwise. So, having in mind this identification, given a sequence $v_\varepsilon \in l^1((y_\varepsilon + \mathbb{Z}_{\tau_\varepsilon}^2) \cap A)$ and a function $v \in L^1(A)$, we will often write, with a slight abuse of notation, $v_\varepsilon \rightarrow v$ instead of $\tilde{v}_\varepsilon \rightarrow v$ in $L^1(A)$. Given a vector τ we will denote $\hat{\tau} := \frac{\tau}{|\tau|}$. For all the notations about spaces of special functions of bounded variation we refer to the book [3].

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with Lipschitz boundary and for every $\varepsilon > 0$ consider the following functional

$$F_\varepsilon(u) := \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbb{Z}^2} \sum_{\substack{\xi \in \mathbb{Z}^2 \\ x + \varepsilon \xi \in \Omega}} \frac{1}{a_\varepsilon |\xi|} \log \left(1 + a_\varepsilon |\xi| \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2 |\xi|^2} \right) \rho(\xi) \quad (2.3)$$

if $u \in l^1(\varepsilon \mathbb{Z}^2 \cap \Omega)$, and $F_\varepsilon(u) := +\infty$ otherwise in $L^1(\Omega)$, where $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$ and $\rho : \mathbb{Z}^2 \rightarrow [0, +\infty)$ satisfies

$$0 < \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) < +\infty \quad \text{and} \quad \rho(\xi) = \rho(\xi^\perp) \quad \forall \xi \in \mathbb{Z}^2. \quad (2.4)$$

In this chapter we will prove the following theorem.

Theorem 2.1 *The functionals F_ε Γ -converge (as $\varepsilon \rightarrow 0$) with respect to the L^1 -norm to the anisotropic Mumford-Shah functional F given by*

$$F(u) := \begin{cases} c_\rho \int_\Omega |\nabla u|^2 dx + \int_{S_u} \phi(\nu_u) d\mathcal{H}^1 & \text{if } u \in GSBV(\Omega), \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

where

$$c_\rho := \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \quad \text{and} \quad \phi(\nu) := \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) |\nu \cdot \hat{\xi}|. \quad (2.5)$$

Moreover, every sequence (u_ε) satisfying $\sup_\varepsilon (F_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty) < +\infty$ is strongly precompact in $L^p(\Omega)$, for every $p \geq 1$.

Remark 2.2 It will be clear from the proof that F_ε Γ -converges with respect to the L^p -norm to the functional

$$F(u) := \begin{cases} c_\rho \int_\Omega |\nabla u|^2 dx + \int_{S_u} \phi(\nu_u) d\mathcal{H}^1 & \text{if } u \in GSBV(\Omega) \cap L^p(\Omega), \\ +\infty & \text{otherwise in } L^p(\Omega), \end{cases}$$

for every $1 \leq p < +\infty$.

The proof of the theorem will be split in the next sections.

3 Estimate from below of the Γ -limit for $N = 1$

In this section we study the one-dimensional version of the functionals defined above. Given a bounded open subset $I \subset \mathbb{R}$ we define

$$I_\varepsilon := \{x \in I \cap \varepsilon\mathbb{Z} : x + \varepsilon \in I\},$$

and, for every $u : I \cap \varepsilon\mathbb{Z} \rightarrow \mathbb{R}$, we define

$$\mathcal{F}_\varepsilon(u, I) := \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon} \log \left(1 + a_\varepsilon \frac{|u(x + \varepsilon) - u(x)|^2}{\varepsilon^2} \right),$$

where, as above, $a_\varepsilon = \varepsilon \log \frac{1}{\varepsilon}$. As usual we will identify every function $u : I \cap \varepsilon\mathbb{Z} \rightarrow \mathbb{R}$ (briefly $u \in l^1(I \cap \varepsilon\mathbb{Z})$) with the piecewise constant function u of $L^1(I)$ given by

$$u(x) := \begin{cases} u\left(\varepsilon \left[\frac{x}{\varepsilon}\right]\right) & \text{if } \varepsilon \left[\frac{x}{\varepsilon}\right] \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Our aim is to prove the following proposition.

Proposition 3.1 *Let $u_\varepsilon \in l^1(I \cap \varepsilon\mathbb{Z})$ such that $u_\varepsilon \rightarrow u$ in $L^1(I)$ as $\varepsilon \rightarrow 0^+$ and $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < +\infty$. Then $u \in SBV(I)$ and*

$$\liminf_{\varepsilon \rightarrow 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq \int_I |u'|^2 dx + \mathcal{H}^0(S_u).$$

Moreover, any sequence u_ε satisfying $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon, I) < +\infty$ is strongly precompact in $L^1(I)$.

We postpone the proof of the proposition after proving some useful lemmas.

Lemma 3.2 Let $p(\varepsilon) > 0$ be such that $\lim_{\varepsilon \rightarrow 0^+} p(\varepsilon) = 0$ and

$$\lim_{\varepsilon \rightarrow 0^+} \left(p(\varepsilon) \log \frac{1}{\varepsilon} - \log \log \frac{1}{\varepsilon} \right) = +\infty,$$

and set $c_\varepsilon := \varepsilon^{p(\varepsilon)}$. Then the following properties hold true:

- a) $\lim_{\varepsilon \rightarrow 0^+} c_\varepsilon = 0$;
- b) $\lim_{\varepsilon \rightarrow 0^+} \frac{\log c_\varepsilon}{\log \frac{1}{\varepsilon}} = 0$;
- c) $\lim_{\varepsilon \rightarrow 0^+} \frac{c_\varepsilon^2}{\varepsilon} \log \frac{1}{\varepsilon} = +\infty$;
- d) $\lim_{\varepsilon \rightarrow 0^+} c_\varepsilon \log \frac{1}{\varepsilon} = 0$;
- e) $\lim_{\varepsilon \rightarrow 0^+} \frac{\log \left(1 + a_\varepsilon \frac{c_\varepsilon^2}{\varepsilon^2} \right)}{\log \frac{1}{\varepsilon}} = 1$.

Proof. Properties a), b), c), and d) follow immediately. Let us check only e). Recalling the definition of a_ε , we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{\log \left(1 + a_\varepsilon \frac{c_\varepsilon^2}{\varepsilon^2} \right)}{\log \frac{1}{\varepsilon}} &= \lim_{\varepsilon \rightarrow 0^+} \frac{\log \left(1 + \frac{\log \frac{1}{\varepsilon} c_\varepsilon^2}{\varepsilon} \right)}{\log \frac{1}{\varepsilon}} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{\log \left(\frac{\log \frac{1}{\varepsilon} c_\varepsilon^2}{\varepsilon} \right)}{\log \frac{1}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0^+} \left(1 + \frac{\log \log \frac{1}{\varepsilon}}{\log \frac{1}{\varepsilon}} + \frac{2 \log c_\varepsilon}{\log \frac{1}{\varepsilon}} \right) = 1, \end{aligned}$$

where the second equality follows from c) while the last from b). ■

Lemma 3.3 Let $u_\varepsilon \in l^1(I \cap \varepsilon\mathbb{Z})$ be such that $\sup_\varepsilon \mathcal{F}_\varepsilon(u_\varepsilon) < K < +\infty$ and let c_ε be as in the previous lemma. Set $b_\varepsilon^2 := \sqrt{c_\varepsilon \log \frac{1}{\varepsilon}}$ and consider the following set

$$D_\varepsilon := \left\{ x \in I_\varepsilon : \frac{|u_\varepsilon(x + \varepsilon) - u_\varepsilon(x)|}{\varepsilon} > \frac{b_\varepsilon}{\sqrt{a_\varepsilon}} \right\}.$$

Then

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{H}^0(D_\varepsilon) c_\varepsilon = 0.$$

Proof. By our assumptions and recalling the definition of a_ε , we have

$$K > \frac{\varepsilon}{a_\varepsilon} \sum_{x \in D_\varepsilon} \log \left(1 + a_\varepsilon \frac{|u(x + \varepsilon) - u(x)|^2}{\varepsilon^2} \right) \geq \frac{\log(1 + b_\varepsilon^2)}{\log \frac{1}{\varepsilon}} \mathcal{H}^0(D_\varepsilon)$$

so that, substituting the expression of b_ε^2 , if ε is small enough, from $d)$ of Lemma 3.2 we get

$$\mathcal{H}^0(D_\varepsilon)c_\varepsilon \leq K \frac{c_\varepsilon \log \frac{1}{\varepsilon}}{\log \left(1 + \sqrt{c_\varepsilon \log \frac{1}{\varepsilon}}\right)} \leq K' \frac{c_\varepsilon \log \frac{1}{\varepsilon}}{\sqrt{c_\varepsilon \log \frac{1}{\varepsilon}}} = K' \sqrt{c_\varepsilon \log \frac{1}{\varepsilon}}.$$

Again $d)$ implies now the thesis. ■

Lemma 3.4 *Let $v_\varepsilon \in SBV(I)$ such that $\lim_{\varepsilon \rightarrow 0^+} \|v'_\varepsilon\|_\infty \sqrt{a_\varepsilon} = 0$. Then, for every $\delta > 0$ there exists $\bar{\varepsilon} > 0$ such that*

$$\frac{1}{a_\varepsilon} \int_I \log(1 + a_\varepsilon |v'_\varepsilon|^2) dx \geq (1 - \delta) \int_I |v'_\varepsilon|^2 dx,$$

for every $\varepsilon \leq \bar{\varepsilon}$.

Proof. Fix $\delta > 0$ and note that there exists $T_\delta > 0$ such that

$$\log(1 + a_\varepsilon t^2) \geq (1 - \delta) a_\varepsilon t^2 \quad \forall t \in \left[0, \frac{T_\delta}{\sqrt{a_\varepsilon}}\right];$$

by assumption if ε is small enough we have $\|v'_\varepsilon\|_\infty \leq T_\delta / \sqrt{a_\varepsilon}$ and therefore

$$\frac{1}{a_\varepsilon} \int_I \log(1 + a_\varepsilon |v'_\varepsilon|^2) dx \geq (1 - \delta) \int_I |v'_\varepsilon|^2 dx.$$

■

We are now in a position to prove Proposition 3.1.

Proof of Proposition 3.1. Let b_ε and c_ε be as in Lemma 3.3 and set

$$B_\varepsilon(u_\varepsilon) := \left\{ x \in I_\varepsilon : \frac{b_\varepsilon}{\sqrt{a_\varepsilon}} < \frac{|u_\varepsilon(x + \varepsilon) - u_\varepsilon(x)|}{\varepsilon} < \frac{c_\varepsilon}{\varepsilon} \right\} = \{x_\varepsilon^1, x_\varepsilon^2, \dots, x_\varepsilon^{m_\varepsilon}\},$$

where $x_\varepsilon^1 < x_\varepsilon^2 < \dots < x_\varepsilon^{m_\varepsilon}$ and $m_\varepsilon := \mathcal{H}^0(B_\varepsilon(u_\varepsilon))$. Now we want to replace the sequence u_ε with a new one \tilde{u}_ε , still converging to u , such that $B_\varepsilon(\tilde{u}_\varepsilon)$ is empty and $\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \leq \mathcal{F}_\varepsilon(u_\varepsilon)$. Setting $v_\varepsilon^0 = u_\varepsilon$ and for $k = 1, \dots, m_\varepsilon - 1$ we define by induction the functions

$$v_\varepsilon^{k+1}(t) := \begin{cases} v_\varepsilon^k(t) & \text{for } t < x_\varepsilon^{k+1}, \\ v_\varepsilon^k(t) - [v_\varepsilon^k(x_\varepsilon^{k+1}) - v_\varepsilon^k(x_\varepsilon^k)] & \text{for } t \geq x_\varepsilon^{k+1}, \end{cases}$$

and finally we set $\tilde{u}_\varepsilon := v_\varepsilon^{m_\varepsilon}$ (see Figure 1).

First of all, using the fact that for every $\varepsilon > 0$ and for every $i = 1, \dots, m_\varepsilon$ we get

$$\int_I |v_\varepsilon^i - v_\varepsilon^{i-1}| dx \leq |v_\varepsilon^{i-1}(x_\varepsilon^i) - v_\varepsilon^{i-1}(x_\varepsilon^{i-1})| |I| = |u_\varepsilon(x_\varepsilon^i) - u_\varepsilon(x_\varepsilon^{i-1})| |I| \leq c_\varepsilon |I|,$$

then we can estimate

$$\int_I |\tilde{u}_\varepsilon - u_\varepsilon| dx \leq \sum_{i=1}^{m_\varepsilon} \int_I |v_\varepsilon^i - v_\varepsilon^{i-1}| dx \leq \sum_{i=1}^{m_\varepsilon} c_\varepsilon |I| \leq \mathcal{H}^0(B_\varepsilon(u_\varepsilon)) c_\varepsilon |I| \leq \mathcal{H}^0(D_\varepsilon) c_\varepsilon |I|.$$

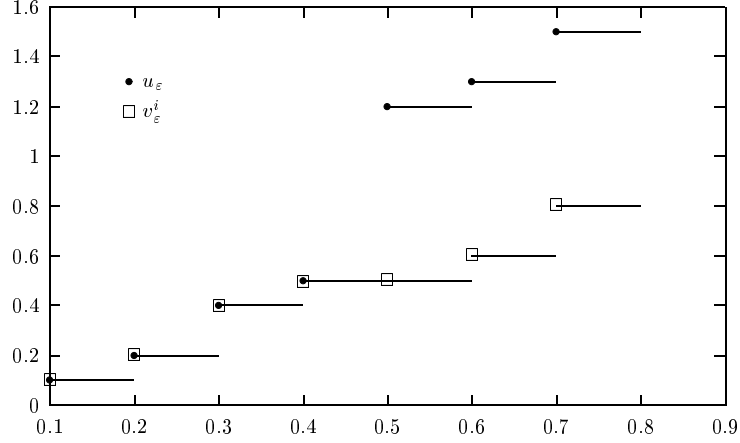


Figure 1: The construction of \tilde{u}_ε .

Therefore, by Lemma 3.3, we get $\tilde{u}_\varepsilon \rightarrow u$ in $L^1(I)$. Moreover, by construction, we clearly have that $\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon) \leq \mathcal{F}_\varepsilon(u_\varepsilon)$. We set

$$I_\varepsilon^b := \left\{ x \in I_\varepsilon : \frac{|u_\varepsilon(x + \varepsilon) - u_\varepsilon(x)|}{\varepsilon} \leq \frac{b_\varepsilon}{\sqrt{a_\varepsilon}} \right\}$$

$$I_\varepsilon^\sharp := \left\{ x \in I_\varepsilon : \frac{|u_\varepsilon(x + \varepsilon) - u_\varepsilon(x)|}{\varepsilon} \geq \frac{c_\varepsilon}{\varepsilon} \right\}$$

and we call w_ε the function belonging to $SBV(I)$ defined by

$$w_\varepsilon(x) := \tilde{u}_\varepsilon\left(\varepsilon \left[\frac{x}{\varepsilon}\right]\right) + \frac{\tilde{u}_\varepsilon\left(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon\right) - \tilde{u}_\varepsilon\left(\varepsilon \left[\frac{x}{\varepsilon}\right]\right)}{\varepsilon} \left(x - \varepsilon \left[\frac{x}{\varepsilon}\right]\right) \quad \text{if } \varepsilon \left[\frac{x}{\varepsilon}\right] \in I_\varepsilon^b,$$

$$w_\varepsilon(x) := \tilde{u}_\varepsilon\left(\varepsilon \left[\frac{x}{\varepsilon}\right]\right) \quad \text{if } \varepsilon \left[\frac{x}{\varepsilon}\right] \in I_\varepsilon^\sharp \text{ or } \varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon \notin (\varepsilon\mathbb{Z} \cap I),$$

and $w_\varepsilon(x) := 0$ otherwise. Roughly speaking w_ε coincides with the affine interpolation of \tilde{u}_ε in the intervals $(y, y + \varepsilon)$ with $y \in I_\varepsilon^b$ while takes the constant value $\tilde{u}_\varepsilon(y)$ in the intervals $(y, y + \varepsilon)$ with $y \in I_\varepsilon^\sharp$; it is clear that

$$w_\varepsilon \rightarrow u \text{ in } L^1 \quad \sqrt{a_\varepsilon} \|w'_\varepsilon\|_\infty \leq b_\varepsilon \rightarrow 0 \quad \text{and} \quad S_{w_\varepsilon} = I_\varepsilon^\sharp + \varepsilon. \quad (3.1)$$

Now we can estimate

$$\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \geq \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon^b} \log \left(1 + a_\varepsilon \frac{|\tilde{u}_\varepsilon(x + \varepsilon) - \tilde{u}_\varepsilon(x)|^2}{\varepsilon^2} \right) \quad (3.2)$$

$$+ \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon^\sharp} \log \left(1 + a_\varepsilon \frac{|\tilde{u}_\varepsilon(x + \varepsilon) - \tilde{u}_\varepsilon(x)|^2}{\varepsilon^2} \right)$$

$$\geq \frac{1}{a_\varepsilon} \int_I \log(1 + a_\varepsilon |w'_\varepsilon|^2) dx + \mathcal{H}^0(I_\varepsilon^\sharp) \frac{\varepsilon}{a_\varepsilon} \log \left(1 + a_\varepsilon \frac{c_\varepsilon^2}{\varepsilon^2} \right). \quad (3.3)$$

Fix $\delta \in (0, 1)$; recalling (3.1) and the definition of a_ε , by Lemma 3.4 and by $e)$ of Lemma 3.2, from (3.3) we deduce the existence of $\bar{\varepsilon}$ such that for $\varepsilon \leq \bar{\varepsilon}$

$$\mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \geq (1 - \delta) \left(\int_I |w'_\varepsilon|^2 dx + \mathcal{H}^0(S_{w_\varepsilon}) \right);$$

by Ambrosio Semicontinuity Theorem we therefore obtain that $u \in SBV(I)$ and

$$\liminf_{\varepsilon \rightarrow 0^+} \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq \liminf_{\varepsilon \rightarrow 0^+} \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon, I) \geq (1 - \delta) \left(\int_I |u'|^2 dx + \mathcal{H}^0(S_u) \right),$$

which gives the desired inequality since δ is arbitrary.

Concerning the last part of the statement, notice that for any energy-bounded sequence u_ε it is possible to construct as above a sequence $w_\varepsilon \in SBV(I)$ satisfying

$$\|u_\varepsilon - w_\varepsilon\|_{L^1} \rightarrow 0 \quad \text{and} \quad \mathcal{F}_\varepsilon(u_\varepsilon, I) \geq (1 - \delta) \left(\int_I |w'_\varepsilon|^2 dx + \mathcal{H}^0(S_{w_\varepsilon}) \right);$$

the precompactness of w_ε and therefore the precompactness of u_ε follows from Ambrosio Compactness Theorem. \blacksquare

We conclude this section with a remark that will be useful in the sequel.

Remark 3.5 Fix $t \in \mathbb{R}$ and for $u \in l^1(\varepsilon\mathbb{Z} \cap I)$ define

$$\mathcal{F}_\varepsilon^t(u, I) := \frac{\varepsilon}{a_\varepsilon} \sum_{x \in I_\varepsilon^t} \log \left(1 + a_\varepsilon \frac{|u(x + \varepsilon) - u(x)|^2}{\varepsilon^2} \right),$$

where

$$I_\varepsilon^t := \{x \in I \cap \varepsilon(t + \mathbb{Z}) : x + \varepsilon \in I\};$$

then we have that Proposition 3.1 is still valid with $\mathcal{F}_\varepsilon^t$ instead of \mathcal{F}_ε (without changes in the proof).

4 Estimate from below of the Γ -limit for $N = 2$

Lemma 4.1 Let $u_\varepsilon \in l^1(\varepsilon\mathbb{Z}^2)$ be such that $u_\varepsilon \rightarrow u$ in $L^1(\mathbb{R}^2)$. For $y, \xi \in \mathbb{Z}^2$, let $v_{\varepsilon, \xi}^y \in l^1(\varepsilon(y + \mathbb{Z}_\xi^2))$ be defined as $v_{\varepsilon, \xi}^y(x) := u_\varepsilon(x)$ for every $x \in \varepsilon(y + \mathbb{Z}_\xi^2)$. Then $v_{\varepsilon, \xi}^y \rightarrow u$ in $L^1(\mathbb{R}^2)$.

Proof. We call Q_ξ the unit cell of the lattice \mathbb{Z}_ξ^2 , i.e.

$$Q_\xi := C_\xi \cap \mathbb{Z}^2 = \{\tau^1, \dots, \tau^k\}, \quad (4.1)$$

where C_ξ is the set defined in (2.2). For $j = 1, \dots, k$ we set $u_\varepsilon^j(x) = u_\varepsilon(x - \varepsilon\tau^j)$. Since

$$\begin{aligned} \int_{\mathbb{R}^2} |u_\varepsilon^j(x) - u(x)| dx &\leq \int_{\mathbb{R}^2} |u_\varepsilon(x - \varepsilon\tau^j) - u(x - \varepsilon\tau^j)| dx + \int_{\mathbb{R}^2} |u(x - \varepsilon\tau^j) - u(x)| dx \\ &= \int_{\mathbb{R}^2} |u_\varepsilon(x) - u(x)| dx + \int_{\mathbb{R}^2} |u(x - \varepsilon\tau^j) - u(x)| dx, \end{aligned}$$

we have that $u_\varepsilon^j \rightarrow u$ in $L^1(I)$ as $\varepsilon \rightarrow 0^+$, for every $j \in \{1, \dots, k\}$; therefore, up to passing to a subsequence, we can suppose that

- there exists $N \subset \mathbb{R}^2$ with $\mathcal{L}^2(N) = 0$ such that $u_\varepsilon^j \rightarrow u$ pointwise in $\mathbb{R}^2 \setminus N$ for $j = 1, \dots, k$;
- $|u_\varepsilon^j| \leq g^j$ almost everywhere where g^j is a L^1 function, for $j = 1, \dots, k$.

Since for every $x \in \mathbb{R}^2 \setminus N$ there exists $j \in \{1, \dots, k\}$ such that $v_{\varepsilon, \xi}^j(x) = u_\varepsilon^j(x)$, we get $v_{\varepsilon, \xi}^j \rightarrow u$ pointwise in $\mathbb{R}^2 \setminus N$; moreover $|v_{\varepsilon, \xi}^j| \leq g_1 + \dots + g_k$ and therefore, by the Dominated Convergence Theorem, $v_{\varepsilon, \xi}^j \rightarrow u$ in L^1 . As the same argument can be repeated for every subsequence, the lemma is proved. ■

We will need also the following lemma, whose proof is elementary (see Figure 2).

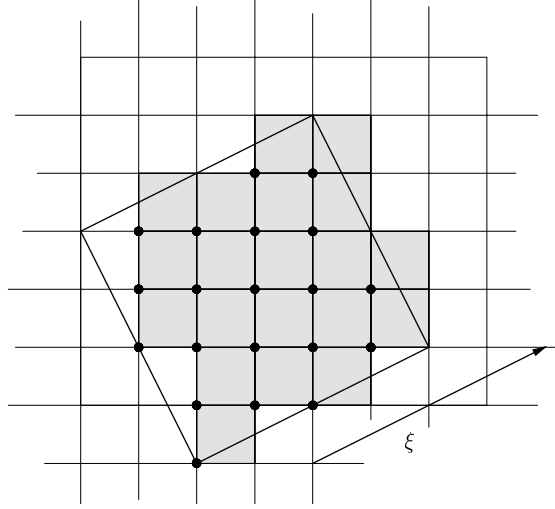


Figure 2: The set Q_ξ and the shaded region.

Lemma 4.2 *Let Q_ξ the unit cell of the lattice \mathbb{Z}_ξ^2 as defined in (4.1). Then $\mathcal{H}^0(Q_\xi) = |\xi|^2$.*

Proof. We refer to Figure 2. We associate every point $x \in Q_\xi$ with the square $x + [0, 1] \times [0, 1]$. The area of the shaded region, which is the union of such squares, coincides with the cardinality of Q_ξ and it is clear from the Figure that it is equal to the area of the set C_ξ (see (2.2)). ■

Before starting the proof of the Γ -liminf inequality it is convenient to rewrite the functional F_ε in a suitable way. After observing that

$$\bigcup_{y \in Q_\xi} (\mathbb{Z}_\xi^2 + y) = \mathbb{Z}^2,$$

we can write, for every $u \in l^1(\varepsilon \mathbb{Z}^2 \cap \Omega)$,

$$\begin{aligned} F_\varepsilon(u) &= \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon \mathbb{Z}^2} \sum_{\substack{\xi \in \mathbb{Z}^2 \\ x + \varepsilon \xi \in \Omega}} \frac{1}{a_\varepsilon |\xi|} \log \left(1 + a_\varepsilon |\xi| \frac{|u(x + \varepsilon \xi) - u(x)|^2}{\varepsilon^2 |\xi|^2} \right) \rho(\xi) \\ &= \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} G_{\varepsilon, \xi}^y(u), \end{aligned} \tag{4.2}$$

where

$$G_{\varepsilon,\xi}^y(u) := \varepsilon^2 \sum_{\substack{x \in \varepsilon(y + \mathbb{Z}_\xi^2) \cap \Omega \\ x + \varepsilon\xi \in \Omega}} \frac{1}{a_\varepsilon|\xi|} \log \left(1 + a_\varepsilon|\xi| \frac{|u(x + \varepsilon\xi) - u(x)|^2}{\varepsilon^2|\xi|^2} \right).$$

Let $u_\varepsilon \rightarrow u$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < +\infty$. Taking u_ε equal to zero outside $(\varepsilon\mathbb{Z}^2 \cap \Omega)$ and u equal to zero outside Ω , we can suppose that $u_\varepsilon \in l^1(\varepsilon\mathbb{Z}^2)$, $u \in L^1(\mathbb{R}^2)$, and $u_\varepsilon \rightarrow u$ in $L^1(\mathbb{R}^2)$. If we are able to prove that $u \in GSBV(\Omega)$ and

$$\liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon,\xi}^y(u_\varepsilon) \geq \frac{1}{|\xi|^2} \left(\int_\Omega |\nabla u \cdot \hat{\xi}|^2 dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| d\mathcal{H}^1 \right), \quad (4.3)$$

for every $\xi \in \mathbb{Z}^2$ and every $y \in Q_\xi$, then, by (4.2), Lemma 4.2, (2.4) and (2.5), we have

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0^+} F_\varepsilon(u_\varepsilon) &\geq \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} \liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon,\xi}^y(u_\varepsilon) \\ &\geq \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \sum_{y \in Q_\xi} \frac{1}{|\xi|^2} \left(\int_\Omega |\nabla u \cdot \hat{\xi}|^2 dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| d\mathcal{H}^1 \right) \\ &= \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) \left(\int_\Omega |\nabla u \cdot \hat{\xi}|^2 dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| d\mathcal{H}^1 \right) \\ &= \int_\Omega \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) (|\nabla u \cdot \hat{\xi}|^2 + |\nabla u \cdot \hat{\xi}^\perp|^2) dx + \int_{S_u} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) |\nu_u \cdot \hat{\xi}| d\mathcal{H}^1 \\ &= c_\rho \int_\Omega |\nabla u|^2 dx + \int_{S_u} \phi(\nu_u) d\mathcal{H}^1. \end{aligned} \quad (4.4)$$

We denote the hyperplane orthogonal to ξ by Π_ξ and Ω_ξ the projection of Ω on Π_ξ . For every $w \in \Pi_\xi$ we set $\Omega_\xi^w := \{t \in \mathbb{R} : w + t\hat{\xi} \in \Omega\}$ and, given a function, we define $f_\xi^w(t) := f(w + t\hat{\xi})$. We also define $O_{\varepsilon,\xi} := \Omega_\xi \cap \varepsilon\mathbb{Z}^2$ (see Figure 3) and for every $x \in \mathbb{R}^2$

$$O_{\varepsilon,\xi}^x := \{y \in x + \varepsilon\xi\mathbb{Z} : y + \varepsilon\xi \in \Omega\}.$$

Note that we can write

$$\begin{aligned} G_{\varepsilon,\xi}^y(u_\varepsilon) &= \varepsilon^2 \sum_{w \in O_{\varepsilon,\xi}} \sum_{O_{\varepsilon,\xi}^{w+\varepsilon y}} \frac{1}{a_\varepsilon|\xi|} \log \left(1 + a_\varepsilon|\xi| \frac{|u_\varepsilon(x + \varepsilon\xi) - u_\varepsilon(x)|^2}{\varepsilon^2|\xi|^2} \right) \\ &= \frac{1}{|\xi|^2} \sum_{w \in O_{\varepsilon,\xi}} \varepsilon|\xi| \sum_{O_{\varepsilon,\xi}^{w+\varepsilon y}} \frac{\varepsilon|\xi|}{a_\varepsilon|\xi|} \log \left(1 + a_\varepsilon|\xi| \frac{|u_\varepsilon(x + \varepsilon\xi) - u_\varepsilon(x)|^2}{\varepsilon^2|\xi|^2} \right) \\ &= \frac{1}{|\xi|^2} \int_{\Omega_\xi} \left[\frac{\varepsilon|\xi|}{a_\varepsilon|\xi|} \sum_{O_{\varepsilon,\xi}^{w+\varepsilon y}} \log \left(1 + a_\varepsilon|\xi| \frac{|v_{\varepsilon,\xi}^y(x + \varepsilon\xi) - v_{\varepsilon,\xi}^y(x)|^2}{\varepsilon^2|\xi|^2} \right) \right] d\mathcal{H}^1(w), \end{aligned} \quad (4.5)$$

where $v_{\varepsilon,\xi}^y$ is the sequence defined in Lemma 4.1. Set $\eta := \varepsilon|\xi|$, $w_{\eta,\xi}^y := v_{\varepsilon,\xi}^y$, $z := y/|\xi|$ and observe that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{a_\eta}{a_\varepsilon|\xi|} = 1. \quad (4.6)$$

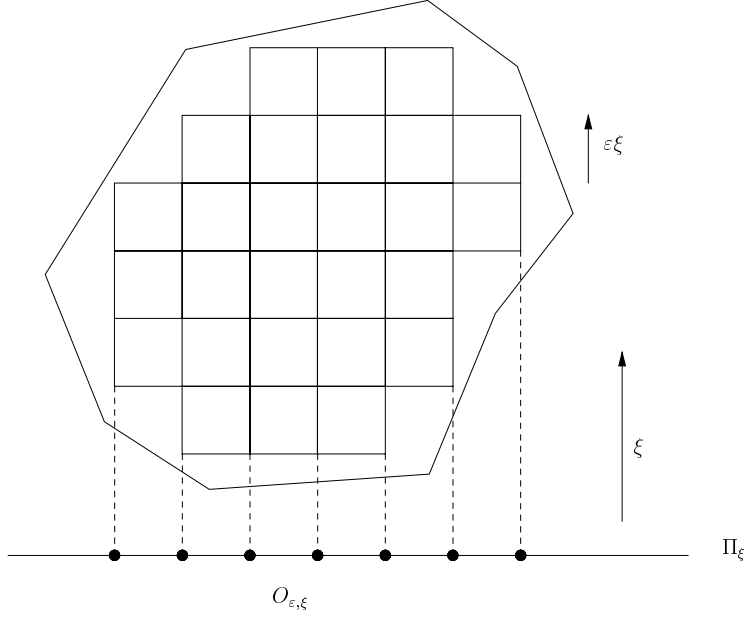


Figure 3: The sets Π_ξ and $O_{\varepsilon, \xi}$.

Fix $\delta \in (0, 1)$; by (4.5), by Fatou's Lemma, and by (4.6), we obtain

$$\begin{aligned}
\liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon, \xi}^y(u_\varepsilon) &\geq \frac{1}{|\xi|^2} \int_{\Omega_\xi} \liminf_{\varepsilon \rightarrow 0^+} \left[\frac{\varepsilon|\xi|}{a_\varepsilon|\xi|} \sum_{O_{\varepsilon, \xi}^{w+\varepsilon y}} \log \left(1 + a_\varepsilon|\xi| \frac{|v_{\varepsilon, \xi}^y(x + \varepsilon\xi) - v_{\varepsilon, \xi}^y(x)|^2}{\varepsilon^2|\xi|^2} \right) \right] d\mathcal{H}^1(w) \\
&\geq \frac{1}{|\xi|^2} \int_{\Omega_\xi} \liminf_{\eta \rightarrow 0^+} \left[\frac{\eta}{a_\eta} \sum_{O_{\eta, \xi}^{w+\eta z}} \log \left(1 + \delta a_\eta \frac{|w_{\eta, \xi}^y(x + \eta\xi) - w_{\eta, \xi}^y(x)|^2}{\eta^2} \right) \right] d\mathcal{H}^1(w) \\
&= \frac{1}{|\xi|^2} \int_{\Omega_\xi} \liminf_{\eta \rightarrow 0^+} \mathcal{F}_\eta^t \left((\sqrt{\delta} w_{\eta, \xi}^y)_\xi^w, \Omega_\xi^w \right) d\mathcal{H}^1(w),
\end{aligned}$$

where $t := z \cdot \hat{\xi}$ and \mathcal{F}_η^t is the functional defined in Remark 3.5. Since $(w_{\eta, \xi}^y)_\xi^w \rightarrow u_\xi^w$ for \mathcal{H}^1 -a.e. $w \in \Pi_\xi$, as $\eta \rightarrow 0$ (thanks to Lemma 4.1), by Proposition 3.1 and Remark 3.5 we deduce

$$\liminf_{\varepsilon \rightarrow 0^+} G_{\varepsilon, \xi}^y(u_\varepsilon) \geq \frac{1}{|\xi|^2} \int_{\Omega_\xi} \left(\int_{\Omega_\xi^w} \delta |(u_\xi^w)'|^2 dt + \mathcal{H}^0(S_{u_\xi^y}) \right) d\mathcal{H}^1(w),$$

from which (4.3) follows by letting $\delta \uparrow 1$ and by applying the well known Slicing Theorem due to Ambrosio (see [2]).

5 Estimate from above of the Γ -limit

The proof of the Γ -limsup inequality will be based on the density result due to Cortesani and Toader. For the convenience of the reader we recall in the following the statement of their theorem. Let Ω be an open bounded subset in \mathbb{R}^N with Lipschitz boundary and denote by $\mathcal{W}(\Omega)$ the space of all function $w \in SBV(\Omega)$ enjoying the following properties:

- i) $\mathcal{H}^{N-1}(\overline{S_w} \setminus S_w) = 0$;
- ii) $\overline{S_w}$ is the intersection of Ω with the union of a finite number of pairwise disjoint $(N-1)$ -simplexes;
- iii) $w \in W^{k,\infty}(\Omega \setminus \overline{S_w})$ for every $k \in \mathbb{N}$.

Cortesani and Toader have proved in [8] the following density result.

Theorem 5.1 *Let $u \in GSBV^p(\Omega)$ ($p > 1$) and let $\phi : \mathbb{R} \times S^{N-1} \rightarrow [0, +\infty)$ be a continuous function increasing in the first variable and even in the second one. Then there exists a sequence w_j in $\mathcal{W}(\Omega)$ such that $w_j \rightarrow u$ strongly in $L^1(\Omega)$, $\nabla w_j \rightarrow \nabla u$ strongly in $L^p(\Omega, \mathbb{R}^N)$, $\lim_j \|w_j\|_\infty = \|u\|_\infty$ and*

$$\limsup_{j \rightarrow \infty} \int_{S_{w_j}} \phi(|w_j^+ - w_j^-|, \nu_{w_j}) d\mathcal{H}^{n-1} \leq \int_{S_u} \phi(|u^+ - u^-|, \nu_u) d\mathcal{H}^{n-1}.$$

Remark 5.2 *Under the additional assumption that $1 < p \leq 2$ the structure of the jump set of the functions w_j given by Theorem 5.1 can be further improved by using a capacitary argument. In particular for $N = 2$ and $p = 2$, we can suppose that $\overline{S_{w_j}}$ is made up of a finite family of pairwise disjoint segments compactly contained in Ω .*

Therefore it will be enough to prove the Γ -limsup inequality for a function $u \in \mathcal{W}(\Omega)$ whose discontinuity set consists of the union of a finite family $\{S_1, \dots, S_k\}$ of disjoint segments compactly contained in Ω . Let $\varepsilon_n \rightarrow 0$ and set, for every $u \in L^1(\Omega)$, $F''(u) := \Gamma\text{-lim sup}_{n \rightarrow \infty} F_{\varepsilon_n}(u)$; we want to prove that

$$F''(u) \leq F(u). \quad (5.1)$$

We begin by assuming that

$$S_i \cap \varepsilon_n \mathbb{Z}^2 = \emptyset \quad \forall n \in \mathbb{N}, \forall i \in \{1, \dots, k\}. \quad (5.2)$$

Let $u_n \in l^1(\varepsilon_n \mathbb{Z} \cap \Omega)$ be such that $u_n(x) = u(x)$ for every $x \in \varepsilon_n \mathbb{Z} \cap \Omega$. Clearly $u_n \rightarrow u$ in $L^1(\Omega)$. As for the proof of the Γ -liminf inequality, the thesis is achieved once we have shown that

$$\limsup_{n \rightarrow \infty} G_{\varepsilon_n, \xi}^y(u_n) \leq \frac{1}{|\xi|^2} \left(\int_{\Omega} |\nabla u \cdot \hat{\xi}|^2 dx + \int_{S_u} |\nu_u \cdot \hat{\xi}| d\mathcal{H}^1 \right) \quad \forall \xi \in \mathbb{Z}^2, y \in Q_\xi. \quad (5.3)$$

To simplify the notation we will prove (5.3) only for $y = 0$. In the sequel, given x_1 and x_2 in \mathbb{R}^2 , we denote by $[x_1, x_2]$ the segment joining the two points. Let us define the following sets:

$$A_n := \{x \in \varepsilon_n \mathbb{Z}_\xi^2 \cap \Omega : x + \varepsilon_n \xi \in \Omega, [x, x + \varepsilon_n \xi] \cap S_j = \emptyset \text{ for } j = 1, \dots, k\},$$

and

$$B_n^j := \{x \in \varepsilon_n \mathbb{Z}_\xi^2 : [x, x + \varepsilon_n \xi] \cap S_j \neq \emptyset\} \quad j = 1, \dots, k.$$

Clearly for n large enough, $B_n^j \cap B_n^i = \emptyset$ if $i \neq j$. Note now that we can write

$$\begin{aligned} G_{\varepsilon_n, \xi}^y(u_n) &= \varepsilon_n^2 \sum_{A_n} \frac{1}{a_{\varepsilon_n} |\xi|} \log \left(1 + a_{\varepsilon_n} |\xi| \frac{|u_n(x + \varepsilon_n \xi) - u_n(x)|^2}{\varepsilon_n^2 |\xi|^2} \right) \\ &\quad + \sum_{j=1}^k \varepsilon_n^2 \sum_{B_n^j} \frac{1}{a_{\varepsilon_n} |\xi|} \log \left(1 + a_{\varepsilon_n} |\xi| \frac{|u_n(x + \varepsilon_n \xi) - u_n(x)|^2}{\varepsilon_n^2 |\xi|^2} \right) \\ &= \underbrace{\frac{1}{|\xi|^2} \int_{\Omega} \frac{1}{a_{\varepsilon_n} |\xi|} \log \left(1 + a_{\varepsilon_n} |\xi| \frac{|v_{n, \xi}^0(x + \varepsilon_n \xi) - v_{n, \xi}^0(x)|^2}{\varepsilon_n^2 |\xi|^2} \right) \chi_{(A_n + \varepsilon_n C_\xi)}}_{I_{n,1}} \\ &\quad + \underbrace{\frac{1}{|\xi|^2} \sum_{j=1}^k \varepsilon_n^2 |\xi|^2 \sum_{B_n^j} \frac{1}{a_{\varepsilon_n} |\xi|} \log \left(1 + a_{\varepsilon_n} |\xi| \frac{|u_n(x + \varepsilon_n \xi) - u_n(x)|^2}{\varepsilon_n^2 |\xi|^2} \right)}_{I_{n,2}}, \end{aligned}$$

where $v_{n, \xi}^0$ is the sequence defined in Lemma 4.1, while C_ξ is the set defined in (2.2). It is immediate to see that

$$\chi_{(A_n + \varepsilon_n C_\xi)} \rightarrow \chi_{\Omega \setminus S_u}. \quad (5.4)$$

Take $x \in \Omega \setminus S_u$ and let $y_n \in \varepsilon_n \mathbb{Z}_\xi^2$ be such that $x \in y_n + \varepsilon_n C_\xi$; by Lagrange's Theorem it turns out that

$$\begin{aligned} \log \left(1 + a_{\varepsilon_n} |\xi| \frac{|v_{n, \xi}^0(x + \varepsilon_n \xi) - v_{n, \xi}^0(x)|^2}{\varepsilon_n^2 |\xi|^2} \right) &= \log \left(1 + a_{\varepsilon_n} |\xi| \frac{|u(y_n + \varepsilon_n \xi) - u(y_n)|^2}{\varepsilon_n^2 |\xi|^2} \right) \\ &= \log \left(1 + a_{\varepsilon_n} |\xi| |\nabla u(\xi_n) \cdot \hat{\xi}|^2 \right) \\ &\leq a_{\varepsilon_n} |\xi| |\nabla u(\xi_n) \cdot \hat{\xi}|^2, \end{aligned}$$

where $\xi_n \in [y_n, y_n + \varepsilon_n \xi]$ and therefore $\xi_n \rightarrow x$. Taking into account the continuity of ∇u and recalling (5.4), we deduce that

$$\limsup_{n \rightarrow \infty} I_{n,1} \leq \frac{1}{|\xi|^2} \int_{\Omega} |\nabla u(x) \cdot \hat{\xi}|^2 dx. \quad (5.5)$$

Moreover, for every $x \in B_n^j$, we have

$$\frac{\varepsilon_n |\xi|}{a_{\varepsilon_n} |\xi|} \log \left(1 + a_{\varepsilon_n} |\xi| \frac{|u_n(x + \varepsilon_n \xi) - u_n(x)|^2}{\varepsilon_n^2 |\xi|^2} \right) \leq \frac{\varepsilon_n}{a_{\varepsilon_n}} \log \left(1 + a_{\varepsilon_n} \frac{4 \|u\|_\infty^2}{\varepsilon_n^2} \right) \rightarrow 1, \quad (5.6)$$

where the last limit follows from the definition of a_{ε_n} . Denote by $l_\xi(S_j)$ the length of the projection of S_j on Π_ξ ; using the fact that $l_\xi(S_j) = \int_{S_j} \nu_u \cdot \hat{\xi} d\mathcal{H}^1$, we easily obtain (see Figure

4 below)

$$\mathcal{H}^0(B_n^j) \leq \frac{l_\xi(S_j)}{\varepsilon_n |\xi|} + 1 \leq \frac{1}{\varepsilon_n |\xi|} \int_{S_j} \nu_u \cdot \hat{\xi} d\mathcal{H}^1 + 1; \quad (5.7)$$

therefore from (5.6) and (5.7) we get

$$\limsup_{n \rightarrow \infty} I_{2,n} \leq \frac{1}{|\xi|^2} \limsup_{n \rightarrow \infty} \sum_{j=1}^k \varepsilon_n |\xi| \mathcal{H}^0(B_n^j) \leq \frac{1}{|\xi|^2} \sum_{j=1}^k \int_{S_j} \nu_u \cdot \hat{\xi} d\mathcal{H}^1 = \frac{1}{|\xi|^2} \int_{S_u} \nu_u \cdot \hat{\xi} d\mathcal{H}^1,$$

which, combined with (5.5), gives (5.3) and therefore (5.1).

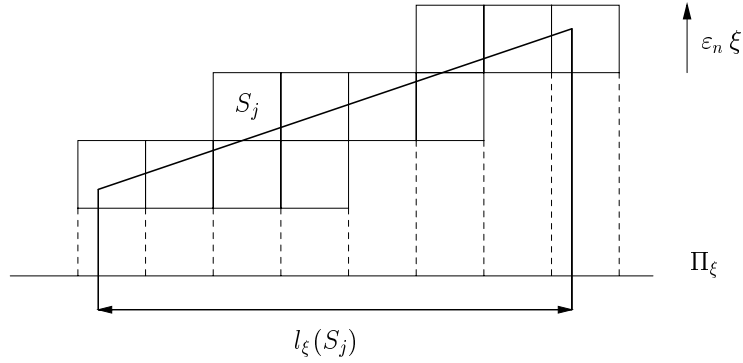


Figure 4: The projection of S_j on Π_ξ .

If (5.2) is not true we can argue in the following way. We first observe that it is possible to find a sequence $(\tau_k) \subset \mathbb{R}^2$ such that $\tau_k \rightarrow 0$ and $S_u + \tau_k$ satisfy (5.2) for every k . Let $u_k(x) := u(x - \tau_k)$, then $u_k \rightarrow u$, $S_{u_k} = S_u + \tau_k$ satisfies (5.2), and $F(u_k) \rightarrow F(u)$; using the previous step and the semicontinuity of F'' , we have

$$F''(u) \leq \liminf_{k \rightarrow \infty} F''(u_k) \leq \lim_{k \rightarrow \infty} F(u_k) = F(u),$$

which concludes the proof.

6 Compactness

In this section we prove the equicoerciveness of the approximating functionals F_ε . We will use the following L^1 -precompactness criterion by slicing introduced by Alberti, Bouchitté & Seppecher (see [1]). Using the notations introduced in section 4, given a family \mathcal{F} of functions, for every $\xi \in S^{N-1}$ and $y \in \Pi_\xi$ we set $\mathcal{F}_\xi^y := \{u_\xi^y : u \in \mathcal{F}\}$; moreover we say that a family \mathcal{F}' is δ -close to \mathcal{F} if \mathcal{F}' is contained in a δ -neighborhood of \mathcal{F} .

Lemma 6.1 *Let \mathcal{F} be a family of equiintegrable functions belonging to $L^1(A)$ and assume that there exists a basis of unit vectors $\{\xi_1, \dots, \xi_N\}$ with the property that for every $i = 1, \dots, N$, for*

every $\delta > 0$, there exists a family \mathcal{F}_δ δ -close to \mathcal{F} such that $(\mathcal{F}_\delta)_{\xi_i}^y$ is precompact in $L^1(A_{\xi_i}^y)$ for \mathcal{H}^{N-1} -a.e $y \in A_{\xi_i}$. Then \mathcal{F} is precompact in $L^1(A)$.

Proposition 6.2 *Let (u_ε) be a sequence of equibounded functions such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < M < +\infty$; then (u_ε) is strongly precompact in $L^p(\Omega)$, for every $p \geq 1$.*

Proof. Clearly it is enough to prove the precompactness in L^1 . Let $\{e_1, e_2\}$ be the canonical basis in \mathbb{R}^2 . Since for $\xi = e_i$ (for $i = 1, 2$) the function $v_{\varepsilon, \xi}^y$ defined in Lemma 4.1 coincides with u_ε , from (4.5) we have

$$\begin{aligned} M &> \sup_\varepsilon F_\varepsilon(u_\varepsilon) \geq \sup_\varepsilon G_{\varepsilon, e_i}^0 \\ &\geq \sup_\varepsilon \int_{\Omega_{e_i}} \mathcal{F}_\varepsilon\left((u_\varepsilon)_{e_i}^w, \Omega_{e_i}^w\right) d\mathcal{H}^1(w) = \sup_\varepsilon \int_{\Omega_{e_i}} f_\varepsilon(w) d\mathcal{H}^1(w), \end{aligned} \quad (6.1)$$

where

$$f_\varepsilon(w) := \mathcal{F}_\varepsilon\left((u_\varepsilon)_{e_i}^w, \Omega_{e_i}^w\right).$$

Fix $\delta \in (0, 1)$ and choose $k > 0$ so large that

$$M \frac{\sup_\varepsilon \|u_\varepsilon\|_\infty}{k} \text{diam}(\Omega) < \delta; \quad (6.2)$$

setting $A_{\varepsilon, i}^k := \{w \in \Omega_{e_i} : f_\varepsilon(w) > k\}$, by Chebychev Inequality and (6.1), we can estimate

$$|A_{\varepsilon, i}^k| \leq \frac{1}{k} \sup_\varepsilon \int_{\Omega_{e_i}} f_\varepsilon(w) d\mathcal{H}^1(w) \leq \frac{M}{k}. \quad (6.3)$$

Let $z_{\varepsilon, \delta}$ be such that $z_{\varepsilon, \delta}(x) = 0$ if the projection of x on Ω_{e_i} belongs to $A_{\varepsilon, i}^k$ and $z_{\varepsilon, \delta}(x) = u_\varepsilon(x)$ otherwise. We clearly have

$$\|z_{\varepsilon, \delta} - u_\varepsilon\|_{L^1} \leq \sup_\varepsilon \|u_\varepsilon\|_\infty |A_{\varepsilon, i}^k| \text{diam}(\Omega) \leq \delta,$$

where the last inequality follows from (6.3) and (6.2). Moreover $\mathcal{F}_\varepsilon\left((z_{\varepsilon, \delta})_{e_i}^w, \Omega_{e_i}^w\right) \leq f_\varepsilon(w)(1 - \chi_{A_{\varepsilon, i}^k}) \leq k$ for every $w \in \Omega_{e_i}$ and therefore $((z_{\varepsilon, \delta})_{e_i}^w)$, by the one dimensional result, is precompact in $L^1(\Omega_{e_i}^w)$ for every $w \in \Omega_{e_i}$. Thus we have constructed a sequence which is δ -close to (u_ε) and such that the one-dimensional sections in the e_i -direction are precompact, for $i = 1, 2$. The thesis follows from Lemma 6.1. \blacksquare

7 Some considerations on the numerical results

The choice of the best algorithm for the solution of the Mumford-Shah functional, and the related convergence properties, are difficult problems which are not going to be tackled here. Nevertheless some heuristic considerations and the numerical results themselves seem to suggest the use of functionals like (1.6) instead of (1.3), independently of the choice of the algorithm.

In detail, let $\Omega = (0, 1)^2$ and $g : \Omega \rightarrow [0, 1]$ be the grey level function of the given image; the results presented here are obtained using the functional

$$F_\varepsilon(u) = \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon\mathbb{Z}^2} \sum_{\substack{\xi \in \mathbb{Z}^2 \\ x + \varepsilon\xi \in \Omega}} \frac{\alpha}{a_\varepsilon|\xi|} \log \left(1 + \frac{\beta a_\varepsilon |\xi| |u(x + \varepsilon\xi) - u(x)|^2}{\alpha \varepsilon^2 |\xi|^2} \right) \rho(\xi) \\ + \varepsilon^2 \sum_{x \in \Omega \cap \varepsilon\mathbb{Z}^2} |u(x) - g(x)|^2, \quad (7.1)$$

for some suitable choice of the coefficients $\alpha > 0$ and $\beta > 0$. The convolution term $\rho : \mathbb{Z}^2 \rightarrow [0, +\infty)$ is not zero only for the nearest neighbors (i.e. $\xi \in \mathbb{Z}^2$ such that $|\xi|_\infty = 1$) and takes constantly the value $(\sqrt{2} - 1)/2$.

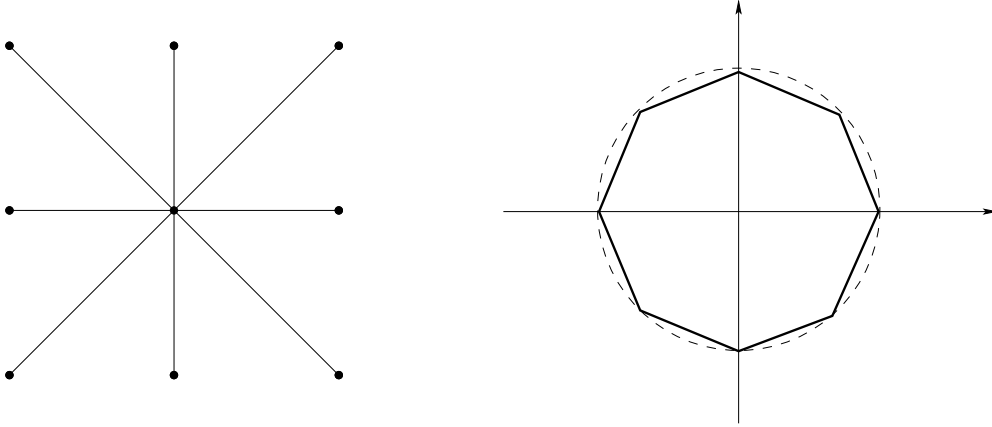


Figure 5: The nearest neighbors and the level curve $\{\phi(v) = 1\}$ compared with the unit circle.

Precisely the functional (7.1) Γ -converges (as $\varepsilon \rightarrow 0$ and in the strong topology of $L^2(\Omega)$) to the anisotropic Mumford-Shah functional

$$F(u) = c_\rho \beta \int_\Omega |\nabla u|^2 dx + \alpha \int_{S_u} \phi(\nu) d\mathcal{H}^1 + \int_\Omega |u - g|^2 dx, \quad (7.2)$$

where the constant c_ρ and the anisotropy function $\phi(\nu)$ are given by

$$c_\rho := \frac{1}{2} \sum_{\xi \in \mathbb{Z}^2} \rho(\xi) = 2(\sqrt{2} - 1),$$

$$\phi(\nu) := \sum_{\xi \in \mathbb{Z}^2} \frac{\rho(\xi)}{|\xi|} |\langle \nu, \xi \rangle| = (\sqrt{2} - 1) \left(|\nu_1| + |\nu_2| + \frac{\sqrt{2}}{2} |\nu_1 + \nu_2| + \frac{\sqrt{2}}{2} |\nu_1 - \nu_2| \right),$$

(note that the level curve $\{\phi(v) = 1\}$ is a regular octagon).

It's well known, see [4], that the minimization of the Mumford-Shah functional produces a filtering effect on the edges of the original image. Indeed it should preserve only the edges

having a contrast greater than a threshold value given (approximatively) by $\sqrt{2\alpha/\sqrt{\beta}}$. On the contrary, as it is noticed in [14], discrete functionals like (1.3) introduce a local threshold on the contrast which is also related to the presence of some local minimizers and stationary points. Indeed, considering in (1.3) the function

$$|\nabla u| \approx |u^+ - u^-|/\varepsilon, \quad f(t) = \frac{2\alpha}{\pi} \arctan\left(\frac{\beta\pi}{2\alpha}t\right) \approx \min\{\beta t, \alpha\}$$

the local behavior of the functional, in the case $|\xi| = 1$, becomes

$$\varepsilon f\left(\frac{|u(x + \varepsilon\xi) - u(x)|^2}{\varepsilon|\xi|}\right) dx \approx \begin{cases} \beta|u(x + \varepsilon\xi) - u(x)|^2 & \text{if } |u(x + \varepsilon\xi) - u(x)|^2 \leq \alpha\varepsilon/\beta \\ \alpha\varepsilon & \text{if } |u(x + \varepsilon\xi) - u(x)|^2 > \alpha\varepsilon/\beta. \end{cases}$$

Hence when the contrast $|u(x + \varepsilon\xi) - u(x)|$ is greater than $\sqrt{\alpha\varepsilon/\beta}$ the functional behaves as in the case of a discontinuity. This is clearly dangerous when ε is small because $\alpha\varepsilon/\beta \rightarrow 0$ and then the functional numerically looks like a constant. This effect justifies the use of special minimizing techniques, like the Graduated Non Convexity algorithm which basically aims at reducing the non convexity of the function $f(t)$, by a recursive minimization of the functional for increasing values of the coefficient β (see [4] for details). From this point of view the function

$$-\frac{\alpha}{\log \varepsilon} \log\left(1 - \frac{\beta \log \varepsilon}{\alpha}t\right)$$

is a better choice than $f(t)$ because its behavior for t large avoids the introduction of a discrete threshold, while its slope, for small values of t , gives a diffusion smaller than β , which can be interpreted as a sort of GNC effect (see Figure 6).

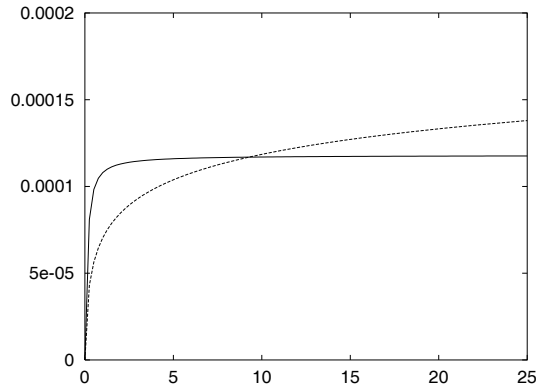


Figure 6: A comparison between $-\frac{\alpha}{\log \varepsilon} \log\left(1 - \frac{\beta \log \varepsilon}{\alpha}t\right)$ and $\frac{2\alpha}{\pi} \arctan\left(\frac{\beta\pi}{2\alpha}t\right)$.

In particular the numerical results reported here were obtained with coefficient $\beta = 5.54 \cdot 10^{-4}$, $\alpha = 1.18 \cdot 10^{-4}$ (corresponding to a threshold 0.1) and $\varepsilon = 1/256$ (since the image has dimension 256×256). The descent direction in both the algorithms is given by $d = (1/C)\nabla F_\varepsilon$, where the constant $C = 2(\varepsilon^2 + \beta)$, is a naive value for the second derivative of the functional.

Then, for every iteration, the successive point is computed by a quadratic backtracking as line search strategy in the segment $[u_k, u_k + td(u_k)]$, for $t \in [0, 2]$ (see [11] or [10]).

The images reported in Figure 8 and the values in Table 1 confirm the previous heuristic considerations. Indeed using the log function the numerical solutions computed by GNC and GD are almost the same, both in terms of energy values and graphical quality of the segmentations. On the contrary there is much difference when the arctan is employed, in particular the solution obtained by GD presents many false edges, generated by the local thresholding, while the one computed by GD is smoother because for small values of t we have

$$-\frac{\alpha}{\log \varepsilon} \log \left(1 - \frac{\beta \log \varepsilon}{\alpha} t \right) < \frac{2\alpha}{\pi} \arctan \left(\frac{\beta \pi}{2\alpha} t \right).$$

	log	arctan		log	arctan
GD	0.0058887	0.0070800	GD	702	1024
GNC	0.0058301	0.0062361	GNC	403	1178

Table 1: Comparison of the performances in terms of energy values (computed in the minimizer), and number of iterations, for Graduated Non Convexity (GNC) and gradient descent (GD).

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Figure 7: The original image.



Figure 8: The numerical results, in the same order as in Table 1.

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