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groups of negatively δ -pinched
manifolds

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Abstract

In this note, we will show that the fundamental group of any negatively δ -pinched ($\delta > \frac{1}{4}$) manifold can't be the fundamental group of a quasi-compact Kähler manifold. As a consequence of our proof, we also show that any nonuniform lattice in $F_{4(-20)}$ cannot be the fundamental group of a quasi-compact Kähler manifold. The corresponding result for uniform lattices was proved by Carlson and Hernández [3]. Finally, we follow Gromov and Thurston [6] to give some examples of negatively δ -pinched manifolds ($\delta > \frac{1}{4}$) of finite volume which, as topological manifolds, admit no hyperbolic metric with finite volume under any smooth structure. This shows that our result for δ -pinched manifolds is a nontrivial generalization of the fact that no nonuniform lattice in $SO(n, 1)$ ($n \geq 3$) is the fundamental group of a quasi-compact Kähler manifold [21].

1 Introduction

In [22], Yau and Zheng (independently, Hernández [7]) studied negatively $\delta(\geq \frac{1}{4})$ -pinched manifolds. In particular, they showed that such manifolds are hermitian negative (for the definition, see Section 2). An interesting consequence of their result (which was not stated explicitly in [22], but implied clearly) is that the fundamental group of a compact negatively $\delta(> \frac{1}{4})$ -pinched manifold cannot be the fundamental group of any compact Kähler manifold (in the sequel, we always assume $\delta > \frac{1}{4}$). On the other hand, in [6], Gromov and Thurston constructed some examples of closed n ($n \geq 4$)-manifolds which admit some negatively δ -pinched metrics, but no metric of constant curvature -1 under any smooth structure. In other words, π_1 of these manifolds can't be a cocompact discrete subgroup of $SO(n, 1)$. A simple homotopical (or cohomological dimension) argument also shows that these π_1 cannot be

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cocompact discrete subgroups of other $SO(m, 1)$, $m \neq n$. So, the above result by Yau-Zheng and Hernández is a nontrivial generalization of a result by Carlson-Toledo [4] and Jost-Yau [8] independently (in [8], although the result was not stated explicitly, it is clear that it is contained in the results of [8]), which asserts that no cocompact lattice in $SO(n, 1)$ ($n \geq 3$) can be π_1 of a compact Kähler manifold. In [21], the second author also showed that the result by Carlson-Toledo and Jost-Yau is still valid for nonuniform lattices: Let \overline{M} be a compact Kähler manifold and D be a normal crossing divisor, denote $\overline{M} \setminus D$ by M . Call M a quasi-compact Kähler manifold. Topologically, this class of manifolds includes quasi-projective varieties by Hironaka's Theorem for resolution of singularities. Then, no nonuniform lattice in $SO(n, 1)$ ($n \geq 3$) can be π_1 of a quasi-compact Kähler manifold. In this note, our purpose is to generalize this to open $-\delta$ -pinched manifolds with finite volume, namely we will show the following

Theorem 1 *Let M be a quasi-compact Kähler manifold and N a complete noncompact $-\delta$ -pinched manifold with finite volume and dimension ≥ 3 . Then $\pi_1(N)$ is not isomorphic to $\pi_1(M)$. Namely, the fundamental group of a complete noncompact $-\delta$ -pinched manifold with finite volume and dimension ≥ 3 cannot be the fundamental group of any quasi-compact Kähler manifold.*

The method of proof is to use harmonic map theory due to Jost and Zuo [9, 10]. One of the ingredients for the proof is to show that a differentiable family of submanifolds of codimension 2 from the harmonic map in question actually is a holomorphic family and it gives rise to a foliation. Also observe that the argument of Theorem 7.1 in [4] does not work in the present noncompact situation. In addition, we need to treat the case of dimension 3 separately. As an interesting corollary of the above theorem, we have

Theorem 2 *Let M_1, M_2, \dots, M_s ($s \geq 2$) be some compact or open Riemann surfaces. Then, the product $M_1 \times M_2 \times \dots \times M_s$ does not admit any complete negatively δ ($\delta > \frac{1}{4}$)-pinched metric with finite volume under any smooth structure.*

This is analogous to the positive curvature case, where by the $\frac{1}{4}$ -sphere theorem, one knows that $S^2 \times S^2 \times \dots \times S^2$ (at least two copies) does not admit any δ -pinched metric.

Combining the technique of the proof of Theorem 1 with the Lie-theoretic analysis of [3], one can also obtain

Theorem 3 *Let Γ be a nonuniform lattice in the exceptional group $F_{4(-20)}$. Then Γ can't be the fundamental group of any quasi-compact Kähler manifold.*

Remark: For the cocompact lattice case of $F_{4(-20)}$, the above result was proved by Carlson and Hernández [3].

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2 Negatively δ -pinched manifolds and harmonic maps

In this section, we will recall some well-known results concerning negatively δ -pinched manifolds [2, 22, 7] and the existence of harmonic maps [10] and give some properties of the harmonic maps in question.

Let N be a complete negatively δ -pinched manifold with finite volume. By means of the Margulis lemma, one gets that as stated and proved in [2], Corollary 1.5.2, N can be considered as the interior of a compact manifold with boundary, and the boundary topologically is the disjoint union of tori up to a finite group action. And concerning the curvature tensor of N , one has

Lemma 1 ([22], Lemma 2, 3 or [7], Theorem 2.5) *N is Hermitian negative, namely, for any two complex vectors Z, W in $T_p(N) \otimes \mathbb{C}$ ($p \in N$),*

$$R(Z, W, \bar{Z}, \bar{W}) \leq 0;$$

and there do not exist complex linearly independent vectors Z, W satisfying

$$R(Z, W, \bar{Z}, \bar{W}) = 0.$$

Now, we turn to harmonic map theory. Let \bar{M} be a compact Kähler manifold of dimension n with a fixed Kähler metric ω_0 , D be a fixed divisor with (at worst) normal crossings and $D = \bigcup_{i=1}^p D_i$. Here, D_i are the irreducible components of D . One may also assume that each irreducible component D_i is free from self intersections. Thus, at each intersection point, precisely at most n components of D meet. Denote $\bar{M} \setminus D$ by M .

Let σ_i ($i = 1, 2, \dots, p$) be a defining section of D_i in $\mathcal{O}(\bar{M}, [D_i])$, which satisfies $|\sigma_i| \leq 1$ for a certain Hermitian metric of $[D_i]$ and defines a holomorphic coordinate system in each small disk transversal to D_i . So, one can get a fibration of a small neighborhood, say $|\sigma_i| \leq \mu \leq 1$, of D_i by small

holomorphic disks which meet D_i transversally. Similarly, for the boundary of such a small neighborhood, denoted by Σ_i^μ , one also gets a fibration by circles. The intersection of two such boundaries is fibered by tori.

Corresponding to the above defining sections σ_i , one can define a complete Kähler metric on M as follows,

$$g := -\frac{\sqrt{-1}}{2} \sum_{i=1}^p \partial\bar{\partial}(\phi(|\sigma_i|)\log|\log|\sigma_i|^2|) + c\omega_0|_M,$$

where ϕ is a suitable C^∞ cut-off function on $[0, \infty)$, so that $\phi(s)$ is identical to one on $[0, \epsilon)$ and to zero on $[2\epsilon, \infty)$, for sufficiently small $\epsilon \geq 0$, and c is taken sufficiently large, so that g is positive definite. Then g is a Kähler metric. One can show that (M, g) is complete and has finite volume [5]. In fact, when restricted to small holomorphic disks transversal to D , this metric looks like the Poincaré metric on the punctured disk (D^*, z)

$$-\frac{\sqrt{-1}}{2} \partial\bar{\partial}\log(-\log|z|^2) = \frac{\sqrt{-1}}{2} \frac{dz \wedge d\bar{z}}{|z|^2(\log|z|^2)^2}.$$

In the sequel, unless stated otherwise, we always assume that M is endowed with the above complete metric g .

As before, let N be a complete noncompact negatively δ -pinched manifold with finite volume, whose universal covering is denoted by \tilde{N} . Denote the isometry group of \tilde{N} by $I(\tilde{N})$. Given a *reductive* homomorphism (for the definition of reductivity, see the Definition 1.1 of [10])

$$\rho : \pi_1(M) \rightarrow I(\tilde{N}),$$

one wants to get a ρ -equivariant harmonic map from the universal covering of M to \tilde{N} . In general, difficulties will arise since the homomorphism ρ may map some small loops around D to some *hyperbolic* or *quasi-hyperbolic* elements (for their definitions and the definitions of *elliptic* and *parabolic elements*, see the Definition 1.2 of [10]) in $I(\tilde{N})$. This is why a ρ -equivariant harmonic map, if it exists, may have infinite energy (here, we use the above metric g to compute the energy). It is however fortunate that this difficulty will not arise for the application in this paper. In the following, we will assume that the image of ρ lies in $\pi_1(N)$. (Actually, in the application, ρ is an isomorphism from $\pi_1(M)$ to $\pi_1(N)$.)

As in [10], one needs to consider only two cases: 1) ρ maps every small loop around a (topological) component of D (as an element in $\pi_1(M)$) to a hyperbolic or elliptic element; 2) ρ maps every small loop around a component of D to a parabolic or quasi-hyperbolic element. It is useful to point out that any two loops in each component of D , as elements of the fundamental group, commute with each other, so ρ maps them simultaneously to

either hyperbolic (elliptic) elements or quasihyperbolic (parabolic) elements. Since N is a complete noncompact negatively δ -pinched manifold with finite volume, so the image elements can not be quasihyperbolic (otherwise, N will be of infinite volume) and therefore only parabolic images may occur in the second case. For the parabolic images case, the problem can be handled as done in pages 85-91 of [9], where all steps can be translated to the present situation after knowing the fact that the Jacobi fields in the present case are also exponentially decaying, which is an easy consequence of Riemannian geometry. We now address the first case. Similar to the parabolic images case, the elliptic images case will also not cause any difficulty and it can also be handled as in [9]. Thus, one only needs to handle the case of hyperbolic images. Since N is a negatively δ -pinched manifold ($\delta > \frac{1}{4}$), so the situation is similar to that of locally symmetric spaces of noncompact type of rank one and slightly simpler than in the situation of general symmetric spaces. In this case, one furthermore has the fact that the element of $\pi_1(\Sigma^\mu)$ (here Σ^μ is the boundary of the μ -tube neighborhood of D) determined by any fibre of $\Sigma_i^\mu \rightarrow D_i$ is central and hence independent of the choice of the fibre and the fact that the centralizer of a hyperbolic element in a discrete subgroup of $I(\tilde{N})$ is virtually cyclic. Using these facts, one can then follow the arguments in pages 477-480 of [10] to get a desired ρ -equivariant map and hence a ρ -equivariant harmonic map. Finally, the argument in pages 481-483 of [10] (Lemma 1.1) shows that the harmonic map is pluriharmonic.

Now, we can state the following theorem on the existence of a ρ -equivariant pluriharmonic map, which is essentially due to Jost and Zuo [9, 10], as follows:

Theorem 4 *Let $M, N, I(\tilde{N})$ and ρ as above. Then there exists a ρ -equivariant pluriharmonic map u from the universal covering \tilde{M} of M with the above metric g to \tilde{N} .*

Remark: Actually, Jost-Zuo's theorem in [9] and [10] is more general in some respects. In particular, if N is a locally symmetric space of noncompact type of rank one with finite volume, the theorem still holds, as will be used in the proof of the Theorem 3.

For simplicity of notation, we shall consider the harmonic map u in the sequel as a map from M to $N' = \tilde{N}/\rho(\pi_1(M))$. (In our applications N' is just N). Let $w \in D_i$ be a regular point of D . Near w , one can choose a coordinate system (z^1, z^2) on \tilde{M} such that z^1 parameterizes small holomorphic discs, which meet D_i transversally near w , z^2 parameterizes D_i (of course, z^2 will have more than one component if the complex dimension of M is greater than 2. In the following, the index 2 will stand for all those z^2 -directions together), and $z^1 = 0$ on a small neighborhood of w in D_i and $z^2(w) = 0$.

One then has some derivative estimates for u (see p.481 of [10]):

$$\left| \frac{\partial u}{\partial z^1}(z^1, z^2) \right|_g \leq \frac{c}{|z^1|}, \quad \left| \frac{\partial u}{\partial z^2}(z^1, z^2) \right|_g \leq c,$$

where c is some positive constant. If w is a singular point of D , i.e., a point at which two irreducible components of D meet, similar estimates can be obtained. One may use $\sigma := \prod_{i=1}^p \sigma_i$ to replace the above coordinate component z^1 . Then, one can get that in the σ -direction, the derivative of u behaves like $\frac{1}{|\sigma|}$, whereas in directions normal to σ , it is bounded.

Now, we shall show that the rank of the harmonic map u has a serious restriction. Let M be as before with the constructed Kähler metric g , the corresponding Kähler form of which is denoted by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta=1}^m g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}$$

where $m = \dim_{\mathbb{C}} M$ and (z^1, z^2, \dots, z^m) is a local coordinate system of M . Introduce a local coordinate system (u^1, u^2, \dots, u^n) on N' . As in [18], we introduce a symmetric $(2, 0)$ -tensor ϕ related to the map u

$$\phi(X, Y) = \langle \partial u(X), \partial u(Y) \rangle, \quad X, Y \in T_x^{1,0} M,$$

which can be locally written as $\sum_{\alpha, \beta=1}^m \phi_{\alpha\beta} dz^\alpha \otimes dz^\beta$. Now, we compute its iterated divergence. By the divergence formula, one has a $(1, 0)$ -form ξ

$$\xi_\alpha = g^{\beta\bar{\gamma}} \phi_{\alpha\beta, \bar{\gamma}}$$

where $(g^{\beta\bar{\gamma}})$ represents the inverse of $(g_{\beta\bar{\gamma}})$ and $_{, \bar{\gamma}}$ denotes the covariant derivative. Then, taking the divergence of ξ again, one obtains, by a direct computation [18],

$$\delta \xi = (|D'' \partial u|^2 - g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R_{iklm} u_\alpha^i u_\gamma^k u_{\bar{\beta}}^l u_{\bar{\delta}}^m)$$

where δ is the codifferential, R_{iklm} is the curvature tensor of N' , and $D'' \partial u$ is the $(0, 1)$ -type covariant derivative of ∂u , which is locally written as

$$(D'' \partial u)_{\alpha\bar{\beta}}^i = u_{\alpha\bar{\beta}}^i + \Gamma_{jk}^i u_\alpha^j u_{\bar{\beta}}^k.$$

Here Γ_{jk}^i are the Christoffel symbols of Y . Then, Jost-Zuo's argument (see p.482 of [10]) shows that the two sides of the above formula are zero pointwise. It should be pointed out that the estimates given above of the derivatives of u near the divisor D are essential in this reasoning. In particular, using the curvature conditions of N' , one obtains that

$$D'' \partial u = 0, \quad g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R_{iklm} u_\alpha^i u_\gamma^k u_{\bar{\beta}}^l u_{\bar{\delta}}^m = 0.$$

Note that the above first formula just represents the pluriharmonicity of u . Taking the holomorphic orthogonal frame e_1, e_2, \dots, e_m on M , one has

$$g^{\alpha\bar{\beta}} g^{\gamma\bar{\delta}} R_{iklm} u_\alpha^i u_\gamma^k u_{\bar{\beta}}^l u_{\bar{\delta}}^m = \langle R(\partial u(e_\alpha), \partial u(e_\beta)) \bar{\partial} u(e_{\bar{\alpha}}), \bar{\partial} u(e_{\bar{\beta}}) \rangle .$$

Thus, by means of the previous lemma, there exist two complex constants a, b (at least one $\neq 0$) satisfying $a\partial u(e_\alpha) + b\partial u(e_\beta) = 0$, namely, $du(T^{1,0}M)$ is complex one-dimensional. So, one obtains

Lemma 2 *Let $u : M \rightarrow N'$ be the pluriharmonic map as in the Theorem 4. Then u has real rank at most 2.*

As a consequence of the above arguments, one has

Theorem 5 *Let M, N be as before, and let $u : M \rightarrow N$ be a smooth map. Then u is harmonic if and only if D'' is the $\bar{\partial}$ -operator of a holomorphic structure on $u^*T^{\mathbb{C}}N$ and ∂u is a holomorphic section of the bundle $\text{Hom}(T^{1,0}M, u^*T^{\mathbb{C}}N)$.*

Proof: It is easy to see that D'' is the $\bar{\partial}$ -operator of a holomorphic structure on $u^*T^{\mathbb{C}}N$ if and only if it satisfies the integrability condition $(D'')^2 = 0$. An easy computation shows

$$(D'')^2(X, Y) = R(du(X), du(Y)),$$

where R is the complex-multilinear extension of the curvature tensor of N and $X, Y \in T^{0,1}M$. Since $du(T^{1,0}M)$ is complex one-dimensional, $(D'')^2(X, Y) = 0$.

3 Factorization of harmonic maps and proofs of theorems

From the argument of the previous section, we know that if f is a harmonic map from M (a quasi-compact Kähler manifold with an appropriate complete Kähler metric constructed as in the previous section) to N (a complete non-compact negatively δ -pinched manifold with finite volume) from the Theorem 4, then it has real rank at most 2. From now on, we assume that $\pi_1(N)$ is isomorphic to $\pi_1(M)$. We want to derive a contradiction, hence the Theorem 1 is proved.

Let f be a harmonic map from M to N from the Theorem 4 which induces an isomorphism from $\pi_1(M)$ to $\pi_1(N)$. From the argument of the previous section, we know that f has real rank at most 2. Obviously, its real rank cannot be zero; if it has real rank one, by a result of J. H. Sampson [19], f maps

M to a closed geodesic in N . So, $\pi_1(N)$ is isomorphic to the ring of integers \mathbb{Z} . Because N is noncompact but of finite volume, its fundamental group must contain some parabolic element. So, each element in $\pi_1(N)$ is parabolic. This is impossible, since N has finite volume. So, we can assume that f has real rank 2 at some point. Actually, it has real rank 2 generically, since it is pluriharmonic (more precisely, $f^*TN \otimes \mathbb{C}$ is a holomorphic bundle under the $(0, 1)$ -part of the induced connection and ∂f is a holomorphic section of the bundle $\text{Hom}(T^{1,0}M, f^*TN \otimes \mathbb{C})$).

In the following, we will show that f gives rise to a foliation on M . Note that the argument of Theorem 7.1 in [4] does not work in the noncompact case. We will make use of a similar (but more general) argument of [8]. Assume f has real rank 2 at the point z_0 , so f has real rank 2 in a neighborhood, say, U . Take a holomorphic coordinate system (z^1, z^2, \dots, z^m) around z_0 (w.l.o.g., one can assume the system covers U). By the previous section's result, $df(T^{1,0}M)$ is complex one-dimensional. Without loss of generality, one can assume $df(\frac{\partial}{\partial z^1}) \neq 0$ and denote it by X . Set $\partial f(\frac{\partial}{\partial z^i}) = q^i X, i = 2, 3, \dots, m$, where $\{q^i\}$ are complex valued functions defined on U . We will show that they are actually holomorphic. Since f has real rank 2, $\bar{X} \neq X$ and $\bar{X} \neq -X$. So, for $i = 2, 3, \dots, m, j = 1, 2, \dots, n$, one has, using the Kählerianity of M and the pluriharmonicity of f

$$\begin{aligned} 0 &= D'' df\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) \\ &= D''_{\frac{\partial}{\partial \bar{z}^j}} df\left(\frac{\partial}{\partial z^i}\right) \\ &= D''_{\frac{\partial}{\partial \bar{z}^j}} \left(df\left(\frac{\partial}{\partial z^i}\right)\right) \\ &= D''_{\frac{\partial}{\partial \bar{z}^j}} (q^i X) \\ &= \frac{\partial q^i}{\partial \bar{z}^j} X + q^i D''_{\frac{\partial}{\partial \bar{z}^j}} \left(df\left(\frac{\partial}{\partial z^1}\right)\right) \\ &= \frac{\partial q^i}{\partial \bar{z}^j} X. \end{aligned}$$

So, $q^i, i = 2, 3, \dots, m$ are holomorphic. Consider the holomorphic distribution on U

$$\left\{ \frac{\partial}{\partial z^2} - q^2 \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^3} - q^3 \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^m} - q^m \frac{\partial}{\partial z^1} \right\}.$$

Obviously, it is the holomorphic kernel of the differential df . Moreover, by the complex version of a standard fact (See, [12], Proposition 1.4.10), $df\left(\left[\frac{\partial}{\partial z^i} - q^i \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^j} - q^j \frac{\partial}{\partial z^1}\right]\right) = 0, i, j = 2, 3, \dots, m, i.e., \left[\frac{\partial}{\partial z^i} - q^i \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^j} - q^j \frac{\partial}{\partial z^1}\right]$ lie in this distribution. So, it is a holomorphic integrable distribution.

Then, the complex version of Frobenius theorem asserts that on a neighborhood of z_0 (assume it still is U), there is a holomorphic coordinate system (w^1, w^2, \dots, w^m) satisfying

$$\left\{ \frac{\partial}{\partial w^2}, \frac{\partial}{\partial w^3}, \dots, \frac{\partial}{\partial w^m} \right\} = \left\{ \frac{\partial}{\partial z^2} - q^2 \frac{\partial}{\partial z^1}, \frac{\partial}{\partial z^3} - q^3 \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^m} - q^m \frac{\partial}{\partial z^1} \right\}.$$

That is to say, $df(\frac{\partial}{\partial w^i}) = 0, i = 2, 3, \dots, m$, but $df(\frac{\partial}{\partial w^1}) \neq 0$. So, when restricted to the hypersurfaces $w^1 = \text{const}$, f is constant. Therefore, one obtains a well-defined foliation \mathcal{F} on a Zariski open set of M , namely, on the set where f has real rank 2. Arguments of Mok (see Proposition (2.2.1) of [15]) imply that \mathcal{F} can be extended as a holomorphic foliation to $M \setminus V$ for some complex analytic variety V of complex codimension at least 2. Then, the study of [16] (see Proposition (2.2) of [16]) shows that the extended foliation actually defines an open analytic equivalence relation, still denoted by \mathcal{F} , on M , and the quotient of M by \mathcal{F} , denoted by S , is an irreducible complex space of complex dimension 1, by a result of Kaup [11]. Therefore, one has a factorization of the harmonic map $f: M \rightarrow N$: $f = h \circ \pi$, where $\pi: M \rightarrow S$ is holomorphic because of the construction of S and $h: S \rightarrow N$ is harmonic since f is pluriharmonic.

Since $f_*: \pi_1(M) \rightarrow \pi_1(N)$ is an isomorphism, $\pi_*: \pi_1(M) \rightarrow \pi_1(S)$ is injective. Therefore, $\pi_1(N)$, as a subgroup of $\pi_1(S)$, acts freely on the universal covering of S , which is contractible as a topological space. Thus, the cohomological dimension of $\pi_1(N)$ [1] is at most 2. But, since N is a negatively δ -pinched manifold with finite volume, it can topologically be regarded as the interior of a manifold with boundary, here the boundary is the disjoint union of tori up to a finite group. So, by means of a result of [1] (p. 211, Corollary 8.3), $\pi_1(N)$ has cohomological dimension $n - 1$, here n is the real dimension of N . So, if $n \geq 4$, we derive a contradiction. In the following, we will treat the case $n = 3$ separately. The idea of the proof was told to us by Professor M. S. Raghunathan.

We now assume that N has dimension 3 and $\pi_1(N)$ is isomorphic to $\pi_1(M)$. So, by means of the previous argument, one has a holomorphic map h from M to an irreducible complex space S of complex dimension 1, which induces an injective map $h_*: \pi_1(M) \rightarrow \pi_1(S)$. Now, we have two cases to discuss: 1. S is noncompact; 2. S is compact.

Case 1: If S is noncompact, a standard argument shows that $\pi_1(S)$ is free, so its cohomological dimension is 1, consequently, the cohomological dimension of its subgroup is also 1. But the cohomological dimension of $\pi_1(N)$ is 2. So only the case 2 may occur.

Case 2: In this case, we also have two cases to discuss: i) the image of h_* has infinite index in $\pi_1(S)$; ii) the image of h_* has finite index in $\pi_1(S)$. If

the case i) is true, one can lift h so that the case can be actually reduced to the case 1). So, that case is impossible; if the case ii) is true, by a finite lifting, one can also assume that h_* is an isomorphism from $\pi_1(M)$ to $\pi_1(S)$. We will also derive a contradiction. Since $\pi_1(M)$ is isomorphic to $\pi_1(N)$ by the assumption, so $\pi_1(N)$ is isomorphic to $\pi_1(S)$. Since N is of negatively δ -pinched curvature and finite volume, by means of Corollary 1.5.2 in [2], one can consider N as the interior of a compact manifold with boundary, here the boundary is the disjoint union of tori up to a finite group action. Then an easy exercise shows that the Euler-Poincaré characteristic $\chi(\pi_1(N))$ of $\pi_1(N)$ is zero by using the homological exact sequence of a space pair $(X, \partial X)$ for a compact manifold X and its boundary ∂X and Poincaré duality theorem for manifolds with boundary (see [14], p. 227).

We now show that the Euler-Poincaré characteristic of $\pi_1(S)$ is not zero and hence a contradiction. Since S is a compact irreducible complex space of complex dimension 1, so, by passing to normalizations, without loss of generality we can assume that S is normal and hence smooth and the above factorization $f = h \circ \pi$ remains valid. Clearly, S can't be the sphere since $\pi_1(N)$ acts freely on the universal covering of S . If S is of genus $g \geq 2$, then the Euler-Poincaré characteristic of $\pi_1(S)$ is $2 - 2g < 0$. This is a contradiction. So, if the assumption is true, S must be a torus. Thus $\pi_1(N) = \mathbb{Z} + \mathbb{Z}$. In the following, using harmonic map theory, we will show that $\pi_1(N)$ cannot be $\mathbb{Z} + \mathbb{Z}$. Assume $\pi_1(N) = \mathbb{Z} + \mathbb{Z}$. Take the standard torus T with a flat metric, so one can get a harmonic map by the well-known theorem of Eells-Sampson for the existence of harmonic maps from this torus to N , which induces an isomorphism from $\pi_1(T)$ to $\pi_1(N)$. Then a standard argument shows that this harmonic map has constant energy density by using the Bochner technique for harmonic maps. Furthermore, it is a totally geodesic map and of real rank 1. Again, by means of a result of Sampson [19], this harmonic map maps T to a geodesic in N . This is a contradiction. This completes the proof of the theorem 1.

Proof of Theorem 3: Let Γ be a nonuniform lattice in $F_{4(-20)}$. (Without loss of generality, one can assume that Γ is torsion-free.) Assume it is the fundamental group of a quasicompact Kähler manifold M . By means of the Theorem 4, one has a harmonic map $f : M \rightarrow H_{\mathbb{O}}^2/\Gamma$, which induces an isomorphism from $\pi_1(M)$ to Γ . Here $H_{\mathbb{O}}^2$ is the Cayley hyperbolic plane. Then a standard argument [18, 21] shows that $df(T_p^{1,0}M)$ ($p \in M$) can be regarded as an Abelian subspace of the complexification $\mathfrak{p}^{\mathbb{C}}$ of the tangent space of $H_{\mathbb{O}}^2$. This tangent space can be identified with the second factor of the Cartan decomposition $\mathfrak{g} = \mathfrak{t} + \mathfrak{p}$, here \mathfrak{g} is the Lie algebra of $F_{4(-20)}$ and \mathfrak{t} is the Lie algebra of the maximal compact subgroup in $F_{4(-20)}$. By the Lie-theoretic analysis in [3], one knows that the complex dimension of an Abelian

subspace in $\mathfrak{p}^{\mathbb{C}}$ is at most 2. So, we have three cases to discuss for $df(T_p^{1,0}M)$.

(a): $\dim_{\mathbb{C}}df(T_p^{1,0}M) = 1$, but $df(T_p^{1,0}M)$ has real points. So the real rank of f is 1. By a result of J. H. Sampson [19], f maps M to a geodesic in $H_{\mathbb{O}}^2/\Gamma$. So, Γ is isomorphic to the ring of integers \mathbb{Z} . This is impossible;

(b): $\dim_{\mathbb{C}}df(T_p^{1,0}M) = 1$, but $df(T_p^{1,0}M)$ does not have real points. So the real rank of f is 2. Similar to the previous proof, we have a decomposition for f : $f = g \circ h$, here h is a holomorphic map from M to an irreducible complex space S of complex dimension 1 and g is a harmonic map. (Note that the discussion of Theorem 7.1 in [4] does not work any more in the present case.) So, the cohomological dimension of π_1 , and hence Γ is at most 2. But, the cohomological dimension of Γ is actually 15 (see [1]). This is a contradiction;

(c): $\dim_{\mathbb{C}}df(T_p^{1,0}M) = 2$. Completely similar to the discussion in [3], one also has a decomposition for f : $f = g \circ h$, here h is a holomorphic map from M to a quotient of the two-ball of finite volume and g is a geodesic immersion. Then, the same cohomological dimension arguments show this is also impossible. This completes the proof of the theorem.

4 Examples of negatively δ -pinched manifolds which are not hyperbolic

In this section, we will give some examples of negatively δ -pinched manifolds of finite volume which admit no hyperbolic metric with finite volume under any smooth structure. The constructing method is from Gromov and Thurston's paper [6]. Actually, Gromov and Thurston constructed some examples of compact manifolds which admit some δ -pinched metric, but no hyperbolic metric under any smooth structure. We only give a sketch here. For detailed constructions, the reader can refer to [6].

Consider a non-singular quadratic form Φ_3 in 4 variables x_2, x_3, x_4, x_5 with coefficients in \mathbb{Q} and real type $(1, 1, 1, -1)$. Let $\Gamma(\Phi_3)$ be the group of automorphisms of the form Φ_3 over the ring of integers \mathbb{Z} . It is well-known that $\Gamma(\Phi_3)$ may be both cocompact and non-cocompact. We assume that $\Gamma(\Phi_3)$ is cocompact. Set the quadratic forms $\Phi_4 = (x_1)^2 + \Phi_3$ and $\Phi_5 = (x_0)^2 + \Phi_4$. Then one knows that $\Gamma(\Phi_4)$ and $\Gamma(\Phi_5)$ are non-cocompact. $\Gamma(\Phi_3)$ ($\Gamma(\Phi_4)$ respectively) can be considered as a subgroup of $\Gamma(\Phi_4)$ ($\Gamma(\Phi_5)$ respectively) and $H^3/(\Gamma(\Phi_3))$ is a compact totally geodesic hypersurface of $H^4/(\Gamma(\Phi_4))$, while $H^4/(\Gamma(\Phi_4))$ is a noncompact totally geodesic hypersurface of $H^5/(\Gamma(\Phi_5))$. Then, Gromov-Thurston's argument (Lemma 1.2 of [6]) can be applied to the space pairs $(H^5/(\Gamma(\Phi_5)), D)$ and $(H^4/(\Gamma(\Phi_4)), H^3/(\Gamma(\Phi_3)))$, here D is a compact subset of $H^4/(\Gamma(\Phi_4))$ containing $H^3/(\Gamma(\Phi_3))$. All these together give the following

Theorem 6 *For every ρ , there are an orientable 5-dimensional complete manifold V of constant curvature -1 and finite volume and a compact totally geodesic orientable submanifold V' of codimension 2 in V such that the normal injectivity radius of V' in V is greater than ρ ; the corresponding homological class of V' in V is trivial.*

Using the above constructed manifolds V, V' , one can get a Z_i -ramified covering \tilde{V}_i of V at V' for any positive integer i . Then, on all these ramified coverings \tilde{V}_i with the induced smooth structure, one can construct some complete negative curvature metrics \tilde{g}_i whose curvature at infinity is constant -1 and of finite volume since V' is compact. Now, consider all \tilde{V}_i as topological manifolds and suppose that there were complete metrics \tilde{g}^i of constant curvature $K = -1$ and finite volume on each \tilde{V}_i under some smooth structure of \tilde{V}_i (not necessarily the above induced smooth structure). The action of the cyclic group Z_i on \tilde{V}_i can be considered as a deck transformation action. Because the dimension of \tilde{V}_i is 5, by means of the Mostow-Prasad rigidity theorem due to G. Prasad [17] (for nonuniform lattices in real hyperbolic spaces), there exists an isometric action of Z_i on \tilde{V}_i whose fixed point set V'' is homeomorphic to V' and whose quotient $\bar{V}_i = \tilde{V}_i/Z_i$ has a natural orbifold structure with constant curvature $K = -1$ and finite volume. Also note that no two orbifolds \bar{V}_i are isometric. On the other hand, by means of the volume estimate argument in [6], the volumes of \bar{V}_i have a uniform bound independent of i . So, Wang's finiteness theorem for locally symmetric orbifolds (see [20], Theorem 8.1), which asserts that there are at most finitely many isometric classes of $n(\geq 4)$ -dimensional complete orbifolds \bar{V} with $K(\bar{V}) = -1$ and $Vol(\bar{V}) \leq$ a fixed constant, implies a contradiction. Namely, there exists a positive integer i_0 , such that for $i \geq i_0$, \tilde{V}_i does not admit a complete metric of constant curvature -1 and finite volume under any smooth structure. Finally, using the arguments of §3 in [6], one can show that there exist some \tilde{V}_i ($i \geq i_0$) which admit no complete metric of constant curvature $K = -1$ and finite volume under any smooth structure, but carry some complete metrics with curvature $-1 \leq K \leq -1 - \epsilon$ and finite volume under the induced smooth structure on \tilde{V}_i from the ramified covering of V at V' for arbitrary small positive ϵ . Thus, combining Gromov and Thurston's examples with ours, we actually obtain the following

Theorem 7 *There exist some complete 5-dimensional topological manifolds which admit no complete metric of constant curvature -1 and finite volume under any smooth structure, but carry some complete metrics with curvature $-1 \leq K < -\frac{1}{4}$ and with finite volume under some smooth structure. In other words, there exist some groups which can be π_1 of some complete (open or closed) 5-dimensionally negatively $\delta(> \frac{1}{4})$ -pinched manifolds of finite volume, but not a (uniform or nonuniform) lattice of $SO(5, 1)$.*

Remarks: A simple homotopical (or cohomological dimension) argument shows that the groups in the above theorem also cannot be (uniform or nonuniform) lattices of $SO(m, 1)$ for any $m \neq 5$; in addition, these examples also show that Theorem 1 is a nontrivial generalization of the theorem in [21]. A natural problem is how to characterize these groups in algebraic terms.

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