

Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

The cohomology of a variation of
polarized Hodge structures over a
quasi-compact Kähler manifold

by

Jürgen Jost, Yi-Hu Yang, and Kang Zuo

Preprint no.: 105

2003



The cohomology of a variation of polarized Hodge structures over a quasi-compact Kähler manifold

J. Jost, Y.-H. Yang*[†] and K. Zuo[†]

1 Introduction

Let (\overline{M}, ω_0) be a compact Kähler manifold of dimension n and D a divisor of \overline{M} with at most normal crossing singularities. Let $D = \cup_{i=1}^l D_i$, where the D_i are smooth divisors of \overline{M} . Denote $\overline{M} \setminus D$ by M , called a quasi-compact Kähler manifold. $j : M \rightarrow \overline{M}$ is the inclusion mapping. According to [3], one can construct a complete Kähler metric ω on M which is a Poincaré-like metric near the divisor and of finite volume.

Let $\{M, \mathbf{H}_{\mathbb{C}} = \mathbf{H}_{\mathbb{Z}} \otimes \mathbb{C}, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{F} = \{\mathbf{F}^p\}_{p=1}^m, \mathbf{S}\}$ be a rational variation of polarized Hodge structures with weight m over M such that each \mathbf{F}^p is a holomorphic subbundle of the local system $\mathbf{H}_{\mathbb{C}}$ and $\nabla^{1,0}$ satisfies the Griffiths' infinitesimal period relation

$$\nabla^{1,0}\mathbf{F}^p \subset \Omega_M^1 \otimes \mathbf{F}^{p-1}.$$

For simplicity, we always assume that all monodromies at the divisor are *unipotent*. Note that by a lemma of Borel (cf. [12], Lemma 4.5), this is not an essential assumption. The polarization \mathbf{S} defines a Hermitian metric

$$h(\cdot, \cdot) = \mathbf{S}(\mathcal{C}\cdot, \bar{\cdot})$$

on $\mathbf{H}_{\mathbb{C}}$ by using the Weil operator \mathcal{C} , called *Hodge metric*, where the bar is the conjugation relative to $\mathbf{H}_{\mathbb{Z}}$. Take the successive quotients $\mathbf{F}^p/\mathbf{F}^{p-1}$, called *Hodge bundles*, denoted by \mathbf{E}^p . Accordingly, we have the induced \mathcal{O} -linear

*The author supported partially by NSF of China (No.10171077) and a direct grant (Title: Cohomology theory with coefficients in local systems and applications to algebraic geometry) from the Chinese University of Hong Kong by the third named author;

[†]The author supported partially by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. CUHK 4034/02P).

map $\theta^p : \mathbf{E}^p \rightarrow \mathbf{E}^{p-1}$ of $\nabla^{1,0}$. Set $\mathbf{E} = \bigoplus \mathbf{E}^p$ and $\theta = \bigoplus \theta^p$. Since $(\nabla^{1,0})^2 = 0$, $\theta \wedge \theta = 0$. So (\mathbf{E}, θ) is a Higgs bundle.

Using the asymptotic behavior of θ (cf. §3), one has the following holomorphic Dolbeault complex of sheaves on \overline{M}

$$\mathbf{E} \xrightarrow{\theta} \mathbf{E} \otimes \Omega_{\overline{M}}^1(\log D) \xrightarrow{\theta} \mathbf{E} \otimes \Omega_{\overline{M}}^2(\log D) \xrightarrow{\theta} \cdots .$$

Furthermore, using the Poincaré-like metric ω , the Hodge metric h induced on \mathbf{E} , and the boundedness of θ (cf. §3), we can define an L^2 -subcomplex on \overline{M}

$$(*) \quad \mathbf{E}_{(2)} \xrightarrow{\theta} (\mathbf{E} \otimes \Omega_{\overline{M}}^1(\log D))_{(2)} \xrightarrow{\theta} (\mathbf{E} \otimes \Omega_{\overline{M}}^2(\log D))_{(2)} \xrightarrow{\theta} \cdots .$$

of the above complex by taking the sheaves of germs of local L^2 sections. Although the definition of the above L^2 -subcomplex depends on the metric on M and the Hodge metric on \mathbf{E} , we will however see that it is essentially independent of both metrics and is obtained by taking local sections satisfying a certain algebraic condition determined by the monodromies of the variation from the arguments in §4. One can then consider the hypercohomology of the L^2 -subcomplex

$$\mathbb{H}^*(\overline{M}, \{(\mathbf{E} \otimes \Omega_{\overline{M}}(\log D))_{(2)}, \theta\}),$$

which is independent of both metrics and depends only on the monodromies of the variation.

On the other hand, by means of a construction of Deligne, we can also define a complex of fine sheaves as follows (for details, cf. §5). Define $[\mathrm{Gr}_F^* A^k(\mathbf{H}_{\mathbb{C}})]_{(2)}$ to be the sheaf obtained by taking local L^2 measurable k -forms on \overline{M} with values in \mathbf{E} , the $\bar{\partial}$ derivatives of which exist in the weak sense and are L^2 ; and put $D'' = \bar{\partial} + \theta$. Then $D''([\mathrm{Gr}_F^* A^k(\mathbf{H}_{\mathbb{C}})]_{(2)}) \subset [\mathrm{Gr}_F^* A^{k+1}(\mathbf{H}_{\mathbb{C}})]_{(2)}$ and $(D'')^2 = 0$ due to ∇ being flat. Thus we have the complex of fine sheaves on \overline{M}

$$(**) \quad [\mathrm{Gr}_F^* A^0(\mathbf{H}_{\mathbb{C}})]_{(2)} \xrightarrow{D''} [\mathrm{Gr}_F^* A^1(\mathbf{H}_{\mathbb{C}})]_{(2)} \xrightarrow{D''} [\mathrm{Gr}_F^* A^2(\mathbf{H}_{\mathbb{C}})]_{(2)} \xrightarrow{D''} \cdots .$$

It is not difficult to see that the complex $(*)$ is actually a subcomplex of $(**)$. Similarly, we can consider the hypercohomology of the above complex of fine sheaves, which, by a standard result, is isomorphic to the cohomology of the corresponding complex of global sections of the sheaves, i.e.

$$H^*(\{\Gamma(\overline{M}), [\mathrm{Gr}_F^* A(\mathbf{H}_{\mathbb{C}})]_{(2)}, D''\}).$$

One of the purpose of this paper is then to show

Theorem A. *The complexes $(*)$ and $(**)$ are quasi-isomorphic and hence have the same hypercohomology, i.e.*

$$\mathbb{H}^*(\overline{M}, \{(\mathbf{E} \otimes \Omega_{\overline{M}}(\log D))_{(2)}, \theta\}) \simeq H^*(\{\Gamma(\overline{M}), [\mathrm{Gr}_F^* A(\mathbf{H}_{\mathbb{C}})]_{(2)}, D''\}).$$

Remark: When M is compact, Theorem A is a formal consequence of some standard homological algebra and the classical $\bar{\partial}$ -Poincaré lemma and was stated by Deligne, but an important point is that he used the Griffiths' infinitesimal period relation to construct the complex $(**)$ (cf. [16], §1). If M is quasi-compact Kählerian, the proof of the theorem becomes highly complicated and was obtained by S. Zucker for the case of curves (cf. [16], Theorem 6.3).

Let $A^k(\mathbf{H}_{\mathbb{C}})_{(2)}$ be the sheaf of germs of local L^2 measurable $\mathbf{H}_{\mathbb{C}}$ -valued k -forms ϕ on \bar{M} for which $D\phi$ exists in the weak sense as a local L^2 form. Here, D is defined as follows: Let ϕ be a smooth k -form on \bar{M} and v a smooth section of $\mathbf{H}_{\mathbb{C}}$, then $D(\phi \otimes v) = d\phi \otimes v + (-1)^k \phi \wedge \nabla v$. It is clear that $D^2 = 0$. $A^k(\mathbf{H}_{\mathbb{C}})_{(2)}$ is a fine sheaf and we then obtain a complex $\{A^k(\mathbf{H}_{\mathbb{C}})_{(2)}, D\}$ of fine sheaves again. The corresponding complex of global sections then computes its hypercohomology, and also the L^2 -de Rham cohomology with values in $\mathbf{H}_{\mathbb{C}}$ on \bar{M} : $H^*_{(2)}(\bar{M}, \mathbf{H}_{\mathbb{C}})$, i.e.

$$H^*_{(2)}(\bar{M}, \mathbf{H}_{\mathbb{C}}) = \mathbb{H}^*(\bar{M}, \{A^k(\mathbf{H}_{\mathbb{C}})_{(2)}, D\}) = H^*(\{\Gamma(\bar{M}, A^k(\mathbf{H}_{\mathbb{C}})_{(2)}), D\}).$$

The standard discussion with the Kähler identity and the harmonic forms of the Laplacians of D'' and D for the situation of variations of Hodge structures (refer to [15], §7) tells us that

$$H^*(\{\Gamma(\bar{M}, [Gr_F^* A^k(\mathbf{H}_{\mathbb{C}})]_{(2)}), D''\}) \simeq H^*(\{\Gamma(\bar{M}, A^k(\mathbf{H}_{\mathbb{C}})_{(2)}), D\})$$

and hence $\simeq H^*_{(2)}(\bar{M}, \mathbf{H}_{\mathbb{C}})$. On the other hand, Cattani-Kaplan-Schmid's theorem [6] tells us that $H^*_{(2)}(\bar{M}, \mathbf{H}_{\mathbb{C}})$ is isomorphic to the intersection cohomology $H^*_{int}(\bar{M}, \mathbf{H}_{\mathbb{C}})$. Therefore, we have

Theorem B. *There exists a natural isomorphism*

$$\mathbb{H}^*(\bar{M}, \{(\mathbf{E} \otimes \Omega_{\bar{M}}(\log D))_{(2)}, \theta\}) \simeq H^*_{int}(\bar{M}, \mathbf{H}_{\mathbb{C}}).$$

We now give a simple application of the above theorem. More applications will be presented in a future paper. Let still $\{M, \mathbf{H}_{\mathbb{C}} = \mathbf{H}_{\mathbb{Z}} \otimes \mathbb{C}, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{F} = \{\mathbf{F}^p\}_{p=1}^m, \mathbf{S}\}$ be a rational variation of polarized Hodge structures with weight m over M , (\mathbf{E}, θ) the corresponding Higgs bundle. Consider all endomorphisms from $\mathbf{H}_{\mathbb{C}}$ to itself, denoted by $\mathcal{E}nd(\mathbf{H}_{\mathbb{C}})$, which obviously has an induced structure of variation of Hodge structures and to which all results in §2–5 can be applied; denote by $(\mathcal{E}nd(\mathbf{E}), \theta^{\mathcal{E}nd})$ the Higgs bundle corresponding to $\mathcal{E}nd(\mathbf{H}_{\mathbb{C}})$. By the properties of the Higgs field θ (cf. §3), it can also be considered as a morphism of sheaves

$$\theta : \mathbf{T}_{\bar{M}}(-\log D) \rightarrow \mathcal{E}nd(\mathbf{E}),$$

where $\mathbf{T}_{\overline{M}}$ is the holomorphic tangent bundle of \overline{M} . By an observation due to K. Zuo [17], one has

Proposition 1 (Zuo) $\theta^{\mathcal{E}nd}(\theta(\mathbf{T}_{\overline{M}}(-\log D))) = 0$, i.e. the above morphism of sheaves is a morphism of Higgs sheaves

$$\theta : (\mathbf{T}_{\overline{M}}(-\log D), 0) \rightarrow ((\mathcal{E}nd(\mathbf{E}), \theta^{\mathcal{E}nd})).$$

Here we consider $\mathbf{T}_{\overline{M}}(-\log D)$ as a Higgs bundle with 0 as its Higgs field.

Therefore, θ induces a morphism between the hypercohomologies,

$$\begin{aligned} \theta : \quad \mathbb{H}^*(\mathbf{T}_{\overline{M}}(-\log D) \xrightarrow{0} \mathbf{T}_{\overline{M}}(-\log D) \otimes \Omega_{\overline{M}}^1(\log D) \xrightarrow{0} \cdots) \\ \rightarrow \mathbb{H}^*(\mathcal{E}nd(\mathbf{E}) \xrightarrow{\theta} \mathcal{E}nd(\mathbf{E}) \otimes \Omega_{\overline{M}}^1(\log D) \xrightarrow{\theta} \cdots). \end{aligned}$$

It is worth noting that the hypercohomology on the left-hand side is just the Čech cohomology $H^*(\overline{M}, \mathbf{T}_{\overline{M}}(-\log D))$. Thus, θ maps $H^*(\overline{M}, \mathbf{T}_{\overline{M}}(-\log D))$ into $\mathbb{H}^*(\mathcal{E}nd(\mathbf{E}) \xrightarrow{\theta} \mathcal{E}nd(\mathbf{E}) \otimes \Omega_{\overline{M}}^1(\log D) \xrightarrow{\theta} \cdots)$. Actually, by the properties of θ (cf. §3), one has a stronger restriction on the image.

Theorem C. The image of θ lies in the hypercohomology

$$\mathbb{H}^*((\mathcal{E}nd(\mathbf{E}))_{(2)} \xrightarrow{\theta} (\mathcal{E}nd(\mathbf{E}) \otimes \Omega_{\overline{M}}^1(\log D))_{(2)} \xrightarrow{\theta} \cdots).$$

Applying Theorem B to the variation of Hodge structure $\mathcal{E}nd(\mathbf{H}_{\mathbb{C}})$, one has

Theorem D. There is a natural map, still denoted by θ ,

$$\theta : H^*(\overline{M}, \mathbf{T}_{\overline{M}}(-\log D)) \rightarrow H_{int}^*(\overline{M}, \mathcal{E}nd(\mathbf{H}_{\mathbb{C}})).$$

Roughly speaking, the above theorem shows that one can transform certain geometric invariants on \overline{M} into topological invariants. We will give much more details and some further applications in a future paper.

The structure of this paper is as follows. In §2 we will review some results about variations of polarized Hodge structures, most of which are due to Schmid [12] and Cattani-Kaplan-Schmid [5] and essential for the development of §3 and §4. §3 is very technical. We first prove an L^2 -adapted theorem for the Hodge filtration of the variation, which, roughly speaking, shows that the meaning of L^2 on \mathbf{F}^p and that on \mathbf{E}^p are the same in some sense; in the process of proving this, we construct an L^2 -adapted basis for \mathbf{F}^p (and hence \mathbf{E}^p), with the help of which we then prove the L^2 -boundedness of θ ; finally, we show the relation of L^2 -adapted bases under different orderings of coordinates, which is used in the proof of the L^2 $\overline{\partial}$ -Poincaré lemma. All the proofs essentially depend on the nilpotent orbit theorem and the SL_2 -orbit theorem. It is worth

pointing out that we always use a 2-dimensional model to discuss details; the general case is more complicated but similar. In §4, we will define the L^2 holomorphic Dolbeault complex of sheaves on \overline{M} and prove that the complex is actually independent of the Hodge metric and the Poincaré-like metric ω of the base manifold and essentially determined by the monodromies of the variation. The proof heavily depends on the estimates of the Hodge metric near the divisor. §5 is devoted to the proof of quasi-isomorphism. In §4, 5, we also use the 2-dimensional model to discuss details.

Acknowledgements: This work began when the second named author was visiting the Institute of Mathematical Sciences and the Department of Mathematics at the Chinese University of Hong Kong in the Winter of 2001; a part (§5) was completed when he visited the Max-Planck-Institute for Mathematics in the Sciences during September 2002–March 2003. He thanks all the above Institutes for hospitality and good working environment.

2 Variation of Hodge structures and the estimate for the Hodge norm

In this section, we will fix some notations and review some results, due to Schmid[12] and Cattani-Kaplan-Schmid[5], which we will use later. The reader can refer to [12, 5] for all these.

Let $\{H^{p,q}\}$ be a Hodge structure with weight k on $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and S a polarization of the Hodge structure. (Call the complex dimension of $H^{p,q}$ the *Hodge number*, denoted by $h^{p,q}$.) Namely, $H_{\mathbb{C}} = \sum_{p+q=k} H^{p,q}$, $H^{p,q} = \overline{H}^{q,p}$ (so $h^{p,q} = h^{q,p}$) and S is symmetric for even k , skew for odd k , and satisfies

$$\begin{aligned} S(H^{p,q}, H^{r,s}) &= 0 \text{ unless } p = s, q = r, \\ S(\mathcal{C}v, \bar{v}) &> 0 \text{ if } v \in H^{p,q}, v \neq 0, \end{aligned}$$

where \mathcal{C} is the Weil operator defined by

$$\mathcal{C}v = \sqrt{-1}^{p-q} v \text{ for } v \in H^{p,q}.$$

To each Hodge structure $H^{p,q}$ on $H_{\mathbb{C}}$ of weight k one can assign the *Hodge filtration*

$$0 \subset \cdots \subset F^{p+1} \subset F^p \subset F^{p-1} \subset \cdots \subset F^0 = H_{\mathbb{C}},$$

with the property

$$H_{\mathbb{C}} = F^p \oplus \overline{F}^{k-p+1}, \text{ for each } p,$$

by setting

$$F^p = \bigoplus_{i \geq p} H^{i, k-i}.$$

Conversely, every decreasing filtration with the above property (Hodge filtration) determines a Hodge structure $\{H^{p,q}\}$ of weight k , by setting

$$H^{p,q} = F^p \cap \overline{F}^q \quad \text{with } p + q = k.$$

Therefore, weighted Hodge structures and Hodge filtrations correspond to each other bijectively. In terms of the Hodge filtration, the conditions on the polarization can be reformulated as

$$\begin{aligned} S(F^p, F^{k-p+1}) &= 0 \quad \text{for all } p, \\ S(\mathcal{C}v, \overline{v}) &> 0 \quad \text{if } v \in H_{\mathbb{C}}, v \neq 0. \end{aligned}$$

Using the polarization S , one can define a positive Hermitian form on $H_{\mathbb{C}}$ by $\langle \cdot, \cdot \rangle = S(\mathcal{C}\cdot, \cdot)$, where \cdot is the conjugation with respect to the real structure $H_{\mathbb{R}}$. Call the polarization S *defined over* \mathbb{Q} if $H_{\mathbb{C}}$ has a lattice structure $H_{\mathbb{Z}}$, which induces the real structure $H_{\mathbb{R}}$, and $S : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \rightarrow \mathbb{Q}$.

Now, assume that $H_{\mathbb{C}}$ is a fixed complex vector space with a lattice structure $H_{\mathbb{Z}}$, which gives rise to the real structure $H_{\mathbb{R}}$, k is a fixed positive integer, and the polarization S on $H_{\mathbb{C}}$ is a nondegenerate bilinear form of $H_{\mathbb{C}}$ defined over \mathbb{Q} relative to the lattice structure $H_{\mathbb{Z}}$, which is symmetric for even k and skew for odd k . Consider the set of all Hodge structures $\{H^{p,q}\}$ relative to the real structure $H_{\mathbb{R}}$ with weight k , fixed Hodge numbers $\{h^{p,q}\}$, and polarized by S , denoted by \mathbf{D} , and the set of all decreasing filtrations $\{F^p\}_{p=0}^k$ of $H_{\mathbb{C}}$ with $\dim_{\mathbb{C}} F^p = \sum_{i \geq k} h^{i, k-i}$ and satisfying

$$S(F^p, F^{k-p+1}) = 0 \quad \text{for all } p,$$

denoted by $\tilde{\mathbf{D}}$. As seen in Schmid's paper[12], \mathbf{D} and $\tilde{\mathbf{D}}$ are complex homogeneous spaces: Set

$$G_{\mathbb{C}} = \{g \in \text{Gl}(H_{\mathbb{C}}) \mid S(gu, gv) = S(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\}$$

and

$$G_{\mathbb{R}} = \{g \in \text{Gl}(H_{\mathbb{R}}) \mid S(gu, gv) = S(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

and fix a point $o \in \mathbf{D}$ (referred to as the reference point or base point). Let the isotropy groups of $\tilde{\mathbf{D}}$ and \mathbf{D} be B and V respectively, then $\tilde{\mathbf{D}}$ and \mathbf{D} can be identified with $G_{\mathbb{R}}/V$ and $G_{\mathbb{C}}/B$ respectively. Furthermore, \mathbf{D} can be considered as an open subset of $\tilde{\mathbf{D}}$, and \mathbf{D} ($\tilde{\mathbf{D}}$) has a $G_{\mathbb{R}}$ ($G_{\mathbb{C}}$)-invariant holomorphic tangent subbundle, denoted by $\mathbf{T}_h(\mathbf{D})$ ($\mathbf{T}_h(\tilde{\mathbf{D}})$) and referred to as the holomorphic *horizontal tangent subbundles*. Let M be a complex manifold. A holomorphic map $\Psi : M \rightarrow \mathbf{D}$ is said to be *horizontal* if the image of the tangent map of Ψ lies in $\mathbf{T}_h(\mathbf{D})$. This definition also applies to $\tilde{\mathbf{D}}$. Finally, set

$$G_{\mathbb{Z}} = \{g \in \text{Gl}(H_{\mathbb{R}}) \mid gH_{\mathbb{Z}} = H_{\mathbb{Z}} \text{ and } S(gu, gv) = S(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\},$$

which is an arithmetic subgroup of $G_{\mathbb{C}}$ (cf. [12]). Let \mathfrak{g} and \mathfrak{g}_0 be the Lie algebras of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$. It is easy to see that \mathfrak{g}_0 is a real form of \mathfrak{g} .

Let M be a complex manifold of dimension n , we consider a family of Hodge structures parameterized by M and with suitable conditions as follows.

Definition 1 *A (rational) variation of (polarized) Hodge structure on the base manifold M consists of the datum*

$$\{M, \mathbf{H}_{\mathbb{Z}} \subset \mathbf{H}_{\mathbb{C}}, \{\mathbf{F}^p\}_{p=0}^k, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{S}\},$$

where 1) $\mathbf{H}_{\mathbb{C}}$ is a flat complex vector bundle of the flat connection ∇ containing a flat lattice $\mathbf{H}_{\mathbb{Z}}$ (and hence a flat real structure $\mathbf{H}_{\mathbb{R}}: \mathbf{H}_{\mathbb{C}} = \mathbf{H}_{\mathbb{R}} \otimes \mathbb{C}$); 2) $\{\mathbf{F}^p\}_{p=0}^k$ is a filtration of $\mathbf{H}_{\mathbb{C}}$, each \mathbf{F}^p is a holomorphic subbundle of $\mathbf{H}_{\mathbb{C}}$ under the holomorphic structure $\nabla^{0,1}$, and the fibres $\{\mathbf{F}_t^p\}_{p=0}^k$ form a Hodge filtration of $(\mathbf{H}_{\mathbb{C}})_t$ with weight k with respect to the real structure $\mathbf{H}_{\mathbb{R}}$ for all $t \in M$; 3) the Griffiths' infinitesimal period relation

$$\nabla^{1,0}\mathbf{F}^p \subset \Omega^1(\mathbf{F}^{p-1})$$

is satisfied; 4) \mathbf{S} is a flat section of $\mathbf{H}_{\mathbb{C}}^* \otimes \mathbf{H}_{\mathbb{C}}^*$ and $\mathbf{S}(t)$ is a polarization defined over \mathbb{Q} , relative to $(H_{\mathbb{Z}})_t$, of the Hodge filtration $\{\mathbf{F}_t^p\}_{p=0}^k$ for all $t \in M$. (Call \mathbf{S} a polarization of the variation.)

Similar to the situation for a single Hodge structure, one can define naturally the Weil operator (a bundle map) $\mathfrak{C}: \mathbf{H}_{\mathbb{C}} \rightarrow \mathbf{H}_{\mathbb{C}}$ on the bundle $\mathbf{H}_{\mathbb{C}}$ such that, for each $t \in M$, \mathfrak{C}_t is the usual Weil operator related to the Hodge filtration $\{\mathbf{F}_t^p\}_{p=0}^k$. Then one can define a Hermitian metric (usually called the *Hodge metric*) as follows

$$h(u, v) = \mathbf{S}(\mathfrak{C}u, \bar{v}),$$

where $\bar{\cdot}$ expresses the conjugation relative to the real structure $\mathbf{H}_{\mathbb{R}}$. In the sequel, we denote the corresponding norm by $\|\cdot\|$. Note that although \mathbf{S} and the real structure $\mathbf{H}_{\mathbb{R}}$ (and hence the conjugation) are flat, the Hodge filtration $\{\mathbf{F}^p\}_{p=0}^k$ (and hence the Weil operator \mathfrak{C}) is not necessarily flat. Therefore, h is not necessarily flat. One of the purpose of this section will be to describe the asymptotic behavior of the Hodge metric h .

Let $\{M, \mathbf{H}_{\mathbb{Z}} \subset \mathbf{H}_{\mathbb{C}}, \{\mathbf{F}^p\}_{p=0}^k, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{S}\}$ be a variation of polarized Hodge structure. As usual, one can assign to the variation a period mapping $\phi: M \rightarrow \mathbf{D}/\Gamma$ or a ρ -equivariant mapping $\tilde{\phi}$ from the universal covering \tilde{M} of M to \mathbf{D} , where ρ is the induced representation of $\pi_1(M)$ into $G_{\mathbb{Z}}$ by the flat connection ∇ , Γ is the image of $\pi_1(M)$ under the representation ρ . Note that in general, \mathbf{D}/Γ is not necessarily a manifold. The following theorem is due to P. Griffiths [9].

Theorem 1 *The period mapping ϕ is holomorphic and comes from the ρ -equivariant mapping $\tilde{\phi}$ which is horizontal.*

In practical problems, the base manifold M of the variation is usually a quasi-projective variety. By Hironaka's theorem, one can always consider M as a Zariski open subset of a smooth projective variety \overline{M} with $D = \overline{M} \setminus M$ being a normal crossing divisor (sometimes called the *singularity* of M). Furthermore, since the main concern in the following is the asymptotic behavior of some objects near the divisor D , one can therefore assume that M is of the form $(\Delta^*)^{n-k} \times (\Delta)^k$, Δ^* being the punctured disk. One can actually assume that M is of the form $(\Delta^*)^n$, since the disk part does not affect the asymptotic behavior. In the remaining part of this paper, we will therefore assume that M is $(\Delta^*)^n$ and $\overline{M} = \Delta^n$. Obviously, $\pi_1(M)$ is generated by n elements, denoted by $\sigma_1, \sigma_2, \dots, \sigma_n$, each σ_i corresponding to the counter-clockwise path around 0 of the i -th component of $(\Delta^*)^n$. It is clear that $\pi_1(M)$ is an Abelian group. The image of σ_i in Γ under ρ is denoted by γ_i , which is possibly trivial and (if nontrivial) referred to as the i -th *Picard-Lefschetz* or *monodromy transformation* of the variation. The universal covering \tilde{M} of M can be regarded as the product U^n , where U is the upper halfplane $\{z \in \mathbb{C} \mid \text{Im}z > 0\}$. Take the standard coordinate systems $\mathbf{z} = (z_1, z_2, \dots, z_n)$ and (t_1, t_2, \dots, t_n) on $\tilde{M} = U^n$ and $M = (\Delta^*)^n$ respectively. Then, the covering mapping is given by

$$\tau : U^n \rightarrow (\Delta^*)^n, \quad \tau(z_1, z_2, \dots, z_n) = (e^{2\pi\sqrt{-1}z_1}, e^{2\pi\sqrt{-1}z_2}, \dots, e^{2\pi\sqrt{-1}z_n});$$

σ_i , as a deck transformation of U^n , corresponds to $z_i \rightarrow z_i + 1$; and the ρ -equivariant condition of $\tilde{\phi}$ can be written as

$$\tilde{\phi}(z_1, \dots, z_i + 1, \dots, z_n) = \gamma_i(\tilde{\phi}(z_1, \dots, z_i, \dots, z_n)),$$

for all $(z_1, \dots, z_n) \in U^n$ and $i = 1, \dots, n$. By Borel's lemma [12], all γ_i are quasi-unipotent. For the sake of simplicity, *throughout this paper we always assume that the γ_i are unipotent, unless stated otherwise*. Denote the logarithm of γ_i by N_i , $i = 1, 2, \dots, n$. It is clear that the $\{N_i\}_{i=1}^n$ are nilpotent and commutative. By the final remark of the Section, *the nilpotency index of each linear combination of $\{N_i\}$ is not greater than k , the weight of the variation*, i.e., $(N_i)^{k+1} = 0$. Define a map by

$$\tilde{\psi}(\mathbf{z}) = \exp\left(-\sum_{i=1}^n z_i N_i\right) \cdot \tilde{\phi}(\mathbf{z}).$$

It is easy to check that $\tilde{\psi}$ remains invariant under the translations $z_i \rightarrow z_i + 1$, $1 \leq i \leq n$. So, $\tilde{\psi}$ drops to a mapping

$$\psi : (\Delta^*)^n \rightarrow \tilde{\mathbf{D}}.$$

Theorem 2 (Nilpotent Orbit Theorem) *The map ψ extends holomorphically to $(\Delta)^n$. Denote the point $\psi(0, 0, \dots, 0) \in \tilde{\mathbf{D}}$ by F , then the map*

$$\mathbf{z} \in \mathbb{C}^n \rightarrow \exp\left(\sum_{i=1}^n z_i N_i\right) \cdot F \in \tilde{\mathbf{D}}$$

is horizontal (equivalently, $N_i F^p \subset F^{p-1}$). Finally, there exist constants $\alpha, \beta, K \geq 0$, such that under the restrictions $\text{Im} z_i \geq \alpha$, $1 \leq i \leq n$, the point $\exp(\sum_{i=1}^n z_i N_i) \cdot F$ lies in \mathbf{D} and satisfies the following inequality

$$d\left(\exp\left(\sum_{i=1}^n z_i N_i\right) \cdot F, \tilde{\phi}(\mathbf{z})\right) \leq K \sum_{i=1}^n (\text{Im}(z_i))^\beta \exp(-2\pi \text{Im}(z_i)),$$

where d again denotes a $G_{\mathbb{R}}$ -invariant Riemannian distance function on \mathbf{D} ; moreover, the constants α, β, K depend only on the choice of d and the weight and Hodge numbers associated to \mathbf{D} .

Definition 2 *A nilpotent orbit is a mapping $\theta : \mathbb{C}^n \rightarrow \tilde{\mathbf{D}}$ of form*

$$\theta(\mathbf{z}) = \exp\left(\sum_{i=1}^n z_i N_i\right) \cdot F$$

where 1) $F = \{F^p\}_{p=0}^k \in \tilde{\mathbf{D}}$; 2) $\{N_i\}_{i=1}^n$ is a commutative set of nilpotent elements of \mathfrak{g}_0 , horizontal at F , i.e., $N_i F^p \subset F^{p-1}$ (hence θ is a horizontal mapping); 3) There exists an $\alpha \in \mathbb{R}$, such that $\theta(\mathbf{z}) \in \mathbf{D}$ for $\text{Im}(z_i) > \alpha$.

Remark. By the definition, the map $\exp(\sum_{i=1}^n z_i N_i) \cdot F$ in the above theorem is a nilpotent orbit.

Concerning nilpotent orbits, one has the *SL₂-orbit theorem*. We now recall it. We first introduce a notion, the so-called (polarized) *mixed Hodge structure* [7], which is very important in Hodge theory. As before, we still let $H_{\mathbb{C}}$ be a complex vector space with a lattice structure $H_{\mathbb{Z}}$, k a positive integer, and S a non-degenerate bilinear form on $H_{\mathbb{C}}$, defined over \mathbb{Q} and such that $S(u, v) = (-1)^k S(v, u)$ for $u, v \in H_{\mathbb{C}}$. We also have corresponding other notations, e.g. $\tilde{\mathbf{D}}, \mathbf{D}, G_{\mathbb{C}}, \mathfrak{g}, G_{\mathbb{R}}, \mathfrak{g}_0$ etc..

Definition 3 *A mixed Hodge structure defined over \mathbb{R} on $H_{\mathbb{C}}$ consists of a pair of finite filtrations of $H_{\mathbb{C}}$*

$$\begin{aligned} W : \quad & \cdots \subset W_l \subset W_{l+1} \subset \cdots \quad (\text{the weight filtration}) \\ F(\in \tilde{\mathbf{D}}) : \quad & \cdots \subset F^p \subset F^{p-1} \subset \cdots \quad (\text{the decreasing filtration}) \end{aligned}$$

such that i) W is defined over \mathbb{R} , ii) the filtration $F(G\mathfrak{r}_l(W_))$ induced on $G\mathfrak{r}_l(W_*) = W_l/W_{l-1}$ by F is a Hodge filtration of weight l .*

Remark. This notion, like the notion of Hodge structure, is compatible with the natural operations, e.g. duality and tensor product etc..

A *splitting* of a mixed Hodge structure (W, F) is a bigrading $H_{\mathbb{C}} = \bigoplus J^{p,q}$ such that

$$W_l = \bigoplus_{p+q \leq l} J^{p,q}, \quad F^p = \bigoplus_{r \geq p} J^{r,s}.$$

An (r, r) -*morphism* (W, F) is an element X of $\mathfrak{gl}(H_{\mathbb{C}})$ satisfying $X(W_l) \subset W_{l+2r}$ and $X(F^p) \subset F^{p+r}$. So the nilpotent element N of a polarized mixed Hodge structure (W, F, N) (see the following definition) is a $(-1, -1)$ -morphism of the mixed Hodge structure (W, F) . Call an (r, r) -morphism X of (W, F) *compatible* with the splitting $\{J^{p,q}\}$ if $X(J^{p,q}) \subset J^{p+r, q+r}$. A construction of Deligne [7, 5] tells us that a mixed Hodge structure always admits some splittings which are compatible with all its morphisms. A mixed Hodge structure (W, F) is said to split over \mathbb{R} if it admits a splitting $\{J^{p,q}\}$ such that $J^{p,q} = \overline{J^{q,p}}$, here $\overline{}$ represents the conjugation relative to the real structure. Such a splitting is then called a *real splitting* of (W, F) . The following proposition will be important for stating the SL_2 -orbit theorem for several variables.

Proposition 2 *Given a mixed Hodge structure (W, F) on $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, there exists a unique $\delta \in \mathfrak{gl}(H_{\mathbb{R}})$ such that $(W, \exp(-\sqrt{-1}\delta)F)$ is a mixed Hodge structure which splits over \mathbb{R} . Furthermore, every morphism of (W, F) commutes with δ .*

The following definition is also important for our exposition.

Definition 4 *A polarized mixed Hodge structure consists of a mixed Hodge structure (W, F) defined over \mathbb{R} on $H_{\mathbb{C}}$ and a nilpotent element $N \in \mathfrak{g}_0$ such that 1) $N^{k+1} = 0$; 2) $W = W(N)[-k]$, where $W(N)$ is the weight filtration induced canonically by N and $(W(N)[-k])_l = (W(N))_{l-k}$; 3) $S(F^p, F^{k-p+1}) = 0$; 4) $NF^p \subset F^{p-1}$; 5) The Hodge structure (of weight $s+l$) on the primitive part $P_{s+l} = \text{Ker}(N^{l+1} : Gr_{s+l}(W_*) \rightarrow Gr_{s+l}(W_*))$ is polarized by the form $S(\cdot, N^l \cdot)$.*

Then, for polarized mixed Hodge structures, one has the following proposition, which is a consequence of the SL_2 -orbit theorem in a single variable [12].

Proposition 3 *Let $\theta(z) = \exp(zN) \cdot F$ be a nilpotent orbit for $z \in \mathbb{C}$, a nilpotent element $N \in \mathfrak{g}_0$ with nilpotent index k , and a filtration $F \in \tilde{\mathbf{D}}$, i.e., satisfying that the mapping θ is horizontal and for some $\alpha > 0$ and $\text{Im}z > \alpha$, $\theta(z) \in \mathbf{D}$. Then $(W(N)[-k], F, N)$ is a polarized mixed Hodge structure.*

Using the above notions and results about (polarized) mixed Hodge structures, one can reformulate the SL_2 -orbit theorem in a single variable in the following manner, which will be helpful for the reformulation of SL_2 -orbit theorem in several variables. From the proposition above, $(W = W(N)[-k], F)$

is a mixed Hodge structure polarized by N (so, N is a $(-1, -1)$ -morphism of (W, F)); on the other hand, by Proposition 1, there exists a $\delta \in \mathfrak{gl}(H_{\mathbb{R}})$ such that $(W, \tilde{F} = \exp(-\sqrt{-1}\delta)F)$ is a mixed Hodge structure splitting over \mathbb{R} . Let $\{\tilde{J}^{p,q}\}$ be a real splitting of (W, \tilde{F}) , i.e.,

$$\tilde{F}^p = \bigoplus_{r \geq p} \tilde{J}^{r,s}, \quad W_l = \bigoplus_{r+s \leq l} \tilde{J}^{r,s}, \quad \tilde{J}^{r,s} = \overline{\tilde{J}^{s,r}}.$$

Define a semisimple transformation \tilde{Y} as follows,

$$\tilde{Y}v = (p + q - k)v \quad \text{for } v \in \tilde{J}^{p,q}.$$

Then, some additional discussion (see §3 of [5]) shows that $\{\tilde{Y}, N\}$ can be completed to an \mathfrak{sl}_2 -triple $\{\tilde{N}^+, \tilde{Y}, N\}$: Actually, the proposition 1 tells us that N is a $(-1, -1)$ -morphism of (W, \tilde{F}) and compatible with the above splitting $\{\tilde{J}^{p,q}\}$ (this needs some additional argument), while \tilde{Y} is a $(0, 0)$ -morphism; and hence \tilde{N}^+ is a $(1, 1)$ -morphism of (W, \tilde{F}) . Like N , the transformations $\delta, \tilde{Y}, \tilde{N}^+$ are infinitesimal isometries of the polarization S and belong to \mathfrak{g}_0 . Thus, we have constructed a representation $\tilde{\rho}_*$ of $\mathfrak{sl}_2(\mathbb{C})$ into \mathfrak{g} with

$$\begin{aligned} \tilde{\rho}_*(Z) &= \sqrt{-1}(\tilde{N}^+ - N), \\ \tilde{\rho}_*(X_+) &= \frac{1}{2}(\tilde{N}^+ + N + \sqrt{-1}\tilde{Y}), \\ \tilde{\rho}_*(X_-) &= \frac{1}{2}(\tilde{N}^+ + N - \sqrt{-1}\tilde{Y}) \end{aligned}$$

(or $\tilde{\rho}_*(\mathbf{y}) = \tilde{Y}, \tilde{\rho}_*(\mathbf{n}_+) = \tilde{N}^+, \tilde{\rho}_*(\mathbf{n}_-) = N$), which lifts to a homomorphism

$$\tilde{\rho} : SL_2(\mathbb{C}) \rightarrow G_{\mathbb{C}}.$$

It is clear that $\tilde{\rho}$ is defined over \mathbb{R} . Again since \tilde{N}^+ and \tilde{Y} are $(1, 1)$ and $(0, 0)$ -morphisms of (W, \tilde{F}) respectively, so they fix the filtration \tilde{F} . Let B be the isotropy subgroup in $G_{\mathbb{C}}$ of $\tilde{\mathbf{D}}$ fixing \tilde{F} , the Lie algebra \mathfrak{b} of which clearly contains \tilde{N}^+ and \tilde{Y} ; let L be the isotropy subgroup in $SL_2(\mathbb{C})$ of \mathbf{P}^1 fixing 0, the Lie algebra \mathfrak{l} of which clearly contains \mathbf{n}_+ and \mathbf{y} . Thus, the homomorphism $\tilde{\rho}$ induces an equivariant embedding

$$\mathbf{P}^1 \simeq SL_2(\mathbb{C})/L \rightarrow \tilde{\mathbf{D}}$$

by $g \mapsto \tilde{\rho}(g)\tilde{F}$, which can be written as, for $z \in \mathbb{C}$,

$$z \rightarrow \exp(zN) \cdot \tilde{F}.$$

A further argument shows that $\exp(zN) \cdot \tilde{F} \in \mathbf{D}$ as $\text{Im}z > 0$ (in particular, $e^{\sqrt{-1}N}\tilde{F} \in \mathbf{D}$), namely the upper half plane is mapped into \mathbf{D} [5].

Write $\exp(zN) \cdot F = \tilde{g}(z) \exp(zN) \cdot \tilde{F}$. Since for some $\alpha > 0$, as $\text{Im}z > 0$, $\exp(zN) \cdot F \in \mathbf{D}$, so $\tilde{g}(z) \in G_{\mathbb{R}}$ for $\text{Im}z > 0$. In particular, $\tilde{g}(\sqrt{-1}y) \in G_{\mathbb{R}}$. In the following, write $\tilde{g}(y)$ instead of $\tilde{g}(\sqrt{-1}y)$. Then, the SL_2 -orbit theorem in a single variable applies to \tilde{g} here. Especially, $\tilde{g}(\infty) \in G_{\mathbb{R}}$. Setting $\mu = \log \tilde{g}(\infty)$ and $\tilde{F}_0 = \tilde{g}(\infty)\tilde{F}$, we obtain a polarized mixed Hodge structure split over \mathbb{R} (W, \tilde{F}_0, N) [5]. The most important is that

$$(W, \tilde{F}_0) \text{ is canonically attached to } (W, F),$$

although defined in terms of the nilpotent orbit $\exp(zN) \cdot F$, and $\tilde{g}(\infty)$ depends on δ (equivalently, on the mixed Hodge structure (W, F) , but not on N). This point will be very useful in the following reformulation for the SL_2 -orbit theorem for several variables.

We can now begin with reformulating the SL_2 -orbit theorem for several variables. Let $F \in \tilde{\mathbf{D}}$, α a fixed positive constant, and $N_1, N_2, \dots, N_n \in \mathfrak{g}_0$ some commuting nilpotent elements of nilpotent indices k , such that for $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{C}^n$

- 1) the mapping $\theta(\mathbf{z}) = \exp(\sum_{i=1}^n z_i N_i) \cdot F$ is horizontal;
- 2) $\exp(\sum_{i=1}^n z_i N_i) \cdot F \in \mathbf{D}$ for $\text{Im}z_i > \alpha, 1 \leq i \leq n$,

i.e., $\theta(\mathbf{z}) = \exp(\sum_{i=1}^n z_i N_i) \cdot F$ is a nilpotent orbit. In the remaining part of this section and the next sections, we will have to fix an ordering of variables z_1, z_2, \dots, z_n , and correspondingly the commuting nilpotent elements N_1, N_2, \dots, N_n . First of all, we state a purely algebraic result, which is due to Cattani and Kaplan [4], as follows:

Lemma 1 *Let N_1, N_2, \dots, N_m be some commutative nilpotent linear transformations in \mathfrak{g}_0 with nilpotent indices k . Set $C(N_1, N_2, \dots, N_m) = \{\sum_{j=1}^m \lambda_j N_j \mid \lambda_j \in \mathbb{R}, \lambda > 0\}$. Then the weight filtration on $H_{\mathbb{C}}$ canonically attached to each nilpotent element $N \in C(N_1, N_2, \dots, N_m)$ is the same.*

Thus, for our present situation, $C_{\mathbf{r}} = C(N_1, \dots, N_r) = \{\sum_{j=1}^r \lambda_j N_j \mid \lambda_j \in \mathbb{R}, \lambda > 0\}$, $1 \leq r \leq n$, determines a unique weight filtration defined over \mathbb{R} on $H_{\mathbb{C}}$, denoted by $W^{\mathbf{r}}$, as follows

$$0 \subset W_{-k}^{\mathbf{r}} \subset W_{-k+1}^{\mathbf{r}} \subset \dots \subset W_{k-1}^{\mathbf{r}} \subset W_k^{\mathbf{r}} = H_{\mathbb{C}}.$$

So, by the proposition 2, we have that $(W^{\mathbf{n}}[-k], F)$ is a mixed Hodge structure polarized by each $N \in C_{\mathbf{n}}$. The previous discussion then shows that one can attach canonically to $(W^{\mathbf{n}}[-k], F)$ another mixed Hodge structure $(W^{\mathbf{n}}[-k], \tilde{F}_{\mathbf{n}})$

also polarized by each $N \in C_{\mathbf{n}}$ and splitting over \mathbb{R} for some $\tilde{F}_{\mathbf{n}} \in \tilde{\mathbf{D}}$. Since $(W^{\mathbf{n}}[-k], \tilde{F}_{\mathbf{n}})$ is polarized by every $N \in C_{\mathbf{n}}$, the mapping

$$(z_1, z_2, \dots, z_{n-1}) \rightarrow \exp\left(\sum_{i=1}^{n-1} z_i N_i\right) \cdot (e^{\sqrt{-1}N_n} \tilde{F}_{\mathbf{n}})$$

is a nilpotent orbit, namely satisfies the properties mentioned above. For this nilpotent orbit, we can do the same argument as before: $(W^{\mathbf{n}-1}[-k], e^{\sqrt{-1}N_n} \tilde{F}_{\mathbf{n}})$ is a mixed Hodge structure polarized by each $N \in C_{\mathbf{n}-1}$ and one can attach canonically to it another \mathbb{R} -split mixed Hodge structure $(W^{\mathbf{n}-1}[-k], \tilde{F}_{\mathbf{n}-1})$ polarized by each $N \in C_{\mathbf{n}-1}$ for some $\tilde{F}_{\mathbf{n}-1} \in \tilde{\mathbf{D}}$. *Inductively, one has that, for $1 \leq r \leq n$, there exists some $\tilde{F}_{\mathbf{r}} \in \tilde{\mathbf{D}}$ such that $(W^{\mathbf{r}}[-k], \tilde{F}_{\mathbf{r}})$ is a \mathbb{R} -split mixed Hodge structure polarized by each $N \in C_{\mathbf{r}}$ and canonically attached to $(W^{\mathbf{r}}[-k], e^{\sqrt{-1}N_{r+1}} \tilde{F}_{\mathbf{r}+1})$; and*

$$(z_1, z_2, \dots, z_r) \rightarrow \exp\left(\sum_{i=1}^r z_i N_i\right) \cdot (e^{\sqrt{-1}N_{r+1}} \tilde{F}_{\mathbf{r}+1})$$

is a nilpotent orbit.

Letting $\{J_{\mathbf{r}}^{p,q}\}$ be the real splitting of $(W^{\mathbf{r}}[-k], \tilde{F}_{\mathbf{r}})$, for $1 \leq r \leq n$, we define some semisimple transformations $\tilde{Y}_{\mathbf{r}}$

$$\tilde{Y}_{\mathbf{r}} v = (p + q - k)v, \quad \text{for } v \in J_{\mathbf{r}}^{p,q}.$$

One can show that $\{\tilde{Y}_{\mathbf{r}}\}_{r=1}^n$ is a commuting set of semisimple endomorphisms in \mathfrak{g}_0 . We also define $\tilde{N}_{\mathbf{r}}^-$ as the component of $N_{\mathbf{r}}$ in the subspace $\cap_{j=1}^{r-1} \ker(\text{ad} \tilde{Y}_j)$ relative to the decomposition of \mathfrak{g}_0 in eigenspaces of the commuting set of semisimple endomorphisms $\{\text{ad}(\tilde{Y}_j)\}_{j=1}^{r-1}$. Set $\tilde{N}_{\mathbf{r}}^- = \sum_{j=1}^r \tilde{N}_j^-$. Then, one can show that $\tilde{Y}_{\mathbf{r}}, \tilde{N}_{\mathbf{r}}^-$ can be expanded to an \mathfrak{sl}_2 -triple $(\tilde{Y}_{\mathbf{r}}, \tilde{N}_{\mathbf{r}}^-, \tilde{N}_{\mathbf{r}}^+)$ and $\{(\tilde{Y}_{\mathbf{r}}, \tilde{N}_{\mathbf{r}}^-, \tilde{N}_{\mathbf{r}}^+)\}_{r=1}^n$ induces a representation ρ_* of $(\mathfrak{sl}_2(\mathbb{C}))^n$ into \mathfrak{g} , which has a particular property. This can be explained as follows. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ admits a natural Hodge structure of weight 0, with

$$\begin{aligned} (\mathfrak{sl}_2(\mathbb{C}))^{-1,1} &= \overline{(\mathfrak{sl}_2(\mathbb{C}))^{1,-1}} = \mathbb{C}(\sqrt{-1}\mathbf{y} + \mathbf{n}_+ + \mathbf{n}_-) \\ (\mathfrak{sl}_2(\mathbb{C}))^{0,0} &= \mathbb{C}(\mathbf{n}_+ - \mathbf{n}_-), \end{aligned}$$

which gives rise to a Hodge structure of weight 0 on $(\mathfrak{sl}_2(\mathbb{C}))^n$. Here,

$$\mathbf{y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{n}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{n}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Fix a point $o \in \mathbf{D}$, called the *reference point* or *base point*, the corresponding Hodge structure of which is denoted by $\{H_0^{p,q}\}$. One can then define a Hodge

structure $\mathfrak{g}^{p,-p}$ of weight 0 on \mathfrak{g} : $\mathfrak{g}^{p,-p} = \{X \in \mathfrak{g} \mid XH_0^{r,s} \subset H_0^{r+p,s-p}\}$. We say that a Lie algebra homomorphism $\xi : (\mathfrak{sl}_2(\mathbb{C}))^n \rightarrow \mathfrak{g}$ is *Hodge* at o if ξ is a $(0,0)$ -morphism from the Hodge structure on $(\mathfrak{sl}_2(\mathbb{C}))^n$ to the induced Hodge structure above on \mathfrak{g} , i.e., $\xi(((\mathfrak{sl}_2(\mathbb{C}))^n)^{r,-r}) \subset \mathfrak{g}^{r,-r}$. Then the following theorem shows that ρ_* above is Hodge. We now are in the position of stating the SL_2 -orbit theorem for several variables.

Theorem 3 (*SL_2 -orbit theorem for several variables*) *For the above fixed ordering of the variables $\mathbf{z} = (z_1, z_2, \dots, z_n)$ in \mathbb{C}^n and the nilpotent orbit $\theta(\mathbf{z})$, there exists a unique Lie group homomorphism*

$$\rho : (SL_2(\mathbb{C}))^n \rightarrow G_{\mathbb{C}}$$

with the following properties:

i) The induced homomorphism ρ_* is Hodge at the point $\exp(\sqrt{-1}N_1) \cdot \tilde{F}_1 \in \mathbf{D}$ and $\tilde{F}_{\mathbf{r}} = \exp(-\sqrt{-1}\tilde{N}_{\mathbf{r}}^-)(\exp(\sqrt{-1}N_1) \cdot \tilde{F}_1)$.

ii) The image of the r -th \mathbf{n}_- (\mathbf{y}, \mathbf{n}_+) factor of $(\mathfrak{sl}_2(\mathbb{C}))^n$ under ρ_* is $\tilde{N}_{\mathbf{r}}^-$ ($\tilde{Y}_{\mathbf{r}}, \tilde{N}_{\mathbf{r}}^+$ respectively) and $W(\tilde{N}_{\mathbf{r}}^-) = W^{\mathbf{r}}$.

In addition, setting $z_i = x_i + \sqrt{-1}y_i$, there exist $G_{\mathbb{R}}$ -valued functions $g_{\mathbf{r}}(y_1, \dots, y_r)$ defined for $y_i > 0$ if $1 \leq r \leq n-1$ and for $y_i > \alpha$, $\alpha \in \mathbb{R}$, if $r = n$, such that

iii) For $j < r \leq n$, $g_{\mathbf{j}}(y_1, \dots, y_j)$ commutes with $\tilde{Y}_{\mathbf{r}}$ and leaves the point $\tilde{F}_{\mathbf{r}}$ fixed. In particular if $j < r$, $g_{\mathbf{j}}(y_1, \dots, y_j)$ is a $(0,0)$ -morphism of the mixed Hodge structure $(W^{\mathbf{r}}, \tilde{F}_{\mathbf{r}})$.

iv) $\sum_{s=1}^r y_s N_s = \text{Ad}(\prod_{j=r-1}^1 g_{\mathbf{j}}(\frac{y_1}{y_{j+1}}, \dots, \frac{y_j}{y_{j+1}})) \sum_{s=1}^r y_s \tilde{N}_s^-$.

v) $\exp \sqrt{-1}(\sum_{j=1}^r y_j N_j) \cdot (\exp(\sqrt{-1}N_{r+1})\tilde{F}_{\mathbf{r}+1}) =$

$(\prod_{j=r}^1 g_{\mathbf{j}}(\frac{y_1}{y_{j+1}}, \dots, \frac{y_j}{y_{j+1}})) \cdot ((\exp \sqrt{-1} \sum_{j=1}^r y_j \tilde{N}_j^-)\tilde{F}_{\mathbf{r}})$, $1 \leq r \leq n$ and $y_{r+1} = 1$.

vi) The functions $g_{\mathbf{r}}(y_1, \dots, y_r)$ and $g_{\mathbf{r}}(y_1, \dots, y_r)^{-1}$ have power series expansions in nonpositive powers of $\frac{y_1}{y_2}, \frac{y_2}{y_3}, \dots, \frac{y_{r-1}}{y_r}, y_r$ with constant term 1 and convergent in any region of the form $\frac{y_1}{y_2} > \beta, \dots, \frac{y_{r-1}}{y_r} > \beta, y_r > \beta, \beta > 0$.

Using the nilpotent orbit theorem and the SL_2 -orbit theorem, one can describe the asymptotic behavior of the Hodge metric. Let $\{M, \mathbf{H}_{\mathbb{Z}} \subset \mathbf{H}_{\mathbb{C}}, \{\mathbf{F}^p\}_{p=0}^k, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{S}\}$ be a variation of Hodge structures, where M is $(\Delta^*)^n$. Let h be the corresponding Hodge metric of the variation. We still use the previous notations. Fix a point $s \in (\Delta^*)^n$. N_i can be considered as a nilpotent linear transformation of the complex vector space $(\mathbf{H}_{\mathbb{C}})_s$. So, by Lemma 1, for $C(N_1, N_2, \dots, N_j)$, $1 \leq j \leq n$, there exists a unique weight filtration of $(\mathbf{H}_{\mathbb{C}})_s$, denoted by $W^{\mathbf{j}}$, with some standard properties (see Section 6 of [12]). Namely,

$$0 \subset W_{-k}^{\mathbf{j}} \subset W_{-k+1}^{\mathbf{j}} \subset \dots \subset W_{k-1}^{\mathbf{j}} \subset W_k^{\mathbf{j}} = (\mathbf{H}_{\mathbb{C}})_s$$

such that, for all $N \in C(N_1, N_2, \dots, N_j)$, $N(W_l^{\mathbf{j}}) \subset W_{l-2}^{\mathbf{j}}$ and, for $l \geq 0$

$$N^l : Gr_l(W_*^{\mathbf{j}}) \rightarrow Gr_{-l}(W_*^{\mathbf{j}})$$

is an isomorphism, where $\text{Gr}_l^{W_*^j} = W_l^j/W_{l-1}^j$. This filtration is called *monodromy weight filtration*, according to Deligne. On the other hand, relative to the flat structure of the bundle $\mathbf{H}_{\mathbb{C}} \rightarrow (\Delta^*)^n$, the Picard-Lefschetz transformations γ_i , and hence the logarithms N_i , can be regarded as flat sections of the bundle $\mathbf{H}_{\mathbb{C}}^* \otimes \mathbf{H}_{\mathbb{C}}$, so one can construct, from the above filtrations, the corresponding filtrations of $\mathbf{H}_{\mathbb{C}}$ by some locally constant sheaves (i.e., some sheaves of locally flat sections), denoted by $\{\mathbf{W}_*^j\}$, which are $\{\gamma_1, \dots, \gamma_j\}$ -invariant. Note that since $\mathbf{H}_{\mathbb{C}}$ has a flat lattice $\mathbf{H}_{\mathbb{Z}} \subset \mathbf{H}_{\mathbb{C}}$, so the above filtrations of locally constant sheaves are defined over \mathbb{Q} ; in general, if $\mathbf{H}_{\mathbb{C}}$ is defined only over \mathbb{R} , the filtrations in question are defined only over \mathbb{R} .

Now, we consider a canonical sheaf extension $\overline{\mathbf{H}}_{\mathbb{C}}$ of $\mathbf{H}_{\mathbb{C}}$ across the singularity, when considering $\mathbf{H}_{\mathbb{C}}$ as a holomorphic vector bundle under the holomorphic structure $\nabla^{0,1}$, as follows. As a sheaf, the germs of the sections of $\overline{\mathbf{H}}_{\mathbb{C}}$ at the singularity are generated by the elements of $\Gamma(M, \mathbf{H}_{\mathbb{C}})$ which are of the form

$$\tilde{v} = \exp\left(\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^n N_i \log t_i\right)v,$$

where v is a multivalued parallel section of $\mathbf{H}_{\mathbb{C}}$. It is clear that \tilde{v} is holomorphic. Similarly, we have the canonical extensions $\{\overline{\mathbf{W}}_l^j\}_{l=-k}^k$ of $\{\mathbf{W}_l^j\}_{l=-k}^k$, which induce a filtration $\overline{\mathbf{H}}_{\mathbb{C}}$, for $1 \leq j \leq n$. We can now state the asymptotic behavior near the singularity of the Hodge metric [5] as follows

Theorem 4 *All the notations as above. If $v \in \cap_j \mathbf{W}_{l_j}^j$ is a flat section and the projection of v in each $\text{Gr}_{l_j}^{W_*^j}$ is nontrivial, then the Hodge norm of \tilde{v} satisfies*

$$\|\tilde{v}\| \sim \left(\frac{\log |t_1|}{\log |t_2|}\right)^{\frac{l_1}{2}} \left(\frac{\log |t_2|}{\log |t_3|}\right)^{\frac{l_2}{2}} \cdots (-\log |t_n|)^{\frac{l_n}{2}}$$

on any region of the form

$$D_{\epsilon} = \{(t_1, t_2, \dots, t_n) \in (\Delta^*)^n \mid \frac{\log |t_1|}{\log |t_2|} > \epsilon, \frac{\log |t_2|}{\log |t_3|} > \epsilon, \dots, -\log |t_n| > \epsilon\}$$

for any $\epsilon > 0$. The same estimate holds for $\|v\|$ when restricted to any sector $|\arg t_j| < \delta$, $j = 1, 2, \dots, n$ for any $\delta > 0$.

To conclude this section, we establish some notation, that will be used in the next two sections. Let \mathbf{H} be a Hermitian vector bundle over a Riemannian manifold M . Let $\mathbf{v} = \{v_1, v_2, \dots, v_q\}$ be a global frame field of \mathbf{H} . Then \mathbf{v} is said to be *L^2 -adapted* if that $\sum_{i=1}^q f_i v_i$ is square integrable implies that each $f_i v_i$ is square integrable with respect to the Hermitian and Riemannian metrics, where the f_i are smooth functions on M . Obviously, the L^2 -adaptedness condition is invariant under constant matrix transformations. Similarly, it is

invariant under scaling, so one can normalize the frame without changing the L^2 -adaptedness of \mathbf{v} . The following lemma will be useful in the sequel (for the proof, see [16], Lemma 4.5).

Lemma 2 *Let \mathbf{v} be a frame for \mathbf{H} with $\sup \|v_i\| < \infty$ for all i . A sufficient condition that \mathbf{v} be L^2 -adapted is that the matrix of the inner products $\langle v_i, v_j \rangle$ has a bounded inverse.*

3 L^2 -adaptedness of the filtration on $\Omega^\bullet(\mathbf{H}_{\mathbb{C}})_{(2)}$

Let $(M, \mathbf{H}_{\mathbb{C}} = \mathbf{H}_{\mathbb{Z}} \otimes \mathbb{C}, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{F} = \{\mathbf{F}^p\}, \mathbf{S})$ be a rational variation of polarized Hodge structure with weight k defined over \mathbb{Z} on M . For the sake of simplicity, we always assume that M is complex 2-dimensional, unless stated otherwise, because the case of higher dimension is similar. Thus, as in §2, M can be seen as $(\Delta^*)^2$ and $\overline{M} = \Delta^2$. Let ω be the Poincaré-like metric on M , which is quasi-isometric to, near the singularity,

$$\eta = \frac{\sqrt{-1}}{2} \left[\frac{dt_1 \wedge d\bar{t}_1}{|t_1|^2 (-\log|t_1|)^2} + \frac{dt_2 \wedge d\bar{t}_2}{|t_2|^2 (-\log|t_2|)^2} \right],$$

and of finite volume, where (t_1, t_2) , as in §2, are the standard coordinates of $(\Delta^*)^2$. Let N_1 and N_2 be the logarithmic monodromies of the variation, which are nilpotent. We fix an ordering (N_1, N_2) from now on as in §2. (Here, we again assume that the monodromies γ_1 and γ_2 are unipotent as in §2.)

Using the Poincaré-like metric ω on M and the Hodge metric h on $\mathbf{H}_{\mathbb{C}}$, we define $\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$ to be the sheaf of germs of local L^2 holomorphic r -forms valued in $j_*\mathbf{H}_{\mathbb{C}}$ on \overline{M} , where j is the inclusion map of M into \overline{M} . In the next section, we will study in detail the asymptotic behavior near the divisor of its sections and show that it is independent of both metrics, but only depends on the logarithmic monodromies of the variation. According to the Hodge filtration $\{\mathbf{F}^p\}$ of the variation, one can then construct a filtration of $\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$, denoted by $F^p\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$, which is the sheaf of germs of local L^2 -holomorphic r -forms on \overline{M} with values in $j_*\mathbf{F}^{p-r}$. Then, using the projection of \mathbf{F}^{p-r} to $\mathbf{E}^{p-r} = \mathbf{F}^{p-r}/\mathbf{F}^{p-r+1}$, one can also do a projection from $F^p\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$ to $(\Omega^r \otimes \mathbf{E}^{p-r})_{(2)}$, the sheaves of germs of local L^2 -holomorphic forms on \overline{M} with values in $j_*\mathbf{E}^{p-r}$, which, as seen in the next section, is actually $(j_*\mathbf{E}^{p-r} \otimes \Omega_{\overline{M}}^r(\log D))_{(2)}$, the sheaf of germs of local L^2 -sections in $j_*\mathbf{E}^{p-r} \otimes \Omega_{\overline{M}}^r(\log D)$. (Note that here we use the induced Hodge metric on \mathbf{E}^p .) One of the main purposes of this section is then to show the following

Theorem 5 *The sequence*

$$0 \rightarrow F^{p+1}\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)} \rightarrow F^p\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)} \rightarrow (\Omega^r \otimes \mathbf{E}^{p-r})_{(2)} \rightarrow 0$$

is exact.

Proof. We will freely use the notations established in §2. In the following discussion, we sometimes return to the general dimension case, i.e. assuming $M = (\Delta^*)^n$, since this will not complicate the discussion.

By the Nilpotent Orbit Theorem, corresponding to the variation, there exists a nilpotent orbit

$$\mathbf{z}(\in \mathbb{C}^n) \rightarrow \exp\left(\sum_{i=1}^n z_i N_i\right) \cdot F \in \tilde{\mathbf{D}}, \quad \text{for some } F \in \tilde{\mathbf{D}},$$

which sufficiently approximates the period map near the divisor of the variation. From §2, we can produce a series of \mathbb{R} -split mixed Hodge structures $(W^r[-k], \tilde{F}_r)$ which are canonically attached to $(W^r[-k], e^{\sqrt{-1}N_{r+1}} \tilde{F}_{r+1})$ satisfying, for some $G_{\mathbb{C}}$ -value functions $g_r(\frac{y_1}{y_{r+1}}, \dots, \frac{y_r}{y_{r+1}})$ (which map into $G_{\mathbb{R}}$ as $\frac{y_r}{y_{r+1}}$ sufficiently large),

$$\begin{aligned} & \exp\left(\sqrt{-1} \frac{y_r}{y_{r+1}} (N_r + \sum_{j=1}^{r-1} \frac{y_j}{y_r} N_j)\right) e^{\sqrt{-1}N_{r+1}} \tilde{F}_{r+1} \\ &= g_r\left(\frac{y_1}{y_{r+1}}, \dots, \frac{y_r}{y_{r+1}}\right) \exp\left(\sqrt{-1} \frac{y_r}{y_{r+1}} (N_r + \sum_{j=1}^{r-1} \frac{y_j}{y_r} N_j)\right) \tilde{F}_r, \end{aligned}$$

where $W^r[-k] = W(N_r + \sum_{j=1}^{r-1} \frac{y_j}{y_r} N_j)[-k]$, $y_j = \text{Im} z_j$, and $0 \leq r \leq n$; here by k we mean the nilpotent index of $\{N_j\}$, we will see at the end of this section that it does not actually exceed the weight of the variation, i.e. k , that is why we here just use k . (Essentially, g_r is determined by the following two 1-dimensional nilpotent orbits $\theta_r(\mathbf{z}) = \exp(z(N_r + \sum_{j=1}^{r-1} \frac{y_j}{y_r} N_j)) e^{\sqrt{-1}N_{r+1}} \tilde{F}_{r+1}$ and $\tilde{\theta}_r(\mathbf{z}) = \exp(z(N_r + \sum_{j=1}^{r-1} \frac{y_j}{y_r} N_j)) \tilde{F}_r$ with

$$\theta_r(\mathbf{z}) = g_{(\frac{y_1}{y_r}, \dots, \frac{y_{r-1}}{y_r}, 1)}(\mathbf{z}) \tilde{\theta}_r(\mathbf{z})$$

so that $g_r(\frac{y_1}{y_{r+1}}, \dots, \frac{y_r}{y_{r+1}}) = g_{(\frac{y_1}{y_r}, \dots, \frac{y_{r-1}}{y_r}, 1)}(\frac{y_r}{y_{r+1}})$. For details, cf. §2.) By the splitting of $(W^r[-k], \tilde{F}_r)$, as in §2, one has a $(0, 0)$ -morphism \tilde{Y}_r , which together with $N_r + \sum_{j=1}^{r-1} \frac{y_j}{y_r} N_j$ determines an sl_2 -triple; in addition, by the SL_2 -orbit theorem for several variables, \tilde{Y}_r and \tilde{N}_r^- together determine another sl_2 -triple, where $\tilde{N}_r^- = \sum_{j=1}^r \tilde{N}_j^-$ and \tilde{N}_j^- is the component of N_j in the subspace $\cap_{k=1}^{j-1} \text{Ker}(\text{ad} \tilde{Y}_k)$.

We now consider

$$\text{Adexp}\left(-\sqrt{-1} \frac{y_r}{y_{r+1}} (N_r + \sum_{j=1}^{r-1} \frac{y_j}{y_r} N_j)\right) \left(g_r\left(\frac{y_1}{y_{r+1}}, \dots, \frac{y_r}{y_{r+1}}\right)\right), \quad 1 \leq r \leq n,$$

denoted by $A_{\mathbf{r}}$, where $y_{n+1} = 1$ and $y_r > 0$. As in the proof of 6.20 of [12], fixing $\frac{y_1}{y_r}, \dots, \frac{y_{r-1}}{y_r}$, the limit $\lim_{\frac{y_r}{y_{r+1}} \rightarrow \infty} A_{\mathbf{r}}$ exists, which is denoted by $g_{\mathbf{r}}(\infty)$, and since $e^{\sqrt{-1}N_{r+1}}\tilde{F}_{\mathbf{r}+1} = A_{\mathbf{r}}\tilde{F}_{\mathbf{r}}$, so

$$e^{\sqrt{-1}N_{r+1}}\tilde{F}_{\mathbf{r}+1} = g_{\mathbf{r}}(\infty)\tilde{F}_{\mathbf{r}}.$$

It should be pointed out that although the limit may depend on the variables $\frac{y_1}{y_r}, \dots, \frac{y_{r-1}}{y_r}$, this will not affect the following arguments. Important is that $g_{\mathbf{r}}(\infty)$ preserves $W_l^{\mathbf{r}}$ and acts as the identity on $Gr_l^{W^{\mathbf{r}}}$ for all $(\frac{y_1}{y_r}, \dots, \frac{y_{r-1}}{y_r})$ (cf. [12], Lemma 6.20), as will be used later in this section, so one can fix some $\frac{y_1}{y_r}, \dots, \frac{y_{r-1}}{y_r}$ after taking the limit each time. Thus we have

$$\begin{aligned} F &= g_{\mathbf{n}}(\infty)\tilde{F}_{\mathbf{n}} = g_{\mathbf{n}}(\infty)e^{-\sqrt{-1}N_n}g_{\mathbf{n}-1}(\infty)\tilde{F}_{\mathbf{n}-1} \\ &= \dots = \prod_{j=n}^1 (g_j(\infty)e^{-\sqrt{-1}N_j})(e^{\sqrt{-1}N_1}\tilde{F}_{\mathbf{1}}) := g(\infty)(e^{\sqrt{-1}N_1}\tilde{F}_{\mathbf{1}}). \end{aligned}$$

Denoting $\prod_{j=n}^1 (g_j(\infty)e^{-\sqrt{-1}N_j})$ by $g(\infty)$, we have

$$F = g(\infty)(e^{\sqrt{-1}N_1}\tilde{F}_{\mathbf{1}});$$

In addition, we also know that the point $e^{\sqrt{-1}N_1}\tilde{F}_{\mathbf{1}}$ of $\tilde{\mathbf{D}}$ actually lies in \mathbf{D} , i.e. corresponds to a pure Hodge structure on $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$.

Let now $G_{\mathbb{C}}$ be the complex group corresponding to the classifying space $\tilde{\mathbf{D}}$ and \mathfrak{g} its Lie algebra. Then the SL_2 -orbit theorem for several variables in §2 tells us that there exists a Lie algebra homomorphism $\rho_* : (\mathfrak{sl}(2, \mathbb{C}))^n \rightarrow \mathfrak{g}$, which is *Hodge* at $e^{\sqrt{-1}N_1}\tilde{F}_{\mathbf{1}}$ (called *horizontal* in [12]). This can be explained as follows. Using the pure Hodge structure $e^{iN_1}\tilde{F}_{\mathbf{1}}$, one can define a pure Hodge structure $\{\mathfrak{g}^{p,-p}\}$ of weight 0 on \mathfrak{g} relative to \mathfrak{g}_0 ; on the other hand, let $\mathbf{n}_j^+, \mathbf{n}_j^-, \mathbf{y}_j$ be the generators of the j -th factor of $(\mathfrak{sl}(2, \mathbb{C}))^n$ as in §2. Then their images under ρ_* are $\tilde{N}_j^+, \tilde{N}_j^-, \tilde{Y}_j$ respectively, and

$$\tilde{N}_j^+ + \tilde{N}_j^- + \sqrt{-1}\tilde{Y}_j \in \mathfrak{g}^{-1,1}, \tilde{N}_j^+ - \tilde{N}_j^- \in \mathfrak{g}^{0,0}.$$

Now, we will temporarily digress from the proof of the theorem and establish a slightly abstract setting, which can be just applied to the above situation. Let $\{H^{p,q}\}$ be a pure Hodge structure of weight k on $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, which has an $(\mathfrak{sl}(2, \mathbb{C}))^n$ -action defined over \mathbb{R} satisfying

$$X_j^+ H^{p,q} \subset H^{p-1,q+1}, X_j^- H^{p,q} \subset H^{p+1,q-1}, Z_j H^{p,q} \subset H^{p,q},$$

where $X_j^+ = \mathbf{n}_j^+ + \mathbf{n}_j^- + \sqrt{-1}\mathbf{y}_j$, $X_j^- = \mathbf{n}_j^+ + \mathbf{n}_j^- + \sqrt{-1}\mathbf{y}_j$, $Z_j = \mathbf{n}_j^+ - \mathbf{n}_j^-$, and the action of the i -th factor and action of the j -th factor commute, $i \neq j$. We

call the $(\mathfrak{sl}(2, \mathbb{C}))^n$ -action *Hodge* (or *horizontal*) at $\{H^{p,q}\}$. When the Hodge structure happens to be polarized by a bilinear form S on $H_{\mathbb{C}}$, one will say that the $(\mathfrak{sl}(2, \mathbb{C}))^n$ -action is *compatible with the polarization* if $(\mathfrak{sl}(2, \mathbb{C}))^n$ acts as a Lie algebra of infinitesimal isometries of S . Similar to what Schmid did in [12], p. 258, one can define the invariance and irreducibility with respect to the given Hodge structure and horizontal $(\mathfrak{sl}(2, \mathbb{C}))^n$ -action.

In the following, we will give an analogous result to Lemma 6.24 of [12]. First of all, we give some basic Hodge structures which are irreducible with respect to a certain horizontal $(\mathfrak{sl}(2, \mathbb{C}))^n$ -action. For simplicity of the statement, *we just return to the case of $n = 2$ again*, as for the higher dimensional case, one can give a similar statement. The first basic irreducible structure is denoted by $S(m) \otimes S(n)$ which is of weight $m + n$, where $S(m)$ is the m -th symmetric power of $S(1)$; the latter is defined as follows: Take $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ with the usual $\mathfrak{sl}(2, \mathbb{C})$ -action so that $n^-e_2 = e_1, n^-e_1 = 0$ and the Hodge structure

$$\begin{aligned} v^+ &= e_1 + \sqrt{-1}e_2 && \text{of type } (0, 1), \\ v^- &= e_1 - \sqrt{-1}e_2 && \text{of type } (1, 0), \end{aligned}$$

where v^+ and v^- are the eigenvectors of $\sqrt{-1}(n^+ - n^-)$ with eigenvalues 1 and -1 respectively. One can also polarize \mathbb{C}^2 over \mathbb{R} as follows: $S(v^+, v^-) = 2\sqrt{-1}$. It is easy to check that the $\mathfrak{sl}(2, \mathbb{C})$ -action on \mathbb{C}^2 is horizontal with respect to the above Hodge structure and compatible with the polarization; moreover $\{e_1, e_2\}$ is orthonormal with respect to the hermitian norm corresponding to this polarization (cf. §2). It is well-known that (polarized) Hodge structures are compatible with the operations of tensor products and symmetric products. So, $S(m) \otimes S(n)$ inherits all structures, including Hodge structure, $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action, and polarization; moreover the action is horizontal and irreducible with respect to the induced Hodge structure and compatible with the polarization; in particular, it follows that $\{e_1^r e_2^{m-r} \otimes e_3^s e_4^{n-s}\}$ is an orthonormal basis of $S(m) \otimes S(n)$ with respect to the corresponding norm, where e_1, e_2 corresponds to the first factor of $S(m) \otimes S(n)$, while $\{e_3, e_4\}$ to the second factor. A direct computation shows that the induced Hodge structure $\{H^{p,q}, p + q = m + n\}$ on $S(m) \otimes S(n)$ is

$$H^{p,q} = \bigoplus_{r=1}^p \bigoplus_{s=1}^q \mathbb{C}(v_1^-)^r (v_1^+)^s \otimes (v_2^-)^{p-r} (v_2^+)^{q-s},$$

where $\{v_1^+, v_1^-\}$ and $\{v_2^+, v_2^-\}$ correspond to the first and second factor of $S(m) \otimes S(n)$ respectively, namely

$$\begin{aligned} v_1^+ &= e_1 + \sqrt{-1}e_2 && \text{and } v_2^+ = e_3 + \sqrt{-1}e_4 && \text{of type } (0, 1) \\ v_1^- &= e_1 - \sqrt{-1}e_2 && \text{and } v_2^- = e_3 - \sqrt{-1}e_4 && \text{of type } (1, 0). \end{aligned}$$

In the following we continue to consider some trivial Hodge structures on which $(\mathfrak{sl}(2, \mathbb{C}))^2$ acts horizontally and irreducibly. The one-dimensional complex vector space \mathbb{C} , with the obvious real structure, carries a unique Hodge structure of weight 2. Deligne denotes it by $H(1)$ and calls it the "Hodge structure of Tate". Take the standard basis $h^{1,1}$, which is of type $(1, 1)$, so that $H(1) = \mathbb{C}h^{1,1}$. For $n \geq 0$, $H(n)$ shall be the n -th symmetric power of $H(1)$, and $H(-n)$ the dual of $H(n)$, the standard of which will be denoted by $h^{n,n}$. $(\mathfrak{sl}(2, \mathbb{C}))^2$ trivially acts on $H(n)$ for all integers n . Also, one can polarize $H(1)$ naturally: the standard basis $h^{1,1}$ enables us to identify $H(1)$ with \mathbb{C} , namely $h^{1,1}$ with 1; then the nondegenerate bilinear form S on $H(1)$ is taken as $S(1, 1) = 1$, from which $H(n)$ inherits the standard polarization, still denoted by S , for all integers n . It is easy to see that the $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action on $H(n)$ is horizontal and irreducible and compatible with the polarization. The second slight nontrivial Hodge structure is defined on \mathbb{C}^2 as follows: let e_1, e_2 be the standard basis vectors of \mathbb{C}^2 , which gives the standard real structure on \mathbb{C}^2 . For $p \neq q$, we define a Hodge structure $E(p, q)$ of weight $p + q$ on \mathbb{C}^2 by requiring that

$$e^{p,q} = e_1 - \sqrt{-1}e_2$$

be of type (p, q) , and

$$e^{q,p} = e_1 + \sqrt{-1}e_2$$

of type (q, p) . Again $(\mathfrak{sl}(2, \mathbb{C}))^2$ trivially acts on \mathbb{C}^2 and the action is horizontal and irreducible with respect to the Hodge structure $E(p, q)$. The bilinear form S on \mathbb{C}^2 , which is described by the identities

$$\begin{aligned} S(e^{p,q}, e^{p,q}) &= 0, & S(e^{q,p}, e^{q,p}) &= 0 \\ S(e^{p,q}, e^{q,p}) &= 2\sqrt{-1}^{q-p}, & S(e^{q,p}, e^{p,q}) &= 2\sqrt{-1}^{p-q}, \end{aligned}$$

polarizes $E(p, q)$. It is easy to see that the polarization is compatible with the action. Now, we are in the position to state the following generalization of a result of Schmid (cf. [12], Theorem 6.24) to arbitrary dimensions (stated here for two dimension for simplicity).

Theorem 6 *Let $H_{\mathbb{C}} = H_{\mathbb{R}} \otimes \mathbb{C}$ be a complex space with a Hodge structure $\{H^{p,q}\}$ of weight k and a horizontal $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action. Then $H_{\mathbb{C}}$ can be decomposed into a direct sum of subspaces which are invariant and irreducible with respect to the given structures: the Hodge structure and the horizontal $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action. Every irreducible factor is isomorphic - relative to the Hodge structure and the horizontal action - to one of the following types: $H(l) \otimes S(m) \otimes S(n)$, with $l \in \mathbb{Z}, m, n \geq 0$, and $k = 2l + m + n$; or $E(p, q) \otimes S(m) \otimes S(n)$, with $p > q, m, n \geq 0$, and $k = p + q + m + n$. If the Hodge structure of $H_{\mathbb{C}}$ happens to have a polarization which is compatible with the $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action, then the decomposition can be chosen to be orthogonal with respect to the polarization,*

and the isomorphisms between the irreducible factors and the irreducible structures of special type can be chosen with the further restriction that they should preserve the polarizations.

Proof. Since $\{Z_1, Z_2\}$ commute and act semisimply on each Hodge subspace $H^{p,q}$, with integral eigenvalues, one can find a basis of $H^{p,q}$, each element of which is an eigenvector of both Z_1 and Z_2 with integral eigenvalue. Then, we can define a linear $SO(2)$ -action on $H_{\mathbb{C}}$ as follows: let $v \in H^{p,q}$ be an eigenvector of both Z_1 and Z_2 with eigenvalues l_1 and l_2 respectively, then the element $e^{\sqrt{-1}\theta} \in SO(2)$ acts on v as multiplication by $e^{\sqrt{-1}(l_1+l_2+p-q)\theta}$, denoted by $e^{\sqrt{-1}\theta}.v$. Obviously the action is orthogonal with respect to the polarization and trivial on $S(m) \otimes S(n)$. (Since $v = (v_1^-)^r (v_1^+)^{m-r} \otimes (v_2^-)^s (v_2^+)^{n-s} \in H^{r+s, m+n-r-s} \subset S(m) \otimes S(n)$ is an eigenvector of Z_1^+ and Z_2^+ with eigenvalues $m-2r$ and $n-2s$ respectively, so $e^{\sqrt{-1}\theta}$ acts on v by multiplying $e^{\sqrt{-1}((m-2r)+(n-2s)+(r+s)-(m+n-r-s))\theta} = 1$). Considering $SO(2)$ as the group of real points of an algebraic 1-torus T , which is defined and *anisotropic* over \mathbb{R} , one can extend the $SO(2)$ -action to a representation of T on $H_{\mathbb{C}}$ over \mathbb{R} . In addition, the $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action on $H_{\mathbb{C}}$ also determines a representation on $H_{\mathbb{C}}$ of the algebraic group $(SL(2, \mathbb{C}))^2$ again defined over \mathbb{R} . Since the $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action is horizontal, the representations of T and $(SL(2, \mathbb{C}))^2$ commute: let $v \in H^{p,q}$ be an eigenvector of both Z_1 and Z_2 with the eigenvalues l_1 and l_2 respectively, then X_1^+v (X_2^+v) is an eigenvector of both Z_1 and Z_2 with the eigenvalues $l_1 + 2$ (l_1) and l_2 ($l_2 + 2$) respectively. So,

$$\begin{aligned} & e^{\sqrt{-1}\theta}.(X_1^+v)(e^{\sqrt{-1}\theta}.(X_2^+v)) \\ &= e^{\sqrt{-1}((l_1+2)+l_2+(p-1)-(q+1))\theta} X_1^+v(e^{\sqrt{-1}(l_1+(l_2+2)+(p-1)-(q+1))\theta} X_2^+v) \\ &= e^{\sqrt{-1}(l_1+l_2+p-q)\theta} X_1^+v(e^{\sqrt{-1}(l_1+l_2+p-q)\theta} X_2^+v) \\ &= X_1^+(e^{\sqrt{-1}\theta}.v)(X_2^+(e^{\sqrt{-1}\theta}.v)) \end{aligned}$$

and similarly one can show the other commutativity, where we use the horizontality of the $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action. Hence one has a representation of the product $T \times SL(2, \mathbb{C})$. One can also show that a $\{Z_1, Z_2\}$ -stable subspace of $H_{\mathbb{C}}$ carries a sub-Hodge structure if and only if it is self-conjugate and invariant under the $SO(2)$ -action: the proof of necessity is obvious; we prove the sufficiency. Let $H_{\mathbb{C}}^1$ be a $\{Z_1, Z_2\}$ -stable subspace of $H_{\mathbb{C}}$ which is self-conjugate and invariant under the $SO(2)$ -action, $v \in H_{\mathbb{C}}^1$. Decompose $v = \sum_{p+q=k, l_1, l_2 \in \mathbb{Z}} v_{l_1, l_2}^{p, q}$, $v_{l_1, l_2}^{p, q} \in H^{p,q}$ is an eigenvector of both Z_1 and Z_2 with eigenvalue l_1 and l_2 respectively, since $e^{\sqrt{-1}\theta}.v \in H_{\mathbb{C}}^1$ for all $\theta \in \mathbb{R}$, $\sum_{l_1+l_2+p-q=s} v_{l_1, l_2}^{p, q} \in H_{\mathbb{C}}^1$. Using the $\{Z_1, Z_2\}$ -stability of $H_{\mathbb{R}}^1$, one can then show that each $v_{l_1, l_2}^{p, q}$ lies in $H_{\mathbb{C}}^1$, and hence $H_{\mathbb{R}}^1$ carries a sub-Hodge structure. Thus, one gets that the subspaces of $H_{\mathbb{C}}$ which are invariant and irreducible with respect to the Hodge structure

and the $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action correspond bijectively to those invariant and irreducible subrepresentations of the representation of $T \times (SL(2, \mathbb{C}))^2$, that are defined over \mathbb{R} . Because the reductivity of the product $T \times (SL(2, \mathbb{C}))^2$, this proves the first part of the theorem.

An irreducible representation defined over \mathbb{R} of the group $T \times (SL(2, \mathbb{C}))^2$ either remains irreducible under $(SL(2, \mathbb{R}))^2$, in which case T acts trivially, or splits into two conjugate subspaces, each of which is T -stable and $(SL(2, \mathbb{C}))^2$ -irreducible, with T acting nontrivially. The first case corresponds to an irreducible Hodge structure with respect to a horizontal $(\mathfrak{sl}(2, \mathbb{C}))^2$ -action of the type $H(l) \otimes S(m) \times S(n)$ with $k = 2l + m + n$, $l \in \mathbb{Z}, m, n \geq 0$, which has dimension $(m + 1)(n + 1)$; the second case corresponds to an irreducible Hodge structure of the type $E(p, q) \otimes S(m) \otimes S(n)$, with $p > q, m, n \geq 0$, and $k = p + q + m + n$, which has dimension $2(m + 1)(n + 1)$. We omit their proofs and the proof of the remaining part of the theorem, since they are standard. \square

We now return to the proof of the theorem at the beginning of this section. Actually, we only need to prove the case of $r = 0$. By the previous discussions, $e^{\sqrt{-1}N_1} \tilde{F}_1$ (briefly denoted by F_0 below) has an $(\mathfrak{sl}(2, \mathbb{C}))^2$ -horizontal action, which is compatible with the polarization. So, by the above decomposition theorem, we can assume that $e^{\sqrt{-1}N_1} \tilde{F}_1$ is irreducible. Since $H(l)$ and $E(p, q)$ are trivial, without loss of generality, we assume that $e^{\sqrt{-1}N_1} \tilde{F}_1 = S(m) \otimes S(n)$. We choose

$$\alpha_{k,l} = (\tilde{N}_1^-)^{m-k} (\tilde{N}_2^-)^{n-l} ((v_1^-)^m \otimes (v_2^-)^n)$$

as a basis of $S(m) \otimes S(n)$. It is easily to check that this basis satisfies

$$\alpha_{k,l} \in F_0^{k+l} \cap W_{2k}^1 \cap W_{2(k+l)}^2$$

and $\alpha_{k,l}$ projects nontrivially in $Gr_{F_0}^s = F_0^s / F_0^{s+1}$, $Gr_{2k}^{W^1} = W_{2k}^1 / W_{2k-1}^1$, and $Gr_{2s}^{W^2} = W_{2s}^2 / W_{2s-1}^2$ with $s = k + l$, where $\{W_k^1\}$ and $\{W_s^2\}$ are the weight filtrations corresponding to $\tilde{N}_1^- = N_1$ and $\tilde{N}_2^- = \tilde{N}_1^- + \tilde{N}_2^-$ (or $c_1 N_1 + c_2 N_2$) respectively; W_k^1 and W_s^2 can actually be described as the subspaces spanned by eigenvectors of \tilde{Y}_1 and $\tilde{Y}_2 = \tilde{Y}_1 + \tilde{Y}_2$ with eigenvalues not greater than $k - m$ and $s - m - n$ respectively [12]. (Note that the notations W^j here are just $W^j[-k]$ in §2 and the same for the remaining part of this section.) Since $F = g(\infty)F_0$, one has $g(\infty)\alpha_{k,l} \in F^{k+l}$ and it nontrivially projects in $Gr_F^s = F^s / F^{s+1}$ with $s = k + l$; by the means of the construction of $g(\infty)$, one also has that $g(\infty)(\alpha_{k,l})$ can be considered as the basis of $Gr_{2k}^{W^1} \cap Gr_{2(k+l)}^{W^2} = W_{2k}^1 \cap W_{2(k+l)}^2 / (W_{2k-1}^1 \cap W_{2(k+l)}^2 + W_{2k}^1 \cap W_{2(k+l)-1}^2)$, since $g(\infty)$ acts as the identity on $Gr_{2k}^{W^1} \cap Gr_{2(k+l)}^{W^2}$.

On the other hand, by the nilpotent orbit theorem, using the notation in the previous sections, one can easily show the following assertion: *Fixing $x \in M$, for $v \in (\mathbf{H}_{\mathbb{C}})_{\mathbf{x}}$, $v \in F^p$ if and only if there exist two holomorphic*

sections w_1 and w_2 of $\overline{\mathbf{H}}_{\mathbb{C}}$ such that $\tilde{v} + t_1 w_1 + t_2 w_2$ is a section of \mathbf{F}^p . So, for $g(\infty)\alpha_{k,l}$, there exist two sections $\beta_{k,l}^1$ and $\beta_{k,l}^2$ of $\overline{\mathbf{H}}_{\mathbb{C}}$ such that

$$\sigma_{k,l} = g(\infty)(\widetilde{\alpha_{k,l}}) + t_1 \beta_{k,l}^1 + t_2 \beta_{k,l}^2$$

is a section of \mathbf{F}^s and, under the natural projection, represents a generator for \mathbf{E}^s for $s = k + l$.

Summing up all the above, one has that $\{\sigma_{k,l} | k + l = s\}$, under the natural projection, represents a set of generators for \mathbf{E}^s , while $\{\sigma_{k,l} | k + l \geq s\}$ represents a set of generators for \mathbf{F}^s . Furthermore, by the norm estimates of the previous section, since $g(\infty)(\widetilde{\alpha_{k,l}}) \in \overline{\mathbf{W}}_{2k}^1 \cap \overline{\mathbf{W}}_{2(k+l)}^2$ and $\|t_j \beta_{k,l}^j\|^2 = O(|t_j|^2 |\log|t_1||^m |\log|t_2||^n)$, one has

$$\begin{aligned} \|\sigma_{k,l}\|^2 &\sim \left(\frac{\log|t_1|}{\log|t_2|}\right)^{2k-m} \cdot (-\log|t_2|)^{2(k+l-m-n)} \\ &= (-\log|t_1|)^{2k-m} (-\log|t_2|)^{2l-n}, \end{aligned}$$

on the domains of the form $D_\epsilon = \{(t_1, t_2) \in (\Delta^*)^2 \mid \frac{\log|t_1|}{\log|t_2|} > \epsilon, -\log|t_2| > \epsilon\}$. (Note again that the notations $\overline{\mathbf{W}}^j$ here are just $\overline{\mathbf{W}}^j[-k]$ in §2.) Again since, by the previous section, one can verify that for $v \in \mathbf{W}_k^1 \cap \mathbf{W}_s^2$

$$\begin{aligned} L(\mathbf{z})(\tilde{v}) &= \tilde{h}(\mathbf{z}) \exp\left(\frac{1}{2} \log \frac{y_1}{y_2} \tilde{Y}_1 + \frac{1}{2} \log y_2 \tilde{Y}_2\right) \exp\left(-\sum_{j=1}^2 x_j N_j\right) \tilde{v} \\ &= \tilde{h}(\mathbf{z}) \left[\left(\frac{y_1}{y_2}\right)^{\frac{k}{2}} y_2^{\frac{s}{2}} \left(\sum_{j=0}^{m+n} \frac{\sqrt{-1}^j}{j!} (N_1 + N_2)^j\right) v + (\text{lower order terms}) \right] \\ &= \tilde{h}(\mathbf{z}) \left[\left(\frac{y_1}{y_2}\right)^{\frac{k}{2}} y_2^{\frac{s}{2}} v + \dots \right], \end{aligned}$$

where $t_j = e^{2\pi i z_j}$, $z_j = x_j + iy_j$ for $j = 1, 2$, and $\tilde{h}(\mathbf{z})$ is strongly asymptotic to the identity as $\frac{y_1}{y_2}, y_2 \rightarrow \infty$. Thus, $L(\mathbf{z})(g(\infty)(\widetilde{\alpha_{k,l}}))$ and $L(\mathbf{z})(\sigma_{k,l})$ are asymptotically the same. So, applying Lemma 2, one has that $\{\sigma_{k,l} | k + l \geq s\}$ gives an L^2 -adapted basis for \mathbf{F}^s . Therefore, by the above norm estimates of $\sigma_{k,l}$, $(\mathbf{F}^s)_{(2)}$ is freely generated by the sections $\{t_1^{\epsilon_k} t_2^{\eta_l} \sigma_{k,l} | k + l \geq s\}$ with $\epsilon_k = 0$ if $2k \leq m$ and $\epsilon_k = 1$ otherwise and $\eta_l = 0$ if $2l \leq n$ and $\eta_l = 1$ otherwise. On the other hand, by the above expression of $L(\mathbf{z})(\tilde{v})$, one knows that the $(k + l, m + n - k - l)$ component of $L(\mathbf{z})(\sigma_{k,l})$ is asymptotic to $y_1^{\frac{2k-m}{2}} y_2^{\frac{2l-n}{2}} \alpha_{k,l}$ (Here, one needs to use the expansion of $g(\infty)$, see [12], the proof of Theorem 6.20), namely $\sigma_{k,l}$ carries much of its norm in this Hodge component. Therefore, the projections in \mathbf{E}^s of the elements in the set $\{t_1^{\epsilon_k} t_2^{\eta_l} \sigma_{k,l} | k + l = s\}$ form a set of generators of \mathbf{E}^s . So, one naturally has $(\mathbf{F}^s)_{(2)}/(\mathbf{F}^{s+1})_{(2)} = \mathbf{E}_{(2)}$. This completes the proof of the theorem. \square

From the above arguments, we also obtain the following byproducts.

Theorem 7 $\nabla^{1,0}$ (hence also θ) is a bounded operator under the Poincaré-like metric on $(\Delta^*)^2$ and the Hodge metric on $\mathbf{H}_{\mathbb{C}}$ (the induced Hodge metric on the Hodge bundle).

Proof. From the previous arguments, we know that $\{\sigma_{k,l} | k+l \geq 0\}$ represents a basis for $\mathbf{H}_{\mathbb{C}}$ and each $\sigma_{k,l}$ can be expressed as

$$\sigma_{k,l} = g(\infty)(\widetilde{\alpha_{k,l}}) + t_1\beta_{k,l}^1 + t_2\beta_{k,l}^2$$

for some holomorphic sections $\beta_{k,l}^1$ and $\beta_{k,l}^2$ of $\overline{\mathbf{H}}_{\mathbb{C}}$. By the definition of $\overline{\mathbf{H}}_{\mathbb{C}}$ in the Section 2 and 3, we have

$$\nabla^{1,0}(\sigma_{k,l}) = \frac{1}{2\pi i} \left(\frac{dt_1}{t_1} \otimes N_1 + \frac{dt_2}{t_2} \otimes N_2 \right) g(\infty)(\widetilde{\alpha_{k,l}}) + \nabla^{1,0}(t_1\beta_{k,l}^1) + \nabla^{1,0}(t_2\beta_{k,l}^2).$$

So, if we neglect the higher order terms, we can consider $\nabla^{1,0}$ as $\frac{1}{2\pi i} \left(\frac{dt_1}{t_1} \otimes N_1 + \frac{dt_2}{t_2} \otimes N_2 \right)$ acting on $\mathbf{H}_{\mathbb{C}}$ by multiplication. On the other hand, by means of the commutativity of N_1, N_2 and the definitions of $\{W_{l_1}^1\}$ and $\{W_{l_2}^2\}$ in the Sections 2 and 3, one has that N_1 lowers the weights of $\{W_{l_1}^1\}$ and $\{W_{l_2}^2\}$ by 2, while N_2 lowers the weight of $\{W_{l_2}^2\}$ by 2 and preserves $\{W_{l_1}^1\}$. Thus, again by Theorem 5.21 of [5], one has that, if $\tilde{v} \in \overline{W}_{l_1}^1 \cap \overline{W}_{l_2}^2$ and the parts of the highest weights of v in $\{W_{l_1}^1\}$ and $\{W_{l_2}^2\}$ are nonzero,

$$\left\| \left(\frac{1}{2\pi i} \frac{dt_1}{t_1} \otimes N_1 \right) \tilde{v} \right\|^2 \sim \log^2 |t_1| \left(\frac{\log |t_1|}{\log |t_2|} \right)^{l_1-2} (\log |t_2|)^{l_2-2}$$

and

$$\left\| \left(\frac{1}{2\pi i} \frac{dt_2}{t_2} \otimes N_2 \right) \tilde{v} \right\|^2 \sim \log^2 |t_2| \left(\frac{\log |t_1|}{\log |t_2|} \right)^{l_1} (\log |t_2|)^{l_2-2},$$

on D_ϵ . Here, we use $\left\| \frac{dt_1}{t_1} \right\|^2 \sim \log^2 |t_1|$ and $\left\| \frac{dt_2}{t_2} \right\|^2 \sim \log^2 |t_2|$, under the Poincaré-like metric of M as defined in the beginning of this section. Hence, $\|\nabla^{1,0}\| \sim 1$ on D_ϵ , namely, $\nabla^{1,0}$ is a bounded operator on D_ϵ , though the bound may depend on ϵ . Note that $\nabla^{1,0}$ is independent of the choice of order (t_1, t_2) , so by applying the same argument to the order (t_2, t_1) , we can also show that $\nabla^{1,0}$ is a bounded operator on the domain $D'_\epsilon = \{(t_1, t_2) \mid \frac{\log |t_2|}{\log |t_1|} > \epsilon, -\log |t_1| > \epsilon\}$. It is clear that for a sufficient small $\epsilon > 0$, $D_\epsilon \cup D'_\epsilon$ contains a neighborhood of the divisor, so $\nabla^{1,0}$ is bounded. The boundedness of θ is obtained by using the projection to the Hodge bundle and the L^2 -adaptedness theorem. \square

Remarks. It is easy to see that the residue $res(\nabla^{1,0})$ of $\nabla^{1,0}$ (according to the notations of [13]) at the singularity is $\{N_1, N_2\}$; again, $\{\sigma_{k,l} | k+l \geq s\}$ represent

a basis for \mathbf{F}_s and the suitable projections of the elements of $\{\sigma_{k,l} | k+l=s\}$ represents a basis of \mathbf{E}_s , so, by the above formula of the action of $\nabla^{1,0}$ on $\sigma_{k,l}$, the residue $res(\theta)$ of the induced map θ (Higgs field) of $\nabla^{1,0}$ on \mathbf{E}_s (Hodge bundles, $= \mathbf{F}_s/\mathbf{F}_{s+1}$) is also $\{N_1, N_2\}$. Namely, we have the following corollary, which is essentially due to Schmid and Simpson [12, 13].

Corollary 1 *The three weight filtrations corresponding to $res(\nabla^{1,0})$ of $\nabla^{1,0}$ (at the singularity, considered as acting on $\mathbf{H}_{\mathbb{C}}$), $res(\theta)$ of θ (at the singularity, considered as acting on the Hodge bundles) and $\{N_1, N_2\}$ coincide under some suitable identifications.*

In the above argument, we use the ordering (N_1, N_2) to construct an L^2 -adaptedness basis $\{\sigma_{k,l} | k+l=s\}$ for the Hodge bundle \mathbf{E}^s ; similarly, one can also use the other ordering (N_2, N_1) to construct the corresponding basis of \mathbf{E}^s . As a conclusion of this section, we will compare the two bases. In the following, a monodromy weight filtration is again considered as some translation (as in the previous) of the usual one as defined in the §2. From the §2 and the above discussion, without loss of generality, we will assume in the following that $(F, W = W(N_1 + N_2))$ is an \mathbb{R} -split mixed Hodge structure. We have two different orderings: (N_1, N_2) and (N_2, N_1) corresponding to the orderings of the coordinates t_1, t_2 on $(\Delta^*)^2$. In the previous discussion, we used the first ordering (N_1, N_2) and constructed a basis $\{\alpha_{k,l}\}$ (up to the action of some elements in $G_{\mathbb{C}}$) of the generic fiber $H_{\mathbb{C}}$ of the pull-back of $\mathbf{H}_{\mathbb{C}}$ to the universal covering of $(\Delta^*)^2$, which flags F , $W(N_1)$, and W ; actually, under the assumption of (F, W) being \mathbb{R} -split, one can furthermore assume that $\{\alpha_{k,l}\}$ corresponds to simultaneous \mathbb{Z}^2 -gradings of $W(N_1)$ and W over \mathbb{R} , namely $\alpha_{k,l}$ being the eigenvector of \tilde{Y}_1 and \tilde{Y}_2 with eigenvalues $2k-m$ and $2(k+l)-m-n$ respectively. Similarly, for the ordering (N_2, N_1) , we can also construct a basis $\{\alpha'_{k,l}\}$ of $H_{\mathbb{C}}$, which flags F , and grades simultaneously $W(N_2)$ and W in the same way as above (of course for a certain other semisimple element \tilde{Y}'_1 corresponding to N_2 , and \tilde{Y}_2). In the following, we will first discuss the relation between the both bases. For convenience, we here state the following

Proposition 4 ([4], Theorem 3.3(2); [2], Theorem 3)¹ *Let $W(N_2, Gr_k^{W(N_1)})$ be the weight filtration on $Gr_k^{W(N_1)}$ corresponding to the induced nilpotent endomorphism of N_2 from $Gr_k^{W(N_1)}$ to itself (still denoted by N_2). Then the projection of W to $Gr_k^{W(N_1)}$ just induces $W(N_2, Gr_k^{W(N_1)})[-k]$.*

By the previous construction, $\alpha_{k,l} \in (W(N_1))_{2k} \cap W_{2(k+l)}$ (resp. $\alpha'_{k,l} \in (W(N_2))_{2l} \cap W_{2(k+l)}$), so $\oplus_l \mathbb{C}\alpha_{k,l}$ can be identified with $Gr_{2k}^{W(N_1)}$. Accordingly,

¹This is a correction to the original statement in [4] by S. Zucker; for this, see *Mathematical Reviews*, 84a:32046 by S. Zucker.

$H_{\mathbb{C}}$ can be identified with $\bigoplus_k \text{Gr}_{2k}^{W(N_1)}$. Applying the above proposition to each piece $(\text{Gr}_{2k}^{W(N_1)}, N_2)$ and then summing up for k , we have $W_{2l}(N_2) = \bigoplus_{l' \leq l, k} \mathbb{C} \alpha_{k, l'}$; on the other hand, the above construction tells us that $W_{2l}(N_2) = \bigoplus_{l' \leq l, k} \mathbb{C} \alpha'_{k, l'}$. Thus, each $\alpha_{k, l}$ can be expressed linearly by $\{\alpha'_{k', l'}; l' \leq l, 0 \leq k' \leq m\}$ and $\alpha'_{k, l}$ also by $\{\alpha_{k', l'}; l' \leq l, 0 \leq k' \leq m\}$. The same discussion also applies to the ordering (N_2, N_1) and one has that each $\alpha_{k, l}$ can be expressed linearly by $\{\alpha'_{k', l'}; k' \leq k, 0 \leq l' \leq n\}$ and $\alpha'_{k, l}$ also by $\{\alpha_{k', l'}; k' \leq k, 0 \leq l' \leq n\}$. Therefore, each $\alpha_{k, l}$ can be expressed linearly by $\{\alpha'_{k', l'}; k' \leq k, l' \leq l\}$ and $\alpha'_{k, l}$ also by $\{\alpha_{k', l'}; k' \leq k, l' \leq l\}$. Namely, we have $\alpha'_{k, l} = \sum_{k' \leq k, l' \leq l} c_{k', l'} \alpha_{k', l'}$ and $\alpha_{k, l} = \sum_{k' \leq k, l' \leq l} c'_{k', l'} \alpha'_{k', l'}$ for some complex number $c_{k', l'}, c'_{k', l'}$. Denote the flat sections generated by $\alpha_{k, l}$ and $\alpha'_{k, l}$ by the same symbols. Denote the monodromized sections of $\alpha_{k, l}$ and $\alpha'_{k, l}$ by $\sigma_{k, l}$ and $\sigma'_{k, l}$. (Here and in the following, we always neglect the higher order terms.) It is clear by the previous construction of an L^2 -adapted basis for the Hodge bundles that $\{\sigma_{k, l}\}$ and $\{\sigma'_{k, l}\}$ are two L^2 -adapted bases. Since $\alpha_{k, l} \in (W(N_1))_{2k} \cap W_{2(k+l)}$ (resp. $\alpha'_{k, l} \in (W(N_2))_{2l} \cap W_{2(k+l)}$), by the estimate of the Hodge norm, we have

$$\|\sigma_{k, l}\|^2 \sim (\log|t_1|)^{2k-m} (\log|t_2|)^{2l-n}, \quad \text{on } D_{\epsilon}$$

and

$$\begin{aligned} \|\sigma'_{k, l}\|^2 &\sim \left(\frac{\log|t_2|}{\log|t_1|}\right)^{2l-n} (\log|t_1|)^{2(l+k)-m-n} \\ &= (\log|t_1|)^{2k-m} (\log|t_2|)^{2l-n}, \quad \text{on } D'_{\epsilon}. \end{aligned}$$

On the other hand, since $\alpha_{k, l} = \sum_{k' \leq k, l' \leq l} c'_{k', l'} \alpha'_{k', l'}$ for some complex number $c'_{k', l'}$, so $\sigma_{k, l} = \sum_{k' \leq k, l' \leq l} c'_{k', l'} \sigma'_{k', l'}$ (since both the parallel translation and the monodromization are linear). Thus, one has, on D'_{ϵ} ,

$$\begin{aligned} \|\sigma_{k, l}\|^2 &\sim \sum_{k' \leq k, l' \leq l} (c'_{k', l'})^2 \|\sigma'_{k', l'}\|^2 \\ &\sim \left(\sum_{k' \leq k, l' \leq l} (c'_{k', l'})^2 \log|t_1|\right)^{2k'-m} (\log|t_2|)^{2l'-n} \\ &\sim (\log|t_1|)^{2k-m} (\log|t_2|)^{2l-n}. \end{aligned}$$

Thus, we obtain the following

Proposition 5 *Near the singularity, there exists a holomorphic basis $\{\sigma_{k, l} \mid k+l = s\}$ for each Hodge bundle \mathbf{E}^s that satisfies*

$$\|\sigma_{k, l}\|^2 \sim (\log|t_1|)^{2k-m} (\log|t_2|)^{2l-n}$$

near the singularity.

4 L^2 -holomorphic Dolbeault complex

In this section, we still assume that $M = (\Delta^*)^2$ and $\overline{M} = (\Delta)^2$, $j : M \rightarrow \overline{M}$ is the inclusion map. Let $(M, \mathbf{H}_{\mathbb{C}} = \mathbf{H}_{\mathbb{Z}} \otimes \mathbb{C}, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{F} = \{\mathbf{F}^p\}, \mathbf{S})$ be a variation of polarized Hodge structure with weight m defined over \mathbb{Q} such that each \mathbf{F}^p is a holomorphic subbundle of the local system $\mathbf{H}_{\mathbb{C}}$ and

$$\nabla^{1,0}\mathbf{F}^p \subset \mathbf{F}^{p-1} \otimes \Omega^1(M).$$

Let γ_1 and γ_2 be the monodromy transformations of the variation, which are always assumed to be unipotent. Let N_1 and N_2 be the logarithmic monodromies of γ_1 and γ_2 respectively, which are nilpotent. Note that in the following, we consider $\mathbf{H}_{\mathbb{C}}$ as a flat bundle, a local system, a holomorphic vector bundle relative to $\nabla^{0,1}$, or a sheaf of local holomorphic sections, this depends on the context.

In the previous section, using the Poincaré-like metric on M and the Hodge metric on $\mathbf{H}_{\mathbb{C}}$, we define $\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$ to be the sheaf of local L^2 holomorphic r -forms valued in $j_*\mathbf{H}_{\mathbb{C}}$ on \overline{M} . One of the purposes of this section will be to show that the sheaf $\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$ can be defined algebraically, just using the logarithmic monodromies N_1 and N_2 , and lies in $j_*\mathbf{H}_{\mathbb{C}} \otimes \Omega_{\overline{M}}^r(\log D)$. As a consequence of this fact together with the asymptotic behavior of θ and the L^2 -adaptedness theorem of §3, we obtain the L^2 holomorphic Dolbeault complex on \overline{M} in §1 (for the precise definition, see this §)

$$(*) \quad \mathbf{E}_{(2)} \xrightarrow{\theta} (\mathbf{E} \otimes \Omega_{\overline{M}}^1(\log D))_{(2)} \xrightarrow{\theta} (\mathbf{E} \otimes \Omega_{\overline{M}}^2(\log D))_{(2)} \xrightarrow{\theta} \cdots,$$

which does not depend on the two metrics, but is defined algebraically.

$\mathbf{H}_{\mathbb{C}}$ has a canonical extension $\overline{\mathbf{H}}_{\mathbb{C}}$ on $\overline{M} = \Delta^2$ across the singularity as follows. As a sheaf, the germs of sections of $\overline{\mathbf{H}}_{\mathbb{C}}$ at the singularity are generated by the elements of $\Gamma(M, \mathbf{H}_{\mathbb{C}})$ which are of the form

$$\tilde{v} = \exp\left(\frac{1}{2\pi\sqrt{-1}}(N_1 \log t_1 + N_2 \log t_2)\right)v,$$

where v is a multivalued flat section of $\mathbf{H}_{\mathbb{C}}$. Again fixing an ordering (N_1, N_2) and a point $p \in M$, as in §2, one has two monodromy weight filtrations $\{W_k^1\}$ and $\{W_s^2\}$ defined over \mathbb{Q} corresponding to N_1 and $c_1 N_1 + c_2 N_2$, $c_1 > 0$, $c_2 > 0$ on $(\mathbf{H}_{\mathbb{C}})_p$ respectively, and they naturally determine two filtrations of $\mathbf{H}_{\mathbb{C}}$ by locally constant systems, denoted by $\{\mathbf{W}_k^1\}$ and $\{\mathbf{W}_s^2\}$ respectively; moreover, one also has the canonical extensions $\{\overline{\mathbf{W}}_k^1\}$ and $\{\overline{\mathbf{W}}_s^2\}$ of $\{\mathbf{W}_k^1\}$ and $\{\mathbf{W}_s^2\}$ across the singularity, which form two filtrations of $\overline{\mathbf{H}}_{\mathbb{C}}$ respectively.

Proposition 6 *If restricting everything to D_ϵ for $\epsilon > 0$, one has the following*

equivalences.

$$\begin{aligned}
\Omega^0(\mathbf{H}_\mathbb{C})_{(2)} &= t_1 t_2 \overline{\mathbf{H}}_\mathbb{C} + t_1 \bigcup_{l_2 - l_1 \leq 0} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2 + t_2 \overline{\mathbf{W}}_0^1 + \bigcup_{l_1 \leq 0, l_2 \leq l_1} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2; \\
\Omega^1(\mathbf{H}_\mathbb{C})_{(2)} &= \\
&\frac{dt_1}{t_1} \otimes (t_1 t_2 \overline{\mathbf{H}}_\mathbb{C} + t_1 \bigcup_{l_2 - l_1 \leq 0} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2 + t_2 \overline{\mathbf{W}}_{-2}^1 + \bigcup_{l_1 \leq -2, l_2 \leq l_1} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2) + \\
&+ \frac{dt_2}{t_2} \otimes (t_1 t_2 \overline{\mathbf{H}}_\mathbb{C} + t_1 \bigcup_{l_2 \leq l_1 - 2} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2 + t_2 \overline{\mathbf{W}}_0^1 + \bigcup_{l_1 \leq 0, l_2 \leq l_1 - 2} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2); \\
\Omega^2(\mathbf{H}_\mathbb{C})_{(2)} &= \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \otimes \\
&\otimes (t_1 t_2 \overline{\mathbf{H}}_\mathbb{C} + t_1 \bigcup_{l_2 \leq l_1 - 2} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2 + t_2 \overline{\mathbf{W}}_{-2}^1 + \bigcup_{l_1 \leq -2, l_2 \leq l_1 - 2} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2).
\end{aligned}$$

Here, $\tilde{v} \in \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2$ for $l_2 - l_1 \leq 0$ implies that v has nontrivial projections to both $Gr_{l_1}^{\mathbf{W}^1}$ and $Gr_{l_2}^{\mathbf{W}^2}$; the other terms have the same explanation. Similarly, one can consider the other ordering (N_2, N_1) and restrict $\Omega^i(\mathbf{H}_\mathbb{C})_{(2)}$, $i = 0, 1, 2$, to D'_ϵ to get the same characterization of the sheaves as above.

Remarks: 1) As ϵ is sufficiently small, $D_\epsilon \cup D'_\epsilon$ covers a neighborhood of the singularity, so the above proposition actually gives a description near the singularity of the sheaves $\Omega^i(\mathbf{H}_\mathbb{C})_{(2)}$, $i = 0, 1, 2$, in terms of the weight filtrations of $N_1, N_2, N_1 + N_2$. 2) From the formulae above, we also know that the sheaves $\Omega^i(\mathbf{H}_\mathbb{C})_{(2)}$ are contained in $j_* \mathbf{H}_\mathbb{C} \otimes \Omega_{\overline{M}}^r(\log D)$, $i = 0, 1, 2$. 3) Although the proposition and the following proof are presented for the case of dimension 2, they still work for general dimension.

Proof. We will first show that one can choose a suitable (flat sections) basis $\{v_i\}$ of $\mathbf{H}_\mathbb{C}$ near the singularity such that it flags $\mathbf{H}_\mathbb{C}$ according to the weight filtrations \mathbf{W}_*^1 and \mathbf{W}_*^2 and $\{\tilde{v}_i\}$ is L^2 -adapted in the sense of §2 on any domain of the form D_ϵ , $\epsilon > 0$. We will use freely the notations and arguments in §2. By the nilpotent orbit theorem and the Proposition 2, we know that $(W^2[-m], \psi(0))$ is a (polarized) mixed Hodge structure; again by Proposition 1, there exists a unique $\delta \in \mathfrak{gl}(H_\mathbb{R})$ such that $(W^2[-m], F = \exp(-\sqrt{-1}\delta)\psi(0))$ is a (polarized) mixed Hodge structure splitting over \mathbb{R} , where $H_\mathbb{R}$ can be considered as the real structure of the standard fibre $H_\mathbb{C}$ of $\mathbf{H}_\mathbb{C}$ as lifted to the universal covering of M . Furthermore, there exists an element $\tilde{g}(\infty) \in G_\mathbb{C}$ such that $\log \tilde{g}(\infty) \in \text{Ker}(\text{ad}(N_j))$ and $(W^2[-m], \tilde{F}_2 = \tilde{g}(\infty)F)$ is canonically attached to $(W^2[-m], \psi(0))$. A simple computation shows that the nilpotent

orbit $\theta(\mathbf{z})$ corresponding to the variation can be expressed as

$$\begin{aligned}\theta &= \exp\left(\sum z_j N_j\right) \cdot \psi(0) \\ &= \exp\left(\sum x_j N_j\right) \cdot e^{-1} \cdot \exp(\sqrt{-1}\text{Ad}(e)\left(\sum y_j N_j + \delta\right)) \cdot eF,\end{aligned}$$

where $t_j = e^{2\pi\sqrt{-1}z_j}$, $z_j = x_j + \sqrt{-1}y_j$, and $e = \exp(\frac{1}{2}\log y_1 \tilde{Y}_1 + \frac{1}{2}\log y_2 \tilde{Y}_2) = \exp(\frac{1}{2}\log s_1 \tilde{Y}_1 + \frac{1}{2}\log s_2 \tilde{Y}_2)$, here $s_1 = \frac{y_1}{y_2}$ and $s_2 = y_2$, while Ad is the adjoint representation of $G_{\mathbb{R}}$. By the SL_2 -orbit theorem (i), $\exp(\sqrt{-1}\tilde{N}_2^-) \cdot \tilde{F}_2 = \exp(\sqrt{-1}N_1) \cdot \tilde{F}_1 \in \mathbf{D}$. So, one has

$$F = \exp(-\sqrt{-1}\tilde{N}_2^-) \cdot \tilde{g}^{-1}(\infty) \cdot \exp(\sqrt{-1}N_1) \cdot \tilde{F}_1.$$

Set $F_0 = \tilde{g}^{-1}(\infty) \cdot \exp(\sqrt{-1}N_1) \cdot \tilde{F}_1 \in \mathbf{D}$ and $\tilde{p} = \exp(\sqrt{-1}\text{Ad}(e)\left(\sum y_j N_j + \delta\right))$. By the definitions of F and e , it is not difficult to see that $eF = F$. So, one has

$$\theta = \exp\left(\sum x_j N_j\right) \cdot e^{-1} \cdot \tilde{p} \cdot \exp(-\sqrt{-1}\tilde{N}_2^-) \cdot F_0.$$

Thus, by the nilpotent orbit theorem again, there exists a $G_{\mathbb{R}}$ -valued function $h(\mathbf{z})$ on \mathbb{C}^2 such that $h(\mathbf{z})$ goes to the identity as $\text{Im}(\mathbf{z}) \rightarrow \infty$ and

$$F_0 = h(\mathbf{z}) \cdot \exp(\sqrt{-1}\tilde{N}_2^-) \cdot \tilde{p}^{-1} \cdot e \cdot \exp\left(-\sum x_j N_j\right) \cdot \tilde{\phi}(\mathbf{z}).$$

On the other hand, the argument in Lemma 5.12 of [6] also tells us that

$$\tilde{p} \rightarrow \exp(\sqrt{-1}\tilde{N}_2^-), \quad \text{as } s_1, s_2 \rightarrow \infty.$$

More precisely, \tilde{p} and its inverse are polynomials in $\{s_1^{-\frac{1}{2}}, s_2^{-\frac{1}{2}}\}$ with constant coefficients and the constant terms being $\exp(\sqrt{-1}\tilde{N}_2^-)$ and $\exp(-\sqrt{-1}\tilde{N}_2^-)$ respectively; moreover, $\exp(\sqrt{-1}\tilde{N}_2^-) \cdot \tilde{p}^{-1}$ can be expressed as $\exp(-\sqrt{-1}\delta(s_2)) \cdot \exp(-\sqrt{-1}Z(s_1))$, where $\delta(s_2)$ and $Z(s_1)$ are some polynomials in s_2^{-1} and s_1^{-1} respectively with coefficients in $\mathfrak{g}_{\mathbb{R}}$. Denote $h(\mathbf{z}) \cdot \exp(\sqrt{-1}\tilde{N}_2^-) \cdot \tilde{p}^{-1}$ by $\tilde{h}(\mathbf{z})$, which, by the behavior of $h(\mathbf{z})$ as $\text{Im}(\mathbf{z}) \rightarrow \infty$, goes to the identity as $s_1, s_2 \rightarrow \infty$. (Note that $s_1, s_2 \rightarrow \infty$ implies $\text{Im}(\mathbf{z}) \rightarrow \infty$, but not the converse. That's why we take the domains D_ϵ in the estimate of the Hodge norm.) Since the image of $\tilde{\phi}$ for enough large $\text{Im}(\mathbf{z})$ and F_0 lie in \mathbf{D} , so $\tilde{h}(\mathbf{z}) \in G_{\mathbb{R}}$ for large enough $\text{Im}(\mathbf{z})$. Thus, we establish an isometry $\tilde{h}(\mathbf{z}) \cdot e \cdot \exp(-\sum x_j N_j)$ between the fiber over \mathbf{z} and $F_0 \in \mathbf{D}$ for large enough $\text{Im}(\mathbf{z})$, denoted by $L(\mathbf{z})$.

Let $v \in W_{l_1}^1 \cap W_{l_2}^2$ and assume that its projections on $Gr_{l_1}^{W_1^1}$ and $Gr_{l_2}^{W_2^2}$ are nontrivial, i.e., the components of v with eigenvalues l_1 and l_2 w.r.t. \tilde{Y}_1 and \tilde{Y}_2 respectively are nontrivial. (Note that $W_{l_j}^j$ is characterized by the property that $W_{l_j}^j$ is linearly spanned by all eigenvectors of \tilde{Y}_j of eigenvalue not greater than l_j . Afterwards, denote the eigenvector space \tilde{Y}_j for the eigenvalue l_j by $H_{l_j}(\tilde{Y}_j)$)

In the following, we also denote by v the multivalued flat section generated by v , this will be clear from the context. Write \tilde{v} as $\tilde{v}(\mathbf{z}) = \exp(\sum z_j N_j)v$ (i.e. lift \tilde{v} to the universal covering of M) and then consider

$$L(\mathbf{z})\tilde{v}(\mathbf{z}) = \tilde{h}(\mathbf{z}) \cdot \exp\left(\frac{1}{2} \log s_1 \tilde{Y}_1 + \frac{1}{2} \log s_2 \tilde{Y}_2\right) \cdot \exp\left(-\sum x_j N_j\right) \cdot \tilde{v}(\mathbf{z}).$$

A direct computation shows

$$\begin{aligned} & \exp\left(\frac{1}{2} \log s_1 \tilde{Y}_1 + \frac{1}{2} \log s_2 \tilde{Y}_2\right) \cdot \exp\left(-\sum x_j N_j\right) \cdot \tilde{v}(\mathbf{z}) \\ = & \exp\left(\frac{1}{2} \log s_1 \tilde{Y}_1 + \frac{1}{2} \log s_2 \tilde{Y}_2\right) \exp(\sqrt{-1}s_2(N_2 + s_1 N_1)) \cdot v \\ = & \sum_{k=0}^m \frac{(\sqrt{-1})^k}{k!} \sum_{j=0}^k C_k^j s_1^{k-j} s_2^k \exp\left(\frac{1}{2} \log s_1 \tilde{Y}_1 + \frac{1}{2} \log s_2 \tilde{Y}_2\right) \cdot N_1^{k-j} N_2^j v. \end{aligned}$$

On the other hand, by the definitions of W_*^1 and W_*^2 , it is not difficult to see that

$$N_1^{k-j} N_2^j \cdot v \in W_{l_1-2k+2j}^1 \cap W_{l_2-2k}^2,$$

and its $l_1 - 2k + 2j$ -eigenvector part and $l_2 - 2k$ -eigenvector part w.r.t. \tilde{Y}_1 and \tilde{Y}_2 respectively are nontrivial. So,

$$\begin{aligned} & \exp\left(\frac{1}{2} \log s_1 \tilde{Y}_1 + \frac{1}{2} \log s_2 \tilde{Y}_2\right) \cdot \exp\left(-\sum x_j N_j\right) \cdot \tilde{v}(\mathbf{z}) \\ = & \sum_{k=0}^m \frac{(\sqrt{-1})^k}{k!} \sum_{j=0}^k C_k^j s_1^{k-j} s_2^k s_1^{\frac{l_1}{2}-k+j} s_2^{\frac{l_2}{2}-k} N_1^{k-j} N_2^j v + (\text{lower order terms}) \\ = & s_1^{\frac{l_1}{2}} s_2^{\frac{l_2}{2}} \sum_{k=0}^m \frac{(\sqrt{-1})^k}{k!} (N_1 + N_2)^k v + (\text{lower order terms}). \end{aligned}$$

Thus, if $\{v_l\}$ is a basis of $(\mathbf{H}_{\mathbb{C}})_p$, which flags $(\mathbf{H}_{\mathbb{C}})_p$ according to the filtrations W_*^1 and W_*^2 (this can always be done by the characterization of W_*^1 and W_*^2 using $\tilde{Y}_j, j = 1, 2$), then $\{s_1^{-\frac{l_1}{2}} s_2^{-\frac{l_2}{2}} L(\mathbf{z})\tilde{v}_l(\mathbf{z})\}$ goes to a basis of $H_{\mathbb{C}}$ as $s_1, s_2 \rightarrow \infty$. So, by the definition of L^2 -adapted basis and the lemma 2 in §2, $\{s_1^{-\frac{l_1}{2}} s_2^{-\frac{l_2}{2}} L(\mathbf{z})\tilde{v}_l(\mathbf{z})\}$, and hence $\{\tilde{v}_l(\mathbf{z})\}$, is an L^2 -adapted basis on any domain of the form

$$\tilde{D}_\epsilon = \{\mathbf{z} = (z_1, z_2) \in \mathbb{C}^2 \mid \frac{\text{Im}z_1}{\text{Im}z_2} > \epsilon, \text{Im}z_2 > \epsilon\}, \quad \epsilon > 0.$$

Dropping $\{\tilde{v}_l(\mathbf{z})\}$ and \tilde{D}_ϵ back to M , $\{\tilde{v}_l\}$ becomes an L^2 -adapted basis on D_ϵ . This shows the claim in the beginning of the proof.

We now turn to prove the proposition. We will prove only the second identity, the others are similar. We also restrict all the following discussion to D_ϵ for some fixed $\epsilon > 0$, if not specified. Due to all objects considered being holomorphic under $\nabla^{0,1}$ and the argument above of getting an L^2 -adapted basis, we only need to show when the elements of the form

$$\frac{dt_1}{t_1} \otimes t_1^{n_1} t_2^{n_2} \tilde{v}_1 + \frac{dt_2}{t_2} \otimes t_1^{n'_1} t_2^{n'_2} \tilde{v}_2$$

are L^2 for $v_1 \in \mathbf{W}_{l_1}^1 \cap \mathbf{W}_{l_2}^2$ and $v_2 \in \mathbf{W}_{l'_1}^1 \cap \mathbf{W}_{l'_2}^2$ satisfying that the projections of both v_1 to $Gr_{l_1}^{\mathbf{W}^1}$ and $Gr_{l_2}^{\mathbf{W}^2}$ and v_2 to $Gr_{l'_1}^{\mathbf{W}^1}$ and $Gr_{l'_2}^{\mathbf{W}^2}$ are nontrivial. By the estimate of the Hodge norm of Theorem 4, we have

$$\left| \frac{dt_1}{t_1} \otimes t_1^{n_1} t_2^{n_2} \tilde{v}_1 \right|^2 \sim |\log |t_1||^2 |t_1|^{2n_1} |t_2|^{2n_2} |\log |t_1||^{l_1} |\log |t_2||^{l_2-l_1}$$

on D_ϵ . So, $\frac{dt_1}{t_1} \otimes t_1^{n_1} t_2^{n_2} \tilde{v}_1$ is L^2 iff the following integral

$$\int_{D_\epsilon} |\log |t_1||^2 |t_1|^{2n_1} |t_2|^{2n_2} |\log |t_1||^{l_1} |\log |t_2||^{l_2-l_1} dV_M$$

is finite, where dV_M is the volume element of M under the Poincaré-like metric. A simple computation shows that the above integral is finite iff

$$\begin{aligned} n_1 &\geq 1 \quad \text{or} \quad n_1 = 0 \quad \text{and} \quad l_1 \leq -2; \\ n_2 &\geq 1 \quad \text{or} \quad n_2 = 0 \quad \text{and} \quad l_2 - l_1 \leq 0. \end{aligned}$$

Equivalently, $\frac{dt_1}{t_1} \otimes t_1^{n_1} t_2^{n_2} \tilde{v}_1$ belongs to

$$\frac{dt_1}{t_1} \otimes (t_1 t_2 \overline{\mathbf{H}}_{\mathbb{C}} + t_1 \bigcup_{l_2-l_1 \leq 0} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2 + t_2 \overline{\mathbf{W}}_{-2}^1 + \bigcup_{l_1 \leq -2, l_2 \leq l_1} \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2).$$

From the discussion above, the above formula should be understood as follows. $\tilde{v}_1 \in \overline{\mathbf{W}}_{l_1}^1 \cap \overline{\mathbf{W}}_{l_2}^2$ for $l_2 - l_1 \leq 0$ implies that v_1 has nontrivial projections to both $Gr_{l_1}^{\mathbf{W}^1}$ and $Gr_{l_2}^{\mathbf{W}^2}$; the other terms have a similar explanation.

The same computation shows that $\frac{dt_2}{t_2} \otimes t_1^{n'_1} t_2^{n'_2} \tilde{v}_2$ is L^2 iff

$$\begin{aligned} n'_1 &\geq 1 \quad \text{or} \quad n'_1 = 0 \quad \text{and} \quad l'_1 \leq 0; \\ n'_2 &\geq 1 \quad \text{or} \quad n'_2 = 0 \quad \text{and} \quad l'_2 - l'_1 \leq -2. \end{aligned}$$

Namely, $\frac{dt_2}{t_2} \otimes t_1^{n'_1} t_2^{n'_2} \tilde{v}_2$ belongs to

$$\frac{dt_2}{t_2} \otimes (t_1 t_2 \overline{\mathbf{H}}_{\mathbb{C}} + t_1 \bigcup_{l'_2 \leq l'_1 - 2} \overline{\mathbf{W}}_{l'_1}^1 \cap \overline{\mathbf{W}}_{l'_2}^2 + t_2 \overline{\mathbf{W}}_0^1 + \bigcup_{l'_1 \leq 0, l'_2 \leq l'_1 - 2} \overline{\mathbf{W}}_{l'_1}^1 \cap \overline{\mathbf{W}}_{l'_2}^2).$$

Clearly, the terms of the formula above should have the same explanation as above. Thus, we finish the proof of the proposition. \square

In the following, we turn to the case of general dimension. Let

$$\mathbf{E} = \bigoplus_{p=0}^m \mathbf{E}^p, \quad \theta = \sum_p \theta^p$$

be the Higgs bundle corresponding to the variation as defined in the introduction. From §3, we know θ is an L^2 -bounded holomorphic 1-form operator and

$$F^{p+1}\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}/F^p\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)} \cong (\Omega^r \otimes \mathbf{E}^{p-r})_{(2)},$$

where $F^p\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$ is the filtration of $\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$ according to $\{\mathbf{F}^p\}$ (cf. §3). On the other hand, by the previous proposition, $\Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)} \subset j_*\mathbf{H}_{\mathbb{C}} \otimes \Omega_{\overline{M}}^r(\log D)$ and can be described by just using the weight filtrations of $N_1, N_2, N_1 + N_2$, so $(\Omega^r \otimes \mathbf{E}^{p-r})_{(2)}$ is actually the set of local L^2 -sections of $j_*\mathbf{E}^{p-r} \otimes \Omega_{\overline{M}}^r(\log D)$ and can essentially be described in terms of the weight filtrations of $N_1, N_2, N_1 + N_2$. Omitting j_* , we from now on denote $(\Omega^r \otimes \mathbf{E}^{p-r})_{(2)}$ by $(\mathbf{E}^{p-r} \otimes \Omega_{\overline{M}}^r(\log D))_{(2)}$, $\bigoplus_p(\mathbf{E}^{p-r} \otimes \Omega_{\overline{M}}^r(\log D))_{(2)}$ by $(\mathbf{E} \otimes \Omega_{\overline{M}}^r(\log D))_{(2)}$. The L^2 boundedness of θ and $\theta \wedge \theta = 0$ tell us that the following sequence

$$(*) \quad \mathbf{E}_{(2)} \xrightarrow{\theta} (\mathbf{E} \otimes \Omega_{\overline{M}}^1(\log D))_{(2)} \xrightarrow{\theta} (\mathbf{E} \otimes \Omega_{\overline{M}}^2(\log D))_{(2)} \xrightarrow{\theta} \dots$$

is a complex, called the L^2 holomorphic Dolbeault complex.

Remark: By the above argument and Proposition 6, one can actually write down $(\mathbf{E} \otimes \Omega_{\overline{M}}^r(\log D))_{(2)}$ explicitly in terms of the basis constructed in §3. Let $\{\sigma_{k,l} | k+l=s\}$ be the basis of \mathbf{E}^s constructed in §3. By the construction, we know that $\sigma_{k,l} \in \overline{\mathbf{W}}_{2k}^1 \cap \overline{\mathbf{W}}_{2(k+l)}^2$ (under a suitable identification, cf. Corollary 1) up to some higher order terms. So, using the estimate of the Hodge norm, we can show which holomorphic r -forms valued in the line bundle generated by $\sigma_{k,l}$ are L^2 .

5 L^2 $\overline{\partial}$ -Poincaré lemma

Let $(M, \mathbf{H}_{\mathbb{C}} = \mathbf{H}_{\mathbb{Z}} \otimes \mathbb{C}, \nabla = \nabla^{1,0} + \nabla^{0,1}, \mathbf{F} = \{\mathbf{F}^p\}, \mathbf{S})$ be a variation of polarized Hodge structure with weight m defined over \mathbb{Q} . In the previous section, we defined the L^2 holomorphic Dolbeault complex $\{(\mathbf{E} \otimes \Omega_{\overline{M}}^r(\log D))_{(2)}, \theta\}$ of the variation on \overline{M} , which is independent of both metrics and essentially determined by the logarithmic monodromies. For uniformity of notations of this section, we from now on denote $(\mathbf{E}^{p-r} \otimes \Omega_{\overline{M}}^r(\log D))_{(2)}$ by $\mathrm{Gr}_F^p \Omega^r(\mathbf{H}_{\mathbb{C}})_{(2)}$, which is a piece of $\{(\mathbf{E} \otimes \Omega_{\overline{M}}^r(\log D))_{(2)}\}$ as seen at the end of §4.

By the infinitesimal period relation of $\nabla^{1,0}$ and the definition and the boundedness of θ , we have

$$\theta(\{Gr_F^p \Omega^r(\mathbf{H}_C)_{(2)}\}) \subset \{Gr_F^p \Omega^{r+1}(\mathbf{H}_C)_{(2)}\}.$$

(More precisely, we should restrict to a piece of θ above.) Thus, we obtain a holomorphic Dolbeault subcomplex of $\{(\mathbf{E} \otimes \Omega_{\overline{M}}(\log D))_{(2)}, \theta\}$ on \overline{M}

$$\{Gr_F^p \Omega(\mathbf{H}_C)_{(2)}, \theta\}.$$

The main purpose of this section is then to show that the hypercohomology $\mathbb{H}^*(\overline{M}, \{Gr_F^p \Omega(\mathbf{H}_C)_{(2)}, \theta\})$ is equal to a certain L^2 -cohomology on \overline{M} . Thus, it is computable from a certain complex of L^2 -differential forms.

To this end, we first define some fine sheaves on \overline{M} . Let $F^p A^k(\mathbf{H}_C)_{(2)} = \bigoplus_{r+s=k} (A^{r,s} \otimes \mathbf{F}^{p-r})_{(2)}$ and $D_1'' = \overline{\partial} + \nabla^{1,0}$ for $p \geq 0$. Here, $A^{r,s}$ is the sheaf of germs of local forms of type (r, s) (not necessarily smooth) on \overline{M} and $(A^{r,s} \otimes \mathbf{F}^{p-r})_{(2)}$ is the sheaf of germs of local L^2 \mathbf{F}^{p-r} -valued forms ϕ of type (r, s) on \overline{M} for which $\overline{\partial}\phi$ are L^2 in the weak sense. Here the action of $\overline{\partial}$ is defined as follows: Let ϕ be a form of type (r, s) and v a holomorphic section of \mathbf{F}^p , then

$$\overline{\partial}(\phi \otimes v) = \overline{\partial}\phi \otimes v.$$

Note that $\{F^p A^k(\mathbf{H}_C)_{(2)}, p \geq 0\}$ is in general not a filtration of $A^k(\mathbf{H}_C)_{(2)}$ (for the definition, cf. §1). It is clear that $D_1''(F^p A^k(\mathbf{H}_C)_{(2)}) \subset F^{p+1} A^{k+1}(\mathbf{H}_C)_{(2)}$ by the boundedness of $\nabla^{1,0}$ (§3). By the Hodge filtration $\{\mathbf{F}^p\}$, take the successive quotients

$$[Gr_F^p A^k(\mathbf{H}_C)]_{(2)} := F^p A^k(\mathbf{H}_C)_{(2)} / F^{p+1} A^k(\mathbf{H}_C)_{(2)},$$

which, by the L^2 -adaptedness theorem (which is clearly true for the differentiable case by the proof), can be identified with $\bigoplus_{r+s=k} (A^{r,s} \otimes \mathbf{E}^{p-r})_{(2)}$. Here, $(A^{r,s} \otimes \mathbf{E}^{p-r})_{(2)}$ is the sheaf on \overline{M} of germs of local L^2 \mathbf{E}^{p-r} -valued forms ϕ of type (r, s) for which $\overline{\partial}\phi$ are L^2 in the weak sense. Denote the induced map of D_1'' by D'' , which is actually $\overline{\partial} + \theta$ and clearly satisfies $(D'')^2 = 0$. We now obtain a complex of fine sheaves on \overline{M}

$$\{[Gr_F^p A(\mathbf{H}_C)]_{(2)}, D''\},$$

for $p \geq 0$ and the holomorphic Dolbeault subcomplex $\{Gr_F^p \Omega(\mathbf{H}_C)_{(2)}, \theta\}$ is obviously its subcomplex. Similar to the holomorphic Dolbeault subcomplex above, $([Gr_F^p A(\mathbf{H}_C)]_{(2)}, D'')$ can also be considered as a piece of a larger complex of fine sheaves: Let \mathbf{E} be the Higgs bundle of the variation with the Higgs field θ , denote by $[Gr_F^* A^k(\mathbf{H}_C)]_{(2)}$ the sheaf of germs of local L^2 k -forms ϕ (not necessarily smooth) with values in \mathbf{E} on \overline{M} , for which $\overline{\partial}\phi$ are also L^2 in the weak sense. It is clear that $\{[Gr_F^* A(\mathbf{H}_C)]_{(2)}, D''\}$ is a complex of fine sheaves and $\{[Gr_F^p A(\mathbf{H}_C)]_{(2)}, D''\}$ is a piece of it.

The purpose of the remaining part in this section will be to show the following

Theorem 8 *The holomorphic Dolbeault complex $\{Gr_F^p \Omega(\mathbf{H}_\mathbb{C})_{(2)}, \theta\}$ is quasi-isomorphic to the complex $\{[Gr_F^p A(\mathbf{H}_\mathbb{C})]_{(2)}, D''\}$ under the inclusion map for $p \geq 0$.*

Concerning the definition of *quasi-isomorphism* and relative notations, one can refer to [10] or the appendix in this paper. Summing up for $p \geq 0$, the above theorem implies

Corollary 2 *The holomorphic Dolbeault complex $(*) \{(\mathbf{E} \otimes \Omega_{\overline{M}}(\log D))_{(2)}, \theta\}$ is quasi-isomorphic to the complex $(**) \{[Gr_F^* A(\mathbf{H}_\mathbb{C})]_{(2)}, D''\}$ under the inclusion map.*

The standard result (cf. [9]) then tells us that the hypercohomologies of $(*)$ and $(**)$ are isomorphic:

$$\mathbb{H}^*(\overline{M}, \{(\mathbf{E} \otimes \Omega_{\overline{M}}(\log D))_{(2)}, \theta\}) \simeq \mathbb{H}^*(\overline{M}, \{[Gr_F^* A(\mathbf{H}_\mathbb{C})]_{(2)}, D''\});$$

on the other hand, since $\{[Gr_F^* A(\mathbf{H}_\mathbb{C})]_{(2)}, D''\}$ is a complex of fine sheaves, so its hypercohomology is just the cohomology $H^*(\{\Gamma(\overline{M}), [Gr_F^* A(\mathbf{H}_\mathbb{C})]_{(2)}, D''\})$ of the corresponding complex of global sections of the sheaves (cf. [9] or the appendix). Thus we obtain Theorem A

$$\mathbb{H}^*(\overline{M}, \{(\mathbf{E} \otimes \Omega_{\overline{M}}(\log D))_{(2)}, \theta\}) \simeq H^*(\{\Gamma(\overline{M}), [Gr_F^* A(\mathbf{H}_\mathbb{C})]_{(2)}, D''\}).$$

In order to prove that the inclusion map is a quasi-isomorphism, we need to prove the $\overline{\partial}$ -Poincaré lemma under the present situation. More concretely, we need to prove:

$L^2 \overline{\partial}$ -Poincaré Lemma valued in a hermitian vector bundle: *Let $\tilde{\psi}^{r,k-r} \in (A^{r,k-r} \otimes \mathbf{E}^{p-r})_{(2)} (\subset [Gr_F^p A^k(\mathbf{H}_\mathbb{C})]_{(2)})$ be a local section near the divisor with $\overline{\partial} \tilde{\psi}^{r,k-r} = 0$. Then there exists a local section $\psi^{r,k-r-1} \in (A^{r,k-r-1} \otimes \mathbf{E}^{p-r})_{(2)}$ near the divisor (possibly on a smaller defining domain) with*

$$\overline{\partial} \psi^{r,k-r-1} = \tilde{\psi}^{r,k-r}.$$

Note that for a neighborhood not containing the divisor, the $L^2 \overline{\partial}$ -Poincaré lemma is just the classical $\overline{\partial}$ -Poincaré lemma. As seen in §3, near the divisor, \mathbf{E}^s is generated by some holomorphic sections $\{\sigma_{k,l} | k+l = s\}$, which are linearly independent everywhere and satisfy the norm estimates of the following form (cf. Prop. 4)

$$\|\sigma_{k,l}\|^2 \sim (-\log|t_1|)^{2k-m} (-\log|t_2|)^{2l-n}.$$

So, we can reduce the above $L^2 \overline{\partial}$ -Poincaré lemma to the following

$L^2 \overline{\partial}$ -Poincaré Lemma valued in a line bundle: *Let L be a holomorphic line bundle on $(\Delta^*)^n$ with a generating section σ , and with a Hermitian metric satisfying, for some fixed integers k, l ,*

$$\|\sigma\|^2 \sim (-\log|t_1|)^k (-\log|t_2|)^l$$

near the divisor. Then for each L -valued L^2 (r, s) -form $\phi \otimes \sigma$ near the divisor with $\bar{\partial}(\phi \otimes \sigma) = 0$, there exists an L -valued L^2 $(r, s - 1)$ -form $\psi \otimes \sigma$ (possibly on a smaller defining domain) with $\bar{\partial}(\psi \otimes \sigma) = \phi \otimes \sigma$, i.e., $\bar{\partial}\psi = \phi$.

Very unfortunately, we can prove the above L^2 $\bar{\partial}$ -Poincaré lemma only for $k \neq k(r)$ and $l \neq l(r)$, where $k(r)$ and $l(r)$ are some integers depending on r (for details, see the following remark). It is however lucky enough that the proof of quasi-isomorphism can be obtained just from the above L^2 $\bar{\partial}$ -Poincaré lemma for $k \neq k(r)$ and $l \neq l(r)$ and the properties of θ established in §3. We now begin to prove the L^2 $\bar{\partial}$ -Poincaré lemma above for $k \neq k(r)$ and $l \neq l(r)$. Actually we only need to prove it for $r = 0$, while at the time $k(0) = l(0) = 1$ (for details, see the following remark). For emphasis, we restate it as

Proposition 7 *Let L be a holomorphic line bundle on $(\Delta^*)^n$ with a generating section σ , and with a Hermitian metric satisfying*

$$\|\sigma\|^2 \sim (-\log|t_1|)^k (-\log|t_2|)^l \quad \text{for } k, l \neq 1$$

near the divisor. Then for each L -valued L^2 $(0, s)$ -form ($s \geq 1$) $\phi \otimes \sigma$ with $\bar{\partial}(\phi \otimes \sigma) = 0$ near the divisor, there exists an L -valued L^2 $(0, s - 1)$ -form $\psi \otimes \sigma$ (possibly on a smaller defining domain) with $\bar{\partial}(\psi \otimes \sigma) = \phi \otimes \sigma$, i.e., $\bar{\partial}\psi = \phi$.

Proof. For the sake of simplicity, we again assume $n = 2$. In order to prove the proposition, we actually need to prove only the following assertion: *Let ϕ be a form of type $(0, s)$ on $(\Delta^*)^2$ with*

$$\|\phi \otimes \sigma\|_{(2)}^2 =: \iint_{(\Delta^*)^2} \|\phi\|^2 \|\sigma\|^2 \frac{dt_1 \wedge d\bar{t}_1}{|t_1|^2 (-\log|t_1|)^2} \frac{dt_2 \wedge d\bar{t}_2}{|t_2|^2 (-\log|t_2|)^2} < \infty$$

and $\bar{\partial}\phi = 0$. Then there exists a form ψ of type $(0, s - 1)$ on $(\Delta^*)^2$ with

$$\|\psi \otimes \sigma\|_{(2)}^2 =: \iint_{(\Delta^*)^2} \|\psi\|^2 \|\sigma\|^2 \frac{dt_1 \wedge d\bar{t}_1}{|t_1|^2 (-\log|t_1|)^2} \frac{dt_2 \wedge d\bar{t}_2}{|t_2|^2 (-\log|t_2|)^2} < \infty$$

and $\bar{\partial}\psi = \phi$. Here, the norms $\|\phi\|$ and $\|\psi\|$ are measured under the Poincaré-like metric. We will use Fourier series to construct a formal solution ψ to $\bar{\partial}\psi = \phi$ for the given ϕ satisfying, for some constant $C > 0$,

$$\|\psi \otimes \sigma\|_{(2)}^2 \leq C \|\phi \otimes \sigma\|_{(2)}^2,$$

which shows that ψ is indeed a real solution to the $\bar{\partial}$ -problem. Using the technique of approximation, one can furthermore assume that ϕ has compact support in $(\Delta^*)^2$.

²In general the defining domain of ψ will be smaller than that of ϕ . Afterwards, we will not specify this, since it is clear from the context.

The proof of the assertion: Using polar coordinates (r_i, θ_i) for coordinate components t_i with $|t_i| = r_i$, $i = 1, 2$, the Poincaré-like metric can then be rewritten as (neglecting the higher term which essentially plays no role)

$$\frac{dr_1^2 + r_1^2 d\theta_1^2}{r_1^2 \log^2 r_1} + \frac{dr_2^2 + r_2^2 d\theta_2^2}{r_2^2 \log^2 r_2}.$$

Since we assume that the dimension of the base manifold is 2, we have only two cases to consider: 1) ϕ is a $(0, 1)$ -form; 2) ϕ is a $(0, 2)$ -form.

Case 1: Write ϕ as $f^1 d\bar{t}_1 + f^2 d\bar{t}_2$. The conditions on ϕ are then equivalent to the following integrability condition

$$\frac{\partial f^1}{\partial \bar{t}_2} = \frac{\partial f^2}{\partial \bar{t}_1}$$

and

$$\begin{aligned} & \iint_{(\Delta^*)^2} (|f^1|^2 |t_2|^{-2} |\log |t_2||^{-2} \|\sigma\|^2 dt_1 \wedge d\bar{t}_1 \wedge dt_2 \wedge d\bar{t}_2 + \\ & \iint_{(\Delta^*)^2} (|f^2|^2 |t_1|^{-2} |\log |t_1||^{-2} \|\sigma\|^2 dt_1 \wedge d\bar{t}_1 \wedge dt_2 \wedge d\bar{t}_2 < \infty; \end{aligned}$$

and we need to prove that there exists a function u on $(\Delta^*)^2$ satisfying

$$\iint_{(\Delta^*)^2} |u|^2 |t_1|^{-2} |t_2|^{-2} (-\log |t_1|)^{-2} (-\log |t_2|)^{-2} \|\sigma\|^2 dt_1 \wedge d\bar{t}_1 \wedge dt_2 \wedge d\bar{t}_2 < \infty$$

and

$$\frac{\partial u}{\partial \bar{t}_1} = f^1; \quad \frac{\partial u}{\partial \bar{t}_2} = f^2.$$

In order to do this, write f^1 , f^2 , and u as (r_1, r_2) -dependent Fourier series

$$\begin{aligned} f^1 &= \sum f_{m,n}^1(r_1, r_2) \exp(\sqrt{-1}m\theta_1 + \sqrt{-1}n\theta_2), \\ f^2 &= \sum f_{m,n}^2(r_1, r_2) \exp(\sqrt{-1}m\theta_1 + \sqrt{-1}n\theta_2), \\ u &= \sum u_{m,n}(r_1, r_2) \exp(\sqrt{-1}m\theta_1 + \sqrt{-1}n\theta_2). \end{aligned}$$

Since $\frac{\partial}{\partial \bar{t}_i} = \frac{1}{2} e^{\sqrt{-1}\theta_i} \left[\frac{\partial}{\partial r_i} + \frac{\sqrt{-1}}{r_i} \frac{\partial}{\partial \theta_i} \right]$, $i = 1, 2$, the integrability condition $\frac{\partial f^1}{\partial \bar{t}_2} = \frac{\partial f^2}{\partial \bar{t}_1}$ becomes

$$\frac{\partial f_{m+1,n}^1}{\partial r_2} - \frac{n}{r_2} f_{m+1,n}^1 = \frac{\partial f_{m,n+1}^2}{\partial r_1} - \frac{m}{r_1} f_{m,n+1}^2;$$

while the equations $\frac{\partial u}{\partial \bar{t}_1} = f^1$, $\frac{\partial u}{\partial \bar{t}_2} = f^2$ become

$$\frac{1}{2} \left[\frac{\partial u_{m,n}}{\partial r_1} - \frac{m}{r_1} u_{m,n} \right] = f_{m+1,n}^1, \quad \frac{1}{2} \left[\frac{\partial u_{m,n}}{\partial r_2} - \frac{n}{r_2} u_{m,n} \right] = f_{m,n+1}^2,$$

for all $m, n \in \mathbb{Z}$; or

$$\begin{aligned} \frac{\partial}{\partial r_2}(r_1^{-m}r_2^{-n}f_{m+1,n}^1) &= \frac{\partial}{\partial r_1}(r_1^{-m}r_2^{-n}f_{m,n+1}^1); \\ \frac{\partial}{\partial r_1}(r_1^{-m}r_2^{-n}u_{m,n}) &= 2r_1^{-m}r_2^{-n}f_{m+1,n}^1, \quad \frac{\partial}{\partial r_2}(r_1^{-m}r_2^{-n}u_{m,n}) = 2r_1^{-m}r_2^{-n}f_{m,n+1}^2. \end{aligned}$$

Put, for all $m, n \in \mathbb{Z}$,

$$u_{m,n}(r_1, r_2) = 2r_1^m r_2^n \int_L \rho_1^{-m} \rho_2^{-n} f_{m+1,n}^1(\rho_1, \rho_2) d\rho_1 + \rho_1^{-m} \rho_2^{-n} f_{m,n+1}^2(\rho_1, \rho_2) d\rho_2,$$

where the integral is a curve integral along the following oriented curve (for some positive constant $A < 1$)

$$L = \begin{cases} (0, 0) \rightarrow (r_1, r_2) & \text{if } (m < 0; n < 0), (m < 0; n = 0, l > 1), \\ & (m = 0, k > 1; n < 0) \text{ or } (m = 0, k > 1; n = 0, l > 1); \\ (0, A) \rightarrow (r_1, r_2) & \text{if } (m < 0; n > 0), (m < 0; n = 0, l > 1), \\ & (m = 0, k > 1; n > 0) \text{ or } (m = 0, k > 1; n = 0, l < 1); \\ (A, 0) \rightarrow (r_1, r_2) & \text{if } (m > 0; n < 0), (m > 0; n = 0, l > 1), \\ & (m = 0, k < 1; n < 0) \text{ or } (m = 0, k < 1; n = 0, l > 1); \\ (A, A) \rightarrow (r_1, r_2) & \text{if } (m > 0; n > 0), (m > 0; n = 0, l < 1), \\ & (m = 0, k < 1; n > 0) \text{ or } (m = 0, k < 1; n = 0, l < 1), \end{cases}$$

and formally

$$u = \sum u_{m,n}(r_1, r_2) \exp(\sqrt{-1}m\theta_1 + \sqrt{-1}n\theta_2).$$

We will next show that

$$\begin{aligned} & \iint_{(\Delta^*)^2} |u|^2 \|\sigma\|^2 \frac{dt_1 \wedge d\bar{t}_1}{|t_1|^2 (-\log|t_1|)^2} \frac{dt_2 \wedge d\bar{t}_2}{|t_2|^2 (-\log|t_2|)^2} \\ & \leq C \iint_{(\Delta^*)^2} \|\phi\|^2 \|\sigma\|^2 \frac{dt_1 \wedge d\bar{t}_1}{|t_1|^2 (-\log|t_1|)^2} \frac{dt_2 \wedge d\bar{t}_2}{|t_2|^2 (-\log|t_2|)^2} \end{aligned}$$

for some constant C , i.e. $\|u\sigma\|_{(2)}^2 \leq C\|\phi \otimes \sigma\|_{(2)}^2$; in terms of polar coordinates and the Fourier series of u, f^1, f^2 and using the asymptotic behavior of $\|\sigma\|$, it is just

$$\begin{aligned} & \sum_{m,n} \iint_{[0,A]^2} |u_{m,n}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\ & \leq C \left\{ \sum_{m,n} \iint_{[0,A]^2} |f_{m+1,n}^1(r_1, r_2)|^2 r_1 r_2^{-1} |\log r_1|^k |\log r_2|^{l-2} dr_1 dr_2 \right. \\ & \quad \left. + \sum_{m,n} \iint_{[0,A]^2} |f_{m,n+1}^2(r_1, r_2)|^2 r_1^{-1} r_2 |\log r_1|^{k-2} |\log r_2|^l dr_1 dr_2 \right\}. \end{aligned}$$

If this is done, then u is the required solution, i.e. $\bar{\partial}u = \phi$.

We first do this for the oriented curves L from $(0, 0)$ to (r_1, r_2) . In this case, we choose a particular path which is from $(0, 0)$ to $(0, r_2)$ (resp. $(r_1, 0)$), then to (r_1, r_2) . (More precisely, the segment from $(0, 0)$ to $(0, r_2)$ (resp. $(r_1, 0)$) should be substituted by a path sufficiently closed to the r_2 -axis (resp. r_1 -axis) from $(0, 0)$ to (s, r_2) (resp. (r_1, s)) for some sufficiently small $s > 0$, but since we assume that ϕ have compact support, this path plays essentially no role.) $u_{m,n}$ can then be rewritten as

$$\begin{aligned} u_{m,n}(r_1, r_2) &= 2r_1^m \int_0^{r_1} \rho_1^{-m} f_{m+1,n}^1(\rho_1, r_2) d\rho_1 \\ (\text{resp. } &= 2r_2^n \int_0^{r_2} \rho_2^{-n} f_{m,n+1}^2(r_1, \rho_2) d\rho_2). \end{aligned}$$

In the following, without loss of generality, we always assume that $\|\sigma\|^2 = |\log |t_1||^k |\log |t_2||^l$. Letting $m < 0$ and $n < 0$, one then has

$$\begin{aligned} &\iint_{[0,A]^2} |u_{m,n}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\ &= \iint_{[0,A]^2} \left(2r_1^m \int_0^{r_1} \rho_1^{-m} f_{m+1,n}^1(\rho_1, r_2) d\rho_1 \right)^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \end{aligned}$$

$$\begin{aligned}
&= \iint_{[0,A]^2} 4r_1^{2m} \left(\int_0^{r_1} \rho_1^{-2m} |f_{m+1,n}^1(\rho_1, r_2)|^2 d\rho_1 \right) \left(\int_0^{r_1} d\rho_1 \right) r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&= \iint_{[0,A]^2} 4r_1^{2m} \left(\int_0^{r_1} \rho_1^{-2m} |f_{m+1,n}^1(\rho_1, r_2)|^2 d\rho_1 \right) r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&= \int_0^A \left\{ \int_0^A 4 \left(\int_0^{r_1} \rho_1^{-2m} |f_{m+1,n}^1(\rho_1, r_2)|^2 d\rho_1 \right) r_1^{2m} |\log r_1|^{k-2} dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&= \int_0^A \left\{ \lim_{\epsilon \rightarrow 0} \left(-4 \int_r^A \rho_1^{2m} |\log \rho_1|^{k-2} d\rho_1 \cdot \int_0^{r_1} \rho_1^{-2m} |f_{m+1,n}^1(\rho_1, r_2)|^2 d\rho_1 \right) \right. \\
&\quad \left. + 4 \int_0^A r_1^{-2m} |f_{m+1,n}^1(r_1, r_2)|^2 \left(\int_{r_1}^A \rho_1^{2m} |\log \rho_1|^{k-2} d\rho_1 \right) dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&= 4 \int_0^A \left\{ \int_0^A r_1^{-2m} |f_{m+1,n}^1(r_1, r_2)|^2 \left(\int_{r_1}^A \rho_1^{2m} |\log \rho_1|^{k-2} d\rho_1 \right) dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&\leq 4 \int_0^A \left\{ \int_0^A r_1^{-2m} |f_{m+1,n}^1(r_1, r_2)|^2 r_1^{2m+1} |\log r_1|^{k-2} \log \frac{A}{r_1} dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&\quad \text{(Here, we used that } \rho^n |\log \rho|^k \text{ is a decreasing function in } \rho \text{ for } n < 0.) \\
&\sim \int_0^A \left\{ \int_0^A r_1 |f_{m+1,n}^1(r_1, r_2)|^2 |\log r_1|^{k-1} dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&\leq \int_0^A \left\{ \int_0^A |f_{m+1,n}^1(r_1, r_2)|^2 |\log r_1|^k dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&= \iint_{[0,A]^2} |f_{m+1,n}^1(r_1, r_2)|^2 |\log r_1|^k |\log r_2|^{l-2} r_2^{-1} dr_1 dr_2;
\end{aligned}$$

similarly, by using the other path mentioned above, one also has

$$\begin{aligned}
&\iint_{[0,A]^2} |u_{m,n}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&\leq \iint_{[0,A]^2} |f_{m,n+1}^2(r_1, r_2)|^2 |\log r_1|^{k-2} |\log r_2|^l r_1^{-1} dr_1 dr_2.
\end{aligned}$$

Letting $m = 0$, $k > 1$ and $n = 0$, $l > 1$, one then has

$$\begin{aligned}
u_{0,0}(r_1, r_2) &= 2 \int_0^{r_1} f_{1,0}^1(\rho_1, r_2) d\rho_1 \\
&\quad \text{(resp. } = 2 \int_0^{r_2} f_{0,1}^2(r_1, \rho_2) d\rho_2),
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{[0,A]^2} |u_{0,0}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&= \iint_{[0,A]^2} \left(2 \int_0^{r_1} f_{1,0}^1(\rho_1, r_2) d\rho_1 \right)^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2
\end{aligned}$$

$$\begin{aligned}
&\leq 4 \int_0^A \left\{ \int_0^A \left(\int_0^{r_1} |f_{1,0}^1(\rho_1, r_2)|^2 \rho_1 |\log \rho_1|^{1+\eta} d\rho_1 \right) \right. \\
&\quad \left. \left(\int_0^{r_1} \rho_1^{-1} |\log \rho_1|^{-1-\eta} d\rho_1 \right) r_1^{-1} |\log r_1|^{k-2} dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&\quad \text{(Here, } \eta \text{ is a positive constant } < k - 1.) \\
&\leq \frac{4}{\eta} \int_0^A \left\{ \int_0^A \left(\int_0^{r_1} |f_{1,0}^1(\rho_1, r_2)|^2 \rho_1 |\log \rho_1|^{1+\eta} d\rho_1 \right) r_1^{-1} |\log r_1|^{k-2-\eta} dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&= \frac{4}{\eta} \int_0^A \left\{ \lim_{\epsilon \rightarrow 0} \left(- \left(\int_0^{r_1} |f_{1,0}^1(\rho_1, r_2)|^2 \rho_1 |\log \rho_1|^{1+\eta} d\rho_1 \right) \left(\int_{r_1}^A |\log \rho_1|^{k-2-\eta} \rho_1^{-1} d\rho_1 \right) \right) \right\}^\epsilon \\
&\quad + \int_0^A |f_{1,0}^1(r_1, r_2)|^2 r_1 |\log r_1|^{1+\eta} \left(\int_{r_1}^A |\log \rho_1|^{k-2-\eta} \rho_1^{-1} d\rho_1 \right) dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&= \frac{4}{\eta(k-1-\eta)} \int_0^A \left\{ \int_0^A |f_{1,0}^1(r_1, r_2)|^2 r_1 |\log r_1|^k dr_1 \right\} r_2^{-1} |\log r_2|^{l-2} dr_2 \\
&= \frac{4}{\eta(k-1-\eta)} \iint_{[0,A]^2} |f_{1,0}^1(r_1, r_2)|^2 |\log r_1|^k |\log r_2|^{l-2} r_1 r_2^{-1} dr_1 dr_2;
\end{aligned}$$

similarly, using the other path, one can also get

$$\begin{aligned}
&\iint_{[0,A]^2} |u_{0,0}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&\leq \frac{4}{\eta(k-1-\eta)} \iint_{[0,A]^2} |f_{0,1}^2(r_1, r_2)|^2 |\log r_1|^{k-2} |\log r_2|^l r_1^{-1} r_2 dr_1 dr_2.
\end{aligned}$$

For the other two subcases ($m < 0$; $n = 0$, $l > 1$ and $m = 0$, $k > 1$; $n < 0$), we can get the same estimates as above.

For the case of the oriented curves from (A, A) to (r_1, r_2) , we have a similar estimate. In this case, the curve integral above, by two different paths, can be rewritten as

$$\begin{aligned}
u_{m,n}(r_1, r_2) &= -2r_2^n \int_{r_2}^A \rho_2^{-n} f_{m,n+1}^2(r_1, \rho_2) d\rho_2 \\
&\quad \text{(resp. } = -2r_1^m \int_{r_1}^A \rho_1^{-m} f_{m+1,n}^1(\rho_1, r_2) d\rho_1).
\end{aligned}$$

Supposing $m > 0$ and $n > 0$, one has the following estimate

$$\begin{aligned}
&\iint_{[0,A]^2} |u_{m,n}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&\leq 4 \iint_{[0,A]^2} r_2^{2n} \left(\int_{r_2}^A \rho_2^{-n} f_{m,n+1}^2(r_1, \rho_2) d\rho_2 \right)^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2
\end{aligned}$$

$$\begin{aligned}
&= 4 \int_0^A \left\{ \int_0^A r_2^{2n} \left(\int_{r_2}^A \rho_2^{-n} f_{m,n+1}^2(r_1, \rho_2) d\rho_2 \right)^2 r_2^{-1} |\log r_2|^{l-2} dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
&= 4 \int_0^A \left\{ \int_0^A r_2^{2n} \left(\int_{r_2}^A \rho_2^{-2n+1} f_{m,n+1}^2(r_1, \rho_2) d\rho_2 \right) \left(\int_{r_2}^A \frac{1}{\rho_2} d\rho_2 \right) r_2^{-1} |\log r_2|^{l-2} dr_2 \right\} \cdot \\
&\quad r_1^{-1} |\log r_1|^{k-2} dr_1 \\
&\sim 4 \int_0^A \left\{ \int_0^A r_2^{2n-1} |\log r_2|^{l-1} \left(\int_{r_2}^A \rho_2^{-2n+1} f_{m,n+1}^2(r_1, \rho_2) d\rho_2 \right) dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
&= 4 \int_0^A \left\{ \lim_{\epsilon \rightarrow 0} \left(\int_0^{r_2} \rho_2^{2n-1} |\log \rho_2|^{l-1} d\rho_2 \cdot \int_{r_2}^A \rho_2^{-2n+1} f_{m,n+1}^2(r_1, \rho_2) d\rho_2 \right) \Big|_{\epsilon}^A \right. \\
&\quad \left. + \int_0^A \left(\int_0^{r_2} \rho_2^{2n-1} |\log \rho_2|^{l-1} d\rho_2 \right) r_2^{-2n+1} f_{m,n+1}^2(r_1, r_2) dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
&= 4 \int_0^A \left\{ \int_0^A \left(\int_0^{r_2} \rho_2^{2n-1} |\log \rho_2|^{l-1} d\rho_2 \right) r_2^{-2n+1} f_{m,n+1}^2(r_1, r_2) dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
&\leq 4 \int_0^A \left\{ \int_0^A r_2^{2n} |\log r_2|^{l-1} \cdot r_2^{-2n+1} |f_{m,n+1}^2(r_1, r_2)|^2 dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
&\quad (\rho^n |\log \rho|^l \text{ is an increasing function as } n > 0.) \\
&\leq 4 \int_0^A \left\{ \int_0^A |f_{m,n+1}^2(r_1, r_2)|^2 r_2 |\log r_2|^l dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
&= 4 \iint_{[0,A]^2} |f_{m,n+1}^2(r_1, r_2)|^2 r_1^{-1} r_2 |\log r_1|^{k-2} |\log r_2|^l dr_1 dr_2;
\end{aligned}$$

similarly, using the other path, one can also show

$$\begin{aligned}
&\iint_{[0,A]^2} |u_{m,n}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&= 4 \iint_{[0,A]^2} |f_{m+1,n}^1(r_1, r_2)|^2 r_1 r_2^{-1} |\log r_1|^k |\log r_2|^{l-2} dr_1 dr_2.
\end{aligned}$$

Supposing $m = 0$, $k < 1$ and $n = 0$, $l < 1$, one then has

$$\begin{aligned}
&\iint_{[0,A]^2} |u_{0,0}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&\leq 4 \iint_{[0,A]^2} \left(\int_{r_2}^A f_{0,1}^2(r_1, \rho_2) d\rho_2 \right)^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
&= 4 \int_0^A \left\{ \int_0^A \left(\int_{r_2}^A f_{0,1}^2(r_1, \rho_2) d\rho_2 \right)^2 r_2^{-1} |\log r_2|^{l-2} dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
&\leq 4 \int_0^A \left\{ \int_0^A \left(\int_{r_2}^A |f_{0,1}^2(r_1, \rho_2)|^2 \rho_2 |\log \rho_2|^{1-\eta} d\rho_2 \right) \cdot \right.
\end{aligned}$$

$$\begin{aligned}
& \left(\int_{r_2}^A \rho_2^{-1} |\log \rho_2|^{-1+\eta} d\rho \right) r_2^{-1} |\log r_2|^{l-2} dr_2 \} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
& (\eta \text{ is some positive constant } < 1.) \\
\sim & \frac{4}{\eta} \int_0^A \left\{ \int_0^A \left(\int_{r_2}^A |f_{0,1}^2(r_1, \rho_2)|^2 \rho_2 |\log \rho_2|^{1-\eta} d\rho_2 \right) \cdot r_2^{-1} |\log r_2|^{l-2+\eta} dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
= & \frac{4}{\eta} \int_0^A \left\{ \left(\lim_{\eta \rightarrow 0} \left(\int_{r_2}^A |f_{0,1}^2(r_1, \rho_2)|^2 \rho_2 |\log \rho_2|^{1-\eta} d\rho_2 \right) \left(\int_0^{r_2} \rho_2^{-1} |\log \rho_2|^{l-2+\eta} d\rho_2 \right) \right) \right\}^\eta \\
& + \int_0^A |f_{0,1}^2(r_1, r_2)|^2 r_2 |\log r_2|^{1-\eta} \left(\int_0^{r_2} \rho_2^{-1} |\log \rho_2|^{l-2+\eta} d\rho_2 \right) dr_2 \} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
= & \frac{4}{\eta(1-l-\eta)} \int_0^A \left\{ \int_0^A |f_{0,1}^2(r_1, r_2)|^2 r_2 |\log r_2|^{1-\eta} |\log r_2|^{l-1+\eta} dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
= & \frac{4}{\eta(1-l-\eta)} \int_0^A \left\{ \int_0^A |f_{0,1}^2(r_1, r_2)|^2 r_2 |\log r_2|^l dr_2 \right\} r_1^{-1} |\log r_1|^{k-2} dr_1 \\
= & \frac{4}{\eta(1-l-\eta)} \iint_{[0,A]^2} |f_{0,1}^2(r_1, r_2)|^2 r_1^{-1} r_2 |\log r_1|^{k-2} |\log r_2|^l dr_1 dr_2;
\end{aligned}$$

similarly, using the other path, one also has

$$\begin{aligned}
& \iint_{[0,A]^2} |u_{0,0}(r_1, r_2)|^2 r_1^{-1} r_2^{-1} |\log r_1|^{k-2} |\log r_2|^{l-2} dr_1 dr_2 \\
= & \frac{4}{\eta(1-l-\eta)} \iint_{[0,A]^2} |f_{1,0}^1(r_1, r_2)|^2 r_1 r_2^{-1} |\log r_1|^k |\log r_2|^{l-2} dr_1 dr_2.
\end{aligned}$$

One can deal similarly with the two other subcases of this case ($m > 0$; $n = 0$, $l < 1$ and $m = 0$, $k < 1$; $n > 0$).

For the two other cases for the curve integral, one can obtain the same estimates as above, we omit these.

Case 2: Writing ϕ as $f d\bar{t}_1 \wedge d\bar{t}_2$, the condition of ϕ can then be restated as

$$\iint_{(\Delta^*)^2} |f|^2 \|\sigma\|^2 dt_1 \wedge d\bar{t}_1 \wedge dt_2 \wedge d\bar{t}_2 < \infty.$$

We now need to prove that there exists a form $\psi = u^1 d\bar{t}_1 + u^2 d\bar{t}_2$ of type $(0, 1)$ satisfying

$$\frac{\partial u^2}{\partial \bar{t}_1} - \frac{\partial u^1}{\partial \bar{t}_2} = f$$

and

$$\begin{aligned}
& \iint_{(\Delta^*)^2} (|u^1|^2 |t_2|^{-2} (-\log |t_2|)^{-2}) \|\sigma\|^2 dt_1 \wedge d\bar{t}_1 \wedge dt_2 \wedge d\bar{t}_2 + \\
& \iint_{(\Delta^*)^2} (|u^2|^2 |t_1|^{-2} (-\log |t_1|)^{-2}) \|\sigma\|^2 dt_1 \wedge d\bar{t}_1 \wedge dt_2 \wedge d\bar{t}_2 < \infty.
\end{aligned}$$

As in the case 1, we still use the method of Fourier series. Write f , u^1 , and u^2 as (r_1, r_2) -dependent Fourier series as follows

$$\begin{aligned} f &= \sum f_{m,n}(r_1, r_2) \exp(\sqrt{-1}m\theta_1 + \sqrt{-1}n\theta_2), \\ u^1 &= \sum u_{m,n}^1(r_1, r_2) \exp(\sqrt{-1}m\theta_1 + \sqrt{-1}n\theta_2), \\ u^2 &= \sum u_{m,n}^2(r_1, r_2) \exp(\sqrt{-1}m\theta_1 + \sqrt{-1}n\theta_2). \end{aligned}$$

The above equation of u^1 and u^2 can then be rewritten as

$$\frac{\partial}{\partial r_1}(r_1^{-m}r_2^{-n}u_{m,n+1}^2) - \frac{\partial}{\partial r_2}(r_1^{-m}r_2^{-n}u_{m+1,n}^1) = 2r_1^{-m}r_2^{-n}f_{m+1,n+1}.$$

Put

$$u_{m,n}^1(r_1, r_2) = \begin{cases} -r_2^n \int_0^{r_2} \rho_2^{-n} f_{m,n+1}(r_1, \rho_2) d\rho_2, & n < 0 \text{ or } n = 0, l > 1; \\ r_2^n \int_{r_2}^A \rho_2^{-n} f_{m,n+1}(r_1, \rho_2) d\rho_2, & n > 0 \text{ or } n = 0, l < 1 \end{cases}$$

and

$$u_{m,n}^2(r_1, r_2) = \begin{cases} r_1^m \int_0^{r_1} \rho_1^{-m} f_{m+1,n}(\rho_1, r_2) d\rho_1, & m < 0 \text{ or } m = 0, k > 1; \\ -r_1^m \int_{r_1}^A \rho_1^{-m} f_{m+1,n}(\rho_1, r_2) d\rho_1, & m > 0 \text{ or } m = 0, k < 1. \end{cases}$$

We now want to show that

$$\begin{aligned} & \sum_{m,n} \iint_{[0,A]^2} |u_{m,n}^1(r_1, r_2)|^2 |\log r_1|^k |\log r_2|^{l-2} r_1 r_2^{-1} dr_1 dr_2 \\ & + \sum_{m,n} \iint_{[0,A]^2} |u_{m,n}^2(r_1, r_2)|^2 |\log r_1|^{k-2} |\log r_2|^l r_1^{-1} r_2 dr_1 dr_2 \\ & \leq C \sum_{m,n} \iint_{[0,A]^2} |f_{m,n}(r_1, r_2)|^2 |\log r_1|^k |\log r_2|^l r_1 r_2 dr_1 dr_2 \end{aligned}$$

for some positive constant C . We perform the estimate only for $u_{m,n}^1$ with $n < 0$, the other cases are similar,

$$\begin{aligned} & \iint_{[0,A]^2} |u_{m,n}^1(r_1, r_2)|^2 |\log r_1|^k |\log r_2|^{l-2} r_1 r_2^{-1} dr_1 dr_2 \\ & = \iint_{[0,A]^2} \left(-r_2^n \int_0^{r_2} \rho_2^{-n} f_{m,n+1}(r_1, \rho_2) d\rho_2 \right)^2 |\log r_1|^k |\log r_2|^{l-2} r_1 r_2^{-1} dr_1 dr_2 \\ & = \int_0^A \left\{ \int_0^A r_2^{2n} \left(\int_0^{r_2} \rho_2^{-2n} |f_{m,n+1}(r_1, \rho_2)|^2 d\rho_2 \right) \left(\int_0^{r_2} d\rho_2 \right) |\log r_2|^{l-2} r_2^{-1} dr_2 \right\} |\log r_1|^k r_1 dr_1 \end{aligned}$$

$$\begin{aligned}
&= \int_0^A \left\{ \int_0^A r_2^{2n} \left(\int_0^{r_2} \rho_2^{-2n} |f_{m,n+1}(r_1, \rho_2)|^2 d\rho_2 \right) |\log r_2|^{l-2} dr_2 \right\} |\log r_1|^k r_1 dr_1 \\
&\sim \int_0^A \left\{ \int_0^A |f_{m,n+1}(r_1, r_2)|^2 |\log r_2|^{l-1} r_2 dr_2 \right\} |\log r_1|^k r_1 dr_1 \\
&\leq \iint_{[0,A]^2} |f_{m,n+1}(r_1, r_2)|^2 |\log r_1|^k |\log r_2|^l r_1 r_2 dr_1 dr_2.
\end{aligned}$$

Thus, we complete the proof of the assertion, and hence the proposition. \square

Remarks. 1) Near the divisor, denote by L_1 (resp. L_2) the line bundle generated by $\frac{dt_1}{t_1}$ (resp. $\frac{dt_2}{t_2}$); under the Poincaré-like metric, $\|\frac{dt_1}{t_1}\|^2 \sim |\log |t_1||^2$ and $\|\frac{dt_2}{t_2}\|^2 \sim |\log |t_2||^2$. So, by the previous proposition, if $k \neq -1$ and $l \neq 1$ (resp. $k \neq 1$ and $l \neq -1$; $k \neq -1$ and $l \neq -1$), the L^2 $\bar{\partial}$ -Poincaré lemma is true for $L^2(0, s)$ -forms ($s \geq 1$) with values in $L_1 \otimes L$ (resp. $L_2 \otimes L$; $L_1 \otimes L_2 \otimes L$). On the other hand, by the proposition in §4, any $L^2(1, s)$ -form valued in \mathbf{E}^p can be written as the sum of some forms valued in $L_1 \otimes \mathbf{E}^p$ or $L_2 \otimes \mathbf{E}^p$ and any $L^2(2, s)$ -forms valued in \mathbf{E}^p can be written as a form valued in $L_1 \wedge L_2 \otimes \mathbf{E}^p$, that is why we proved the L^2 $\bar{\partial}$ -Poincaré lemma only for $r = 0$. All these will be used in the following proof of quasi-isomorphism. 2) The classical L^2 existence theorem for the $\bar{\partial}$ -problem [1, 11] (refer to [8]) works only for partial cases. This can be seen from the following discussion. Let L be a holomorphic line bundle on $(\Delta^*)^2$ with a generating section σ and a Hermitian metric satisfying, near the divisor,

$$\|\sigma\|^2 = (-\log |t_1|)^k (-\log |t_2|)^l \text{ for some real numbers } k \text{ and } l.$$

Take the Poincaré-like metric on $(\Delta^*)^2$ as before

$$\omega = \frac{\sqrt{-1}}{2} \left(\frac{dt_1 \wedge d\bar{t}_1}{|t_1|^2 (-\log |t_1|)^2} + \frac{dt_2 \wedge d\bar{t}_2}{|t_2|^2 (-\log |t_2|)^2} \right).$$

A standard computation tells us that the curvature of the metric on the line bundle L is

$$\sqrt{-1}\Theta(L) = \frac{\sqrt{-1}}{2} \left(\frac{k dt_1 \wedge d\bar{t}_1}{|t_1|^2 (-\log |t_1|)^2} + \frac{l dt_2 \wedge d\bar{t}_2}{|t_2|^2 (-\log |t_2|)^2} \right).$$

Using the notations of page 30 of [8], the curvature eigenvalues, relative to ω , are $\gamma_1 = k$ and $\gamma_2 = l$. (Here, without loss of generality, we have assumed that $k \leq l$.) Then for any L -valued (p, q) -form $u = \sum u_{JK} dz_J \wedge d\bar{z}_K \otimes \sigma$,

$$\begin{aligned}
\langle [\sqrt{-1}\Theta(L), \Lambda]u, u \rangle &= \sum_{|J|=p, |K|=q} \left(\sum_{j \in J} \gamma_j + \sum_{j \in K} \gamma_j - \sum_{1 \leq j \leq n} \gamma_j \right) |u_{JK}|^2 \\
&\geq (\gamma_1 + \cdots + \gamma_q - \gamma_{p+1} - \cdots - \gamma_2) |u|^2.
\end{aligned}$$

So, by the L^2 existence theorem for the $\bar{\partial}$ -problem (refer to Theorem 5.1 of [8]), one has that if $\gamma_1 + \cdots + \gamma_q - \gamma_{p+1} - \cdots - \gamma_2 > 0$, $q \geq 1$, then for each L -valued $L^2(p, q)$ form ψ on $(\Delta^*)^2$ with $\bar{\partial}\psi = 0$, there exists an L -valued $L^2(p, q-1)$ form ϕ on $(\Delta^*)^2$ such that $\bar{\partial}\phi = \psi$ and

$$\|\phi\|_{(2)} \leq C\|\psi\|_{(2)}.$$

Here, $C = C(k, l, p, q)$ is a constant depending only on k, l, p, q . In particular, if u is a $(0, 1)$ -form, the condition above is $k - k - l = -l > 0$; if u is a $(0, 2)$ -form, the condition above is $k + l - k - l = 0$, which is empty; if u is a $(1, 1)$ -form, the condition above is $k - l \leq 0$, which is also empty; if u is a $(1, 2)$ -form, the condition above is $k + l - l = k > 0$; if u is a $(2, 1)$ -form, the condition above is $k > 0$; if u is a $(2, 2)$ -form, the condition above is $k + l > 0$. For the higher dimensional case, we have a similar argument.

The proof of quasi-isomorphism: Fix $p \geq 0$, we now prove that the complex $\{\mathrm{Gr}_F^p \Omega(\mathbf{H}_{\mathbb{C}})_{(2)} = (\Omega \otimes \mathbf{E}^{p-r})_{(2)}, \theta\}$ is quasi-isomorphic to the complex $\{[\mathrm{Gr}_F^p A(\mathbf{H}_{\mathbb{C}})]_{(2)} = \bigoplus_{r+s=p} (A^{r,s} \otimes \mathbf{E}^{p-r})_{(2)}, D'' = \bar{\partial} + \theta\}$. It is sufficient to show that the corresponding cohomological sheaves of the two complexes are the same near the singularity. So, in the following, we always restrict our discussion to some small neighborhood near the singularity. We first review some facts which are established (though which may not be stated explicitly, but are clear from the context) in §3. We know that, if neglecting the higher order terms, $\theta \sim N_1 \otimes \frac{dt_1}{t_1} + N_2 \otimes \frac{dt_1}{t_1}$; on the other hand, the higher order terms essentially play no role, so, we assume that $\theta = N_1 \otimes \frac{dt_1}{t_1} + N_2 \otimes \frac{dt_1}{t_1}$ in the following discussion. From the construction of $\sigma_{k,l}$, as an element of some L^2 -adapted basis of $(\mathbf{E}^{k+l})_{(2)}$, one has that, if both $\frac{dt_1}{t_1} \otimes N_1 \sigma_{k,l}$ and $\frac{dt_2}{t_2} \otimes N_2 \sigma_{k,l}$ are nonzero,

$$\left\| \frac{dt_1}{t_1} \otimes N_1 \sigma_{k,l} \right\|^2 \sim \|\sigma_{k,l}\|^2 \quad \text{and} \quad \left\| \frac{dt_2}{t_2} \otimes N_2 \sigma_{k,l} \right\|^2 \sim \|\sigma_{k,l}\|^2,$$

and they can be considered as two elements in an L^2 adapted basis $(\Omega_M^1 \otimes (\mathbf{E}^{k+l-1}))_{(2)}$; if $\|\sigma_{k,l}\|^2 \sim |\log |t_1|| |\log |t_2||^{2l-n}$ (resp. $|\log |t_1||^{2k-m} |\log |t_2||$, $|\log |t_1|| |\log |t_2||$), then $N_1 \sigma_{k,l}$ (resp. $N_2 \sigma_{k,l}$, $N_1 N_2 \sigma_{k,l}$) is not equal zero; if $\left\| \frac{dt_1}{t_1} \otimes \sigma_{k,l} \right\|^2 \sim \log |t_1|| |\log |t_2||^{2l-n}$ (resp. $\left\| \frac{dt_2}{t_2} \otimes \sigma_{k,l} \right\|^2 \sim \log |t_1||^{2k-m} |\log |t_2||$), then there exists an element $\sigma_{k+1,l}$ (resp. $\sigma_{k,l+1}$) in some L^2 -adapted basis $\{\sigma_{s,t}\}^3$ of $(\mathbf{E}^{k+l+1})_{(2)}$ with $N_1 \sigma_{k+1,l} = \sigma_{k,l}$ (resp. $N_2 \sigma_{k,l+1} = \sigma_{k,l}$) and

$$\|\sigma_{k+1,l}\|^2 \sim \left\| \frac{dt_1}{t_1} \otimes \sigma_{k,l} \right\|^2 \quad (\text{resp.} \quad \|\sigma_{k,l+1}\|^2 \sim \left\| \frac{dt_2}{t_2} \otimes \sigma_{k,l} \right\|^2).$$

³which, very probably, is slightly different from, but is equal to, up to a higher order term and hence in the following is viewed as, the one constructed in the previous section.

Let U be an open subset containing the origin $(0,0) \in (\Delta)^2$. In the following argument, we sometimes need to shrink U to a smaller open subset containing the origin, this will not be pointed out explicitly since it will be obvious from the context. In order to prove that the two cohomological sheaves are the same near the singularity, it is sufficient to show that *for any D'' -closed form $\phi \in [\mathrm{Gr}_F^p A^k(\mathbf{H}_\mathbb{C})]_{(2)}$ on U , there are a θ -closed form $\eta \in \mathrm{Gr}_F^p \Omega^k(\mathbf{H}_\mathbb{C})_{(2)}$ and a form $\psi \in [\mathrm{Gr}_F^p A^{k-1}(\mathbf{H}_\mathbb{C})]_{(2)}$ on U satisfying $\phi = \eta + D''\psi$, $k = 0, 1, 2, 3, 4$.* In the following, we only consider the case of $k = 2$, the other cases are similar, even much easier. It is easy to see that the complex $\{[\mathrm{Gr}_F^p A^k(\mathbf{H}_\mathbb{C})]_{(2)}, D''\}$ is actually a double complex with the differentials $\{\bar{\partial}, \theta\}$. Putting the two complexes together, we have the following diagram

$$\begin{array}{ccccccccc}
\mathbf{E}_{(2)}^p & \longrightarrow & (A^{0,0} \otimes \mathbf{E}^p)_{(2)} & \xrightarrow{\bar{\partial}} & (A^{0,1} \otimes \mathbf{E}^p)_{(2)} & \xrightarrow{\bar{\partial}} & (A^{0,2} \otimes \mathbf{E}^p)_{(2)} & \longrightarrow & 0 \\
\downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \\
(\Omega^1 \otimes \mathbf{E}^{p-1})_{(2)} & \longrightarrow & (A^{1,0} \otimes \mathbf{E}^{p-1})_{(2)} & \xrightarrow{\bar{\partial}} & (A^{1,1} \otimes \mathbf{E}^{p-1})_{(2)} & \xrightarrow{\bar{\partial}} & (A^{1,2} \otimes \mathbf{E}^{p-1})_{(2)} & \longrightarrow & 0 \\
\downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \\
(\Omega^2 \otimes \mathbf{E}^{p-2})_{(2)} & \longrightarrow & (A^{2,0} \otimes \mathbf{E}^{p-2})_{(2)} & \xrightarrow{\bar{\partial}} & (A^{2,1} \otimes \mathbf{E}^{p-2})_{(2)} & \xrightarrow{\bar{\partial}} & (A^{2,2} \otimes \mathbf{E}^{p-2})_{(2)} & \longrightarrow & 0
\end{array}$$

Let $\phi \in [\mathrm{Gr}_F^p A^2(\mathbf{H}_\mathbb{C})]_{(2)}$ be a D'' -closed form on U . By the consideration of type, it can be decomposed into $\phi = \phi^{2,0} + \phi^{1,1} + \phi^{0,2}$ with $\phi^{r,2-r} \in (A^{r,2-r} \otimes \mathbf{E}^{p-r})_{(2)}$ satisfying $\bar{\partial}\phi^{r,2-r} + \theta\phi^{r-1,3-r} = 0$, $r = 0, 1, 2, 3$ ($\phi^{3,-1}, \phi^{-1,3} = 0$ automatically). We first consider the parts of types $(0,2)$ and $(1,1)$: $\phi^{0,2}$ and $\phi^{1,1}$. *We want to eliminate the part $\phi^{0,2}$ from ϕ by the solutions for the $\bar{\partial}$ -problem and the properties of θ* , more precisely, we want to find some $\psi \in [\mathrm{Gr}_F^p A^1(\mathbf{H}_\mathbb{C})]_{(2)}$ satisfying $\phi - D''\psi$ not containing the part of type $(0,2)$. By the construction in the previous section, near the singularity, \mathbf{E}^p can be decomposed into the holomorphic direct sum of some trivial line bundles according to an L^2 -adapted basis of $\mathbf{E}_{(2)}^p \{\sigma_{k,l}^p\}$, satisfying

$$\|\sigma_{k,l}^p\|^2 \sim |\log|t_1||^{k'} |\log|t_2||^{l'}$$

for some integers k', l' depending on k, l respectively. So, $\phi^{0,2} = \sum \phi_{p,k,l}^{0,2} \otimes \sigma_{k,l}^p$, here $\{\phi_{p,k,l}^{0,2}\}$ are some $(0,2)$ -forms and each summand is L^2 . If for a $\sigma_{k,l}$, $k', l' \neq 1$, then by the previous proposition, there exists a local section $\psi_{p,k,l}^{0,1} \otimes \sigma_{k,l}^p \in (A^{0,1} \otimes \mathbf{E}^p)_{(2)}(U)$ satisfying $\bar{\partial}(\psi_{p,k,l}^{0,1} \otimes \sigma_{k,l}^p) = \phi_{p,k,l}^{0,2} \otimes \sigma_{k,l}^p$. Thus the part of type $(0,2)$ of $\phi - D''(\psi_{p,k,l}^{0,1} \otimes \sigma_{k,l}^p)$ does not contain a summand with value in the line bundle generated by $\sigma_{k,l}^p$. Therefore we can assume that $\phi^{0,2}$ has only summands of form $\phi_{p,k,l}^{0,2} \otimes \sigma_{k,l}^p$ satisfying

$$\|\sigma_{k,l}^p\|^2 \sim |\log|t_1|| |\log|t_2||^{l'} \quad \text{or} \quad |\log|t_1||^{k'} |\log|t_2||.$$

Let $\phi_{p,k,l}^{0,2} \otimes \sigma_{k,l}^p$ be such a summand with $\|\sigma_{k,l}^p\|^2 \sim |\log |t_1|| |\log |t_2||'$ (the other case is similar). From the above review, we know that $\frac{dt_1}{t_1} \otimes N_1 \sigma_{k,l}^p$ and $\frac{dt_2}{t_2} \otimes N_2 \sigma_{k,l}^p$ (if $\neq 0$) are two elements of some L^2 -adapted basis of $(\Omega^1 \otimes \mathbf{E}^{p-1})_{(2)}$. So, by $\theta \phi^{0,2} + \bar{\partial} \phi^{1,1} = 0$, we have that there exist two summands $\phi_1^{0,1} \wedge \frac{dt_1}{t_1} \otimes N_1 \sigma_{k,l}^p$ and $\phi_2^{0,1} \wedge \frac{dt_2}{t_2} \otimes N_2 \sigma_{k,l}^p$ of $\phi^{1,1}$ (when decomposed according to some L^2 -adapted basis containing $\frac{dt_1}{t_1} \otimes N_1 \sigma_{k,l}^p$ and $\frac{dt_2}{t_2} \otimes N_2 \sigma_{k,l}^p$ of $(\Omega^1 \otimes \mathbf{E}^{p-1})_{(2)}$) with $\bar{\partial} \phi_1^{0,1} = \bar{\partial} \phi_2^{0,1} = \phi_{p,k,l}^{0,2}$. Set $\psi = \phi_1^{0,1} \otimes \sigma_{k,l}^p$. Since $\|\sigma_{k,l}^p\| \sim \|\frac{dt_1}{t_1} \otimes N_1 \sigma_{k,l}^p\|$ and $\bar{\partial} \psi = \phi_{p,k,l}^{0,2} \otimes \sigma_{k,l}^p$, so $\psi \in (A^{0,1} \otimes \mathbf{E}^p)_{(2)}$. Thus again the part of type $(0, 2)$ of $\phi - D''\psi$ does not contain $\phi_{p,k,l}^{0,2} \otimes \sigma_{k,l}^p$ as a summand when decomposed according to some L^2 -adapted basis of $\mathbf{E}_{(2)}^p$ containing $\sigma_{k,l}^p$. Similarly, we can eliminate other such terms of ϕ .

Summing the above argument up, we now assume that ϕ does not contain the part of type $(0, 2)$. Namely, $\phi = \phi^{2,0} + \phi^{1,1}$. We now want to eliminate the term $\phi^{1,1}$ in ϕ by using the same method as above. But as seen in the following, the situation is slightly different. Choose an L^2 -adapted basis $\{\sigma_{k,l}^{p-1}\}$ of $\mathbf{E}_{(2)}^{p-1}$ with $\|\sigma_{k,l}^{p-1}\|^2 \sim |\log |t_1||^{k'} |\log |t_2||^{l'}$, where k', l' depends on k, l respectively. It is easy to see that $\{\frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}, \frac{dt_2}{t_2} \otimes \sigma_{k,l}^{p-1}\}$ can be considered as an L^2 -adapted basis of $(\Omega^1 \otimes \mathbf{E}^{p-1})_{(2)}$. According to the basis, ϕ can then be decomposed as follows

$$\phi^{1,1} = \sum \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1} + \sum \phi_{p-1,k,l,2}^{0,1} \wedge \frac{dt_2}{t_2} \otimes \sigma_{k,l}^{p-1},$$

for some $(0, 1)$ -forms $\phi_{p-1,k,l,1}^{0,1}$ and $\phi_{p-1,k,l,2}^{0,1}$. Since $\phi^{0,2} = 0$, each term above is zero under $\bar{\partial}$. So, if $\|\frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}\|^2 \sim |\log |t_1||^{k_1} |\log |t_2||^{l_1}$ with $k_1, l_1 \neq 1$, then the $\bar{\partial}$ -problem tells us that there exists a local section $\psi_{p-1,k,l,1}^{0,0} \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1} \in (A^{1,0} \otimes \mathbf{E}^{p-1})_{(2)}(U)$ satisfying $\bar{\partial}(\psi_{p-1,k,l,1}^{0,0} \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}) = \phi_{p-1,k,l,1}^{0,1} \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}$; thus the part of type $(1, 1)$ of $\phi - D''(\psi_{p-1,k,l,1}^{0,0} \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1})$ does not contain a $(0, 1)$ -form with value in the line bundle generated by $\frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}$. So, we can assume that the above decomposition of $\phi^{1,1}$ contains only the terms satisfying

$$\|\frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}\|^2 \sim |\log |t_1|| |\log |t_2||^{l_1} \quad \text{or} \quad |\log |t_1||^{k_1} |\log |t_2||$$

and

$$\|\frac{dt_2}{t_2} \otimes \sigma_{k,l}^{p-1}\|^2 \sim |\log |t_1|| |\log |t_2||^{l_1} \quad \text{or} \quad |\log |t_1||^{k_1} |\log |t_2||.$$

We can do a further reduction as follows: Let the decomposition of $\phi^{1,1}$ have a term satisfying $\|\frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}\|^2 \sim |\log |t_1|| |\log |t_2||^{l_1}$. By the review before, there exists an element $\sigma_{k+1,l}^p$, lying in some L^2 -adapted basis of $\mathbf{E}_{(2)}^p$, satisfying $N_1 \sigma_{k+1,l}^p = \sigma_{k,l}^{p-1}$ and $\|\sigma_{k+1,l}^p\|^2 \sim |\log |t_1|| |\log |t_2||^{l_1}$. Set $\psi = \phi_{p-1,k,l,1}^{0,1} \otimes \sigma_{k+1,l}^p$,

which is obviously a local section in $(A^{0,1} \otimes \mathbf{E}^p)_{(2)}$. Since $\phi^{0,2} = 0$, one has $\bar{\partial}(\phi_{p-1,k,l,1}^{0,1}) = 0$. So $\bar{\partial}(\psi) = 0$. Consider $\phi - D''\psi = \phi - \theta\psi = \phi - \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1} - \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_2}{t_2} \otimes N_2\sigma_{k+1,l}^p$. We know that, if $\frac{dt_2}{t_2} \otimes N_2\sigma_{k+1,l}^p \neq 0$, $\|\frac{dt_2}{t_2} \otimes N_2\sigma_{k+1,l}^p\|^2 \sim |\log|t_1|| |\log|t_2||^{l_1}$. So, if doing the same argument for the terms containing $\frac{dt_2}{t_2}$, we can always assume that in the decomposition of $\phi^{1,1}$ there exist only the terms satisfying

$$\|\frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}\|^2 \sim |\log|t_1||^{k_1} |\log|t_2||$$

and

$$\|\frac{dt_2}{t_2} \otimes \sigma_{k,l}^{p-1}\|^2 \sim |\log|t_1|| |\log|t_2||^{l_1}.$$

In the following, we first discuss the terms containing $\frac{dt_1}{t_1}: \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}$ with $\|\frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}\|^2 \sim |\log|t_1||^{k_1} |\log|t_2||$ and $k_1 \neq 1$. The terms containing $\frac{dt_2}{t_2}$ with $l_1 \neq 1$ can be discussed similarly. As pointed out in the review before, $N_2\sigma_{k,l}^{p-1} \neq 0$. So $\frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \otimes N_2\sigma_{k,l}^{p-1}$ can be considered as an element of some L^2 -adapted basis of $(\Omega^2 \otimes \mathbf{E}^{p-2})_{(2)}$. Again since $\bar{\partial}\phi^{2,0} + \theta\phi^{1,1} = 0$, so $\phi^{2,0}$ contains a part of form $\phi^{0,0} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \otimes N_2\sigma_{k,l}^{p-1} \in (A^{2,0} \otimes \mathbf{E}^{p-2})_{(2)}$ with $\bar{\partial}\phi^{0,0} = \phi_{p-1,k,l,1}^{0,1}$. Set $\psi = \phi^{0,0} \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}$, which is clearly a local section in $(A^{1,0} \otimes \mathbf{E}^{p-1})_{(2)}$ and satisfies $\bar{\partial}\psi = \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_1}{t_1} \otimes \sigma_{k,l}^{p-1}$. It is clear that $\phi - D''\psi$ does not contain the term discussed. So we can assume that $\phi^{1,1}$ contains only two terms ⁴ $\phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_1}{t_1} \otimes \sigma_{k,l,1}^{p-1}$ and $\phi_{p-1,k,l,2}^{0,1} \wedge \frac{dt_2}{t_2} \otimes \sigma_{k,l,2}^{p-1}$ satisfying

$$\|\frac{dt_1}{t_1} \otimes \sigma_{k,l,1}^{p-1}\|^2 \sim |\log|t_1|| |\log|t_2||$$

and

$$\|\frac{dt_2}{t_2} \otimes \sigma_{k,l,2}^{p-1}\|^2 \sim |\log|t_1|| |\log|t_2||,$$

where $\sigma_{k,l,1}^{p-1}$ and $\sigma_{k,l,2}^{p-1}$ are two elements of the L^2 -adapted basis of $\mathbf{E}_{(2)}^{p-1}$ chosen before. By the above asymptotic behavior of $\sigma_{k,l,1}^{p-1}$ and $\sigma_{k,l,2}^{p-1}$ and the construction of an L^2 -adapted basis of $\mathbf{E}_{(2)}^{p-1}$ in the previous section, one can actually choose the above L^2 -adapted basis $\{\sigma_{k,l}^{p-1}\}$ in the way that there exist an element $\sigma_{k,l}^p$ in some L^2 -adapted basis of $\mathbf{E}_{(2)}^p$ satisfying $\|\sigma_{k,l}^p\|^2 \sim |\log|t_1|| |\log|t_2||$ and $N_1\sigma_{k,l}^p = \sigma_{k,l,1}^{p-1}$ and $N_2\sigma_{k,l}^p = \sigma_{k,l,2}^{p-1}$. We now go on doing a reduction for ϕ . As above, we still discuss the term containing $\frac{dt_1}{t_1}: \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_1}{t_1} \otimes \sigma_{k,l,1}^{p-1}$.

⁴Here and in the sequel, w.l.o.g., we always assume that the L^2 -adapted basis $\{\sigma_{k,l}^{p-1}\}$ is constructed using an irreducible representation of $(\mathfrak{sl}_2(\mathbb{C}))^2$ by the method of §3.

As before, one still has $\bar{\partial}\phi_{p-1,k,l,1}^{0,1} = 0$. By the above choice of the L^2 -adapted basis, one has

$$\begin{aligned}
& \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_1}{t_1} \otimes \sigma_{k,l,1}^{p-1} \\
&= \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_1}{t_1} \otimes N_1 \sigma_{k,l}^p \\
&= -\theta(\phi_{p-1,k,l,1}^{0,1} \otimes \sigma_{k,l}^p) - \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_2}{t_2} \otimes N_2 \sigma_{k,l}^p \\
&= -\theta(\phi_{p-1,k,l,1}^{0,1} \otimes \sigma_{k,l}^p) - \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_2}{t_2} \otimes \sigma_{k,l,2}^{p-1} \\
&= -D''(\phi_{p-1,k,l,1}^{0,1} \otimes \sigma_{k,l}^p) - \phi_{p-1,k,l,1}^{0,1} \wedge \frac{dt_2}{t_2} \otimes \sigma_{k,l,2}^{p-1}.
\end{aligned}$$

Obviously, $\phi_{p-1,k,l,1}^{0,1} \otimes \sigma_{k,l}^p$ is a local section of $(A^{0,1} \otimes \mathbf{E}^p)_{(2)}$. So, one can furthermore assume that $\phi^{1,1}$ has only the term $\phi_{p-1,k,l,2}^{0,1} \wedge \frac{dt_2}{t_2} \otimes \sigma_{k,l}^{p-1}$ with

$$\left\| \frac{dt_2}{t_2} \otimes \sigma_{k,l}^{p-1} \right\|^2 \sim |\log |t_1|| |\log |t_2||.$$

Again using the relation $\bar{\partial}\phi^{2,0} + \theta\phi^{1,1} = 0$, we can finally assume that $\phi^{1,1} = 0$, namely, ϕ has only the part of type $(2, 0)$ up to an exact D'' -form. Obviously, a D'' -closed form of type $(2, 0)$ is holomorphic and θ -closed. This completes the proof of quasi-isomorphism, and hence Theorem A. \square

APPENDIX: In this appendix, the aim is two-fold. One is to prove that the complex $\{Gr_F^p \Omega(\mathbf{H}_{\mathbb{C}})_{(2)}, \theta\}$ is quasi-isomorphic to the complex $([Gr_F^p A(\mathbf{H}_{\mathbb{C}})]_{(2)}, D'')$ under the inclusion map for $p \geq 0$, if the L^2 $\bar{\partial}$ -Poincaré lemma is always true. This is actually an application of a general result to this special case; from the following proof, we can easily abstract this. The proof is standard. We first establish some notations.

On a topological space X , a *complex of sheaves* (K^*, d) is given by sheaves of Abelian groups K^p together with sheaf maps d

$$K^0 \xrightarrow{d} K^1 \xrightarrow{d} \dots \xrightarrow{d} K^p \xrightarrow{d} K^{p+1} \xrightarrow{d} \dots$$

satisfying $d^2 = 0$. Associated to a complex of sheaves (K^*, d) are the cohomology sheaves $H^q(K^*)$: Setting, for an open subset $U \subset X$, $K^q(U) = H^0(U, K^q)$, the presheaf

$$U \rightarrow \frac{\text{Ker}\{d : K^q(U) \rightarrow K^{q+1}(U)\}}{dK^{q-1}(U)}$$

gives rise to a sheaf $H^q(K^*)$, whose stalk is

$$(H^q(K^*))_x = \lim_{x \in U} \frac{\text{Ker}\{d : K^q(U) \rightarrow K^{q+1}(U)\}}{dK^{q-1}(U)}.$$

A section σ of $H^q(K^*)$ over an open set $U \subset X$ is given by a covering $\{U_\alpha\}$ of U and $\sigma_\alpha \in K^q(U_\alpha)$ such that

$$\begin{aligned} d\sigma &= 0, \\ \sigma_\alpha - \sigma_\beta &= d\eta_{\alpha\beta}, \quad \eta_{\alpha\beta} \in K^{q-1}(U_\alpha \cap U_\beta); \end{aligned}$$

the section is zero in case

$$\sigma_\alpha = d\eta_\alpha, \quad \eta_\alpha \in K^{q-1}(U_\alpha),$$

after perhaps refining the given covering. We note that, by the above definitions, *the cohomology sheaves $H^q(K^*) = 0$ for $q > 0$ iff the Poincaré lemma holds for the complex of sheaves (K^*, d) .*

Definition *A map*

$$j : L^* \rightarrow K^*$$

between complexes of sheaves is a quasi-isomorphism if it induces an isomorphism on cohomology sheaves:

$$j_* : H^q(L^*) \rightarrow H^q(K^*), \quad q \geq 0.$$

We now turn to the proof of quasi-isomorphism, provided that the $L^2 \bar{\partial}$ -Poincaré lemma be true. By the definition, we need to show that the two sheaves, $H^k(\text{Gr}_F^p \Omega(\mathbf{H}_\mathbb{C})_{(2)})(U)$ and $H^k([\text{Gr}_F^p A(\mathbf{H}_\mathbb{C})]_{(2)})(U)$, have the same germs for all $k \geq 0$ and $p \geq 0$. Let U be an open subset of \bar{M} , then the definition tells us

$$\begin{aligned} H^k(\text{Gr}_F^p \Omega(\mathbf{H}_\mathbb{C})_{(2)})(U) &= \frac{\text{Ker}(\Omega^k \otimes \mathbf{E}^{p-k} \rightarrow \Omega^{k+1} \otimes \mathbf{E}^{p-k-1})(U)}{\text{Im}(\Omega^{k-1} \otimes \mathbf{E}^{p-k+1} \rightarrow \Omega^k \otimes \mathbf{E}^{p-k})(U)}, \\ H^k([\text{Gr}_F^p A(\mathbf{H}_\mathbb{C})]_{(2)})(U) &= \frac{\text{Ker}([\text{Gr}_F^p A^k(\mathbf{H}_\mathbb{C})]_{(2)} \rightarrow [\text{Gr}_F^p A^{k+1}(\mathbf{H}_\mathbb{C})]_{(2)})(U)}{\text{Im}([\text{Gr}_F^p A^{k-1}(\mathbf{H}_\mathbb{C})]_{(2)} \rightarrow [\text{Gr}_F^p A^k(\mathbf{H}_\mathbb{C})]_{(2)})(U)}. \end{aligned}$$

Here, "→" in the first formula is under θ , while → in the second formula is under D'' . Let $\phi = \sum_{r+s=k} \phi^{r,s} \in \text{Ker}([\text{Gr}_F^p A^k(\mathbf{H}_\mathbb{C})]_{(2)} \rightarrow [\text{Gr}_F^p A^{k+1}(\mathbf{H}_\mathbb{C})]_{(2)})(U)$ with $\phi^{r,s} \in (A^{r,s} \otimes \mathbf{E}^{p-r})_{(2)}$, which represents the germ of an element in

$H^k([\mathrm{Gr}_F^p A(\mathbf{H}_\mathbb{C})]_{(2)})$. Equivalently, $D''\phi = 0$, i.e.,

$$\begin{aligned}\theta\phi^{k,0} &= 0, \\ \bar{\partial}\phi^{k,0} + \theta\phi^{k-1,1} &= 0, \\ \bar{\partial}\phi^{k-1,1} + \theta\phi^{k-2,2} &= 0, \\ \dots \\ \bar{\partial}\phi^{k-r,r} + \theta\phi^{k-r-1,r+1} &= 0, \\ \dots \\ \bar{\partial}\phi^{1,k-1} + \theta\phi^{0,k} &= 0, \\ \bar{\partial}\phi^{0,k} &= 0.\end{aligned}$$

By means of the last formula and the Poincaré lemma for $\bar{\partial}$, one has that on U (possibly a smaller open set), there exists a local section $\psi^{0,k-1} \in (A^{0,k-1} \otimes \mathbf{E}^p)_{(2)}$ satisfying

$$\phi^{0,k} = \bar{\partial}\psi^{0,k-1}.$$

Substituting the above formula into $\bar{\partial}\phi^{1,k-1} + \theta\phi^{0,k} = 0$ and using $\theta\bar{\partial} + \bar{\partial}\theta = 0$, one has

$$\bar{\partial}(\phi^{1,k-1} - \theta\psi^{0,k-1}) = 0.$$

The same reasoning derives that there exists a local section $\psi^{1,k-2} \in (A^{1,k-2} \otimes \mathbf{E}^{p-1})_{(2)}$ on a (possibly smaller than U) open subset, satisfying

$$\phi^{1,k-1} = \bar{\partial}\psi^{1,k-2} + \theta\psi^{0,k-1}.$$

Substituting the above formula into $\bar{\partial}\phi^{2,k-2} + \theta\phi^{1,k-1} = 0$ and using $\theta\bar{\partial} + \bar{\partial}\theta = 0$ and $\theta \wedge \theta = 0$, one has

$$\bar{\partial}(\phi^{2,k-2} - \theta\psi^{1,k-2}) = 0.$$

Inductively, one has that there exists a sequence of germs of local sections $\psi^{r,k-r-1} \in (A^{r,k-r-1} \otimes \mathbf{E}^{p-r})_{(2)}$, $2 \leq r \leq k-1$, satisfying

$$\phi^{r,k-r} = \bar{\partial}\psi^{r,k-r-1} + \theta\psi^{r-1,k-r}.$$

Setting $r = k-1$ and substituting the above formula into $\bar{\partial}\phi^{k,0} + \theta\phi^{k-1,1} = 0$, one has

$$\bar{\partial}(\phi^{k,0} - \theta\psi^{k-1,0}) = 0.$$

So, $\phi^{k,0} - \theta\psi^{k-1,0}$ is a local holomorphic k -form with values in \mathbf{E}^{p-k} (if $p-k \geq 0$), denoted by h^k , i.e.,

$$\phi^{k,0} = \theta\psi^{k-1,0} + h^k,$$

and hence $\theta h^k = 0$, which implies that h^k represents the germ of some element in $H^k(\text{Gr}_F^p \Omega(\mathbf{H}_\mathbb{C})_{(2)})$. Summing up $\sum_{r=1}^k \phi^{r,k-r}$, one has

$$\begin{aligned} \phi &= \bar{\partial}\psi^{0,k-1} + \theta\psi^{0,k-1} + \bar{\partial}\psi^{1,k-2} + \dots + \theta\psi^{k-2,1} + \bar{\partial}\psi^{k-1,0} + \theta\psi^{k-1,0} + h^k \\ &= D''(\psi^{0,k-1} + \psi^{1,k-2} + \dots + \psi^{k-2,1} + \psi^{k-1,0}) + h^k, \end{aligned}$$

which implies that the germ represented by ϕ in the sheaf $H^k([\text{Gr}_F^p A(\mathbf{H}_\mathbb{C})]_{(2)})$ can be identified with the germ represented by h^k in the sheaf $H^k(\text{Gr}_F^p \Omega(\mathbf{H}_\mathbb{C})_{(2)})$. We need to show that this identification is well-defined, namely, if the germ in $H^k([\text{Gr}_F^p A(\mathbf{H}_\mathbb{C})]_{(2)})$ represented by ϕ can also be identified with the germ in $H^k(\text{Gr}_F^p \Omega(\mathbf{H}_\mathbb{C})_{(2)})$ represented by another local holomorphic k -form with values in $\mathbf{E}^{p-k} h_1^k$ (that is to say, there exists a local section $\psi_1 \in [\text{Gr}_F^p A^{k-1}(\mathbf{H}_\mathbb{C})]_{(2)}$ with $\phi = D''\psi_1 + h_1^k$), then h^k and h_1^k represent the same germ in $H^k(\text{Gr}_F^p \Omega(\mathbf{H}_\mathbb{C})_{(2)})$. The proof of this assertion is completely similar to the previous arguments—the same steps let us obtain a local holomorphic $k-1$ -form h^{k-1} with values in \mathbf{E}^{p-k+1} satisfying

$$h_1^k - h^k = \theta h^{k-1}.$$

Thus, we get a natural map j^* from $H^k([\text{Gr}_F^p A(\mathbf{H}_\mathbb{C})]_{(2)})$ to $H^k(\text{Gr}_F^p \Omega(\mathbf{H}_\mathbb{C})_{(2)})$. The assertion of uniqueness also shows that the map j^* is surjective. The proof of injectivity is easy: Suppose that $j^*([\phi_1]) = j^*([\phi_2])$ for $\phi_i \in \text{Ker}([\text{Gr}_F^p A^k(\mathbf{H}_\mathbb{C})]_{(2)} \rightarrow [\text{Gr}_F^p A^{k+1}(\mathbf{H}_\mathbb{C})]_{(2)})(U)$, $i = 1, 2$. By the above arguments, there exist some local sections $\psi_i \in [\text{Gr}_F^p A^{k-1}(\mathbf{H}_\mathbb{C})]_{(2)}$ and some local holomorphic k -forms h_i^k with values in \mathbf{E}^{p-k} satisfying $\phi_i = D''\psi_i + h_i^k$ for $i = 1, 2$. On the other hand, since $j^*([\phi_1]) = j^*([\phi_2])$, so there exists a local holomorphic $k-1$ -form h^{k-1} with values in \mathbf{E}^{p-k+1} satisfying $h_1^k - h_2^k = \theta h^{k-1}$. So, one has

$$\phi_1 - \phi_2 = D''(\psi_1 - \psi_2) + (h_1^k - h_2^k) = D''(\psi_1 - \psi_2 + h^{k-1}).$$

Therefore, as germs, $[\phi_1] = [\phi_2]$. This completes the proof of the quasi-isomorphism theorem.

The second aim of the appendix is to state that the hypercohomology of a complex of fine sheaves is isomorphic to the cohomology of the corresponding complex of global sections. First, we need to show what the hypercohomology of a sheaves complex is. As before, let (K^*, d) be a complex of sheaves with differentials d on a topological space X . Take a covering $\underline{U} = \{U_\alpha\}$ of X and let $C^p(\underline{U}, K^q)$ be the Čech cochains of degree p with values in K^q . The two operators

$$\begin{aligned} \delta &: C^p(\underline{U}, K^q) \rightarrow C^{p+1}(\underline{U}, K^q) \\ d &: C^p(\underline{U}, K^q) \rightarrow C^p(\underline{U}, K^{q+1}), \end{aligned}$$

satisfy $\delta^2 = d^2 = 0$ and $d\delta + \delta d = 0$; and one has a double complex

$$\{C^{p,q} := C^p(\underline{U}, K^q); \delta, d\}.$$

Put $C^n(\underline{U}) = \bigoplus_{p+q=n} C^p(\underline{U}, K^q)$ and $D = \delta + d$, then $\{C^*, D\}$ is a complex. A refinement \underline{U}' of \underline{U} induces mappings

$$\begin{aligned} C^p(\underline{U}, K^q) &\rightarrow C^p(\underline{U}', K^q), \\ H^*(C^*(\underline{U}), D) &\rightarrow H^*(C^*(\underline{U}'), D), \end{aligned}$$

and then we define the hypercohomology of (K^*, d) as

$$\mathbb{H}^* X, \{K^*, d\} = \varinjlim_{\underline{U}} H^*(C^*(\underline{U}), D).$$

For the complex of sheaves (K^*, d) , one can consider the complex of its global sections. Putting $\Gamma(K^p) = H^0(X, K^p)$, one then has a complex $\{\Gamma(K^p), d\}$ and its usual cohomology $H^*(\{\Gamma(K^p), d\})$. Then one has

Theorem. *If the complex of sheaves (K^*, d) is a complex of fine sheaves, then*

$$\mathbb{H}^*(X, \{K^*, d\}) \cong H^*(\{\Gamma(K^p), d\}).$$

References

- [1] A. Andreotti and E. Vesentini, Carleman estimates for the Laplace-Beltrami equation in complex manifolds. *Publ. Math. I.H.E.S.*, **25** (1965), 81-130.
- [2] E. Cattani, Mixed Hodge structures, compactifications and monodromy weight filtration, in *Topics in Transcendental Algebraic Geometry*, ed P. Griffiths, *Annals of Mathematics Studies*, Vol. **106**, 75-100.
- [3] Cornalba and Griffiths, Analytic cycles and vector bundles on non-compact algebraic varieties, *Invent. Math.* **28** (1975), 1-106.
- [4] E. Cattani, A. Kaplan, Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structures, *Inventiones Math.* **67** (1982), 101-115.
- [5] E. Cattani, A. Kaplan and W. Schmid, Degeneration of Hodge structures, *Annals of Mathematics*, **123** (1986), 457-535.
- [6] E. Cattani, A. Kaplan and W. Schmid, L^2 and intersection cohomologies for a polarizable variation of Hodge structure, *Inventiones Math.*, **87** (1987), 217-252.
- [7] P. Deligne, Théorie de Hodge. II, *Publ. Math. IHES* **40** (1972), 5-57

- [8] J.-P. Demailly, L^2 Vanishing Theorems for Positive Line Bundles and Adjunction Theory, LNM, **1646**, 1994.
- [9] P. Griffiths, Periods of integrals on algebraic manifolds. III. Some global differential-geometric properties of the period mapping, Inst. Hautes Etudes Sci. Publ. Math. **38** (1970) 125–180.
- [10] P. Griffiths and Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978.
- [11] L. Hörmander L^2 estimates and existence theorems for the $\bar{\partial}$ operator, Acta Mathematica, **113** (1965), 89-152.
- [12] W. Schmid, Variation of Hodge structure: The singularities of period mapping, Inventiones Math., **22**(1973),211-319.
- [13] C. Simpson, Harmonic bundles on noncompact curves, J. Amer. Math. Soc., **3**(1990), 713-770.
- [14] C. Simpson, Higgs bundles and local systems, IHES Publ. **75** (1992), 5-95.
- [15] S. Zucker, Generalized intermediate Jacobians and the theorem on normal functions, Inventiones Math., **33** (1976), 185-222.
- [16] S. Zucker, Hodge theory with degenerating coefficients: L^2 -cohomology in the Poincaré metric, Annals of Mathematics, **109**(1979), 415-476.
- [17] K. Zuo, On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications, *Kodaira's issue*, Asian J. Math. **4** (2000), no. 1, 279–301.

Juergen Jost,
 Max-Planck Institute for Mathematics in the Sciences, Inselstr. 22-26, 04103
 Leipzig, Germany;
 Yi-Hu Yang,
 Department of Applied Mathematics, Tongji University Shanghai 200092, China;
 Kang Zuo,
 Department of Mathematics, the Chinese University of Hong Kong, Hong
 Kong, China.