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by

Hông Vân Lê

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Hông-Vân Lê
Max-Planck-Institute for Mathematics in Sciences
Inselstraße 22-26
D-04103 Leipzig, Germany
and
Hanoi Institute of Mathematics
P.O.Box 10000, Boho, Hanoi, Vietnam
email: hvle@mis.mpg.de

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Abstract

We give a detailed proof of Thom's theorem on the existence of a complex vector bundle over a compact oriented manifold M^m whose top Chern class lies in a given positive ray in $H^{even}(M^m, \mathbb{Z})$.

MSC: 55F10

1 Introduction.

The problem of representing a homology class $A \in H_*(V, \mathbb{Z})$ by a smooth submanifold $W \subset V$ was investigated by R. Thom in his famous paper [Thom1954]. The main idea of Thom is to study the relations between 3 objects: smooth submanifolds, (co)homology classes and smooth vector bundles. Let us now recall some important notions and results in [Thom1954].

- B_G: - the classifying space of a group $G \subset O(k)$ and $E_G \to B_G$ the corresponding universal fibration,
- A_G : - the associated fibration $E_G \times_G D^k \to B_G$, where D^k is the unit ball in $\mathbb{R}^k$,
- M_G: - the Thom space of the fibration $D^k \to A_G \to B_G$.

Recall that the algebra $H^*(M(G))$ is obtained from that one of $H^*(B_G)$ by shifting all dimension by k. Namely we have the following Thom's isomorphism

$$\phi_G : H^{r-k}(B_G) \to H^r_K(A'_G = A_G \setminus \partial A_G) \cong H^r(M(G)),$$

here H^*_K denotes the cohomologies with compact support. The class $H^k(M(G)) \ni U = \phi_G(u^0)$ is called the **fundamental (Thom's) class of** $M(G)$, where $u_G \in H^0(B_G)$ is the unit of the ring $H^*(B_G)$. We observe that there is sequence of natural mappings

$$B_G \xrightarrow{i} A_G \xrightarrow{e} M_G$$

where i is the embedding via the zero section, and e is the map which is identity in $A_G \setminus \partial A_G$ and which maps the boundary ∂A_G to a point. It is easy to see that

$$i^* \circ e^*(U) = e_k \tag{1.1}$$

where e_k is the Euler class of the universal bundle $V_G = E_G \times_G V^k \to B_G$.

A cohomology class $u \in H^k(A)$ of a space A is called **realizable w.r.t. a group** $G \subset O(k)$, if there exists a map $f : A \to M(G)$ such that $f^*(U) = u$.

1.1. Thom Criterion. [Thom1954 Theorem II.1] *Suppose that V^n is a compact orientable smooth manifold. A class $z \in H_{n-k}(V^n)$ can be realized by a submanifold W^{n-k} whose normal bundle has a G -structure, if and only if the Poincare dual class $PD(z) \in H^k(V^n)$ is realizable w.r.t. G.*

We shall say that a cohomology class $u \in H^k(A)$ of a space A is **strongly realizable w.r.t. a group** $G \subset O(k)$, if there exists a map $g : A \to B_G$ such that $g^*(e_k) = u$. Because of (1.1) the strong realizability of u implies the realizability of u (w.r.t. G). Clearly the strong realizability of u is equivalent to the condition that u is the Euler class of a G-vector bundle over M^m.

In [Thom1954] Thom also proved the following existence theorem.

1.2. Theorem [Thom1954 Theorem II.25]. *For each cohomology class $z \in H^k(M^n, \mathbb{Z})$ there exists a number $N(k,n)$ such that the class $N(k,n)\cdot z$ is strongly realizable w.r.t. $G = SO(k)$. If $k = 2l$, then there exists a number $N_1(k,n) \geq N(k,n)$ such that the class $N_1(k,n) \cdot z$ is strongly realizable w.r.t. $G = U(k)$.*

Thom gave a detailed proof of Theorem 1.2 for $G = SO(k)$. He noticed that his proof also works for $G = U(k)$ or $Sp(k)$. Since we use Thom's theorem for $G = U(k)$ in [Le2004] and [Le2005] we feel a need for a detailed proof of Thom's theorem 1.2 in this case.

I thank Dietmar Salamon for stimulating discussions on Thom's theorems in [Thom1954].

2 Proof of Theorem 1.2 for $G = U(k)$.

2.1. The strategy to find a map $g : M^m \to B_{U(k)}$. Suppose that $u \in H^{2k}(M^m, \mathbb{Z})$. Then there is a map

$$f : M^m \to K(\mathbb{Z}, 2k)$$

such that $f^*(\tau) = u$, where τ is the fundamental class of $H^k(K(\mathbb{Z}, 2k), \mathbb{Z})$. Moreover we can assume that $f(M^m) \subset K^m(\mathbb{Z}, 2k)$, where $K^q(\mathbb{Z}, 2k)$ is the q-skeleton of the Eilenberg-McLane space $K(\mathbb{Z}, 2k)$. To prove Theorem 2.1 it suffices to find a map

$$h : K^m(\mathbb{Z}, 2k) \to B_{U(k)}$$

such that for some positive number $N_1(k,m)$ we have

$$h^*(e_{2k}) = N_1(k,m) j^*(\tau), \tag{2.2}$$

where $e_{2k} = c_k$ is the top Chern class of the universal bundle $V_{U(k)}$ over $B_{U(k)}$ and j is the embedding $K^m(\mathbb{Z}, 2k) \to K(\mathbb{Z}, 2k)$.

In oder to find the map h satisfying (2.2) we first notice that $\pi_{2k}(B_{U(k)}) \otimes \mathbb{Q} = 1$ (see Proposition 2.6), so we can define a map $G_{2k} : K^{2k}(\mathbb{Z}, 2k) = S^{2k} \to B_{U(k)}$ by sending S^{2k} to the generator of the free part of $\pi_{2k}(B_{U(k)})$. Next we shall extend map G_q inductively on q by applying the following

2.3. Lemma. ([Thom1954, Lemma 2.24]) *Suppose that Y is a space such that the free component of $\pi_k(Y)$ is isomorphic to $\mathbb{Z}$ with a generator t. If for all $q \ge k$ the group $\pi_q(Y)$ is of finite type, and $H^{q+1}(K(\mathbb{Z}, k), \pi_q(Y))$ is finite, then for each q there exists a map $G_q : K(\mathbb{Z}, k) \supset K^q \to Y$ such that $(G_q)_*(\pi_k(K(\mathbb{Z}, k)) =< N(q, k)t >_{\otimes \mathbb{Z}}$.*

In order to apply Lemma 2.3 to $Y = B_{U(k)}$ we must verify that

$$\pi_{2k}(B_{U(k)}) \otimes \mathbb{Q} = \mathbb{Q}, \tag{2.3.1}$$

$$\pi_q(B_{U(k)}) \text{ is of finite type, } \forall q \ge k, \tag{2.3.2}$$

$$H^{q+1}(K(\mathbb{Z}, 2k), \pi_q(B_{U(k)})) \otimes \mathbb{Q} = 0, \ \forall q \ge k. \tag{2.3.3}$$

Once the conditions (2.3.1), (2.3.2) and (2.3.3) hold, Lemma 2.3 gives us a map

$$h : K^m(\mathbb{Z}, 2k) \to B_{U(k)}$$

such that $h_*(w) = N(m, k)t$, where w is a generator of $\pi_{2k}(K(\mathbb{Z}, 2k)) = \mathbb{Z}$. In Lemma 2.8 we shall show that the pairing of the Chern class c_k with t is positive. Hence we get

$$h^*(c_k) = N'(m, k)\tau, \tag{2.4}$$

where $N'(m, k) > 0$. So the map h satisfies (2.2).

2.5. Homotopy type of $B_{U(k)}$ and verifying conditions (2.3.1), (2.3.2).

Clearly the conditions (2.3.1) and (2.3.2) are consequences of the following

2.6. Proposition. *We have*

$$\pi_{2i}(B_{U(k)}) \otimes \mathbb{Q} = \mathbb{Q}, \text{ if } i \le k$$

$$\pi_j(B_{U(k)}) \otimes \mathbb{Q} = 0, \text{ for all other } j.$$

Proof of Proposition 2.6. There are two proofs of Proposition 2.6. In the first proof we use Thom's argument based on applying the $\mathcal{C}$-version of the Whitehead theorem to a map $f : B_{U(k)} \to K(\mathbb{Z}, 2) \times \cdots \times$

$K(\mathbb{Z}, 2k)$ and we use a computation of $H^*(B_{U(k)}, \mathbb{Z})$, see the proof of Lemma 2.8 below. In the second proof we use an induction argument. The Proposition 2.6 is clearly true for $B_{U(1)} = \mathbb{C}P^\infty$. Suppose that the Proposition is valid till k. To show its validity for $k+1$ we consider the following exact homotopy sequence

$$\to \pi_q(S^{2k+1}) \to \pi_q(B_{U(k)}) \to \pi_q(B_{U(k+1)}) \to \pi_{q-1}(S^{2k+1}) \to ...$$

related to the fibration $S^{2k+1} \to B_{U(k)} \to B_{U(k+1)}$. Next we use the well-known fact that $\pi_q(S^{2k+1})$ is zero, if $q \le 2k$, it is finite if $q \ge 2k+2$, and $\pi_{2k+1}(S^{2k+1}) = \mathbb{Z}$, (see e.g. [Spanier1966, 9.7]). Applying this fact and our induction assumption to the above exact sequence after tensoring with $\mathbb{Q}$ we get the induction statement for $k+1$. □

Now we shall see that condition (2.3.3) is a consequence of a computation of the cohomology ring $H^*(K(\mathbb{Z}, k), \mathbb{Q})$, obtained by Serre and Cartan. Namely we have

2.7. Lemma. (see e.g.[F-F, 3.25]) *a) If n is odd, then*

$$H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \Lambda_{\mathbb{Q}}(x), \dim x = n.$$

b) If n is even, then

$$H^*(K(\mathbb{Z}, n), \mathbb{Q}) = \mathbb{Q}[x], \dim x = n.$$

Lemma 2.7 can be easily proved by using induction and the cohomology spectral sequence associated with the fibration $K(\mathbb{Z}, n-1) \cong \Omega K(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$, whose fiber is contractible.

2.8. Lemma. *The pairing of $c_k \in H^k(B_{U(k)}, \mathbb{Z})$ with the generator t of $\pi_{2k}(B_{U(k)} \otimes \mathbb{Q})$ is positive.*

Proof. First we notice that the cohomology ring $H^*(B_{U(k)}, \mathbb{Z})$ is a polynomial ring generated by the Chern classes $c_1(V_{U(k)}), \cdots, c_k(V_{U(k)})$, moreover there are no polynomial relations between these generators (see e.g. [M-S1974, 14.5]). Now we use the existence of a map

$$f : B_{U(k)} \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \times \cdots \times K(\mathbb{Z}, 2k)$$

such that

$$f^*(\tau_{2l}) = c_l, \tag{2.9}$$

where τ_{2l} is the fundamental class of $H^{2l}(K(\mathbb{Z}, 2l), \mathbb{Z})$. By Lemma 2.7.b the induced map $f^* : H_*(B_{U(k)}, \mathbb{Z}) \to H_*(K(\mathbb{Z}, 2) \times \cdots \times K(\mathbb{Z}, 2k), \mathbb{Z})$ is a $\mathcal{C}$-isomorphism, where $\mathcal{C}$ is the class of finite groups. Hence the generalized Whitehead theorem of Serre [Serre1953] gives us an isomorphism

$$f^* : \pi_{2k}(B_{U(k)}) \otimes \mathbb{Q} \to \pi_{2k}(K(\mathbb{Z}, 2k)) \otimes \mathbb{Q} = Q.$$

Now using (2.9) we get immediately that

$$< c_k, t > = < \tau_{2k}, f_*(t) > > 0.$$

□

To make a good feeling of completeness we shall outline the Thom proof of Lemma 2.3 here. Lemma 2.3 shall be proved by induction on the dimension q. Clearly Lemma 2.3 for $q = k$ is trivial. To construct a map $G_g : K^{q+1}(\mathbb{Z}, 2k) \to Y$ from a map $G_q : K^q(\mathbb{Z}, 2k) \to Y$ we use the existence of a "covering map" $F_N : K(\mathbb{Z}, n) \to K(\mathbb{Z}, n)$ with the following property

$$F_N^*(\tau) = N\tau.$$

By theorem of simplicial approximation we can assume that F_N sends $K^q(\mathbb{Z}, n)$ to $K^q(\mathbb{Z}, n)$ for each $q \geq n$.

The idea of Thom is that by using this covering map we can modify map G_q to a map

$$G_q' : K^q(\mathbb{Z}, 2k) \stackrel{F_m}{\to} K^q(\mathbb{Z}, 2k) \to Y$$

such that G_q' can be extended to K^{q+1}, since the obstruction to this extension lies in the group $F_N^*(H^{q+1}(K(\mathbb{Z}, 2k), \pi_q(Y))$ which is trivial under the condition of Lemma 2.3 and taking into account the following Lemma.

2.10. Lemma. [Thom1954, Lemma 2.23]. *Let G be a finitely generated abelian group and suppose that all the elements of $H^r(K(\mathbb{Z}, n), G)$ is of finite order N. Then there exists a number $m \neq 0$ such that then endomorphism $(F_m)^* : H^r(K(\mathbb{Z}, n), G)$ is trivial.*

In closing this note we remark that the proof of main technical Lemma 2.3 is somewhat similar to the proof of Serre of the following

2.11. Proposition. [Serre 1953, V.2.2]. *Let K be a finite polyhedral, n be an odd number and $x \in H^n(K, \mathbb{Z})$. Then there are a* *number $N \neq 0$ and a map $K \to S^n$ such that $f^*(u) = Nx$, where u is the fundamental class of $H^n(S^n, \mathbb{Z})$.*

Since n is odd we get that for $i > n$ the group $\pi_i(S^n)$ is finite. Hence any map $K^q \to S^n$ after composing with a certain covering map $F_N : S^n \to S^n$ can be extended to $K^{q+1} \to S^n$. So in the Serre proof we compose the old map with a covering map of the target space S^n, while in the Thom proof we compose the old map with a covering map on the domain space $K(\mathbb{Z}, 2k)$.

Since S^{2k+1} is the Thom space for $G = e \in SO(2k+1)$ we remark that the proof of Theorem 1.2 for $G = U(k)$ together with Proposition 2.11 and Proposition 1.1 are sufficient to get the following Thom's stability statement. For each $x \in H^*(M^m, \mathbb{Z})$ there is a positive number N such that Nx is the Poincare dual to the fundamental class of a closed submanifold in M^m.

References

[F-F1986] A. Fomenko and B. Fuks, *Course in homotopic topology*, (in Russian) Moscow 1986.

[Le2004] H. V. Le, *Realizing homology classes by symplectic submanifolds*, preprint MPI MIS 62, 2004.

[Le2005] H. V. Le, *Holomorphic vector bundles over projective algebraic manifolds,* in preperation.

[M-S1972] J. Milnor and J. Stasheff, *Characteristic classes*, Princeton Univ. Press 1974.

[Serre1953] J. P. Serre, *Groupes d'homotopie et classes de groupes abeliens,* Ann. Math., 58 (1953), 258-294.

[Spanier1966] E. H. Spanier, *Algebraic Topology*, Springer-Verlag 1989.

[Thom1954] R. Thom, *Quelques properties globales des variates differentiables*, Comm. Math. Helv. vol 28, (1954), 17-86.