Max-Planck-Institut
für Mathematik
in den Naturwissenschaften
Leipzig

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by

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Preprint no.: 46 2005

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1 Introduction

Let $\overline{S}$ be a compact Riemann surface (holomorphic curve) of genus g. Let $p_1, p_2, \cdots, p_s$ be $s > 0$ points on it; these points define a divisor, and we denote the open Riemann surface $\overline{S} \setminus \{p_1, \ldots, p_s\}$ by S. When $3g - 3 + s > 0$, it carries a complete hyperbolic metric of finite volume, the so-called Poincaré metric; the points $p_1, p_2, \cdots, p_s$ then become cusps at infinity. Even in the remaining cases, that is, for a once or twice punctured sphere, we can equip S with a metric that is hyperbolic in the vicinity of the cusp(s), and for our purposes, the behavior of the metric there is all what counts, and we call such a metric Poincaré-like. In any case, our metric on S is denoted by ω. Denote the inclusion map of S in $\overline{S}$ by j. Let $\rho : \pi_1(S) \to Sl(n, \mathbb{C})$ be a semisimple linear representation of $\pi_1(S)$ which is unipotent near the cusps (for the precise definition, cf. §2.1). Corresponding to such a representation ρ, one has a local system L_ρ over S and a ρ-equivariant harmonic map $h : S \to Sl(n, \mathbb{C})/SU(n)$ with a certain special growth condition near the divisor. For the present case of complex dimension 1, this is elementary; it also follows from the general result of [6], see also the remark in §2.2). This harmonic map can be considered as a Hermitian metric on L_ρ—harmonic metric—so that we have a so-called harmonic bundle (L_ρ, h) [13]. Such a bundle carries interesting structures, e.g. a Higgs bundle structure (E, θ), where $\theta = \partial h$, and it has a log-singularity at the divisor.

The purpose of this note is to investigate various cohomologies of $\overline{S}$ with degenerating coefficients L_ρ (considered as a local system — a flat vector bundle, a Higgs bundle, or a $\mathcal{D}$-module, depending on the context): **the Čech cohomology of** $j_* L_\rho$ (note that in the higher dimensional case, one needs to

*Supported partially by NSF of China (No. 10471105) and "Shuguang Project" of Committee of Education of Shanghai (04SG21)

consider the corresponding intersection cohomology [3]), **the L^2-cohomology, the L^2-Dolbeault cohomology, and the L^2-Higgs cohomology, and the relationships between them**. Here, L^2 is defined by using the Poincaré(-like) metric ω and the harmonic metric h. We want to generalize the results [15] valid for the case of variations of Hodge structures (VHS) to the case of harmonic bundles, as was suggested by Simpson [13]; in principle, in view of our assumption on the representations in question being unipotent, the situation should be similar to the case of VHS.

This paper is meant to be a part of the general program of studying cohomologies with degenerating coefficients on quasiprojective varieties and their Kählerian generalizations. The general aim here is not restricted to the case of curves nor to the one of representations that are unipotent near the divisor. The purpose of this note therefore is to illuminate at this particular case where many of the (analytic and geometric) difficulties of the general case are not present what differences will appear when we consider unipotent harmonic bundles instead of VHSs; for the case of VHSs, the various cohomologies have been considered by various authors [1, 10, 14, 9] and are well understood by now.

Acknowledgements: This work was begun when the second author was visiting Mainz University, and its main part was finished when he stayed at the Max Planck Institute for Mathematics in the Sciences in Leipzig. He thanks the above Institutes for hospitality and a good working environment.

2 The geometry associated with representations of fundamental groups

2.1 The decomposition of a flat connection

In order to make this note an introduction into our general program, we describe some background material here. The knowledgable reader may skip this and perhaps also the next §§. Until the noncompact case is addressed explicitly below, the space X will be compact in order to avoid difficulties with the analysis, in particular with the existence of the harmonic map.

Let X be a Riemannian manifold, V a $Gl(n, \mathbb{C})$ bundle on X with a flat connection D or, equivalently, a representation

$$\rho : \pi_1(X) \to Gl(n, \mathbb{C}).$$

A metric $\langle \cdot, \cdot \rangle$ on V is equivalent to a ρ-equivariant map

$$h : \tilde{X} \to Gl(n, \mathbb{C})/U(n) =: \tilde{Y}$$

$(\langle v, v\rangle = h_{ij}v^i v^j)$.

The simplest case is, of course, $n = 1$. Then V is a line bundle, and

$$Gl(1, \mathbb{C})/U(1) = \mathbb{C}^*/S^1 \simeq \mathbb{R}^+.$$

Using the isomorphism

$$\log : \mathbb{R}^+ \to \mathbb{R},$$

a metric can be written as

$$h = e^\lambda, \text{ for } \lambda : \tilde{X} \to \mathbb{R},$$

and so $h^{-1}dh = d\lambda$.[1]

Given a(ny) metric h on V, the flat connection D will not preserve it in general, and so, we split D as

$$D = D_h + \theta,$$

where D_h preserves h, i.e.,

$$d\langle v, v\rangle = \langle D_h v, v\rangle + \langle v, D_h v\rangle.$$

Thus, we have $\theta = 0$ iff D is unitary iff h is constant. The derivative dh thus measures the deviation of D from being unitary, i.e., $dh = \theta$. Thus, the energy of the map h is given by $\int \|\theta\|^2$, and h is harmonic iff

$$D_h^*\theta = 0. \tag{1}$$

For $n = 1$, reverting to our $h^{-1}dh$ notation, we have

$$D_h = d - h^{-1}dh, \text{ i.e. } \theta = h^{-1}dh = d\lambda.$$

The harmonic map equation (1) becomes

$$0 = (d^* - h^{-1}dh)d\lambda = d^*d\lambda - d\lambda \wedge d\lambda = d^*d\lambda,$$

and so λ is a harmonic function on $\tilde{X}$. λ is not well defined as a function on X, but only as a map from X into its Albanese variety. $d\lambda$, however, is well defined, and a harmonic 1-form, and λ thus is the period map of a harmonic 1-form.

[1] $h^{-1}dh$ is of course the derivative of the map $h : \tilde{X} \to \tilde{Y}$; usually, we should write dh, but the problem is that d has two meanings, namely on one hand, the exterior derivative d, leading to the notation $h^{-1}dh$, and on the other hand, the differential d of a map between Riemannian manifolds, suggesting to write dh. In the sequel, we use the notation $h^{-1}dh$ only for the case $n = 1$ and write dh else.

2.2 Variations of Hodge structures

Let Z and X be Kähler manifolds, with $\dim Z > \dim X$, $f : Z \to X$ holomorphic with smooth fibers. For the fiber Z_x over $x \in X$, we have the Hodge decomposition

$$H^k(Z_x) = \bigoplus_{p+q=k} H^{p,q}(Z_x).$$

This induces a filtration ($n = \dim_{\mathbb{C}} Z_x = \dim_{\mathbb{C}} Z - \dim_{\mathbb{C}} X$)

$$0 \subset F_x^n \subset \cdots \subset F_x^{q+1} \subset F_x^q \subset \cdots \subset H^k(Z_x).$$

This filtration defines an element of a subdomain of a Grassmannian of flags. From this, one obtains the Griffiths period map

$$X \to D \tag{2}$$

into the Griffiths period domain that is obtained by also imposing a polarization, i.e., a Hermitian form $\langle u, v\rangle$ s.t.

$$H^{p,q} \perp H^{r,s} \text{ for } (p,q) \neq (r,s) \tag{3}$$

and

$$(-1)^p \langle v, v\rangle > 0 \text{ for } v \in H^{p,q}. \tag{4}$$

This Griffiths period domain admits a holomorphic map

$$D \to G/K \tag{5}$$

onto a Hermitian symmetric space of noncompact type. D is itself not Kähler because the natural metric coming from the polarization is indefinite, see (4). However, the image of the period map is tangential to a holomorphic distribution that is not integrable, but has nonpositive curvature in the tangential directions (those correspond to the directions in G/K).

There is a natural flat connection D on the above decomposition, obtained by translating cohomology classes topologically or, equivalently, by considering the flat vector bundle V with fiber

$$V_x = H^k(Z_x)$$

over $x \in X$. If we relate D to the above decomposition, we obtain

$$D = \partial + \bar\partial + \theta + \bar\theta : H^{p,q} \to \Omega^{1,0}(H^{p,q}) \oplus \Omega^{0,1}(H^{p,q}) \oplus \Omega^{1,0}(H^{p-1,q+1}) \oplus \Omega^{0,1}(H^{p+1,q-1}). \tag{6}$$

∂ and $\bar\partial$ simply come from the complex structure. θ and $\bar\theta$ have to shift the degree because in contrast to ∂ and $\bar\partial$, they do not operate by differentiation, but rather by multiplication with a 1-form – as always when we split a connection as $D = d + A$.

Now an abstract complex variation of Hodge structures (VHS) is defined as a complex vector bundle V over X with a decomposition

$$V = \bigoplus_{p+q=k} V^{p,q}$$

as in (3, 4) and a flat connection D satisfying (6).

This leads to a holomorphic vector bundle

$$E = \bigoplus_{p+q=k} E^{p,q},$$

using the above $\bar{\partial}$, with an endomorphism valued 1-form

$$\theta : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1(X)$$

with

$$\theta \wedge \theta = 0$$

(this is the flatness condition on D).

By a result of Griffiths, E is stable, and the flatness implies $c_i(E) = 0$ for all i.

2.3 Harmonic bundles and Higgs bundles

It was then Simpson's fundamental idea ([12]) to revert this construction. A Higgs bundle (E, θ) over X consists of a holomorphic vector bundle E and $\theta : E \to E \otimes \Omega^1(X)$ with $\theta \wedge \theta = 0$. The understanding of the relationship between harmonic bundles and Higgs bundles also owes much to Hitchin's important paper [4].

By a theorem of Narasimhan-Seshadri for curves and by Donaldson, Uhlenbeck-Yau, Simpson for higher dimensional X, for a stable Higgs bundle, one can construct a Hermitian Yang-Mills connection D_0, and $D_0 + \theta$ is flat if all $c_i(E) = 0$. Such a D_0 then defines a harmonic metric on E, i.e., a harmonic map into a symmetric space G/K as in (2, 5). Conversely, from a semisimple representation

$$\rho : \pi_1(X) \to G,$$

G a linear algebraic group, one obtains a ρ-equivariant harmonic map

$$h : \tilde{X} \to G/K,$$

and this defines a Higgs bundle (E, θ) with $\theta = dh$ ($\theta \wedge \theta = 0$ follows from the pluriharmonicity of h as originally discovered in a somewhat different context by Jost-Yau [5]; in fact, in our case of a Riemann surface, $\theta \wedge \theta = 0$ is trivial because θ is a $(1,0)$-form, see below). More precisely, decompose our flat connection $D = d' + d''$ into operators of type $(1,0)$ and $(0,1)$ respectively. Let δ' and δ'' be the unique operators of type $(1,0)$ and $(0,1)$ such that the connections $\delta' + d''$ and $\delta'' + d'$ preserve the metric h. Let $\partial = (d' + \delta')/2$ and $\overline{\partial} = (d'' + \delta'')/2$, and let $\theta = (d' - \delta')/2$ and $\overline{\theta} = (d'' - \delta'')/2$. It is clear that $\mu = \partial + \overline{\partial}$ also preserves the metric and $\overline{\theta}$ is the conjugate adjoint of θ w.r.t. h. Also, as mentioned, $(\overline{\partial} + \theta)^2 = 0$. Thus we obtain a structure of Higgs bundle on the bundle L_ρ defined by the representation ρ with $\overline{\partial}$ being the holomorphic structure and θ the Higgs field; later on we denote it by $(E, \overline{\partial} + \theta)$.

We also have the **Kähler identities:** Set $D''_h = \overline{\partial} + \theta$, $D'_h = \partial + \overline{\theta}$ and $D^c_h = D''_h - D'_h$. Note that $D = D' + D''$ and $D''_h = (D + D^c_h)/2$. Let Λ be the adjoint of the operation of wedging with the Kähler form ω. Then one has the first order Kähler identities

$$\begin{aligned} (D'_h)^* = \sqrt{-1}[\Lambda, D''_h], \quad (D''_h)^* = -\sqrt{-1}[\Lambda, D'_h] \\ (D^c_h)^* = -\sqrt{-1}[\Lambda, D], \quad (D)^* = \sqrt{-1}[\Lambda, D^c_h], \end{aligned}$$

where * represents the adjoint of the respective operator. Set $\Delta = DD^* + D^*D$ and $\Delta'' = D''_h(D''_h)^* + (D''_h)^* D''_h$. Using the above first order identities, one then has

$$\Delta = 2\Delta''.$$

This shows that spaces of Δ-harmonic forms valued in the local system L_ρ can be identified with that of Δ''-harmonic forms valued in the Higgs bundle E.

2.4 The noncompact case

When X is no longer compact, the geometry works as before, but there arise difficulties with the existence of the harmonic map h. On the other hand, the geometry of the bundle near a compactifying divisor leads to very interesting structures which are, in fact, our main interest. We first turn to the analytic aspect. Even though in the present paper we shall be mainly concerned with the case of curves where the existence result is essentially elementary, in order to put the paper into a proper perspective, we shall describe the equivariant harmonic maps (equivalently harmonic metrics) due to Jost-Zuo [6] when the representations of π_1 are linearly semisimple and unipotent and some consequences, all of which will be used in the next section. The construction of the most general equivariant maps corresponding to general linearly reductive representations will be given in [8]; especially, there we will see that the construction of Jost-Zuo [6, 7] corresponds to taking the trivial filtration structure on the corresponding local systems (cf. [13]).

Throughout this note, we will take the Poincaré-like metric on the base manifolds, namely on punctured disks Δ^* near the divisor (puncture) the metric is isometric to

$$\frac{dz \wedge d\overline{z}}{|z|^2(\log|z|)^2}.$$

Such a metric is complete, of finite volume and bounded geometry.

Let $\rho : \pi_1(S) \to GL(n, \mathbb{C})$ be a semisimple linear representation, and restrict ρ to a neighborhood of $p_i \in D$, say a disk around p_i, which we call the *boundary representation* of ρ. Throughout this note, *we assume that all such boundary representations of ρ are unipotent*. This means that if denoting the image under ρ of the generator $\pi_1(\Delta^*)$ by the matrix γ, then γ, under a suitable basis, can be represented by a upper-triangle matrix with all the diagonal entries being 1. It is worth pointing out that in [6] the authors used a geometric definition since their construction needs to apply to more general target spaces; if the representations of π_1 are linear, it is easy to see that their geometric condition reduces to our unipotent condition. In any case, from a geometric perspective, the essence of this condition is the following. Let c be a closed curve in Δ^* representing ρ, for example the circle $r = r_0$, r again being the Euclidean radius in $\Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$ (0 being the puncture, cusp, divisor,...). Thus c is freely homotopic to arbitrarily short curves as we can move it closer to 0. In geometric terms, the unipotency condition means that the image γ of our homotopy class can also be represented by arbitrarily short curves. The prototype is the representation ρ corresponding to the identity map of Δ^*, equipped with the Poincaré metric ω. Since that metric has finite volume, the identity map has finite energy. Conversely, when the lengths of all curves homotopic to the image of c have a positive lower bound, then any map in that homotopy class has infinite energy. An example are maps $f : \Delta^* \to U(1) = S^1$ equivariant w.r.t. the representation $\rho : \pi_1(\Delta^*) \to \mathbb{Z}$ that maps the generator of $\pi_1(\Delta^*)$ to the generator of $\pi_1(U(1)) = \pi_1(S^1) = \mathbb{Z}$. In that case, any map homotopic to f necessarily has infinite energy. In particular, while one can also show the existence of a harmonic map in this – non-unipotent – case, that map does not have finite energy, and the analysis becomes more subtle.

Proposition 1 *Let $\rho : \pi_1(S) \to Sl(n, \mathbb{C})$ be a semisimple representation all boundary representations of which are unipotent. Then there exists a ρ-equivariant harmonic map of finite energy*

$$h : \tilde{S} \to Sl(n, \mathbb{C})/SU(n),$$

where $\tilde{S}$ is the universal covering of S; moreover the norm of the derivative dh of h, when going down to S and measured near the divisor with respect to the Poincaré-like metric and the standard Riemannian symmetric metric *on $Sl(n,\mathbb{C})/SU(n)$, $\leq C|\log r|^2$ for some constant $C > 0$, where r is the Euclidean radius of Δ^*. Therefore, in Simpson's notation [13], the harmonic bundle (L_ρ, h) is tame.*

Let (L_ρ, h) be the harmonic bundle in the Proposition 1 and D the flat connection. Since the harmonic bundle is tame, so our discussions in the following and the next subsection lie in the framework of Simpson [13]. We first describe the norm estimate for flat sections near the divisor w.r.t. the harmonic metric h (for details, cf. [13]). We restrict ourself to the punctured disk Δ^*. Denote the image of the generator of $\pi_1(\Delta^*)$ by γ, called the monodromy of L_ρ, which is by the assumption unipotent; denote its logarithm by N—the logarithmic monodromy, which is nilpotent. Canonically, each fiber of L_ρ has a so-called weight filtration $\{W_l\}_{l=-k}^k$ arising from N (k is the weight of N), which is D-invariant and which therefore determines a filtration of L_ρ by some local subsystems, denoted by $\mathbf{W}_l$. Then a section v of $\mathbf{W}_l$, if not lying in $\mathbf{W}_{l-1}$, has the following norm estimate

$$\|v\|_h^2 \sim |\log r|^l.$$

As explained above, canonically, from the harmonic bundle (L_ρ, h) constructed in the Proposition (using the estimate for the derivative of h), one can derive a structure of Higgs bundle on L_ρ [7].

Again since $(E, \overline{\partial} + \theta)$ comes from a tame harmonic bundle, Theorem 1 of [13] works; in particular, the curvature R_μ corresponding to the connection $\mu = \partial + \overline{\partial}$ is bounded and the Higgs field θ has a log-singularity, and hence by a result of Griffiths-Cornalba [2] one can extend E across the divisor, denoted by j_*E as usual. More importantly, we have an identification between the weight filtration of N (the logarithmic monodromy of L_ρ) on L_ρ and the weight filtration $\{W_l\}_{l=-k}^k$ of the residue of θ on $res_{p_i}(j_*E)$ (Simpson's notation); this derives a similar norm estimate for meromorphic sections of E, i.e. let e be a meromorphic section, if the image in $res_{p_i}(j_*E)$ of e lies in W_l but not in W_{l-1}, then one has

$$\|e\|^2 \leq C|\log r|^l.$$

Remark. Due to the construction of the harmonic maps of [6, 7] and the assumption that the representation is unipotent, we have only the trivial filtered structure [13] at the divisor on both the local system L_ρ and the Higgs bundle (E, D''). And hence in the norm estimates for flat sections of L_ρ we do not have factors of the form $r^b, b \in \mathbb{R}$; similarly for the estimates of meromorphic sections of (E, D'') one does not have factors like $r^\alpha, 0 < \alpha < 1$. (For details, cf. [13, 9])

Remark. It should be pointed out that in order that various dualities and integration make sense, one needs to restrict various discussions above to related L^2-spaces, as is done in the next section, since our base manifolds are noncompact; otherwise we will not have nice identities. We omit these details, since they are standard by now.

3 The L^2-cohomology and the L^2-Higgs cohomology

In this section, we continue to assume that the representation ρ is semisimple and unipotent. Therefore we have the corresponding harmonic bundle (L_ρ, h) and the Higgs bundle (E, θ) with $\theta = \partial h$ and the hermitain metric h; we also have various norm estimates w.r.t. the metric h; the most important thing is that one can identify the logarithmic monodromy and the residue of θ.

3.1 The L^2-cohomology: The L^2-Poincaré Lemma

Denoting the inclusion map of S in $\overline{S}$ by j, one has the direct image sheaf j_*L_ρ of the local system L_ρ on $\overline{S}$ and then the Čech cohomology $H^*(\overline{S}, j_*L_\rho)$. On the other hand, using the Poincaré-like metric ω on S and the harmonic metric h on L_ρ, one can define a complex $\{\mathcal{A}^{\cdot}_{(2)}(L_\rho), D\}$ of fine sheaves on $\overline{S}$ as follows. Let U be an open subset of $\overline{S}$. Then $\mathcal{A}^i_{(2)}(L_\rho)(U)$ is defined as the set of

L_ρ-valued i-forms η on $U \cap S$, with measurable coefficients and measurable exterior derivative $D\eta$, such that η and $D\eta$ have finite L^2 norm on $K \cap S$, for any compact subset K of U, where D is the canonical flat connection of L_ρ.

Since the sheaves are fine, so the cohomology of the complex of global sections computes the hypercohomology of $\{\mathcal{A}^{\cdot}_{(2)}(L_\rho), D\}$, namely

$$\mathbb{H}^*(\overline{S}, \{\mathcal{A}^{\cdot}_{(2)}(L_\rho), D\}) \cong H^*(\{\Gamma(\mathcal{A}^{\cdot}_{(2)}(L_\rho)), D\}).$$

We call this cohomology the L^2-*cohomology* of $\overline{S}$ with values in L_ρ, denoted by $H^*_{(2)}(\overline{S}, L_\rho)$. The purpose of this subsection is then to establish the following identification

Theorem 1 *There exists a natural identification*

$$H^*(\overline{S}, j_*L_\rho) \cong H^*_{(2)}(\overline{S}, L_\rho).$$

Remarks. If L_ρ comes from a variation of Hodge structure (VHS), the identification was proved by S. Zucker [15]. In the higher dimensional case, instead of the Čech cohomology, one needs to consider the intersection cohomology [3]; the identification was proved by Cattani-Kaplan-Schmid [1] in the case of VHS. In general, one has the following

Conjecture 1 *There exists a natural identification*

$$H^*_{int}(\overline{S}, j_*L_\rho) \cong H^*_{(2)}(\overline{S}, L_\rho).$$

Canonically, the proof of Theorem 1 is reduced to prove

Theorem 2 *(The L^2-Poincaré lemma) The complex $\{\mathcal{A}^{\cdot}_{(2)}(L_\rho), D\}$ is a resolution of j_*L_ρ. This is equivalent to saying that*
1) $j_*L_\rho = \{\eta \in \mathcal{A}^0_{(2)}(L_\rho) \mid D\eta = 0\}$;
2) the differential D satisfies the Poincaré lemma, i.e., if an i-form η in $\mathcal{A}^i_{(2)}(L_\rho)$ is D-closed, then there exists an $i-1$-form $\sigma \in \mathcal{A}^{i-1}_{(2)}(L_\rho)$ satisfying $D\sigma = \eta$, for $i = 1, 2$.

In order to prove the above theorem, we need the following

Lemma 1 *Let V be a constant one dimensional local system over Δ^*, with generator v, and assume that the corresponding line bundle has a Hermitian metric with $\| v \|^2 \sim |\log r|^k$, where r is the Euclidean radius. Then the cohomology sheaves for $\mathcal{A}^{\cdot}_{(2)}(V)$ have stalks at the origin,*

$$\mathcal{H}^0\big(\mathcal{A}^{\cdot}_{(2)}(V)\big) = \begin{cases} V & \text{if } k \le 0 \\ 0 & \text{if } k > 0, \end{cases}$$

$$\mathcal{H}^1\big(\mathcal{A}^{\cdot}_{(2)}(V)\big) = \begin{cases} \frac{dt}{t} \otimes V & \text{if } k \le -2 \\ 0 & \text{if } k \ge -1, k \ne 1 \\ \mathcal{M}_1 dr \otimes v & \text{if } k = 1, \end{cases}$$

$$\mathcal{H}^2\big(\mathcal{A}^{\cdot}_{(2)}(V)\big) = \begin{cases} 0 & \text{if } k \ne -1 \\ \mathcal{M}_1 dr \wedge \frac{dt}{t} \otimes v & \text{if } k = -1, \end{cases}$$

where $\mathcal{M}_1$ is defined as

$$\frac{\{\textit{measurable functions } f : \int_0^A |f(r)|^2 |\log r| (r dr) < \infty \textit{ for some } A < 1\}}{\{f : f = u' \textit{ weakly with } \int_0^A |u(r)|^2 |\log r|^{-1} (r^{-1} dr) < \infty \textit{ for some } A < 1\}}.$$

Proof. cf. [15], Proposition 6.6.

Proof of Theorem 2. On S, the exactness is canonical. So the trouble comes from the singular points $p_1, \cdots, p_s$, and since we are working with sheaves, from now on we just localize the problem to a punctured disk $\Delta^* \subset \Delta$.

The proof of 1). This is equivalent to showing that an L^2 flat section should be a section of j_*L_ρ—an invariant section of γ, which equivalently lies in the kernel of N. To this end, denote the image of the generator of $\pi_1(\Delta^*)$ under the representation ρ by γ, which, by the assumption, is unipotent, $\log\gamma$ by N, which is nilpotent. For a (multi-valued) flat section v of L_ρ, setting

$$\tilde{v} = \exp\left(\frac{1}{2\pi\sqrt{-1}} N \log t\right) v,$$

which is single-valued and d''-holomorphic ($D = d' + d''$, the $(1,0)$ and $(0,1)$-part respectively), one then has the canonical extension $\overline{L}_\rho$ of L_ρ to $\overline{S}$ ($= \Delta$ locally) when L_ρ is considered as a d''-holomorphic bundle: sections of $\overline{L}_\rho$ at the origin are generated by sections of the form $\tilde{v}$. Since N is nilpotent, each fiber of L_ρ canonically has a so-called weight filtration $\{W_i\}$ which is D-invariant and which therefore determines a filtration of L_ρ by some local subsystems, denoted by $\mathbf{W}_i$ and the corresponding extension by $\overline{\mathbf{W}}_i$.

By the norm estimate of the harmonic metric h, it is easy to see that a d''-holomorphic L^2 section of L_ρ on Δ lies in $\overline{\mathbf{W}}_0 + t\overline{L}_\rho$. On the other hand, it is clear that d'-closed sections of $\overline{\mathbf{W}}_0 + t\overline{L}_\rho$ should be generated by sections of the form $\tilde{v}$ satisfying $Nv = 0$. This finishes the proof of 1).

The proof of 2). Without loss of generality, we can assume that the representation $\rho : \pi_1(\Delta^*) \to Gl(m+1, \mathbb{C})$ is irreducible; equivalently, N acts irreducibly on $\mathbb{C}^{m+1}$. Canonically, one has a weight filtration of $\mathbb{C}^{m+1}$:

$$0 \subset W_{-m} \subset W_{-(m-2)} \subset \cdots \subset W_{m-2} \subset W_m = \mathbb{C}^{m+1},$$

which satisfies that the quotient $\mathrm{Gr}_i^W = W_i/W_{i-2}$ is of dimension 1 and $NW_i = W_{i-2}$. Correspondingly, one has the invariant subbundles of L_ρ,

$$0 \subset \mathbf{W}_{-m} \subset \mathbf{W}_{-(m-2)} \subset \cdots \subset \mathbf{W}_{m-2} \subset \mathbf{W}_m = L_\rho,$$

and hence the corrsponding filtration of $\mathcal{A}^{\cdot}_{(2)}(L_\rho)$

$$0 \subset \mathcal{A}^{\cdot}_{(2)}(\mathbf{W}_{-m}) \subset \mathcal{A}^{\cdot}_{(2)}(\mathbf{W}_{-(m-2)}) \subset \cdots \subset \mathcal{A}^{\cdot}_{(2)}(\mathbf{W}_{m-2}) \subset \mathcal{A}^{\cdot}_{(2)}(L_\rho),$$

which is clearly D-invariant. For simplicity, we denote $\mathcal{A}^{\cdot}_{(2)}(\mathbf{W}_{-i})$ by K_i; one has then a filtered complex

$$K_{-m} \supset K_{-m+2} \supset \cdots \supset K_{m-2} \supset K_m \supset 0;$$

note that $K_{-m+1} = K_{-m+2}, \ldots$. Consider the quotient K_i/K_{i+1}, which is clearly $\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-i}(L_\rho))$. We now have the spectral sequence $(E_r, d_r)_{r\geq 1}$ of the filtered complex $\{K_i, D\}_{i=-m}^m$, which, by the theory of spectral sequences, converges to the cohomology of K_{-m}, namely the sheaf cohomology of $\{\mathcal{A}^{\cdot}_{(2)}(L_\rho), D\}$. To prove Theorem 2, we need to analyze the sequence $(E_r, d_r)_{r\geq 1}$.

By the definition of spectral sequences,

$$E_1^{p,q} = H^{p+q}(\mathrm{Gr}^p K^*) = \mathcal{H}^{p+q}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho))\big),$$

where $\mathrm{Gr}^p K_* = K_p/K_{p+1}$. We will show that the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is trivial, as can be observed as follows. Observe the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathcal{A}^{p+q}_{(2)}(\mathbf{W}_{-p-1}) & \xrightarrow{i} & \mathcal{A}^{p+q}_{(2)}(\mathbf{W}_{-p}) & \xrightarrow{Proj} & \mathcal{A}^{p+q}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho)) & \longrightarrow 0 \\ & \downarrow D & & \downarrow D & & \downarrow D & \\ 0 \longrightarrow & \mathcal{A}^{p+q+1}_{(2)}(\mathbf{W}_{-p-1}) & \xrightarrow{i} & \mathcal{A}^{p+q+1}_{(2)}(\mathbf{W}_{-p}) & \xrightarrow{Proj} & \mathcal{A}^{p+q+1}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho)) & \longrightarrow 0 \end{array}$$

where i and $Proj$ are the inclusion and the projection respectively; the third D is the derived one. Let $\phi \otimes \tilde{v} \in \mathcal{A}^{p+q}_{(2)}(\mathbf{W}_{-p})$ represent a cohomologicy class from $\mathcal{H}^{p+q}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho))\big)$; equivalently, $d\phi = 0$ (since $D\tilde{v} \in \Omega^1(\mathbf{W}_{-p-1})$, more precisely $\in \Omega^1(\mathbf{W}_{-p-2})$). So $D(\phi \otimes \tilde{v}) \in \mathcal{A}^{p+q+1}_{(2)}(\mathbf{W}_{-p-2})$, which represents a trivial cohomology class in $\mathcal{H}^{p+q+1}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p-1}(L_\rho))\big)(= E_1^{p+1,q})$, namely $d_1([\phi \otimes \tilde{v}]) = 0$.

Next, let us consider d_2. Since $d_1 = 0$, $E_2^{p,q} = E_1^{p,q}$. Similar to the above argument, taking a form $\phi \otimes \tilde{v} \in \mathcal{A}^{p+q}_{(2)}(\mathbf{W}_{-p})$, which represents a cohomology class of $\mathcal{H}^{p+q}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho))\big)$, $D(\phi \otimes \tilde{v}) = (-1)^{p+q}\frac{1}{2\pi\sqrt{-1}}\big((\phi \wedge \frac{dt}{t}) \otimes N\tilde{v}\big)$ and then lies in $\mathcal{A}^{p+q+1}_{(2)}(\mathbf{W}_{-p-2})$, which represents a cohomological class of $E_2^{p+2,q-1} = \mathcal{H}^{p+q+1}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p-2}(L_\rho))\big)$. So, d_2 is induced by $\frac{1}{2\pi\sqrt{-1}}\frac{dt}{t}N$.

Let us now consider the kernel and image of d_2. Applying Lemma 1 to $E_2^{p,q} = \mathcal{H}^{p+q}\big(\mathcal{A}^{\cdot}_{(2)}(\mathrm{Gr}^W_{-p}(L_\rho))\big)$, we get that the only (possibly) nontrivial terms at E_2 are $\{E_2^{p,-p}\}_{p\geq 0}$, $\{E_2^{p+2,-p-1}\}_{p\geq 0}$, $E_2^{-1,2}$, and $E_2^{1,1}$. Furthermore, from the above argument together with Lemma 1, we obtain that $d_2 : E_2^{p,-p} \to E_2^{p+2,-p-1}, p \geq 0$ (if $E_2^{p+2,-p-1}$ is nontrivial, i.e. $p \leq m-2$) and $d_2 : E_2^{-1,2} \to E_2^{1,1}$ are isomorphisms and that $d(E_2^{p+2,-p-1}) = 0, p \geq 0$ and $d_2(E_2^{1,1}) = 0$.

Summing all the above up, the only possible nontrivial terms at E_3 are $E_3^{m-1,-m+1}, E_3^{m,-m}$. Thus the spectral sequence $\{E_r, d_r\}_{r\geq 1}$ degenerates at E_3 and the only possible terms are $E_3^{m-1,-m+1}, E_3^{m,-m}$. Therefore, by the theory of spectral sequences of filtered complexes, $\mathcal{H}^i(\mathcal{A}^{\cdot}_{(2)}(L_\rho)) = 0, i = 1, 2$. The proof of 2) is finished. □

3.2 The L^2-Higgs cohomology: The L^2-$\overline{\partial}$-Poincaré Lemma

As seen in §2, the harmonic metric h (the ρ-equivariant harmonic map) with the tame growth condition on L_ρ derives a structure of Higgs bundle on L_ρ: $(E, D'' = \overline{\partial} + \theta)$, satisfying $D = D' + D''$ with $D' = \partial + \overline{\theta}$; moreover E, as a holomorphic vector bundle, has bounded curvature under the harmonic metric h so that E can be analytically extended to $\overline{S}$, as usual just denoted by j_*E. Furthermore, θ has a log-singularity, i.e. $\theta \sim \frac{dt}{t}N$. It is especially worth pointing out that by an argument of Simpson (cf. [13]), *the residue N of θ here coincides with the logarithmic monodromy N in the local system*

L_ρ; so although under different bundle structures, we have the same weight filtration under certain suitable identification. Throughout this subsection, we consider the Higgs bundle $(E, D'' = \overline{\partial} + \theta)$ together with the harmonic metric h, just forgetting that it comes from the local system corresponding to the representation ρ. As in the previous subsection, using the Poincaré-like metric ω on S and the harmonic matric h on (E, D''), one can similarly define a complex $\{\mathcal{A}^{\cdot}_{(2)}(E), D''\}$ of fine sheaves on $\overline{S}$. Let U be an open subset of $\overline{S}$. Then $\mathcal{A}^i_{(2)}(E)(U)$ is defined as the set of

E-valued i-forms η on $U \cap S$, with measurable coeffcients and measurable exterior derivative $\overline{\partial}\eta$, such that η and $D''\eta$ have finite L^2 norm on $K \cap S$, for any compact subset K of U.

Remarks. 1) The Higgs condition $(D'')^2 = 0$ makes $\{\mathcal{A}^{\cdot}_{(2)}(E), D''\}$ a complex, which is actually a complex of certain differential forms valued in j_*E. 2) Due to following lemma, it is actually sufficient to assume that η and $\overline{\partial}\eta$ are L^2 in the above definition. 3) Since the sheaves are fine, so again the hypercohomology $\mathbb{H}^*(\{\mathcal{A}^{\cdot}_{(2)}(E), D''\})$ is computed by the cohomology $H^*(\{\Gamma(\mathcal{A}^{\cdot}_{(2)}(E)), D''\})$ of the complex of global sections; we call it the *L^2-Higgs cohomology* of $\overline{S}$ valued in the Higgs bundle $(E, D = \overline{\partial} + \theta)$, denoted by $H^*_{(2)}(\overline{S}, E)$.

Lemma 2 *θ is an L^2-bounded operator.*

Proof. As mentioned before, $\theta \sim \frac{dt}{t}N$. So it suffices to show that $\frac{dt}{t}N$ is L^2-bounded. Since N lowers weights by 2, so by the estimate of the norm of the harmonic bundle, the norm changes by multiplication with $(\log|t|)^{-2}$; on the other hand, $\|\frac{dt}{t}\|^2_\omega = (\log|t|)^2$. The proof is obtained. □

On the other hand, based on the above lemma one can also define a subcomplex of $\{\mathcal{A}^{\cdot}_{(2)}(E), D''\}$—the *$L^2$-holomorphic Dolbeault complex* $\{\Omega^{\cdot}_{(2)}(E), \theta\}$ as follows: $\Omega^i_{(2)}(E)(U)$ is defined as the set of

E-valued holomorphic i-forms η on $U \cap S$ such that η has finite L^2 norm on $K \cap S$, for any compact subset K of U.

$\theta \wedge \theta = 0$ makes $\{\Omega^{\cdot}_{(2)}(E), \theta\}$ again a complex, which is actually a complex of certain meromorphic differential forms valued in j_*E. We call the hypercohomology $\mathbb{H}^*(\{\Omega^{\cdot}_{(2)}(E), \theta\})$ the *L^2-Dolbeault cohomology* of $\overline{S}$ valued in the Higgs bundle $(E, D = \overline{\partial} + \theta)$.

The purpose of this subsection is then to show that the above two complexes have the same hypercohomologies; more precisely

Theorem 3 *(L^2-$\overline{\partial}$-Poincaré lemma) The inclusion $i : \{\Omega^{\cdot}_{(2)}(E), \theta\} \hookrightarrow \{\mathcal{A}^{\cdot}_{(2)}(E), D''\}$*

is a quasi-isomorphism; and hence one has

$$\mathbb{H}^*(\{\Omega^{\cdot}_{(2)}(E),\theta\}) \cong H^*_{(2)}(\overline{S},E).$$

Remarks. In the case when E comes from a VHS, the theorem was showed by S. Zucker [15] (for the case of curves) and Jost-Yang-Zuo [9] (for the general case).

In order to prove the theorem, we need some preliminaries. First we give the following

Definition 1 *Let V be a Hermitian vector bundle over a Riemannian manifold M, $\mathbf{v} = \{v_1, v_2, \cdots, v_q\}$ be a global frame field of V. Then $\mathbf{v}$ is said to be L^2-adapted if $\sum_{i=1}^q f_i v_i$ is square integrable implies that each $f_i v_i$ is square integrable, where the f_i are smooth functions on M.*

We next show the following general lemma, which will be needed when we consider general semisimple representations (not necessarily being unipotent at infinity), although what we really need here is the special case $\alpha = 0$ which was proved in [15].

Lemma 3 *Let V be a holomorphic line bundle on Δ^* with generating section σ, and with a Hermitian metric satisfying*

$$\|\sigma\|^2 \sim r^\alpha |\log r|^k, \quad 0 \le \alpha < 1, k \in \mathbb{Z}.$$

Then for every germ of an L^2 $(0,1)$-form $\phi = f d\bar{t} \otimes \sigma$ at the origin, if 1) $\alpha = 0$ and $k \neq 1$ or 2) $\alpha \neq 0$, there exists an L^2 section $u \otimes \sigma$ with $\overline{\partial} u = f d\bar{t}$.

Proof. Similar to [15], we also use Fourier series. Using polar coordinates, we write u and f as r-dependent Fourier series

$$u = \sum_{n=-\infty}^{\infty} u_n(r) e^{\sqrt{-1}n\theta}, f = \sum_{n=-\infty}^{\infty} f_n(r) e^{\sqrt{-1}n\theta}.$$

As $\partial/\partial\bar{t} = (1/2)e^{\sqrt{-1}\theta}[\partial/\partial r + (\sqrt{-1}/r)\partial/\partial\theta]$, the equation $\overline{\partial} u = f$ becomes

$$\frac{1}{2}\big(u_n' - \frac{n}{r}u_n\big) = f_{n+1} \quad \text{for all } n \in \mathbb{Z},$$

for C^∞ germs u and f, or

$$\frac{1}{2}\frac{d}{dr}[r^{-n}u_n(r)] = r_{-n} f_{n+1}(r).$$

We are given that for some $A < 1$,

$$\|\phi\|^2_{(2)} = 4\pi \sum_{n=-\infty}^{\infty} \int_0^A |f_n|^2 |\log r|^k r^{1+\alpha} dr < \infty,$$

and we want to obtain some u satisfying

$$\|u\otimes\sigma\|_{(2)}^2 = 2\pi\sum_{n=-\infty}^{\infty}\int_0^A |u_n|^2|\log r|^{k-2}r^{-1+\alpha}dr<\infty.$$

In order to obtain u, we make use of integral representations and try to show that $\|u\|_{(2)}^2\leq C\|f\|_{(2)}^2$ for some positive constant C. Then a standard approximation argument shows that the lemma is true. As pointed out above, the case $\alpha=0$ has been proved in [15], so from now on we assume $\alpha>0$. Set

$$u_n(r)=\begin{cases} 1)\ 2r^n\int_0^r\rho^{-n}f_{n+1}(\rho)d\rho & \text{if } n<0,\\ 2)\ -2\int_r^A f_1(\rho)d\rho & \text{if } n=0,\\ 3)\ -2r^n\int_r^A\rho^{-n}f_{n+1}(\rho)d\rho & \text{if } n>0.\end{cases}$$

(Note that the case of $\alpha>0$ is different from the case $\alpha=0$. In the latter case, if $n=0$, one needs furthermore consider the values of k; in the former case, we need not even consider k, but one needs take $u_0=-2\int_r^A f_1(\rho)d\rho$, since $\int_0^r f_1(\rho)d\rho$ can possible not be integrable.) In order to prove $\|u\|_{(2)}^2\leq C\|f\|_{(2)}^2$, it is sufficient to prove that for all n

$$\int_0^A|u_n|^2|\log r|^{k-2}(r^{-1+\alpha}dr)\leq C\int_0^A|f_{n+1}|^2|\log r|^k(r^{1+\alpha}dr).$$

If $n<0$, one has

$$\begin{aligned} &\int_0^A|u_n|^2r^{-1+\alpha}|\log r|^{k-2}dr\\ \leq\ & 4\int_0^A r^{2n}\Big(\int_0^r\rho^{-2n}|f_{n+1}(\rho)|^2d\rho\Big)\Big(\int_0^r d\rho\Big)r^{-1+\alpha}|\log r|^{k-2}dr\\ =\ & 4\int_0^A\Big(\int_0^r\rho^{-2n}|f_{n+1}(\rho)|^2d\rho\Big)r^{2n+\alpha}|\log r|^{k-2}dr\\ =\ & 4\int_0^A\Big(\int_\rho^A r^{2n+\alpha+1}|\log r|^{k-2}(r^{-1}dr)\Big)\rho^{-2n}|f_{n+1}(\rho)|^2d\rho\\ \leq\ & 4\int_0^A\Big(\int_\rho^A r^{-1}dr\Big)|f_{n+1}(\rho)|^2\rho^{\alpha+1}|\log\rho|^{k-2}d\rho\\ \leq\ & C\int_0^A|f_{n+1}(\rho)|^2\rho^{\alpha+1}|\log\rho|^kd\rho.\end{aligned}$$

In the second inequality above, we used the property of $|\log r|^{k-2}r^{2n+\alpha+1}$ being decreasing since $2n+\alpha+1<0$. If $n=0$, one has

$$\begin{aligned}|u_0(r)|^2\ &=\ 4\Big(\int_r^A|f_1(\rho)|^2|\log\rho|^{1+k}\rho^{1+\alpha}d\rho\Big)\Big(\int_r^A|\log\rho|^{-1-k}\rho^{-1-\alpha}dr\Big)\\ &\leq\ C\Big(\int_r^A|f_1(\rho)|^2|\log\rho|^{1+k}\rho^{1+\alpha}d\rho\Big)r^{-\alpha}|\log r|^{-k},\end{aligned}$$

in the above inequality, we again used the property of $|\log\rho|^{-1-k}\rho^{-\alpha}$ being decreasing. Estimating $\int_0^A |u_0(r)|^2|\log r|^{k-2}r^{-1+\alpha}dr$, we have

$$\begin{aligned}
&\int_0^A |u_0(r)|^2|\log r|^{k-2}r^{-1+\alpha}dr \\
\le\ & C\int_0^A \Big(\int_r^A |f_1(\rho)|^2|\log\rho|^{1+k}\rho^{1+\alpha}d\rho\Big)r^{-1}|\log r|^{-2}dr \\
=\ & C\int_0^A \Big(\int_0^r \rho^{-1}|\log\rho|^{-2}d\rho\Big)|f_1(r)|^2|\log r|^{1+k}r^{1+\alpha}dr \\
=\ & C\int_0^A |f_1(r)|^2|\log r|^{k}r^{1+\alpha}dr.
\end{aligned}$$

If $n > 0$, one has

$$\begin{aligned}
&\int_0^A |u_n|^2 r^{-1+\alpha}|\log r|^{k-2}dr \\
\le\ & 4\int_0^A r^{2n}\Big(\int_r^A \rho^{-2n+1}|f_{n+1}(\rho)|^2d\rho\Big)\Big(\int_r^A \frac{1}{\rho}d\rho\Big)r^{-1+\alpha}|\log r|^{k-2}dr \\
\le\ & C\int_0^A \Big(\int_r^A \rho^{-2n+1}|f_{n+1}(\rho)|^2d\rho\Big)r^{2n+\alpha-1}|\log r|^{k-1}dr \\
=\ & C\int_0^A \Big(\int_0^\rho r^{2n+\alpha-1}|\log r|^{k-1}dr\Big)\rho^{-2n+1}|f_{n+1}(\rho)|^2d\rho \\
\le\ & C\int_0^A |f_{n+1}(\rho)|^2|\log\rho|^{k-1}\rho^{1+\alpha}d\rho.
\end{aligned}$$

In the last inequality above, we use that $r^{2n+\alpha-1}|\log r|^{k-1}$ is increasing, since $2n+\alpha-1>0$. The proof of the lemma is completed. □

Proof of Theorem 3. Similar to the proof of Theorem 2, the difficulty again comes from $\overline{S}\setminus S$, so one can restrict the problem to a small enough punctured disk Δ^* so that we can choose a holomorphic basis of j_*E: $e_1, e_2, \cdots, e_n$, which is compatible with the weight filtation corresponding to N (the residue of θ) at the origin. The compatibility implies that each e_i of the basis has the property

$$\begin{aligned}
\|e_i\|_h &\sim |\log r|^k, \text{ for a certain integer } k; \\
\|\theta e_i\|_{h,\omega} &\sim |\log r|^k, \text{ if } \theta e_i \neq 0,
\end{aligned}$$

where ω is the Poincaré-like metric. In particular, for some section e, if $\|e\|_h \sim |\log r|$, then $\theta e \neq 0$; if $\|e\|_h \sim |\log r|^{-1}$, then there exists a section e' satisfying $\theta e' = \frac{dt}{t}\otimes e$. Using these facts, it is not difficult to prove that the basis $e_1, e_2, \ldots, e_n$ is an L^2-adapted one.

In order to prove the theorem, we need to show that the inclusion i derives an isomorphism between the corresponding cohomology sheaves at the origin; by the argument of standard homological algebra, this is equivalent to showing that for any D''-closed form $\phi \in \mathcal{A}^r_{(2)}(E)$ on a neighborhood U of the origin, there is a θ-closed form $\eta \in \Omega^r_{(2)}(E)$ and a form $\psi \in \mathcal{A}^{r-1}_{(2)}(E)$ on (a possibly smaller) U satisfying $\phi = \eta + D''\psi$, $r = 0, 1, 2$.

When $r = 0$, it is clear that a D''-closed form is a holomorphic section of E and θ-closed; in the case of $r = 2$, the form ϕ can be written as the sum of forms $\phi' \wedge \frac{dt}{t} \otimes e$, ϕ' is a complex-value $(0,1)$-form and e is an element of the basis. Since the basis is L^2-adapted, each summand is still L^2. Considering $\frac{dt}{t} \otimes e$ as a generator of a holomorphic line bundle, its norm satisfies under the harmonic metric and the Poincaré-like metric

$$\|\frac{dt}{t} \otimes e\| = |\log r|^k,$$

for some integer k. When $k \neq 1$, by the above lemma, there exists an L^2-form $u\frac{dt}{t} \otimes e$ satisfying

$$\overline{\partial}(u\frac{dt}{t} \otimes e) = \phi' \wedge \frac{dt}{t} \otimes e,$$

and hence $D''(u\frac{dt}{t} \otimes e) = \phi' \otimes \frac{dt}{t} \otimes e$; assuming $k = 1$, there exists a holomorphic section e' of E satisfying $\theta e' = \frac{dt}{t} \otimes e$, and hence $\theta(-\phi' \otimes e') = D''(-\phi' \otimes e') = \phi' \wedge \frac{dt}{t} \otimes e$. It is clear that $-\phi' \otimes e'$ is L^2 ($\overline{\partial}(-\phi' \otimes e')$ is automatically L^2 since $= 0$) and hence $-\phi' \otimes e' \in \mathcal{A}^1_{(2)}(E)$. Summing all the above, we have that any form of $\mathcal{A}^2_{(2)}(E)$ is D''-coclosed.

We now turn to the case of $r = 1$. In order to make the proof clearer, we can refer from time to time to the following diagram

$$\begin{tikzcd}
\Omega^0_{(2)}(E) \arrow[r, "i"] \arrow[d, "\theta"] & \mathcal{A}^{0,0}_{(2)}(E) \arrow[r, "\overline{\partial}"] \arrow[d, "\theta"] & \mathcal{A}^{0,1}_{(2)}(E) \arrow[r] \arrow[d, "\theta"] & 0 \\
\Omega^1_{(2)}(E) \arrow[r, "i"] & \mathcal{A}^{1,0}_{(2)}(E) \arrow[r, "\overline{\partial}"] & \mathcal{A}^{1,1}_{(2)}(E) \arrow[r] & 0.
\end{tikzcd}$$

Write $\phi = \phi^{1,0} + \phi^{0,1}$, $\phi^{1,0}$ (resp. $\phi^{0,1}$) being the part of type $(1,0)$ (resp. $(0,1)$). The D''-closedness of ϕ is equivalent to

$$\overline{\partial}\phi^{1,0} + \theta\phi^{0,1} = 0.$$

Write $\phi^{0,1}$ as the sum of forms $\phi^{0,1}_i \otimes e_i$, $i = 1, \cdots, n$, which, by the L^2-adaptedness of the basis, are L^2 again. Assume $\|e_{i_0}\| \sim |\log r|$ and for $i \neq i_0$, $\|e_i\| \sim |\log r|^k, k \neq 1$. By the above lemma, for $i \neq i_0$, there exists an L^2 section $u_i e_i$ satisfying

$$\overline{\partial}u_i \otimes e_i = \phi^{0,1}_i \otimes e_i.$$

It is easy to see that the part of type $(0,1)$ of the form $\phi - D''(u_i e_i)$ does not contain any further terms of the form $\phi_i^{0,1} \otimes e_i, i \neq i_0$. So, w.l.o.g., we may assume that $\phi^{0,1} = \phi_{i_0}^{0,1} \otimes e_{i_0}$. In order to deal with the term $\phi_{i_0}^{0,1} \otimes e_{i_0}$, we use the D''-closedness of ϕ, i.e. $\overline{\partial}\phi^{1,0} + \theta\phi^{0,1} = 0$. Considering the holomorphic vector bundle $\frac{dt}{t} \otimes E$, it is not difficult to see that one can extend $\theta e_i, i = 1, \cdots, n$, to an L^2-adapted basis (we do not really need this basis). Due to the relation $\overline{\partial}\phi^{1,0} + \theta\phi^{0,1} = 0$, one can write $\phi^{1,0}$ as $u_{i_0}\theta e_{i_0}$ satisfying $\overline{\partial} u_{i_0} = \phi_{i_0}^{0,1}$. Clearly both $u_{i_0}e_{i_0}$ and $\overline{\partial}(u_{i_0}e_{i_0})$ are L^2, namely $u_{i_0}e_{i_0} \in \mathcal{A}^0_{(2)}(E)$, and $\phi - D''(u_{i_0}e_{i_0})$ does no longer contain the part of type $(0,1)$. Obviously a D''-closed L^2-form of type $(1,0)$ is holomophic and θ-closed. Summing the above argument up, we have that for any D''-closed form $\phi \in \mathcal{A}^1_{(2)}(E)$ on a neighborhood U of the origin, there is a θ-closed form $\eta \in \Omega^1_{(2)}(E)$ and a section $\psi \in \mathcal{A}^0_{(2)}(E)$ on (a possibly smaller) U satisfying $\phi = \eta + D''\psi$. This finishes the proof of the theorem. $\Box$

Using the Kähler identity for harmonic bundles (cf. §2), $H^*_{(2)}(\overline{S}, L_\rho)$ can be identified with $H^*_{(2)}(\overline{S}, E)$, and hence we have the following

Corollary 1

$$H^*(\overline{S}, j_*L_\rho) \cong \mathbb{H}^*(\{\Omega^{\cdot}_{(2)}(E), \theta\}).$$

References

[1] E. Cattani, A. Kaplan, & W. Schmid, L^2 and intersection cohomologies for a polarizable variation of Hodge structure. Inventiones Math., **87**(1987), 217-252.

[2] M. Cornalba & P. Griffiths, Analytic cycles and vector bundles on non-compact algebraic varieties, Invent. Math., **28**(1975), 1-106.

[3] J. Cheeger, M. Goresky, & R. MacPherson, L^2-cohomology and intersection homology of singular algebraic varieties. *Seminar on differential geometry*, ed. by S.-Y. Yau, Ann. Math. Studies, **102**(1982), 303-340.

[4] N. Hitchin, The self-duality equations on a Riemann surface, Proc.London Math.Soc., **76**(1987), 59-126

[5] J. Jost & S.T. Yau, Harmonic maps and Kähler manifolds, Math.Ann., **262**(1983), 145-166

[6] J. Jost & K. Zuo, Harmonic maps and $SL(r, \mathbb{C})$-representations of fundamental groups of quasiprojective manifolds. J. Algebraic Geometry, **5**(1996), 77-106.

[7] J. Jost & K. Zuo, Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasiprojective manifolds. J. Differential Geometry, **47**(1997) 469-503.

[8] J. Jost, Y.-H. Yang, & K. Zuo, Linear representations of π_1 of quasi-projective varieties, constructions of harmonic maps, and parabolic structures. Preprint

[9] J. Jost, Y.-H. Yang, & K. Zuo, The cohomology of a variation of polarized Hodge structures over a quasi-compact Kähler manifold. Preprint, 2003.

[10] E. Looijenga, L^2-cohomology of locally symmetric varieties, Compositio Math., **67**(1988), 3-20.

[11] T. Mochizuki, Asymptotic behavior of tame harmonic bundles with trivial parabolic structures. J. Differential Geometry, **62**(2002), 351-359.

[12] C. Simpson, Higgs bundles and local systems, IHES Publ.Math., **75**(1992), 5-95

[13] C. Simpson, Harmonic bundles on noncompact curves. Journal of Amer. Math. Soc., **3**(1990) 713-770.

[14] L. Saper & M. Stern, L^2-cohomology of arithmetic varieties. Ann. Math., **132**(1989), 1-69.

[15] S. Zucker, Hodge theory with degenerating coefficients: L^2 cohomology in the Poincaré metric. Ann. Math., **109**(1979), 415-476.

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