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On the scaling of the two well problem

by

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ON THE SCALING OF THE TWO WELL PROBLEM

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ABSTRACT. Let $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ for $\sigma > 0$. And let $K := SO(2) \cup SO(2) H$. We establish a sharp relation between the following two minimisation problems.

Firstly the two well problem with surface energy. Let $q \geq 1$. Let

$$I_\epsilon^q(u) = \int_\Omega d(Du(z), K) + \epsilon \left|D^2 u(z)\right|^q dL^2 z$$

and let A_F denote the subspace of functions in $W^{2,q}(\Omega)$ with $\det(Du(z)) \geq 0$ for a.e. $z \in \Omega$ and $\sup_{z\in\Omega} \| [Du(z)]^{-1} \| \leq C$ satisfying the affine boundary condition $u(z) = F(z)$ for $z \in \partial\Omega$, where $F \notin K$. We consider the scaling (with respect to ϵ) of

$$m_\epsilon^q := \inf_{u \in A_F} I_\epsilon^q(u).$$

Secondly the finite element approximation to the two well problem without surface energy. Let $\delta > 0$ be any small number. Let $\mathcal{F}_h(u) = \int_\Omega d\left(Du(z), N_{h^{\frac{\delta}{84}}}(K)\right) dL^2 z$. Let B_F^h denote the space of functions that are piecewise affine on a triangular grid $\{\tau_i\}$ of grid size h satisfying the affine boundary condition $u(z) = F(z)$ for $z \in \partial\Omega$. We consider the scaling of

$$\alpha_h := \inf_{u \in B_F^h} \mathcal{F}_h(u).$$

Let $q \geq 1$. We will show that for any small h, for $\epsilon := h^q$ we have

$$\alpha_h \geq c h^{\frac{1}{3}} \Longrightarrow m_\epsilon^q \geq c' \epsilon^{\frac{1}{3q} + \delta}.$$

Simple examples show $\alpha_h \leq C h^{\frac{1}{3}}$ and $m_\epsilon^q \leq C' \epsilon^{\frac{1}{3q}}$ so our theorem states that optimal (scaling) lower bounds on α_h imply optimal (scaling) lower bounds on m_ϵ^q for any $q \geq 1$.

The main tool we will use to establish this reduction will be an L^q version of the sub-optimal two well Liouville Theorem proved in [22]. We will give a simple proof of this result using the case of equality of the isoperimetric inequality.

In addition for the case $q = 1$ we show optimal (scaling) lower bounds on I_ϵ^1 follow from optimal (scaling) lower bounds on $\mathcal{F}_0$ by applying the optimal two well Liouville Theorem of Conti, Schweizer [6].

1. Introduction

Let H be a diagonal matrix with $\det(H) = 1$. Let $K := SO(2) \cup SO(2) H$. We are concerned with minimising the functional

$$\mathcal{I}(u) = \int_\Omega d(Du(z), K) \, dL^2 z \tag{1}$$

over the space L_F of functions with affine boundary condition $F \notin K$. This functional is a special case of the functional proposed by Ball and James [2], [3] and Chipot, Kinderlehrer [5] in their well known model of solid solid phrase transitions.

Surprisingly, for $F \in \operatorname{int}(K^{qc})$ (see [28] for background and precise definitions) there exists an exact minimiser of $\mathcal{I}$, this follows from work of Müller and Šverák [24], [25], see Sychev [29], [30] and Kirchheim [16], [17] for latter developments and Dacorogna Marcellini [8] for a different approach to some related problems. The approach of Müller and Šverák uses the

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theory of "convex integration" (denoted by CI from this point) developed by Gromov, it is one of the simplest results of the theory.

If we add a small cost to the oscillation of the functional, we have a functional of the form

$$I^q_\epsilon(u) = \int_\Omega d(Du(z), K) + \epsilon \left|D^2 u(z)\right|^q dL^2 z. \tag{2}$$

Nothing is known about the *minimiser* of the functional (however there does now exist a Γ convergence result for the functional $\frac{I^2_\epsilon}{\sqrt{\epsilon}}$ [6]). In particular it is completely unknown if for very small ϵ the minimiser is something like the absolute minimiser of $\mathcal{I}$ provided by CI[1].

This question is best expressed by considering the scaling of

$$m^q_\epsilon := \inf_{u \in W^{2,q}(\Omega) \cap A_F} I^q_\epsilon(u). \tag{3}$$

An upper bound of $m^q_\epsilon \leq c\epsilon^{\frac{1}{3q}}$ is provided by the standard double laminate. This follows from the characterisation of the quasiconvex hull K^{qc} provided by [31].

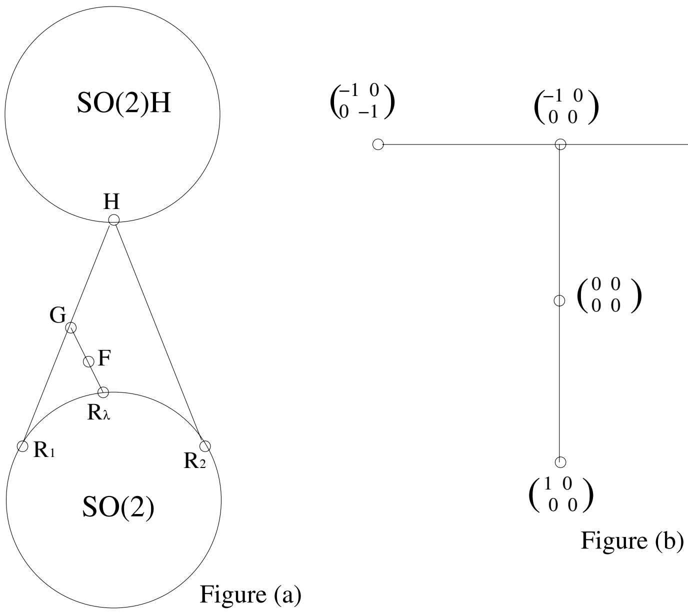

FIGURE 1

If $m^q_\epsilon \sim \epsilon^{\frac{1}{3q}+\alpha}$ for $\alpha > 0$ then the minimiser will have to take a very different form than the double laminate. On the other hand if $\alpha = 0$ then energetically the minimiser does no better than the double laminate.

This question is important because CI solutions are important, many counter examples to natural conjectures in PDE have been achieved via CI, [26], [16], [27]. Minimising functional I^q_ϵ is the simplest problem that constrains oscillation is some slight way where we can hope to see the effect of the existence of exact minimisers of (1).

Following the partial results of [22] we reduce this question to the question of the scaling of a functional similar to $\mathcal{I}$ over the subspace B^h_F of functions that are piecewise affine on a triangular grid (with grid size h, where none of the edges of the triangles are in the set of rank-1

[1] We know it can not *be* a function u with $\mathcal{I}(u) = 0$ because the result of Dolzmann Müller [9], that any u with this property and with the property that Du is a BV has to be laminate

directions of K). Our reduction is sharp in the sense that we will show optimal lower bounds for the finite element approximation implies optimal lower bounds for m^q_ϵ, for any $q \geq 1$.

Our main tool to achieve this is an L^q version of the sub-optimal two well Liouville theorem established in [21], this result may be of independent interest. See [6] for the (scaling) optimal version of the L^1 theorem.

1.1. **Two well Liouville Theorem.**

Theorem 1. *Let $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$ for $\sigma > 0$. Let $p \geq 1$, $q \geq 1$. Let $K = SO(2) \cup SO(2)\, H$. Let $u \in W^{2,q}(B_1(0)) \cap W^{1,p}(B_1(0))$ be a function with the property that $\det(Du(x)) \geq 0$ and $\| [Du(x)]^{-1} \| \leq C$ for a.e. $x \in B_1(0)$ where $C > 0$ is any large constant.*

There exists positive constants $\mathcal{C}_1 << 1, \mathcal{C}_2 >> 1$ depending only on σ, p, q such that if $\epsilon \in (0, \mathcal{C}_1)$ and u satisfies the following inequalities

$$\int_{B_1(0)} d^p(Du(z), K)\, dL^2 z \leq \mathcal{C}_1 \epsilon \tag{4}$$

$$\int_{B_1(0)} \left| D^2 u(z) \right|^q dL^2 z \leq \mathcal{C}_1 \epsilon^{1-q}, \tag{5}$$

then there exist $J \in \{Id, H\}$ and $R \in SO(2)$ such that

$$\int_{B_{\mathcal{C}_1}(0)} |Du(z) - RJ|^p\, dL^2 z \leq \mathcal{C}_2 \epsilon^{\frac{1}{p k_p}}, \tag{6}$$

where $k_p = 4$ when $p > 1$ and $k_p = 5$ when $p = 1$.

We will give a simple proof of this via the case of equality of the isoperimetric inequality. More specifically, it is well known that amongst all bodies B of volume 1 in $\mathbb{R}^n$, the ball minimises $H^{n-1}(\partial B)$, i.e. the ball gives the case of equality of the isoperimetric inequality. A quantitative statement of this kind is given by the following Lemma of Hall, Haymann, Weitsman [14].

Lemma 1 (Hall et al.)**.** *Let E be a set of finite perimeter [2] in $\mathbb{R}^2$, $R := \left(\frac{L^2(E)}{\pi}\right)^{\frac{1}{2}}$ and let the Fraenkel asymmetry $\lambda(E)$ be defined by*

$$\lambda(E) = \inf_{a \in \mathbb{R}^2} \frac{L^2(E \backslash B_R(a))}{\pi R^2}. \tag{7}$$

Then

$$(\mathrm{Per}(E))^2 \geq 4\pi \left(1 + \frac{(\lambda(E))^2}{4}\right) L^2(E).$$

Theorem 1 generalises Theorem 1 of [21] in that hypothesis (5) is an L^q bound on $D^2 u$ instead of an L^1 bound as in [21], [6]. Simple examples show that $1-q$ is the optimal power in ϵ. Additionally the control of Du in (6) is (at least) $\epsilon^{\frac{1}{5p}}$ which improves the $\epsilon^{\frac{1}{800}}$ bound of [21] but is much weaker than the optimal $\epsilon^{\frac{1}{p}}$ bound of [6]. The main reason for the improvement comes from the application the quantitative Liouville Theorem of Friesecke et al. (see Theorem 3 of Section 2) in a efficient way, and this we learned from the work of Conti, Schweizer [6].

The isoperimetric inequality method is the fastest, "calculation free" way to see why the sub optimal theorem is true, it helps to show why this initially surprising result is actually quite natural.

[2] Hall et al. state their Lemma for sets with smooth boundaries. By Theorem 3.41 [1] we can approximate any set A of finite perimeter with a sequence of sets (A_n) that converge in measure to A which have smooth boundaries and for which $\mathrm{Per}(A_n) \to \mathrm{Per}(A)$ as $n \to \infty$, hence its easy to see the lemma holds for sets of finite perimeter.

The conditions that function u is sense preserving (i.e. $\det(Du(x)) > 0$ a.e.) and satisfies $\sup_{x\in B_1(0)} \| [Du(x)]^{-1} \| \leq C$ are technical conditions that we use for convenience, in words, they say that u can not compress small balls into shapes of arbitrary small diameter or reverse their orientation, as such they are not such unnatural conditions for functions describing elastic deformations.

Conjecture 1. *Let H, K be as in Theorem 1. Let $u \in W^{2,q}(B_1(0)) \cap W^{1,p}(B_1(0))$. There exists positive constants $\mathcal{C}_1 << 1$, $\mathcal{C}_2 >> 1$ depending on σ, p, q such that if u satisfies (4) and (5) then for some $J \in \{Id, H\}$, $R \in SO(2)$ we have*

$$\int_{B_{\mathcal{C}_1}(0)} |Du(z) - RJ|^p \, dL^2 z \leq \mathcal{C}_2 \epsilon^{\frac{1}{p}}.$$

For the case $q = 1$ this has been proved in [6].

1.2. **Finite element approximations.** In order to explain our main application of this result we will need to give a bit more background.

A *triangulation* (denoted $\triangle_h$) of Ω of size h is a collection of pairwise disjoint triangles $\{\tau_i\}$ all of diameter h such that $\Omega \subset \bigcup_{\tau_i \in \triangle_h} \tau_i$.

We can approximate any continuous function u uniformly by a function $\tilde{u}$ that is piecewise affine on the triangles of $\triangle_h$ by the following procedure. For each triangle $\tau_i \in \triangle_h$, define $\tilde{u}_{\lfloor \tau_i}$ to be the affine map we get by interpolating u on the corners of τ_i. We will call $\tilde{u}$ the *interpolant* of u.

Let B_F^h denote the space of Lipschitz functions in L_F that are piecewise affine on the triangles of $\triangle_h$. Our interest in this space of functions comes from the fact that minimisation of functionals of the form (1) over B_F^h provides a convenient intermediary problem for the study of surface energy problems: let $\triangle_h$ be a triangulation for which the edges of the triangles are not parallel to the rank-1 connections of the wells K, if $\tilde{u} \in B_F^h$ and $\tau_1, \tau_2 \in \triangle_h$ are such that $d\left(D\tilde{u}_{\lfloor \tau_1}, SO(2)\right) \approx 0$ and $d\left(D\tilde{u}_{\lfloor \tau_2}, SO(2)H\right) \approx 0$ it is easy to see τ_1 can not touch τ_2, i.e. there must be a triangle τ_3 between τ_1 and τ_2 for which $d\left(Du_{\lfloor \tau_3}, K\right) \geq o(1)$.

For example if we have an interpolant of a laminate, if triangle τ cuts through an interface of the laminate the affine map we get from interpolating the laminate on the corners of τ will have its linear part some distance from the wells. See figure 2.

So we can not lower the energy of $\mathcal{I}$ over B_F^h by simply making a laminate type function with finer layers, there is a competition between the "surface energy" as given by the error contributed from the interfaces and the "bulk energy" which in the case of the laminate is the width of the interpolation layer.

Theorem 2. *Let $K = SO(2) \cup SO(2)H$, $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$. Let Ω be a bounded Lipschitz domain. Let $\triangle_h$ be a triangulation of Ω of grid size h with the directions of the edges of the triangles some uniform distance away from the set of rank-1 directions of K. Let $F \in \mathrm{int}(K^{qc})$. Let $\delta > 0$ be some small number.*

Denote by B_F^h the set of functions with affine boundary condition F that are piecewise affine on the triangulation $\triangle_h$. Define $\mathcal{F}_h(u) := \int_\Omega d\left(Du(z), N_{h^{\frac{\delta}{84}}}(K)\right) dL^2 z$.

Let A_F^ζ denote the space of ζ-Lipschitz functions in $W^{2,q}(\Omega)$ with $\det(Du(z)) \geq 0$ for a.e. and $\sup_{x\in\Omega} \| [Du(x)]^{-1} \| \leq C$ with affine boundary condition F.

Let $q > 1$. Let $\mathcal{A} > 0$. There exists $h_0 = h_0(\sigma, q, \delta, \mathcal{A}, \zeta)$ such that if $h \in (0, h_0)$ then

$$\inf_{v \in B_F^h} \mathcal{F}_h(v) \geq \mathcal{A} h^{\frac{1}{3}} \Rightarrow \inf_{u \in A_F^\zeta} I_\epsilon^q(u) \geq \epsilon^{\frac{1}{3q}+\delta} \text{ for } \epsilon = h^q. \tag{8}$$

For $q = 1$. We have for $h \in (0, h_0)$

$$\inf_{v \in B_F^h} \mathcal{F}_0(v) \geq \mathcal{A} h^{\frac{1}{3}} \Rightarrow \inf_{u \in A_F^\zeta} I_\epsilon^1(u) \geq \epsilon^{\frac{1}{3q}+\delta}, \text{ for } \epsilon = h. \tag{9}$$

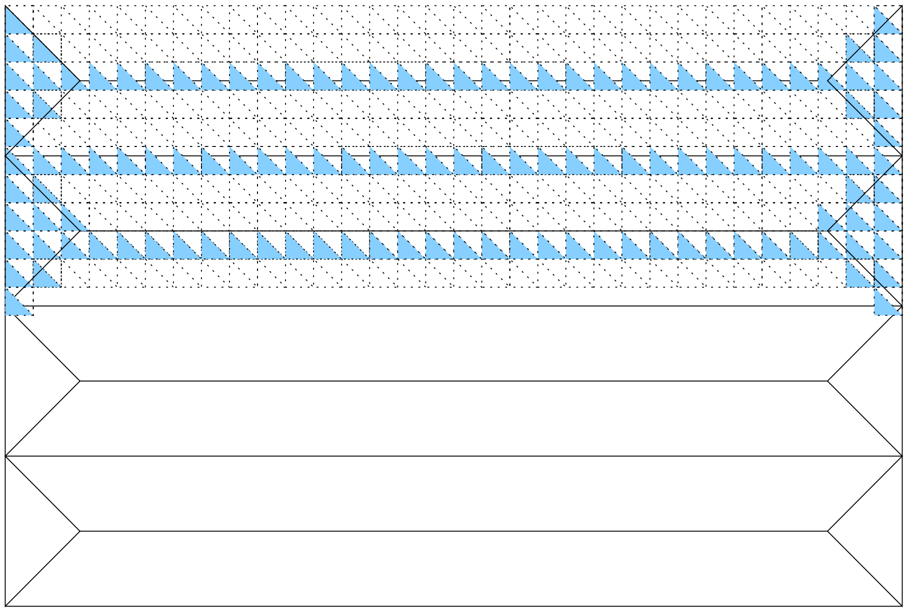

FIGURE 2

We will show in fact that Conjecture 1 implies a cleaner formulation of (8), namely that for any $\delta > 0$ there exists $h_0 = h_0(\sigma, q, \delta, \mathcal{A}, \zeta)$ such that $h \in (0, h_0)$

$$\inf_{v \in B_F^h} \mathcal{F}_0(v) \geq \mathcal{A} h^{\frac{1}{3}} \Rightarrow \inf_{u \in A_F^{\zeta}} I_\epsilon^q(u) \geq \epsilon^{\frac{1}{3q}+\delta} \text{ for } \epsilon = h^q. \tag{10}$$

Recall Conjecture 1 is proved in [6] for the case $q = 1$ and so for this case we can establish (9).

Let $B_1 := \operatorname{diag}(1, 0)$, $B_2 := \operatorname{diag}(-1, 1)$, $B_3 := \operatorname{diag}(-1, 1)$. See figure 1 (b). Define $\widetilde{\mathcal{F}}(u) := \int_\Omega d(Du(z), \{B_1, B_2, B_3\})\, dL^2 z$. F.E. approximations of $\widetilde{\mathcal{F}}$ over $A_{F_0}^h$ (where $F_0 := \operatorname{diag}(0, 0)$) have been studied by Chipot [4] and the author [19]. It has been shown $\inf_{u \in A_F^h} \widetilde{\mathcal{F}}(u) \sim h^{\frac{1}{3}}$. From Šverák's characterisation [31] we know the exact arrangement of rank-1 connections between the matrices in the set $SO(2) \cup SO(2) H$ and a matrix in the interior of the quasiconvex hull, see figure 1 (a). As we can see from figures 1 (a) and (b), the finite well functional $\widetilde{\mathcal{F}}$ precisely mimics these rank-1 connections.

Conjecture 2. *Let K, H be defined as in Theorem 2. Given $F \in \operatorname{int}(K^{qc})$. Let $\delta > 0$. Take B_F^h be as in Theorem 2.*

Define $\mathcal{F}_h(u) := \int_\Omega d\left(Du(z), N_{h^{\frac{\delta}{84}}}(K)\right) dL^2 z$. Then there exists constant c depending on σ such that

$$\inf_{u \in B_F^h} \mathcal{F}_h(u) \geq c h^{\frac{1}{3}}.$$

Informally Theorem 2 says that the optimal scaling for I_ϵ^q would follow from Conjecture 2. This is not simply a matter of replacing ϵ with h. There is no reason to think the existence of an absolute minimiser to $\mathcal{I}$ will cause $\mathcal{F}_h$ to scale to zero faster that at rate $h^{\frac{1}{3}}$. In their most constructive form [24], CI solutions are made as a limit of "laminate within laminate" type functions, and for complicated functions of this type we expect the interpolant to have many triangles with the derivative not close to the wells. For example Chipot ([4] , Theorem 4.3) proved the upper bound of $e^{-c|\operatorname{In} h|^{\frac{1}{2}}}$ for the a functional $\mathcal{B}$ of the form of $\mathcal{F}$ whose wells are the Tartar square; $A_1 = -A_3 = \operatorname{diag}(-1, -3)$ and $A_2 = -A_4 = \operatorname{diag}(-3, 1)$ and F belong to the rank-1 convex hull of A_1, A_2, A_3, A_4. The point being that functions that lower the energy of $\mathcal{B}$ have to be n-th order laminate within laminate type functions and for these functions $\mathcal{B}$ can only be made to scale to zero at a very slow rate.

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2. Sketch of proof of the two well Liouville theorem

The strategy of the proof is to find a radius $r \in \left(\frac{1}{2}, 1\right)$ such that $L^2\left(u\left(B_r(0)\right)\right) \approx \pi r^2$ and $H^1\left(\partial u\left(B_r(0)\right)\right) \approx 2\pi r$. Theorem 1 then implies $u\left(B_r(0)\right)$ is close to a ball in the sense of smallness of $\lambda\left(u\left(B_r(0)\right)\right)$. *In some sense* this is not very far from saying u is close to a rotation on $B_r(0)$, elementary arguments involving (4) and (5) allows us to show this is actually the case.

Now we will go through the steps in detail. For simplicity assume u is an invertible C^1 function which satisfies the following hypotheses, $\int_{B_1(0)} d\left(Du, K\right) dx \leq \epsilon$ and $\int_{B_1(0)} \left|D^2 u\right| dx \leq \mathcal{C}_1$.

Provided $\mathcal{C}_1$ is small enough our estimate on $\left|D^2 u\right|$ means that for most $r > 0$, Du on $\partial B_r(0)$ can not jump from being close to well $SO(2)$ to being close to well $SO(2)\,H$. Formally, if there exists $a, b \in \partial B_r(0)$ with $Du(a)$ close to $SO(2)$ and $Du(b)$ close to $SO(2)\,H$ then by the fundamental theorem of Calculus $\mathcal{C}_1 \geq \int_{\partial B_r(0)} \left|D^2 u(x)\right| dH^1 x \geq \left|Du(a) - Du(b)\right| \gtrsim \mathrm{dist}\left(SO(2), SO(2)\,H\right)$, contradiction.

Thus we must have that for some $J \in \{Id, H\}$, $d\left(Du(z), K\right) = d\left(Du(z), SO(2)\,J\right)$ for all $z \in \partial B_r(0)$. By a change of variables we can assume $J = Id$. So

$$\begin{aligned} H^1\left(\partial u\left(B_r(0)\right)\right) &= \int_{\partial B_r(0)} \left|Du(z)\, t_z\right| dH^1 z \\ &\approx 2\pi r. \end{aligned}$$

And as $\det(M) = 1$ for any $M \in K$ we have that $L^2\left(u\left(B_r(0)\right)\right) = \int_{B_r(0)} \det\left(Du(z)\right) dL^2 z \approx L^2\left(B_r(0)\right) = \pi r^2$. So applying Theorem 1 we know that $u\left(B_r(0)\right)$ is quantitatively close to being a ball of radius r, i.e. the Fraenkel asymmetry $\lambda\left(u\left(B_r(0)\right)\right)$ is small.

Next we will follow the strategy of Conti, Schweizer [6] which is to find a ball $B_c(y) \subset B_1(0)$ such that

$$\int_{B_c(y)} d\left(Du, SO(2)\right) \approx 0 \tag{11}$$

and then apply the the following theorem of Friesecke James and Müller. [3]

Theorem 3 (Friesecke, James, Müller). *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $q > 1$. There exists a constant $C(U, q)$ with the following property. For each $v \in W^{1,q}(U, \mathbb{R}^n)$ there exists an associated rotation $R \in SO(n)$ such that*

$$\|Dv - R\|_{L^q(U)} \leq C(U, q)\, \|\mathrm{dist}\left(Dv, SO(n)\right)\|_{L^q(U)}. \tag{12}$$

[3] Friesecke, James, Müller Theorem was first stated for L^2 but the same result holds for L^q for $q > 1$ with small modifications of the proof

From Theorem 3 and (11) we can conclude there exists $R \in SO(2)$ such that $\|Du - R\|_{L^q(B_c(y))} \approx 0$, and this (possibly after changing variables) gives conclusion (6).

Let $B := \{x : Du(x) \text{ is close to } SO(2)H\}$. Given hypothesis (4), in order to carry out the argument we just need to find a ball $B_c(y) \subset B_1(0)$ such that $L^2(B_c(y) \cap B) \approx 0$. This follows from smallness of $\lambda(u(B_R(0)))$ by the following two steps.

Step 1. We say two points on a circle are *antipodal* if the line segment joining them goes through the centre of the circle. We know[4] $\partial u(B_r(0))$ is roughly a circle and using the fact that $\int_{\partial B_r(0)} d(Du(z), SO(2))\, dH^1 z \approx 0$, we will show any two antipodal points on $\partial B_r(0)$, say a, b will be mapped to antipodal points $u(a)$, $u(b)$ on the "circle" $\partial u(B_r(0))$, and hence $|u(a) - u(b)| \gtrsim r$.

Step 2. We will use the fact there exists a large set of directions $\Theta \subset S^1$ such that for any $v \in \Theta$ we have $|Hv| < 1$ to show that along a line $[a_v, b_v] := B_r(0) \cap \langle v \rangle$ we have, $H^1([a_v, b_v] \cap B) \approx 0$. Then we use a co-area argument to integrate χ_B over $X := \bigcup_{v \in \Theta} [a_v, b_v]$, trivially there then exists $B_c(y) \subset X$ with property (11).

Proof of Step 1. Suppose we let Γ_1 and Γ_2 be the connected components of $\partial B_r(0) \setminus \{a, b\}$. Since $Du(z)$ is close to $SO(2)$ for most of the points $z \in \partial B_r(0)$ it is easy to see that $H^1(u(\Gamma_i)) \lesssim \pi r$ for $i = 1, 2$ which together with the fact that each $u(\Gamma_i)$ must go around the outside of the "ball" $u(B_r(0))$ to connect $u(a)$ to $u(b)$ this implies that $u(a)$ and $u(b)$ must be antipodal.

Proof of Step 2. Let $v \in \Theta$, let a_v, b_v be antipodal points on $\partial B_r(0)$ defined by $\frac{a_v - b_v}{|a_v - b_v|} = v$. By definition of Θ there exists $s_\sigma \in (0, 1)$ such that $|Hv| < s_\sigma$. And by Step 1 we know $|u(a) - u(b)| \gtrsim r$, so using the formula $H^1(u([a_v, b_v])) = \int_{a_v}^{b_v} |Du(z)v|\, dx$, we have

$$\begin{aligned} H^1(u([a_v, b_v])) &= H^1(u([a_v, b_v] \setminus B)) + H^1(u(B \cap [a_v, b_v])) \\ &\lesssim H^1([a_v, b_v] \setminus B) + \left| H \frac{a_v - b_v}{|a_v - b_v|} \right| H^1(B \cap [a_v, b_v]) + \int_{a_v}^{b_v} d(Du, K)\, dx \\ &\lesssim r - (1 - s_\sigma) H^1(B \cap [a_v, b_v]), \end{aligned}$$

and since $H^1(u([a_v, b_v])) \geq |u(a_v) - u(b_v)| \gtrsim r$. So we must have $H^1(B \cap [a_v, b_v]) \approx 0$. Now by the co-area formula we

$$\int_X \frac{1}{|z|} \chi_B(z)\, dL^2 z = \int_\Theta \left(L^1(B \cap [a_v, b_v]) + \int_{a_v}^{b_v} d(Du, K)\, dx \right) dH^1 v \approx 0$$

which gives $L^2(B \cap X) \approx 0$.

3. Fine properties of Sobolev functions and functions of integrable dilation

We will need the following well known lemma.

Lemma 2. *Let Ω be a Lipschitz domain. Let $u \in W^{1,p}(\Omega)$. There exists a Borel $G_u \subset \Omega$ with $H^1(\text{int}(\Omega) \setminus G_u) = 0$ such that for every $x \in G_u$ the limit* $\lim_{r \to 0} \pi^{-1} r^{-2} \int_{B_r(x)} u(z)\, dL^2 z =:$ $\widehat{u}(x)$ *exists and we even have*

$$\lim_{r \to 0} r^{-2} \int_{B_r(x)} |u(z) - \widehat{u}(x)|^{p^*}\, dL^2 z = 0$$

where p^* is the Hölder conjugate, i.e. $\frac{1}{p^*} + \frac{1}{p} = 1$.

This follows from Theorem 1 of Section 4.8 and Theorem 3 of Section 5.6.3 of [10].

[4] By smallness of $\lambda(u(B_r(y)))$, see Lemma 4

Definition 1. *Given* $u \in W^{1,p}(\Omega)$ *we define the* **precise representative** $\widehat{u}$ of u by

$$\widehat{u}(x) := \begin{cases} \lim_{r\to 0}\left(\pi r^2\right)^{-1}\int_{B_r(x)} u(z)\,dL^2 z \text{ if } x\in G_u \\ 0 \text{ if } x\in \Omega\backslash G_u \end{cases}$$

The following lemma is also well known, see for example [12],[10] for convenience of the reader we give some details.

Proposition 1. *Let* $w \in W^{1,p}(\Omega)$. *Suppose* $B_h(0)\subset\Omega$ *then for almost every* $h\in(0,r)$ *the function*

$$w^h(t) := \widehat{u}\left(h\cos\frac{t}{h}, h\sin\frac{t}{h}\right)$$

is absolutely continuous over $[0,2\pi h]$ *and*

$$\int_{h=0}^{r}\int_0^{2\pi}\left|Dw^h(z)\right|^p dL^1 z dL^1 r \le \int_{B_r(0)}|Dw(z)|^p dL^2 z$$

Proof. We define the standard convolution, $w_\epsilon := w*\rho_\epsilon$ where $\rho_\epsilon(x) := \rho\left(\frac{x}{\epsilon}\right)\epsilon^{-2}$ and ρ is a smooth convolution kernel. So we known by the standard theorems $w_\epsilon \overset{W^{1,p}}{\to} w$ as $\epsilon\to 0$.

Let $\delta>0$. By the co-area formula, following arguments of the proof of Theorem 4.50 [12] there must exist $K_\delta\subset(0,r)$ with $L^1((0,r)\backslash K_\delta)\le\delta$ and a sequence $\epsilon_n\to 0$ such that for any $h\in K$

$$\int_{\partial B_h(x_0)}|\hat{w}(z)-w_{\epsilon_n}(z)|^p + |Dw(z)-Dw_{\epsilon_n}(z)|^p dH^1 z\to 0 \tag{13}$$

as $n\to\infty$.

So let $h\in K$ be one of the a.a. radii for such that Dw is defined on all but a set of H^1 zero measure on $\partial B_h(0)$, and the function $x\to Dw(x)$ is L^p integrable on $\partial B_h(0)$.

Let $v^h_{\epsilon_n}(t) := w_{\epsilon_n}(h\cos t, h\sin t)$ for $t\in[0,2\pi)$. Let $L_h:[0,2\pi)$ be the L^p integrable function given by $L_h(t) := \frac{-\partial w}{\partial x_1}(h\cos t,h\sin t)h\sin t + \frac{\partial\hat{w}}{\partial x_2}(h\cos t,h\sin t)h\cos t$. By (13) we have that $Dv^h_{\epsilon_n}\overset{L^p([0,2\pi))}{\to} L_h$ as $n\to\infty$. So for any $\epsilon>0$ there exists $N_\epsilon\in\mathbb{N}$ such that $\|Dv^h_{\epsilon_{n_1}} - Dv^h_{\epsilon_{n_2}}\|_{L^p([0,2\pi))}\le\epsilon$ for all $n_1,n_2\ge N_\epsilon$.

From this by picking a Lebesgue point of w on $\partial B_h(0)$ and using the fundamental theorem of Calculus (as in the proof of Theorem 1, Section 4.9.1 [10]) we can show the sequence $v^h_{\epsilon_n}$ converges uniformly to the limit $v^h(t) := \hat{w}(h\cos t,h\sin t)$ and $v^h\in W^{1,p}((0,2\pi])$ with $Dv^h(t) = L_h(t)$ for a.e. $t\in(0,2\pi]$.

Define $w^h(t) := v^h\left(\frac{t}{h}\right)$ for $t\in[0,2\pi h]$. Then

$$\begin{aligned}\int_{h\in K_\delta}\int_{[0,2\pi h]}\left|Dw^h(t)\right|^p dL^1 t dL^1 h &\le \int_{h\in K_\delta}\int_{[0,2\pi h]}\left|Dw\left(h\cos\frac{t}{h},h\sin\frac{t}{h}\right)\right|^p dL^1 t dL^1 h\\ &\le \int_{B_r(0)}|Dw(z)|^p dL^2 z\end{aligned}$$

by the co-area formula. Since δ was arbitrary we have that the function w^h is defined and absolutely continuous on $\partial B_h(0)$ for every $h\in\bigcup_n K_{n^{-1}}$ and this completes the proof. □

Definition 2. *Given an open set* $\Omega\subset\mathbb{R}^n$. *A function* $f:\Omega\to\mathbb{R}^n$ *is called* **sense preserving** if $\det(Df(z))\ge 0$ for a.e. $z\in\Omega$.

Definition 3. *Given a connected open set* $\Omega\subset\mathbb{R}^n$. *A sense preserving function* $f:\Omega\to\mathbb{R}^n$ *is said to be of* **finite dilation** *if and only if* $\|Df(x)\|^n\le K(x)\,|\det(Df(x))|$ *a.e. where* $1\le K(x)<\infty$. *The function* f *is said to have* **integrable dilation** *if and only if* $\int_\Omega K(x)\,dL^n x<\infty$.

We will need the following theorem [15].

Theorem 4 (Iwaniec, Šverák). *Let $\Omega \subset \mathbb{R}^2$ be a connected open set. Given function $f : \Omega \to \mathbb{R}^2$, $f \in W^{1,2}(\Omega)$ which has integrable dilation then f is open and discrete.*

It is also well known that functions of finite dilation are continuous [13].

Lemma 3. *Let Ω be an open connected set in $\mathbb{R}^2$. Let $q \geq 1$. Let $C > 0$ be some arbitrary large constant. Suppose $u \in W^{2,q}(\Omega)$ is a sense preserving function with the property that $\sup_{x\in\Omega} \| [Du(x)]^{-1} \| \leq C$ In addition u satisfies*

$$\int_\Omega d(Du(z), K)\, dL^2 z \leq \infty \tag{14}$$

then u has integrable dilation and consequently for any $z \in \Omega$ and a.e. $r > 0$ such that $B_r(z) \subset \Omega$ we have that $u(B_r(z))$ is an open set of finite perimeter and

$$\partial u(B_r(z)) \subset u(\partial B_r(z)), \text{ which gives } \mathrm{Per}(u(B_r(z))) \leq H^1(u(\partial B_r(z))). \tag{15}$$

Proof. Now let $R(x) \in SO(2)$, $S(x) \in M^{2\times 2}_{sym}$ be the polar decomposition of the matrix $Du(x)$, i.e. $Du(x) = R(x)S(x)$. Let $\lambda_1(x)$, $\lambda_2(x)$ be the eigenvalues of $S(x)$, by assumption we have $\min\{|\lambda_1(x)|, |\lambda_2(x)|\} \geq C^{-1}$ for a.e. $x \in \Omega$.

Given $x \in \Omega$ for which $Du(x)$ is defined, assume without loss of generality that $|\lambda_1(x)| \geq |\lambda_2(x)|$. Now for any $v \in S^1$,

$$\begin{aligned} |Du(x)v|^2 &\leq |\lambda_1(x)|^2 \\ &\leq C|\lambda_1(x)|^2 |\lambda_2(x)| \\ &\leq c(d(Du(x), K) + 1)\det(Du(x)). \end{aligned} \tag{16}$$

And as $(d(Du(x), K) + 1)$ is integrable so by (16) we know u is a mapping of integrable dilation. By Sobolev embedding theorem we know $u \in W^{1,2}(\Omega)$ and thus by Theorem 4 we have that u is open and discrete, we also know u is continuous.

Since u is open, it is well know (see exercise 9.12, [32]) that $\partial u(B_r(z)) \subset u(\partial B_r(z))$. By Proposition 1 for a.e. $r > 0$ such that $B_r(x) \subset \Omega$ we know $\widehat{Du}$ is absolutely continuous on $\partial B_r(z)$ and so $H^1(u(\partial B_r(z))) = \int_{\partial B_r(z)} \left|\widehat{Du}(y)t_y\right| dH^1 y < \infty$ where t_y is the tangent to $\partial B_t(z)$ at y, which of course implies $H^1(\partial u(B_r(z))) < \infty$. So by Proposition 3.61 [1] $u(B_r(z))$ is a set of finite perimeter and $\mathrm{Per}(u(B_r(z))) \leq H^1(\partial u(B_r(z))) \leq H^1(u(\partial B_r(z)))$. □

4. Details of proof of Theorem 1

4.1. **Preproof.** Following the notation of the introduction, let B be the set of points for which Du is close to $SO(2)H$, we have to split the lemma into cases depending on the proportion of B inside $B_1(0)$.

If B is the majority, we will have to do a change of variables and define $\tilde{u} = u \circ H^{-1}$, then $\tilde{u}$ is defined on a thin ellipse in which we will need to look for circles for which $D\tilde{u}$ stays close to $SO(2)$. In order to find such a circle we will need $D\tilde{u}$ to be "mostly" close to $SO(2)$ in the ellipse. For this we require B to be the "large" majority in $B_1(0)$.

On the other hand if $L^2(B_1(0)\backslash B) > \sqrt{\mathcal{C}_1}$ since we can use the hypotheses (4), (5) to show that on all but a set of radii of measure $\approx \mathcal{C}_1$ we have Du is uniformly close to either $SO(2)$ or $SO(2)H$ on $\partial B_r(0)$, and so we must be able to find at least one for which Du is uniformly close to $SO(2)$.

Hence in our lemmas we will have to argue two cases depending on the sign of

$$L^{\mathfrak{e}}(u) := \int_{B_1(0)} \mathfrak{e} d(Du(z), SO(2)) - d(Du(z), SO(2)H)\, dL^2 z. \tag{17}$$

4.2. **Preliminary lemmas.**

Lemma 4. *Let E be a set of finite perimeter in $\mathbb{R}^2$ with $L^2(E) \geq 1$, let ε be a small number, suppose E has the following properties*

$$\mathrm{Per}(E) \leq 2\pi \left(\frac{L^2(E)}{\pi} \right)^{\frac{1}{2}} + \varepsilon \tag{18}$$

then there exists $a \in \mathbb{R}^2$ such that for $R := \left(\frac{L^2(E)}{\pi} \right)^{\frac{1}{2}}$

$$B_{c_1 \varepsilon^{\frac{1}{4}}}(x) \cap \partial E \neq \emptyset \text{ for each } x \in \partial B_R(a). \tag{19}$$

Proof. Let $\lambda(E)$ be defined as in (7) Lemma 1. So there exists $a \in \mathbb{R}^2$ such that $L^2(E \backslash B_R(a)) \leq 2\lambda(E)\pi R^2$ and by Lemma 1 $(\mathrm{Per}(E))^2 \geq 4\pi^2 \left(1 + \frac{(\lambda(E))^2}{4}\right) R^2$. So by (18)

$$\pi R^2 (\lambda(E))^2 \leq 4\pi\varepsilon R + \varepsilon^2 \leq 20\varepsilon R$$

and so $(\lambda(E))^2 \leq 10\varepsilon R^{-1}$, thus $\lambda(E) \leq \frac{c_0}{2}\sqrt{\varepsilon}$ for some constant $c_0 > 1$ so

$$L^2(E \cap B_R(a)) \geq \left(1 - c_0\sqrt{\varepsilon}\right) \pi R^2. \tag{20}$$

Thus $B_{c_0 \varepsilon^{\frac{1}{4}} R}(x) \cap E \cap B_R(a) \neq \emptyset$ for any $x \in \partial B_R(a)$, since otherwise we contradict (20).

On the other hand if for some $x \in \partial B_R(a)$ we have $B_{c_0 \varepsilon^{\frac{1}{4}} R}(x) \backslash B_R(a) \subset E$ then we have $L^2(E \backslash B_R(a)) \geq \frac{\pi c_0^2}{2}\sqrt{\varepsilon} R^2$ and together with (20) this implies $L^2(E) > \pi R^2$ which contradicts the definition of R. So let $c_1 = c_0 R_1$ for every $x \in \partial B_R(a)$ we have $B_{c_1 \varepsilon^{\frac{1}{4}}}(x) \cap E^c \neq \emptyset$ and $B_{c_1 \varepsilon^{\frac{1}{4}}}(x) \cap E \neq \emptyset$. Hence

$$B_{c_1 \varepsilon^{\frac{1}{4}}}(x) \cap \partial E \neq \emptyset \text{ for any } x \in \partial B_R(a). \quad \square \tag{21}$$

Lemma 5. *Let $p \geq 1$, $q \geq 1$. Suppose $u \in W^{2,p}(B_1(0)) \cap W^{1,q}(B_1(0))$ satisfies properties (4), (5). Let $L^{\mathfrak{e}}(u)$ be defined by (17).*

There exists a small positive constant $\mathfrak{e} = \mathfrak{e}(\sigma)$ such that the following holds true:

If $L^{\mathfrak{e}}(u) \geq 0$ then for any $b \in B_{\frac{\sigma^2}{8}}(0)$ we must be able to find a set $E_b \subset \left(\frac{\sigma}{4}, \frac{\sigma}{2}\right)$ such that $L^1\left(\left(\frac{\sigma}{4}, \frac{\sigma}{2}\right) \backslash E_b\right) < \frac{\mathcal{C}_1 \sigma^{-2}}{\sqrt{\mathfrak{e}}}$ and for any $R \in E_b$

$$\int_{H^{-1}(\partial B_R(b))} d^q\left(\widehat{Du}(z), SO(2) H\right) dH^1 z \leq c\epsilon. \tag{22}$$

If $L^{\mathfrak{e}}(u) < 0$ then for any $b \in B_{\mathfrak{e}^2}(0)$ we can find a set $E_b \subset \left(2\mathfrak{e}^2, 1 - \mathfrak{e}^2\right)$ such that $L^1\left(\left(2\mathfrak{e}^2, 1 - \mathfrak{e}^2\right) \backslash E_b\right) < \frac{\mathcal{C}_1 \sigma^{-2}}{\sqrt{\mathfrak{e}}}$ and for any $R \in E_b$

$$\int_{\partial B_R(b)} d^q\left(\widehat{Du}(z), SO(2)\right) dH^1 z \leq c\epsilon. \tag{23}$$

Proof. First we will deal with the case were $L^{\mathfrak{e}}(u) \geq 0$. Let $b \in B_{\frac{\sigma^2}{8}}(0)$.

Let

$$\Pi = \left\{ r \in \left(0, \frac{\sigma}{2}\right) : \widehat{Du} \text{ is absolutely continuous on } H^{-1}(\partial B_r(b)) \right\}.$$

By a version of Proposition 1 [5] we have $L^1\left(\left(0,\frac{\sigma}{2}\right)\backslash\Pi\right)=0$. For any $r\in\Pi$ let $\widehat{Du}'$ denote the derivative of $\widehat{Du}$ along $H^{-1}\left(\partial B_r\left(b\right)\right)$. Note $\widehat{Du}'\in L^p\left(H^{-1}\left(\partial B_r\left(b\right)\right)\right)$ and

$$\int_{H^{-1}(B_R(b))}\left|D^2u\left(z\right)\right|^p dL^2z\geq\sigma\int_0^R\int_{H^{-1}(\partial B_r(b))}\left|\widehat{Du}'\left(z\right)\right|^p dH^1zdL^1r \tag{24}$$

Since $L^{\mathfrak{e}}\left(u\right)\geq 0$

$$\begin{aligned}\int_{B_1(0)}d\left(Du\left(z\right),SO\left(2\right)H\right)dL^2z &\leq \int_{B_1(0)}\mathfrak{e}d\left(Du\left(z\right),SO\left(2\right)\right)dL^2z\\ &\leq \int_{B_1(0)}\mathfrak{e}\left(d\left(Du\left(z\right),K\right)+\sigma^{-1}\right)dL^2z\\ &\leq 7\mathfrak{e}\sigma^{-1}. \end{aligned}\tag{25}$$

Let

$$G_1:=\left\{r\in\left(\frac{\sigma}{4},\frac{\sigma}{2}\right):\int_{H^{-1}(\partial B_r(b))}d^q\left(\widehat{Du}\left(z\right),K\right)dH^1z\leq\sqrt{\mathfrak{e}}\epsilon\right\} \tag{26}$$

and

$$G_2:=\left\{r\in\left(\frac{\sigma}{4},\frac{\sigma}{2}\right)\cap\Pi:\int_{H^{-1}(\partial B_r(b))}\left|\widehat{Du}'\left(z\right)\right|^p dH^1z\leq\sqrt{\mathfrak{e}}\epsilon^{1-p}\right\}. \tag{27}$$

Now we can define function $\Psi:H^{-1}\left(B_{\frac{\sigma}{2}}\left(b\right)\right)\to\mathbb{R}$ by

$$\Psi\left(z\right):=r \text{ if and only if } z\in H^{-1}\left(\partial B_r\left(b\right)\right).$$

It is easy to see $|D\Psi|\leq\sigma^{-1}$, so by the co-area formula

$$\begin{aligned}&\int_{r\in\left(\frac{\sigma}{4},\frac{\sigma}{2}\right)\backslash G_1}\int_{H^{-1}(\partial B_r(b))}d^q\left(\widehat{Du}\left(z\right),K\right)dH^1zdL^1r\\ &\qquad\leq\int_{H^{-1}\left(B_{\frac{\sigma}{2}}(b)\right)}\left|D\Psi\left(z\right)\right|d^q\left(Du\left(z\right),K\right)dL^2z\\ &\qquad\overset{(4)}{\leq}\mathcal{C}_1\sigma^{-1}\epsilon.\end{aligned}$$

So

$$L^1\left(\left(\frac{\sigma}{4},\frac{\sigma}{2}\right)\backslash G_1\right)\leq\frac{\mathcal{C}_1\sigma^{-1}}{\sqrt{\mathfrak{e}}}. \tag{28}$$

In exactly the same way we have

$$L^1\left(\left(\frac{\sigma}{4},\frac{\sigma}{2}\right)\backslash G_2\right)\leq\frac{\mathcal{C}_1\sigma^{-2}}{\sqrt{\mathfrak{e}}}. \tag{29}$$

Let $p*$ be the Holder conjugate of p, i.e. $\frac{1}{p*}+\frac{1}{p}=1$. Now for any $r\in G_1\cap G_2\cap\Pi$ we have

$$\int_{H^{-1}(\partial B_r(b))}\frac{\left(d^{\frac{q}{p*}}\left(\widehat{Du}\left(z\right),K\right)\right)^{p*}}{p*}+\epsilon^p\frac{\left|\widehat{Du}'\left(z\right)\right|^p}{p}dH^1z\overset{(26),(27)}{\leq}2\sqrt{\mathfrak{e}}\epsilon. \tag{30}$$

By Young's inequality this implies

$$\int_{H^{-1}(\partial B_r(b))}d^{\frac{q}{p*}}\left(\widehat{Du}\left(z\right),K\right)\left|\widehat{Du}'\left(z\right)\right|dH^1z\leq2\sqrt{\mathfrak{e}}. \tag{31}$$

[5] This follows by almost exactly the same proof as Proposition 1, the only difference being we need to use the co-area formula with respect to a function whose level sets are of the form $H^{-1}\left(\partial B_r\left(b\right)\right)$ hence the Jacobean of this function is equal to σ^{-1}

Since $r\in G_1$, (see (26)) we must have a point $x_0\in H^{-1}(\partial B_r(b))$ such that

$$d\left(\widehat{Du}(x_0),SO(2)J_1\right)\le c\left(\sqrt{\mathfrak{e}}\epsilon\right)^{\frac{1}{q}} \text{ for some } J_1\in\{Id,H\}. \tag{32}$$

Step 1. We will show that for any $r\in G_1\cap G_2\cap\Pi$ we have $J_1\in\{Id,H\}$ such that $d\left(\widehat{Du}(z),SO(2)J_1\right)\le d\left(\widehat{Du}(z),K\right)$ for all $z\in H^{-1}(\partial B_r(b))$.

Proof of Step 1. We know there exists $x_0\in H^{-1}(\partial B_r(b))$ such that (32) holds true. Let $J_2\in\{Id,H\}\backslash J_2$. If Step 1 is false we have a point $x_1\in H^{-1}(\partial B_r(b))$ such that

$$d\left(\widehat{Du}(x_1),SO(2)J_2\right)<d\left(\widehat{Du}(x_1),SO(2)J_1\right).$$

Recall we chose $r>0$ so that $\widehat{Du}$ is absolutely continuous on $H^{-1}(\partial B_r(b))$. Define $g:[0,2\pi)\to\mathbb{R}$ by $g(\theta):=d\left(\widehat{Du}\left(H^{-1}\left(re^{i\theta}+b\right)\right),K\right)$. Let $W:=\{\theta\in(0,2\pi]:g(\theta)>0\}$, it is easy to see g is absolutely continuous on W and $|Dg(\theta)|\le c\left|\widehat{Du}'\left(H^{-1}\left(re^{i\theta}+b\right)\right)\right|$ for any $\theta\in W$. Let $d_0:=d(SO(2),SO(2)H)$.

Now $\sup_{z\in[0,2\pi)}g(z)\ge\frac{d_0}{2}$, $\inf_{z\in[0,2\pi)}g(z)\le 2\left(\sqrt{\mathfrak{e}}\epsilon\right)^{\frac{1}{q}}$, so there must be a subinterval $I\subset(0,2\pi]$ with the following properties;

- letting a,b be the end points of I, $|g(a)-g(b)|=\frac{d_0}{4}$.
- $\inf\{g(x):x\in I\}>\frac{d_0}{8}$, $\sup\{g(x):x\in I\}<d_0$.

So

$$\begin{aligned}\frac{d_0}{4} &= |g(a)-g(b)|\\ &= \left|\int_I Dg(x)\,dL^1x\right|\\ &\le c\int_{H^{-1}(\partial B_r(b))}\left|\widehat{Du}'(z)\right|dH^1z.\end{aligned} \tag{33}$$

Let $J:=\left\{H^{-1}\left(re^{i\theta}+b\right):\theta\in I\right\}$. We know $d^{\frac{q}{p^*}}\left(\widehat{Du}(x),K\right)>\left(\frac{d_0}{8}\right)^{\frac{q}{p^*}}$ for all $x\in J$, so

$$\begin{aligned}\int_J\left|\widehat{Du}'(x)\right|d^{\frac{q}{p^*}}\left(\widehat{Du}(x),K\right)dH^1x &\ge \left(\frac{d_0}{8}\right)^{\frac{q}{p^*}}\int_J\left|\widehat{Du}'(x)\right|dH^1x\\ &\overset{(33)}{\ge} \left(\frac{d_0}{8}\right)^{\frac{q}{p^*}}\frac{d_0}{4c}\end{aligned}$$

by (31) assuming constant $\mathfrak{e}$ is small enough we have a contradiction, thus Step 1 is proved.

Step 2. We will show there exists $J_1\in\{Id,H\}$ such that for any $r\in G_1\cap G_2\cap\Pi$ we have

$$d\left(\widehat{Du}(z),SO(2)J_1\right)\le d\left(\widehat{Du}(z),K\right) \text{ for all } z\in H^{-1}(\partial B_r(b)). \tag{34}$$

Proof of Step 2. Suppose not, so we can find $r_1,r_2\in G_1\cap G_2\cap\Pi$ such that

$$d\left(\widehat{Du}(z),SO(2)\right)\le d\left(\widehat{Du}(z),K\right) \text{ for all } z\in H^{-1}(\partial B_{r_1}(b))$$

and

$$d\left(\widehat{Du}(z),SO(2)H\right)\le d\left(\widehat{Du}(z),K\right) \text{ for all } z\in H^{-1}(\partial B_{r_2}(b)).$$

Assume without loss of generality that $r_1\le r_2$. Let

$$W_1:=\left\{z\in H^{-1}(\partial B_{r_1}(b)):d\left(\widehat{Du}(z),SO(2)\right)<\sqrt{\mathfrak{e}}\sqrt{\epsilon}\right\} \tag{35}$$

and let

$$W_2 := \left\{ z \in H^{-1}\left(\partial B_{r_2}(b)\right) : d\left(\widehat{Du}(z), SO(2)H\right) < \sqrt{\mathfrak{e}}\sqrt{\epsilon} \right\}. \tag{36}$$

Since $r_1, r_2 \in G_1$ (see definition (26)) we have that

$$\begin{aligned} H^1\left(H^{-1}\left(\partial B_{r_1}(b)\right) \backslash W_1\right)\sqrt{\mathfrak{e}}\sqrt{\epsilon} &\leq \int_{H^{-1}\left(\partial B_{r_1}(b)\right)} d\left(\widehat{Du}(z), SO(2)\right) dH^1 z \\ &\leq \sqrt{\mathfrak{e}}\epsilon. \end{aligned}$$

So

$$H^1\left(H^{-1}\left(\partial B_{r_1}(b)\right) \backslash W_1\right) \leq \sqrt{\epsilon} \tag{37}$$

and in the same way

$$H^1\left(H^{-1}\left(\partial B_{r_2}(b)\right) \backslash W_2\right) \leq \sqrt{\epsilon}. \tag{38}$$

Let

$$F_1 = \left\{ a \in [-\sigma r_1, \sigma r_1] : \int_{P^{-1}_{e_1^\perp}(a) \cap B_1(0)} d^q\left(\widehat{Du}(z), K\right) dH^1 z \leq \sqrt{\mathfrak{e}}\epsilon \right\}$$

$$F_2 = \left\{ a \in [-\sigma r_1, \sigma r_1] : \int_{P^{-1}_{e_1^\perp}(a) \cap B_1(0)} \left|\widehat{Du}'(z)\right|^p dH^1 z \leq \sqrt{\mathfrak{e}}\epsilon^{1-p} \right\}.$$

In exactly the same way as we established (28) and (29), by Fubini $L^1\left([-\sigma r_1, \sigma r_1] \backslash F_1\right) \leq \frac{\mathcal{C}_1}{\sqrt{\mathfrak{e}}}$ and $L^1\left([-\sigma r_1, \sigma r_1] \backslash F_2\right) \leq \frac{\mathcal{C}_1}{\sqrt{\mathfrak{e}}}$ and note that for any $a \in F_1 \cap F_2$ we have

$$\begin{aligned} &\int_{P^{-1}_{e_1^\perp}(a) \cap B_1(0)} \epsilon d^{\frac{q}{p^*}}\left(\widehat{Du}(z), K\right)\left|\widehat{Du}'(z)\right| dL^1 z \\ &\qquad \leq \int_{P^{-1}_{e_1^\perp}(a) \cap B_1(0)} d^q\left(\widehat{Du}(z), K\right) + \epsilon^p \left|\widehat{Du}'(z)\right|^p dL^1 z \\ &\qquad \leq \sqrt{\mathfrak{e}}\epsilon \end{aligned}$$

where p^* is the Holder exponent of p.

So by an identical argument to that of Step 1 we can show that for any $x \in F_1 \cap F_2$ there exists $J_1 \in \{Id, H\}$ such that for $J_2 \in \{Id, H\} \setminus \{J_1\}$ we have $d\left(\widehat{Du}(z), SO(2)J_1\right) \leq d\left(\widehat{Du}(z), SO(2)J_2\right)$ for all $z \in P^{-1}_{e_1^\perp} \cap B_1(0)$. However by (37), (38) $L^1\left(P_{e_1^\perp}(W_1 \cap W_2)\right) \geq \frac{\sigma r_1}{2}$ so assuming $\mathcal{C}_1$ is chosen small enough we have $P_{e_1^\perp}(W_1 \cap W_2) \cap F_1 \cap F_2 \neq \emptyset$ which contradicts the definition of W_1, W_2, see (36) and (35). This completes the proof Step 2.

Step 3. We complete the proof of Lemma 5 for the case $L^{\mathfrak{e}}(u) > 0$.

Proof of Step 3. We need only show that in (34) we can take, $J_1 = H$ for any $r \in G_1 \cap G_2 \cap \Pi$. Let $\mathbb{A} := \bigcup_{r \in G_1 \cap G_2 \cap \Pi} H^{-1}\left(\partial B_r(b)\right)$.

So suppose not, then

$$\begin{aligned} \int_{\mathbb{A}} d^q\left(\widehat{Du}(z), SO(2)\right) dL^2 z &= \int_{\mathbb{A}} d^q\left(\widehat{Du}(z), K\right) dL^2 z \\ &\overset{(4)}{\leq} c\epsilon \end{aligned}$$

Note that by (28), (29) and the co-area formula we have

$$\begin{aligned} L^2(\mathbb{A}) &\geq \sigma L^1\left(G_1 \cup G_2\right) \\ &\geq \frac{\sigma^2}{16}. \end{aligned}$$

Now we can extract subset $\widetilde{\mathbb{A}} \subset \mathbb{A}$ such that $\sup\left\{d^q\left(Du\left(z\right), SO\left(2\right)\right) : z \in \widetilde{\mathbb{A}}\right\} \leq 2c\epsilon$ with the property that $L^2\left(\widetilde{\mathbb{A}}\right) \geq \frac{L^2(\mathbb{A})}{2}$, but as $d_0 >> \epsilon^{\frac{1}{q}}$ we have $\inf\left\{d\left(Du\left(z\right), SO\left(2\right)H\right) : z \in \widetilde{\mathbb{A}}\right\} \geq \frac{d_0}{2}$ and hence

$$\int_{\widetilde{\mathbb{A}}} d\left(Du\left(z\right), SO\left(2\right)H\right) dL^2 z \geq \frac{d_0}{2}\frac{\sigma^2}{32}$$

and (assuming $\mathfrak{e}$ is small enough) this contradicts (25). Now defining $E_b := G_1 \cap G_2 \cap \Pi$ by (28) and (29) this set satisfies all the properties of the statement. Hence the lemma is proved for the case $L^{\mathfrak{e}}\left(u\right) \leq 0$.

Step 4. We complete the proof of the case where $L^{\mathfrak{e}}\left(u\right) < 0$.
Proof of Step 4. Arguing identically to the case where $L^{\mathfrak{e}}\left(u\right) \geq 0$ we can show there exists a set $E_b \subset \left(2\mathfrak{e}^2, 1-\mathfrak{e}^2\right)$, $J_1 \in \{Id, H\}$ such that $L^1\left(\left(2\mathfrak{e}^2, 1-\mathfrak{e}^2\right) \backslash E_b\right) \leq \frac{2\sigma^{-2}\mathcal{C}_1}{\sqrt{\mathfrak{e}}}$ and

$$\int_{\partial B_r(b)} d^q\left(\widehat{Du}\left(z\right), SO\left(2\right)J_1\right) dH^1 z \leq c\epsilon \text{ for each } r \in E_b.$$

So the set $\mathbb{U} := \bigcup_{r \in E_b} \partial B_r\left(b\right)$ has the property

$$\int_{\mathbb{U}} d^q\left(\widehat{Du}\left(z\right), SO\left(2\right)J_1\right) dH^1 z \leq \mathcal{C}_1\epsilon. \tag{39}$$

We claim $J_1 = Id$. Suppose not, assuming $\mathcal{C}_1$ is small enough $L^2\left(B_1\left(0\right)\backslash\mathbb{U}\right) \leq 5\mathfrak{e}^2$. By Hölder's inequality

$$\begin{aligned}\int_{\mathbb{U}} d\left(Du\left(z\right), SO\left(2\right)H\right) dL^2 z &\leq c\left(\int_{\mathbb{U}} d^q\left(Du\left(z\right), SO\left(2\right)H\right) dL^2 z\right)^{\frac{1}{q}} \\ &\overset{(39)}{\leq} c\epsilon^{\frac{1}{q}}.\end{aligned} \tag{40}$$

Thus

$$\begin{aligned}&\int_{B_1(0)} d\left(Du\left(z\right), SO\left(2\right)H\right) dL^2 z \\ &\qquad \leq \int_{\mathbb{U}} d\left(Du\left(z\right), SO\left(2\right)H\right) dL^2 z + \int_{B_1(0)\backslash\mathbb{U}} d\left(Du\left(z\right), K\right) + \sigma^{-1} dL^2 z \\ &\qquad \overset{(40),(4)}{\leq} c\epsilon^{\frac{1}{q}} + \sigma^{-1} L^2\left(B_1\left(0\right)\backslash\mathbb{U}\right) \\ &\qquad \leq c\mathfrak{e}^2.\end{aligned} \tag{41}$$

However since $L^{\mathfrak{e}}\left(u\right) < 0$ this implies

$$\int_{B_1(0)} d\left(Du\left(z\right), SO\left(2\right)\right) dL^2 z < c\mathfrak{e}. \tag{42}$$

Let

$$\mathbb{D} := \left\{z \in B_1\left(0\right) : d\left(Du\left(z\right), SO\left(2\right)H\right) \leq \sqrt{\mathfrak{e}}, \text{ and } d\left(Du\left(z\right), SO\left(2\right)\right) \leq \sqrt{\mathfrak{e}}\right\},$$

so by (42), (41) $L^2\left(\mathbb{D}\right) \leq \pi - c\sqrt{\mathfrak{e}}$ however as $d\left(SO\left(2\right), SO\left(2\right)H\right) = d_0$, $\mathbb{D}$ should be empty, so this a contradiction. □

Lemma 6. *Let $p \geq 1$, $q \geq 1$. Suppose $u \in W^{2,p}\left(B_1\left(0\right)\right) \cap W^{1,q}\left(B_1\left(0\right)\right)$ is a sense preserving function for which $\sup_{x \in B_1(0)} \|\left[Du\left(x\right)\right]^{-1}\| \leq C$ and u satisfies properties (4), (5). Let $L^{\mathfrak{e}}$ be defined by (17) and let constant $\mathfrak{e} = \mathfrak{e}\left(\sigma\right)$ be as in Lemma 5.*

If $L^{\mathfrak{e}}(u) \geq 0$ then for any $b \in B_{\frac{\sigma^2}{8}}(0)$ there exists a set $\mathcal{Y}_b \subset \left(\frac{\sigma}{4}, \frac{\sigma}{2}\right)$ with $L^1\left(\left(\frac{\sigma}{4}, \frac{\sigma}{2}\right) \backslash \mathcal{Y}_b\right) \leq \frac{\sigma}{100}$ and for any $r \in \mathcal{Y}_b$ we have

$$L^2\left(u\left(H^{-1}\left(B_r(b)\right)\right)\right) \geq \pi r^2 - c\epsilon^{\frac{1}{q}}. \tag{43}$$

If $L^{\epsilon}(u) < 0$ then for any $b \in B_{\mathfrak{e}^2}(0)$ there exists a set $\mathcal{Y}_b \subset \left(2\mathfrak{e}^2, \frac{1}{2}\right)$ with $L^1\left(\left(2\mathfrak{e}^2, \frac{1}{2}\right) \backslash \mathcal{Y}_b\right) \leq \frac{1}{100}$ and any $r \in \mathcal{Y}_b$ is such that

$$L^2\left(u\left(B_r(b)\right)\right) \geq \pi r^2 - c\epsilon^{\frac{1}{q}}.$$

Proof. We will only argue the case $L^{\mathfrak{e}}(u) \geq 0$, the argument for the case $L^{\mathfrak{e}}(u) < 0$ is identical.

Step 1. We will show that for any $b \in B_{\frac{\sigma^2}{8}}(0)$ there exists a set $\mathcal{Y}_b \subset \left(\frac{\sigma}{4}, \frac{\sigma}{2}\right)$ with $L^1\left(\left(\frac{\sigma}{4}, \frac{\sigma}{2}\right) \backslash \mathcal{Y}_b\right) \leq \frac{\sigma}{100}$ such that for some $A \in SO(2)H$ there exists an affine function l_A with derivative A such that

$$\|u - l_A\|_{L^{\infty}(H^{-1}(\partial B_r(b)))} \leq c\sqrt{\mathcal{C}_1 \mathfrak{e}^{-\frac{1}{2}}}. \tag{44}$$

Proof of Step 1. Let $E_b \subset \left(\frac{\sigma}{4}, \frac{\sigma}{2}\right)$ be the set defined in Lemma 5. Let

$$D := \bigcup_{r \in E_b \cap \left(\frac{\sigma}{4}, \frac{\sigma}{2}\right)} H^{-1}\left(\partial B_r(b)\right).$$

Now by definition of E_b, see (22) we have

$$\begin{aligned}\int_D d^q\left(Du(z), SO(2)H\right) dL^2 z &\leq \sigma^{-1} \int_{E_b} \int_{H^{-1}(\partial B_r(b))} d^q\left(Du(z), SO(2)H\right) dH^1 z dL^1 r \\ &\leq c\epsilon \end{aligned} \tag{45}$$

And let $T := \bigcup_{r \in \left(\frac{\sigma}{4}, \frac{\sigma}{2}\right)} H^{-1}\left(\partial B_r(b)\right)$, we know

$$\begin{aligned} L^2(T \backslash D) &\leq 5\sigma^{-1} L^1\left(\left(\frac{\sigma}{4}, \frac{\sigma}{2}\right) \backslash E_b\right) \\ &\leq \frac{5\mathcal{C}_1 \sigma^{-3}}{\sqrt{\mathfrak{e}}}. \end{aligned} \tag{46}$$

Now by Proposition A1 [11] there exists a constant $U = U(T)$ and a function $v : T \to \mathbb{R}^2$ such that $\|Dv\|_{L^{\infty}(T)} \leq U 100\sigma^{-1}$ and

$$\|Dv - Du\|_{L^q(T)} \leq c \int_{\{x \in T : |Du(x)| > 100\sigma^{-1}\}} |Du(x)|^q \, dL^2 x. \tag{47}$$

Now

$$\begin{aligned}\int_{\{x \in T : |Du(x)| > 100\sigma^{-1}\}} |Du(x)|^q \, dL^2 x &\leq 2^q \int_{\{x \in T : |Du(x)| > 100\sigma^{-1}\}} d^q\left(Du(x), K\right) dL^2 x \\ &\overset{(4)}{\leq} 2^q \epsilon. \end{aligned} \tag{48}$$

And

$$\begin{aligned}\int_T d^q\left(Dv(z), SO(2)H\right) dL^2 z &\leq cL^2(T \backslash D) + \int_D d^q\left(Dv(z), SO(2)H\right) dL^2 z \\ &\overset{(46),(47),(48)}{\leq} \int_D d^q\left(Du(z), SO(2)H\right) dL^2 z + \frac{c\mathcal{C}_1}{\sqrt{\mathfrak{e}}} + c\epsilon \\ &\overset{(45)}{\leq} \frac{c\mathcal{C}_1}{\sqrt{\mathfrak{e}}}. \end{aligned}$$

For the case $q>1$ we can (after change of variables) apply Theorem 3 to conclude there exists $A\in SO(2)H$ such that

$$\int_T |Dv(z)-A|^q\, dL^2 z\leq \frac{c\mathcal{C}_1}{\sqrt{\mathfrak{e}}}. \tag{49}$$

For the case $q=1$ note

$$\begin{aligned}\int_T d^2(Dv(z),SO(2)H) &\leq c\int_T d(Dv(z),SO(2)H)\\ &\leq \frac{c\mathcal{C}_1}{\sqrt{\mathfrak{e}}}\end{aligned}$$

and again we apply Theorem 3, so there exists $A\in SO(2)H$ such that

$$\begin{aligned}\int_T |Dv(z)-A|\, dL^2 z &\leq c\left(\int_T |Dv(z)-A|^2\, dL^2 z\right)^{\frac{1}{2}}\\ &\leq c\sqrt{\mathcal{C}_1\mathfrak{e}^{-\frac{1}{2}}}. \end{aligned}\tag{50}$$

Now by applying (47), (48) to (49), (50), for any $q\geq 1$ we have

$$\int_T |Du(z)-A|^q\, dL^2 z\leq c\sqrt{\mathcal{C}_1\mathfrak{e}^{-\frac{1}{2}}}.$$

By Poincaré's inequality there exists and affine map l_A with $Dl_A=A$ such that

$$\int_T |u(z)-l_A(z)|^q\, dL^2 z\leq c\sqrt{\mathcal{C}_1\mathfrak{e}^{-\frac{1}{2}}}.$$

So by the co-area formula there exists a set $\mathcal{Y}_b\subset\left(\frac{\sigma}{4},\frac{\sigma}{2}\right)$ such that $L^1\left(\left(\frac{\sigma}{4},\frac{\sigma}{2}\right)\backslash\mathcal{Y}_b\right)\leq\frac{\sigma}{100}$ such that for each $r\in\mathcal{Y}_b$ we have

$$\int_{H^{-1}(\partial B_r(b))} |u(z)-l_A(z)|^q+|Du(z)-A(z)|^q\, dH^1 z\leq c\sqrt{\mathcal{C}_1\mathfrak{e}^{-\frac{1}{2}}} \tag{51}$$

By the fundamental theorem of Calculus any $r\in\mathcal{Y}_b$ satisfies (44) so this completes the proof of Step 1.

Step 2. We will show we can find $r_1\in\left(\frac{\sigma}{4},\frac{3\sigma}{8}\right)\cap\mathcal{Y}_b$ such that

$$\int_{H^{-1}\left(B_{r_1}(b)\right)}\det(Du(z))\, dL^2 z=L^2\left(u\left(H^{-1}\left(B_{r_1}(b)\right)\right)\right).$$

Proof of Step 2. Following ideas of [6] (Step 1 of the proof Proposition 2.2) we will use some elements of degree theory.

Let $r_0\in\mathcal{Y}_b\cap\left(\frac{22\sigma}{50},\frac{\sigma}{2}\right)$. We consider the homotopy defined by $\mathbb{H}(x,t)=tu(x)+(1-t)\,l_A(x)$ for $t\in[0,1]$, $x\in H^{-1}(B_{r_0}(b))$. Note that for every $t\in[0,1]$, $x\to\mathbb{H}(x,t)$ is C^0. Also note that for $\mathcal{C}_1$ small enough by (44) we have

$$l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)\cap\mathbb{H}\left(\partial H^{-1}(B_{r_0}(b)),t\right)=\emptyset$$

for all $t\in[0,1]$. So by Theorem 2.3 [12] we have that for any $p\in l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)$, $d\left(\mathbb{H}(t,\cdot),H^{-1}(B_{r_0}(b)),p\right)$ is independent of t. As $\det(A)=1$ and we know

$$d\left(l_A(x),H^{-1}(B_{r_0}(b)),p\right)=1$$

this implies $d\left(u,H^{-1}(B_{r_0}(b)),p\right)=1$ for any $p\in l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)$.

Now since by Sobolev embedding $u \in W^{1,2}(B_1(0))$ and $\det(Du(x)) \geq C^{-2}$ for a.e. $x \in B_1(0)$ by Theorem 5.32 [12] we know u satisfies the hypotheses to apply Remark 5.26 [12] and so we know that for a.e. $p \in l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)$ we have

$$d\left(u, H^{-1}\left(B_{r_0}(b)\right), p\right) = \sum_{z \in \left\{y \in H^{-1}\left(B_{r_0}(b)\right): u(y)=p\right\}} \operatorname{sgn}\left(\det\left(Du(z)\right)\right) \tag{52}$$

and as $\det(Du(z)) = 1$ this gives us that $\#\left\{y \in H^{-1}\left(B_{r_0}(b)\right) : u(y) = p\right\} = 1$ for a.e. $p \in l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)$.

Now take $r_1 \in \left(\frac{\sigma}{4}, \frac{5\sigma}{16}\right) \cap \mathcal{Y}_b$, so again assuming $\mathcal{C}_1$ is small enough from (44) we have $u\left(\partial H^{-1}\left(B_{r_1}(b)\right)\right) \subset l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)$ hence $\mathbb{H}\left(\partial H^{-1}\left(B_{r_1}(b)\right), t\right) \subset l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)$ for all $t \in [0,1]$. So for $p \notin l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)$ we know by Theorem 2.3 [12] the degree $d\left(\mathbb{H}(t,\cdot), H^{-1}\left(B_{r_1}(b)\right), p\right)$ is independent of t and so we know $d\left(u, H^{-1}\left(B_{r_1}(b)\right), p\right) = 0$, by (52) this implies that $L^2\left(u\left(H^{-1}\left(B_{r_1}(b)\right)\right) \backslash l_A\left(H^{-1}\left(B_{\frac{3\sigma}{8}}(b)\right)\right)\right) = 0$.

Hence for a.e. $p \in u\left(H^{-1}\left(B_{r_1}(b)\right)\right)$, since $r_1 < r_0$ we have

$$\begin{aligned} 1 &\leq \#\left\{y \in H^{-1}\left(B_{r_1}(b)\right) : u(y) = p\right\} \\ &\leq \#\left\{y \in H^{-1}\left(B_{r_0}(b)\right) : u(y) = p\right\} \\ &= 1. \end{aligned}$$

So by Remark 5.26 [12] we know $d\left(u, H^{-1}\left(B_{r_1}(b)\right), y\right) = 1$ for a.e. $y \in u\left(H^{-1}\left(B_{r_1}(b)\right)\right)$. Now we can apply Theorem 5.35 [12], so

$$\begin{aligned} \int_{H^{-1}\left(B_{r_1}(b)\right)} \det\left(Du(z)\right) dL^2 z &= \int_{u\left(H^{-1}\left(B_{r_1}(b)\right)\right)} d\left(u, H^{-1}\left(B_{r_1}(b)\right), y\right) dL^2 y \\ &= L^2\left(u\left(H^{-1}\left(B_{r_1}(b)\right)\right)\right). \end{aligned} \tag{53}$$

This completes the proof of Step 2.

Let $\mathcal{G} := \left\{z \in H^{-1}\left(B_{r_1}(b)\right) : d\left(Du(z), K\right) \leq 1\right\}$ so by (4) we have $L^2\left(H^{-1}\left(B_{r_1}(b)\right) \backslash \mathcal{G}\right) \leq c\epsilon$. So for each $z \in \mathcal{G}$ let $A(z) \in SO(2) \cup SO(2)H$ such that $d\left(Du(z), K\right) = \left|Du(z) - A(z)\right|$.

$$\begin{aligned} \int_{\mathcal{G}} \det\left(Du(z)\right) dL^2 z &= \int_{\mathcal{G}} \det\left(A(z) + \left(Du(z) - A(z)\right)\right) dL^2 z \\ &= \int_{\mathcal{G}} \det\left(A(z)\right) + \operatorname{cof}\left(A(z)\right) : \left(Du(z) - A(z)\right) + \det\left(Du(z) - A(z)\right) dL^2 z \\ &\overset{(4)}{\geq} L^2(\mathcal{G}) - c\epsilon^{\frac{1}{q}} \\ &\geq \pi r_1^2 - c\epsilon^{\frac{1}{q}}. \end{aligned}$$

And since $\det(Du(z)) > 0$ for a.e. $z \in B_1(0)$, this together with (53) clearly implies (43). □

4.3. Main Proposition.

Proposition 2. *Let $p, q \geq 1$. Suppose $u \in W^{2,p}(B_1(0)) \cap W^{1,q}(B_1(0))$ is a sense preserving function with $\sup_{x \in B_1(0)} \| \left[Du(x)\right]^{-1} \| \leq C$ which satisfies inequalities (4), (5).*

There exists small positive constant $\mathfrak{e} = \mathfrak{e}(\sigma)$ such that if we define $L^{\mathfrak{e}}(u)$ by (27) and define Θ_u by

$$\Theta_u := \begin{cases} H \text{ if } & L^{\mathfrak{e}}(u) \geq 0 \\ Id \text{ if } & L^{\mathfrak{e}}(u) < 0 \end{cases}$$

Then the function $\tilde{u} := u \circ \Theta_u^{-1}$ satisfies the following property.

There exists constant $\mathfrak{c}_1 = \mathfrak{c}_1(\sigma) > 0$ *such that for any* $b \in B_{\mathfrak{c}_1}(0)$ *we can find* $R_1 \geq 2\mathfrak{c}_1$ *with the property that for every* $\beta \in (0, 2\pi]$, $\{a_\beta, b_\beta\} = \left\{\lambda e^{i\beta} + b : \lambda > 0\right\} \cap \partial B_{R_1}(b)$, *then*

$$|\tilde{u}(a_\beta) - \tilde{u}(b_\beta)| \geq 2\left(1 - c\epsilon^{\frac{1}{4q}}\right) R_1. \tag{54}$$

Proof.

We will argue the case $L^{\mathfrak{e}}(u) \geq 0$, the case $L^{\mathfrak{e}}(u) < 0$ can be dealt with in an identical manner.

Now by Lemmas 5, 6 there exists $A \in SO(2)H$ and sets $\mathcal{Y}_b, E_b \subset \left(\frac{\sigma}{4}, \frac{\sigma}{2}\right)$ with

$$L^1\left(\left(\frac{\sigma}{4}, \frac{\sigma}{2}\right) \backslash \mathcal{Y}_b \cap E_b\right) \leq \frac{\sigma}{50}$$

such that for any $r \in \mathcal{Y}_b \cap E_b$, (22) implies that Du is close to A in the $dH^1_{\lfloor \partial H^{-1}(\partial B_R(b))}$ norm, and from (43) we have

$$L^2\left(u\left(H^{-1}\left(B_r(b)\right)\right)\right) \geq \pi r^2 - c\epsilon^{\frac{1}{q}}. \tag{55}$$

Let $\tilde{u} := u \circ H^{-1}$. Let $R_1 \in \mathcal{Y}_b \cap E_b$. So using Hölder for the last inequality

$$\begin{aligned}\int_{\partial B_{R_1}(b)} d\left(\widehat{D\tilde{u}}(z), SO(2)\right) dH^1 z &\leq c\int_{H^{-1}\left(\partial B_{R_1}(b)\right)} d\left(\widehat{Du}(y) \circ H^{-1}, SO(2)\right) dH^1 y \\ &\overset{(22)}{\leq} c\epsilon^{\frac{1}{q}}.\end{aligned} \tag{56}$$

Let $\widetilde{K} = KH^{-1}$, it is easy to see that by Hölder's inequality, from (4)

$$\int_{B_\sigma(0)} d\left(D\tilde{u}(z), \widetilde{K}\right) dL^2 z \leq c\epsilon^{\frac{1}{q}}. \tag{57}$$

Claim. We will show $\tilde{u}(B_{R_1}(b))$ satisfies condition (18) of Lemma 2.

Proof of Claim. From (55) we know

$$\begin{aligned}L^2\left(\tilde{u}\left(B_{R_1}(b)\right)\right) &= L^2\left(u\left(H^{-1}\left(B_{R_1}(b)\right)\right)\right) \\ &\geq \pi R_1^2 - c\epsilon^{\frac{1}{q}}.\end{aligned} \tag{58}$$

So

$$\sqrt{\frac{L^2\left(\tilde{u}\left(B_{R_1}(b)\right)\right)}{\pi}} \geq R_1 - c\epsilon^{\frac{1}{q}}. \tag{59}$$

Now we know $H^1\left(\tilde{u}\left(\partial B_{R_1}(b)\right)\right) = \int_{\partial B_{R_1}(b)} \left|\widehat{D\tilde{u}}(z)\, t_z\right| dH^1 z$. So

$$\begin{aligned}\left|H^1\left(\tilde{u}\left(\partial B_{R_1}(b)\right)\right) - 2\pi R_1\right| &\leq \left|\int_{\partial B_{R_1}(b)} \left|\widehat{D\tilde{u}}(z)\, t_z\right| - 1 dH^1 z\right| \\ &\leq \int_{\partial B_{R_1}(b)} d\left(\widehat{D\tilde{u}}(z), SO(2)\right) dH^1 z \\ &\overset{(56)}{\leq} c\epsilon^{\frac{1}{q}}.\end{aligned} \tag{60}$$

Now we can assume R_1 was chosen to be one of the radii for which we can apply Lemma 3, so we know $u\left(B_{R_1}(b)\right)$ is a set of finite perimeter and so $\mathrm{Per}\left(u\left(B_{R_1}(b)\right)\right) \leq H^1\left(u\left(\partial B_{R_1}(b)\right)\right)$. So putting this together with (59) we have

$$\sqrt{\frac{L^2\left(\tilde{u}\left(B_{R_1}(b)\right)\right)}{\pi}} \geq \frac{\mathrm{Per}\left(u\left(B_{R_1}(b)\right)\right)}{2\pi} - c\epsilon^{\frac{1}{q}}. \tag{61}$$

Hence the set $\tilde{u}\left(B_{R_1}(b)\right)$ has property (18) for $\varepsilon = c\epsilon^{\frac{1}{q}}$, which proves the claim.

Let a_β, b_β be two antipodal points on $\partial B_{R_1}(a)$, i.e. $\{a_\beta, b_\beta\} = \partial B_{R_1}(a) \cap \left\{\langle e^{i\beta}\rangle + a\right\}$. Let Γ_1, Γ_2 be the connected components of $\partial B_{R_1}(a) \setminus \{a_\beta, b_\beta\}$. Note $\tilde{u}(\Gamma_1)$ and $\tilde{u}(\Gamma_2)$ are the connected components of $\tilde{u}(\partial B_{R_1}(a)) \setminus \{\tilde{u}(a_\beta), \tilde{u}(b_\beta)\}$.

Let $R_2 := \sqrt{\frac{L^2(\tilde{u}(B_{R_1}(b)))}{\pi}} + c\epsilon^{\frac{1}{q}}$, i.e. $R_2 \geq R_1$ (see (59)). Now by Lemma 4 for $\varepsilon = c\epsilon^{\frac{1}{q}}$ there exists $a \in \mathbb{R}^2$, such that $\partial B_{R_2}(a)$ has property (21). Let $x_1, x_2, \dots x_{2m}$ be evenly spaced points on $\partial B_{R_2}(a)$ where $|x_k - x_{k+1}| \in \left(1000 c_1 \varepsilon^{\frac{1}{4}}, 2000 c_1 \varepsilon^{\frac{1}{4}}\right)$, see figure 3.

Recall that by Lemma 3 we know that $\partial\tilde{u}(B_{R_1}(a)) \subset \tilde{u}(\partial B_{R_1}(a))$. We start with x_1, by Lemma 2 we can pick $z_1 \in B_{c_1\varepsilon^{\frac{1}{4}}}(x_1) \cap \tilde{u}(\partial B_{R_1}(a))$, suppose without loss of generality that $z_1 \in \tilde{u}(\Gamma_1)$. It will be clear in the forth coming argument that we must have

$$B_{c_1\varepsilon^{\frac{1}{4}}}(x_k) \cap \tilde{u}(\Gamma_1) = \emptyset \text{ for some } k \in \{2, 3, \dots 2m\} \tag{62}$$

we will assume this is the case for the time being and come back to it later.

Let

$$\varphi_1 = \min\left\{k \in \{2, 3, \dots 2m\} : B_{c_1\varepsilon^{\frac{1}{4}}}(x_k) \cap \tilde{u}(\Gamma_1) = \emptyset\right\}$$

and let

$$\varphi_2 = \max\left\{k \in \{2, 3, \dots 2m\} : B_{c_1\varepsilon^{\frac{1}{4}}}(x_k) \cap \tilde{u}(\Gamma_1) = \emptyset\right\}.$$

Now any $k \in \{\varphi_1 + 1, \dots \varphi_2 - 1\}$ has to be such that $B_{c_1\varepsilon^{\frac{1}{4}}}(x_k) \cap \tilde{u}(\Gamma_1) = \emptyset$ since otherwise $\tilde{u}(\Gamma_1)$ would be dis-connected. Now let $\{\tilde{z}_1, \tilde{z}_2, \dots \tilde{z}_{2m}\}$ be a reordering of $\{z_1, \dots z_{2m}\}$ where $\tilde{z}_{k+1}$ is the clockwise nearest neighbour to $\tilde{z}_k$ for each $k \in \{1, 2, \dots 2m-1\}$ and $\tilde{z}_1, \tilde{z}_2, \dots \tilde{z}_{p_1} \in \tilde{u}(\Gamma_1)$, $\tilde{z}_{p_1+1}, \tilde{z}_2, \dots \tilde{z}_{2m} \in \tilde{u}(\Gamma_2)$.

Let θ_k denote the angle between $\tilde{z}_k$ and $\tilde{z}_{k+1}$ for $k = 1, 2, \dots 2m-1$ and θ_{2m} be the angle between z_{2m} and z_1. It is easy to see $|\tilde{z}_k - \tilde{z}_{k+1}| \geq 2\left(R_2 - c\epsilon^{\frac{1}{4q}}\right)\sin\frac{\theta_k}{2}$. Hence

$$\begin{aligned} H^1(\tilde{u}(\Gamma_1)) &\geq \sum_{k=1}^{p_1-1} |\tilde{z}_k - \tilde{z}_{k+1}| \\ &\geq 2\left(R_2 - c\epsilon^{\frac{1}{4q}}\right)\sum_{k=1}^{p_1-1} \sin\frac{\theta_k}{2} \\ &\geq R_2\left(\sum_{k=1}^{p_1-1}\theta_k\right) - c\epsilon^{\frac{1}{4q}}. \end{aligned} \tag{63}$$

And we know from (56) $H^1(\tilde{u}(\Gamma_1)) = \int_{\Gamma_1} |D\tilde{u}(x)\, t_x|\, dH^1 x \leq \pi R_1 + c\epsilon^{\frac{1}{q}}$, which implies

$$\pi R_1 + c\epsilon^{\frac{1}{4q}} \overset{(63)}{\geq} R_2\left(\sum_{k=1}^{p_1-1}\theta_k\right)$$

and hence as $R_2 \geq R_1$ we have $\sum_{k=1}^{p_1-1}\theta_k \leq \pi + c\epsilon^{\frac{1}{4q}}$.

Via exactly the same arguments it is clear (62) must be true, i.e. if (62) was false then $H^1(\tilde{u}(\Gamma_1))$ would be too long. Also by the same argument we can show $\sum_{k=p_1}^{2m-1}\theta_k \leq \pi + c\epsilon^{\frac{1}{4q}}$. Since obviously $\sum_{k=1}^{2m}\theta_k = 2\pi$ so we have

$$\left|\sum_{k=1}^{p_1-1}\theta_k - \pi\right| \leq c\epsilon^{\frac{1}{4q}}. \tag{64}$$

Now $\tilde{z}_1, \tilde{z}_{p_1} \in N_{c\epsilon^{\frac{1}{4q}}}(\tilde{u}(\partial\Gamma_1))$, without loss of generality we can assume $\tilde{z}_1 \in N_{c\epsilon^{\frac{1}{4q}}}(\tilde{u}(a_\beta))$ and $\tilde{z}_{p_1} \in N_{c\epsilon^{\frac{1}{4}}}(\tilde{u}(b_\beta))$. So as $\tilde{u}(a_\beta), \tilde{u}(b_\beta) \in N_{c\epsilon^{\frac{1}{4q}}}(\partial B_{R_2}(a))$ and as by (64) the angle

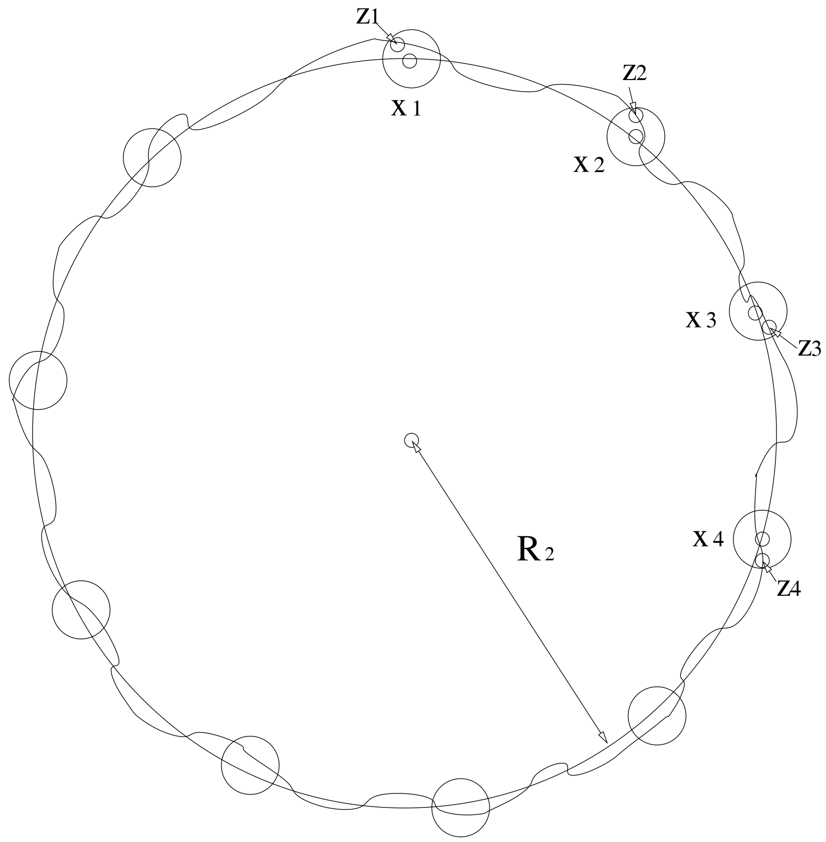

FIGURE 3

between them is within $c\epsilon^{\frac{1}{4q}}$ of π, thus $|\tilde{u}(a_\beta) - \tilde{u}(b_\beta)| \geq 2R_2 - c\epsilon^{\frac{1}{4q}}$. This completes the proof of Proposition 2 in the case where $L^{\mathfrak{e}}(u) \geq 0$.

In the case where $L^{\mathfrak{e}}(u) < 0$ we know there exists $R > 2\mathfrak{c}_1$ satisfying (23). We can then argue in exactly the same way to show $\sqrt{\frac{L^2(u(B_R(b)))}{\pi}} \geq \frac{H^1(\partial u(B_R(b)))}{2\pi} - c\epsilon^{\frac{1}{q}}$ then we can use Lemma 4 to show antipodal points on $\partial B_R(b)$ are mapped to points distance $R - c\epsilon^{\frac{1}{4q}}$ apart. □

4.4. **Proof of Theorem 1 continued.** As in the proof of Proposition 2, we will concentrate on the case where $L^{\mathfrak{e}}(u) \geq 0$.

By Proposition 2, $\tilde{u} := u \circ H^{-1}$ has the property that for every $b \in B_{\mathfrak{c}_1}(0)$ there exists $R_1 > 2\mathfrak{c}_1$ and $a \in \mathbb{R}^2$ such that (54) holds true. As stated before, it is easy to see

$$\int_{B_\sigma(0)} d\left(D\tilde{u}(z), \widetilde{K}\right) dL^2 z \leq c\epsilon^{\frac{1}{q}}. \tag{65}$$

It is a calculation to see that for

$$\phi_1 := \begin{pmatrix} \frac{\sigma}{\sqrt{1+\sigma^2}} \\ \frac{1}{\sqrt{1+\sigma^2}} \end{pmatrix} \text{ and } \phi_2 := \begin{pmatrix} \frac{\sigma}{\sqrt{1+\sigma^2}} \\ \frac{-1}{\sqrt{1+\sigma^2}} \end{pmatrix}$$

we have $\left|H^{-1}\phi_i\right| = 1$. Let

$$\Xi_1 := \left\{\theta \in (0, 2\pi] : e^{i\theta} = \begin{pmatrix} \frac{a}{\sqrt{1+a^2}} \\ \frac{1}{\sqrt{1+a^2}} \end{pmatrix} \text{ for some } a \in \left(\frac{-\sigma}{\sqrt{2}}, \frac{\sigma}{\sqrt{2}}\right)\right\}.$$

And $l_\theta^z := \left(\langle e^{i\theta}\rangle + z\right) \cap B_{R_1}(z)$. Let $V_r(x) := \{l_\theta^x \cap B_r(x) : \theta \in \Xi_1\} \backslash B_{\frac{r}{2}}(x)$.

Using the Fubini argument from Section 2.3 [6] we will show we can find $b \in V_{\mathfrak{c}_1}(0)$ such that

$$\int_{B_{\frac{\sigma}{2}}(b)} d\left(D\tilde{u}(z), \widetilde{K}\right) |z-b|^{-1} dL^2 z \leq c\epsilon, \tag{66}$$

we argue as follows, by Fubini Theorem we have

$$\begin{aligned}
&\int_{V_{\mathfrak{c}_1}(0)} \int_{B_{\frac{\sigma}{2}}(b)} d\left(D\tilde{u}(x), \widetilde{K}\right) |x-y|^{-1} dL^2 x dL^2 y \\
&\leq \int_{V_{\mathfrak{c}_1}(0)} \int_{B_\sigma(0)} d\left(D\tilde{u}(x), \widetilde{K}\right) |x-y|^{-1} dL^2 x dL^2 y \\
&= \int_{B_\sigma(0)} d\left(D\tilde{u}(x), \widetilde{K}\right) \int_{V_{\mathfrak{c}_1}(0)} |x-y|^{-1} dL^2 y dL^2 x \\
&\leq c \int_{B_\sigma(0)} d\left(D\tilde{u}(x), \widetilde{K}\right) dL^2 x \\
&\leq c\epsilon.
\end{aligned}$$

Thus there must exists $y \in V_{\mathfrak{c}_1}(0)$ such that (66) holds true. Note that for some constant $\mathfrak{c}_2 = \mathfrak{c}_2(\sigma) > 0$ we have $B_{\mathfrak{c}_2}(0) \subset V_{\mathfrak{c}_1}(b)$

By Proposition 4 there exists $R_1 > 2\mathfrak{c}_1$ such that for any $\beta \in (0, 2\pi]$, letting $\{a_\beta, b_\beta\} = \left(\langle e^{i\beta}\rangle + b\right) \cap \partial B_{R_1}(b)$ we have

$$|\tilde{u}(a_\beta) - \tilde{u}(b_\beta)| \geq 2\left(1 - c\epsilon^{\frac{1}{4q}}\right) R_1. \tag{67}$$

Let

$$B := \left\{x \in B_{R_1}(b) : d\left(D\tilde{u}(x), SO(2) H^{-1}\right) \leq d\left(D\tilde{u}(x), SO(2)\right)\right\}. \tag{68}$$

Now it is an exercise to see that there exists $s_\sigma \in (0,1)$ such that for any $\theta \in \Xi_1$ we have $\left|H^{-1} e^{i\theta}\right| \leq s_\sigma$. We estimate that

$$\begin{aligned}
|\tilde{u}(a_\theta) - \tilde{u}(b_\theta)| &\leq \int_{l_\theta^b} \left|D\tilde{u}(z) e^{i\theta}\right| dH^1 z \\
&\leq s_\sigma H^1\left(l_\theta^b \cap B\right) + H^1\left(l_\theta^b \backslash B\right) + \int_{l_\theta^b} d\left(D\tilde{u}(z), \widetilde{K}\right) dH^1 z \\
&= H^1\left(l_\theta^b\right) - (1 - s_\sigma) H^1\left(l_\theta^b \cap B\right) + \int_{l_\theta^b} d\left(D\tilde{u}(z), \widetilde{K}\right) dH^1 z.
\end{aligned} \tag{69}$$

Now $H^1\left(l_\theta^b\right) = |a_\theta - b_\theta| = 2R_1$ so putting (67), with (69) we have

$$2\left(1 - c\epsilon^{\frac{1}{4q}}\right) R_1 \leq 2R_1 - (1 - s_\sigma) H^1\left(l_\theta^b \cap B\right) + \int_{l_\theta^b} d\left(D\tilde{u}(z), \widetilde{K}\right) dH^1 z.$$

This implies

$$(1 - s_\sigma) \int_{l_\theta^b} \chi_B(z) dH^1 z \leq \int_{l_\theta^b} d\left(D\tilde{u}(z), \widetilde{K}\right) dH^1 z + c\epsilon^{\frac{1}{4q}}. \tag{70}$$

Since $R_1 \leq \frac{\sigma}{2}$ by the co-area argument of Section 2.3, Case 1 [6].

$$\begin{aligned}\int_{V_{R_2}(b)} (1-s_\sigma)\chi_B(z)|z-b|^{-1} dL^2 z &= \int_{\Xi_1}\int_{l_\theta^b} (1-s_\sigma)\chi_B(z)\, dH^1 z dL^1\theta \\ &\overset{(70)}{\leq} \int_{\Xi_1}\int_{l_\theta^b} d\left(D\tilde{u}(z), \widetilde{K}\right) dH^1 z dL^1\theta + c\epsilon^{\frac{1}{4q}} \\ &\leq \int_{B_{R_2}(b)} d\left(D\tilde{u}(z), \widetilde{K}\right)|z-b|^{-1} dL^2 z + c\epsilon^{\frac{1}{4q}} \\ &\overset{(66)}{\leq} c\epsilon + c\epsilon^{\frac{1}{4q}}.\end{aligned}$$

As $|z-b|^{-1} \geq 1$ for any $z \in B_{R_2}(b)$, and since $B_{\mathfrak{c}_2}(0) \subset V_{R_1}(b)$, this gives

$$L^2(B \cap B_{\mathfrak{c}_2}(0)) \leq c\epsilon^{\frac{1}{4q}}. \tag{71}$$

So (recall definition (68))

$$\begin{aligned}&\int_{B_{\mathfrak{c}_2}(0)} d(D\tilde{u}(z), SO(2))\, dL^2 z \\ &\leq \int_{B_{\mathfrak{c}_2}(0)\setminus B} d\left(D\tilde{u}(z), \widetilde{K}\right) dL^2 z + \int_{B_{\mathfrak{c}_2}(0)\cap B} d\left(D\tilde{u}(z), \widetilde{K}\right) + \sigma^{-1} dL^2 z \\ &\overset{(65)}{\leq} c\epsilon^{\frac{1}{q}} + \sigma^{-1} L^2(B_{\mathfrak{c}_2}(0) \cap B) \\ &\overset{(71)}{\leq} c\epsilon^{\frac{1}{4q}}.\end{aligned} \tag{72}$$

Since $d^q(D\tilde{u}(z), SO(2)) \leq c\left(d(D\tilde{u}(z), SO(2)) + d^q\left(D\tilde{u}(z), \widetilde{K}\right)\right)$ we have

$$\int_{B_{\mathfrak{c}_2}(0)} d^q(D\tilde{u}(z), SO(2))\, dL^2 z \overset{(65),(72)}{\leq} c\epsilon^{\frac{1}{4q}}$$

Now in the case $q > 1$ we can apply Theorem 3 so we have that there exists $A \in K$ such that

$$\int_{B_{\mathfrak{c}_2}(0)} |D\tilde{u}(z) - A|^q\, dL^2 z \leq c\epsilon^{\frac{1}{4q}},$$

which implies

$$\int_{B_{\sigma\mathfrak{c}_2}(0)} |Du(z) - AH|^q\, dL^2 z \leq c\epsilon^{\frac{1}{4q}}. \tag{73}$$

In the case $q = 1$ we have to apply Proposition A1 of [11] which gives us a c-Lipschitz function v such that

$$\|D\tilde{u} - Dv\|_{L^1(B_1(0))} \leq c\epsilon. \tag{74}$$

So using Lipschitzness

$$\begin{aligned}\int_{B_{\mathfrak{c}_2}(0)} d^{\frac{5}{4}}(Dv(z), SO(2))\, dL^2 z &\leq c\int_{B_{\mathfrak{c}_2}(0)} d(Dv(z), SO(2))\, dL^2 z \\ &\overset{(74)}{\leq} c\int_{B_{\mathfrak{c}_2}(0)} d(D\tilde{u}(z), SO(2))\, dL^2 z + c\epsilon \\ &\overset{(72)}{\leq} c\epsilon^{\frac{1}{4}}.\end{aligned}$$

So applying Theorem 3 we have there exists $R \in SO(2)$ such that

$$\int_{B_{\mathfrak{c}_2}(0)} |Dv(z) - R|^{\frac{5}{4}}\, dL^2 z \leq c\epsilon^{\frac{1}{4}}. \tag{75}$$

Thus using Hölder's inequality

$$\begin{aligned}\int_{B_{\mathfrak{c}_2}(0)} |D\tilde{u}(z) - R|\, dL^2 z &\overset{(74)}{\leq} \int_{B_{\mathfrak{c}_2}(0)} |Dv(z) - R|\, dL^2 z + c\epsilon \\ &\leq c\left(\int_{B_{\mathfrak{c}_2}(0)} |Dv(z) - R|^{\frac{5}{4}}\, dL^2 z\right)^{\frac{4}{5}} + c\epsilon \\ &\overset{(75)}{\leq} c\epsilon^{\frac{1}{5}}.\end{aligned}$$

And this implies

$$\int_{B_{\sigma\mathfrak{c}_2}(0)} |Du(z) - RH|\, dL^2 z \leq c\epsilon^{\frac{1}{5}}.$$

In the case where $L^{\mathfrak{e}}(u) < 0$ the argument is identical. □

5. Proof of Theorem 2

With a view to later developments we will prove the following results in more generality than is needed.

Definition 4. *For $p > 1$, $q \geq 1$, $e \geq 1$. We will say we have an (p, q, e)* **Liouville Theorem** for a function class in $W^{1,p}(B_1(0)) \cap W^{2,q}(B_1(0))$ if there exists positive constants $\mathcal{C}_1 << 1$ and $\mathcal{C}_2 >> 1$ depending on p, q, σ such that the inequalities

$$\int_{B_1(0)} d^p(Du(z), K)\, dL^2 z \leq \mathcal{C}_1\epsilon, \quad \int_{B_1(0)} \left|D^2 u(z)\right|^q dL^2 z \leq \mathcal{C}_1\epsilon^{1-q}$$

imply that there exists $J \in \{Id, H\}$, $R \in SO(2)$ such that

$$\int_{B_{\mathcal{C}_1}(0)} |Du(z) - RJ|^p\, dL^2 z \leq \mathcal{C}_2\epsilon^{\frac{1}{ep}}.$$

So in Theorem 1 we established a $(p, q, 4)$ Liouville Theorem for orientation preserving functions in $W^{1,p}(B_1(0)) \cap W^{2,q}(B_1(0))$ with the property $\sup_{x\in B_1(0)} \| [Du(x)]^{-1} \| \leq C$. In Conjecture 1 we conjectured that an (optimal) $(p, q, 1)$ Liouville Theorem holds for functions in $W^{1,p}(B_1(0)) \cap W^{2,q}(B_1(0))$ and recall in [6] a $(p, 1, 1)$ Liouville Theorem has been proved.

We have the following proposition.

Proposition 3. *Let $H = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}$, $K = SO(2) \cup SO(2) H$. Let $p \in [1, 2]$, $q \geq 1$, $e \in [1, 4]$. Let B_F^h be as defined in Theorem 2. Let A denote the space of sense preserving functions in $W^{1,p}(Q_1(0)) \cap W^{2,q}(Q_1(0))$ for which $\sup_{x\in Q_1(0)} \| [Du(x)]^{-1} \| \leq C$. Define $\mathcal{F}_h$ as in Theorem 2.*

Suppose we have a (p, q, e) Liouville Theorem for A. Let A_F^ζ denote the subset of functions in A with affine boundary condition F that are ζ-Lipschitz. Let $\delta > 0$ be a small number. Let $\alpha \in [1, 2]$, if $u \in A_F^\zeta$ is such that

$$I_\epsilon^q(u) \leq \epsilon^{\frac{\alpha}{3q}} \tag{76}$$

then there exists a constant $\mathcal{C}_3 = \mathcal{C}_3(\delta, q, \sigma, \zeta)$ such that for $h = \epsilon^{\frac{1}{q}}$, letting $\tilde{u} \in B_F^h$ denote the piecewise affine interpolant of u we have

$$\mathcal{F}_h(\tilde{u}) \leq \mathcal{C}_3 h^{\frac{\alpha}{3}-\delta}.$$

If $e = 1$ we have the stronger result $\mathcal{F}_0(\tilde{u}) \leq \mathcal{C}_3 h^{\frac{\alpha}{3}-\delta}$.

Proof. Let $p \in (1,2]$ be such that $\frac{2(p-1)\alpha}{q} = \delta$ we assume δ is sufficiently small so that such a p can be found. Let $w_1, w_2 \in S^1$ be two (non-equal) vectors such that w_1, w_2 and $w_1 - w_2$ are not in the set of rank-1 connections. We assume the triangulation $\triangle_h$ is composed of triangles which (in pairs, see figure 4) form the parallelepipeds of the set

$$Q_1(0) \setminus \left(\left\{ k\epsilon^{\frac{1}{q}} w_2 + \langle w_1 \rangle : k \in \mathbb{Z} \right\} \cup \left\{ k\epsilon^{\frac{1}{q}} w_1 + \langle w_2 \rangle : k \in \mathbb{Z} \right\} \right).$$

Let $\varphi > 1$ be some constant we will decided on later. Let $\{c_i : i = 1, 2, \dots N_0\}$ be an ordering of the points

$$\left\{ k_1 w_1 \epsilon^{\frac{1}{q}} + k_2 w_2 \epsilon^{\frac{1}{q}} : k_1, k_2 \in \mathbb{Z},\ k_1 w_1 \epsilon^{\frac{1}{q}} + k_2 w_2 \epsilon^{\frac{1}{q}} \in Q_{1-\varphi\epsilon^{\frac{1}{q}}}(0) \right\}.$$

For each $i \in \{1, 2, \dots N_0\}$ let $v_i : Q_\varphi(0) \to \mathbb{R}^2$ be defined by $v_i(z) := u\left(c_i + z\epsilon^{\frac{1}{q}}\right)\epsilon^{-\frac{1}{q}}$. Let

$$\begin{aligned} \alpha_i &:= \mathcal{C}_1^{-1} \int_{Q_\varphi(0)} d^p\left(Dv_i(z), K\right) dL^2 z \\ &= \epsilon^{-\frac{2}{q}} \mathcal{C}_1^{-1} \int_{Q_{\varphi\epsilon^{\frac{1}{q}}}(c_i)} d^p\left(Du(z), K\right) dL^2 z. \end{aligned} \tag{77}$$

So note

$$\begin{aligned} \sum_{i=1}^{N_0} \epsilon^{\frac{2}{q}} \alpha_i &= \mathcal{C}_1^{-1} \sum_{i=1}^{N_0} \int_{Q_{\varphi\epsilon^{\frac{1}{q}}}(c_i)} d^p\left(Du(z), K\right) dL^2 z \\ &\overset{(76)}{\leq} c\epsilon^{\frac{\alpha}{3q}}. \end{aligned} \tag{78}$$

Let

$$B_1 := \left\{ i : \int_{Q_\varphi(0)} \left| D^2 v_i(z) \right|^q dL^2 z \geq \mathcal{C}_1 \alpha_i^{1-q} \right\}.$$

Let $M = \mathrm{Card}(B_1)$. Now there must exist subset $\widetilde{B_1} \subset B_1$ such that $\mathrm{Card}\left(\widetilde{B_1}\right) \geq \frac{M}{2}$ with the property that for every $i \in \widetilde{B_1}$ we have $\alpha_i \leq c'\epsilon^{\frac{\alpha}{3q} - \frac{2}{q}} M^{-1}$ since otherwise we have that the set $E_1 := \left\{ i \in B_1 : \alpha_i M > c'\epsilon^{\frac{\alpha}{3q} - \frac{2}{q}} \right\}$ is such that $\mathrm{Card}(E_1) \geq \frac{M}{2}$.

So

$$\begin{aligned} \sum_{i \in E_1} \alpha_i &\geq \mathrm{Card}(E_1) \frac{c'}{M} \epsilon^{\frac{\alpha}{3q} - \frac{2}{q}} \\ &\geq \frac{c'}{2} \epsilon^{\frac{\alpha}{3q} - \frac{2}{q}} \end{aligned}$$

which contradicts (78) for constant c' large enough.

So

$$\begin{aligned} \sum_{i \in \widetilde{B_1}} \int_{Q_\varphi(0)} \left| D^2 v_i(z) \right|^q dL^2 z &\geq \sum_{i \in \widetilde{B_1}} \mathcal{C}_1 \alpha_i^{1-q} \\ &\geq \mathcal{C}_1 \mathrm{Card}\left(\widetilde{B_1}\right) \left(c'\epsilon^{\frac{\alpha}{3q} - \frac{2}{q}} M^{-1} \right)^{1-q} \\ &\geq c\epsilon^{\left(\frac{\alpha}{3q} - \frac{2}{q}\right)(1-q)} \mathrm{Card}\left(\widetilde{B_1}\right) M^{q-1} \\ &\geq cM^q \epsilon^{\left(\frac{\alpha}{3q} - \frac{2}{q}\right)(1-q)}. \end{aligned}$$

Then

$$\sum_{i\in\widetilde{B_1}}\int_{Q_{\varphi\epsilon^{\frac{1}{q}}}(c_i)}\left|D^2u(y)\right|^q\epsilon^{1-\frac{2}{q}}dL^2y\geq cM^q\epsilon^{\left(\frac{\alpha}{3q}-\frac{2}{q}\right)(1-q)}.$$

This implies $\epsilon^{\frac{\alpha}{3q}-1}\geq cM^q\epsilon^{\left(\frac{\alpha}{3q}-\frac{2}{q}\right)(1-q)-1+\frac{2}{q}}$ so $\epsilon^{\frac{\alpha}{3q}-1}\geq cM^q\epsilon^{\frac{\alpha}{3q}-\frac{\alpha}{3}+1}$ and thus

$$c\epsilon^{\frac{\alpha}{3q}-\frac{2}{q}}\geq M=\mathrm{Card}(B_1). \tag{79}$$

So if $i\notin B_1$ then by the fact we have an (p,q,e) Liouville Theorem (see Definition 4) there exists $A_i\in K$ such that (recall (77))

$$\int_{Q_{\mathcal{C}_1\varphi}(0)}|Dv_i(z)-A_i|^p\,dL^2z\leq\mathcal{C}_2\alpha_i^{\frac{1}{ep}}. \tag{80}$$

Let $\tau>0$ be some small number we decide on later. We will show that if $A\geq\epsilon^{\frac{\tau ep}{ep-1}}$ then $A^{\frac{1}{ep}}\leq\epsilon^{-\tau}A$. To see this note that

$$A^{\frac{1}{ep}}=A^{\frac{1}{ep}-1}A=\frac{A}{A^{\frac{ep-1}{ep}}}$$

since $A\geq\epsilon^{\frac{\tau ep}{ep-1}}$ we know $A^{\frac{ep-1}{ep}}\geq\epsilon^{\tau}$ so $A^{\frac{1}{ep}}\leq\epsilon^{-\tau}A$.

Let $\Lambda=\epsilon^{\frac{\tau ep}{ep-1}}$. Now if $\alpha_i\in(0,\Lambda)$ then

$$\begin{aligned}\int_{Q_{\mathcal{C}_1\varphi}(0)}|Dv_i(z)-A_i|^p\,dL^2z&\leq\mathcal{C}_2\alpha_i^{\frac{1}{ep}}\\&\leq\mathcal{C}_2\Lambda^{\frac{1}{ep}}\\&=\mathcal{C}_2\epsilon^{\frac{\tau}{ep-1}}.\end{aligned}\tag{81}$$

Let $B_2:=\{i:\alpha_i\in(0,\Lambda)\}$. Let $G:=\{1,2,\dots N_0\}\setminus(B_1\cup B_2)$. So for each $i\in G$ by (80) there exists $A_i\in K$ such that

$$\begin{aligned}\int_{Q_{\mathcal{C}_1\varphi}(0)}|Dv_i(z)-A_i|^p\,dL^2z&\leq\mathcal{C}_2\alpha_i^{\frac{1}{ep}}\\&\leq\epsilon^{-\tau}\mathcal{C}_2\alpha_i.\end{aligned}\tag{82}$$

We assume φ has been chosen big enough so that $\mathrm{diam}(P_i)\leq\frac{\mathcal{C}_1\varphi\epsilon^{\frac{1}{q}}}{4}$ for any $i\in\{1,2,\dots N_0\}$. So if $P_i\cap Q_{\frac{\mathcal{C}_1\varphi}{2}\epsilon^{\frac{1}{q}}}(c_j)\neq\emptyset$ then $P_i\subset Q_{\mathcal{C}_1\varphi\epsilon^{\frac{1}{q}}}(c_j)$.

Let $Z_1:=\bigcup_{i\in G}Q_{\frac{\mathcal{C}_1\varphi}{2}\epsilon^{\frac{1}{q}}}(c_i)$. Let

$$F_1(x):=\sum_{i\in G}\chi_{Q_{\mathcal{C}_1\varphi\epsilon^{\frac{1}{q}}}(c_i)}(x)\,|Du(x)-A_i|^p.$$

So

$$\begin{aligned}\int F_1(x)\,dL^2x&:=c\sum_{i\in G}\int_{Q_{\mathcal{C}_1\varphi\epsilon^{\frac{1}{q}}}(c_i)}|Du(x)-A_i|^p\,dL^2x\\&=c\sum_{i\in G}\epsilon^{\frac{2}{q}}\int_{Q_{\mathcal{C}_1\varphi}(0)}|Dv_i(x)-A_i|^p\,dL^2x\\&\overset{(82)}{\leq}c\epsilon^{\frac{2}{q}}\sum_{i\in G}\epsilon^{-\tau}\alpha_i\\&\overset{(78)}{\leq}c\epsilon^{\frac{\alpha}{3q}-\tau}.\end{aligned}$$

Let $Z_2 := \bigcup_{i\in B_2} Q_{\frac{\mathcal{C}_1\varphi}{2}\epsilon^{\frac{1}{q}}}(c_i)$. Recall for each $i \in B_2$ there exists $A_i \in K$ such that inequality (81) holds true.

Let

$$Y_t^1 := \left\{k\epsilon^{\frac{1}{q}}w_2 + \langle w_1\rangle : k\in\mathbb{Z}\right\} + tw_2, \quad Y_t^2 := \left\{k\epsilon^{\frac{1}{q}}w_1 + \langle w_2\rangle : k\in\mathbb{Z}\right\} + tw_1.$$

Let t_0 be the smallest positive number such that $Y_0^1 = Y_{t_0}^1$ and let t_1 be the smallest positive number such that $Y_0^2 = Y_{t_1}^2$. Let $L_1 : Q_1(0) \to [0,t_0]$ be such that $L_1^{-1}(s) = Y_s^1 \cap Q_1(0)$ and let $L_2 : Q_1(0) \to [0,t_1]$ be such that $L_2^{-1}(s) = Y_s^2 \cap Q_1(0)$. It is easy to see that $|DL_1| \leq c$ and $|DL_2| \leq c$ so by the co-area formula we must be able to find σ_1, σ_2 such that

$$\int_{L_1^{-1}(\sigma_1)} F_1(z)\, dH^1 z \leq c\epsilon^{-\frac{1}{q}}\epsilon^{\frac{\alpha}{3q}-\tau} \tag{83}$$

and

$$\int_{L_1^{-1}(\sigma_2)} F_1(z)\, dH^1 z \leq c\epsilon^{-\frac{1}{q}}\epsilon^{\frac{\alpha}{3q}-\tau}. \tag{84}$$

Let $\{P_i : i = 1,2,\dots N_1\}$ be an ordering of the set of (complete) parallelograms formed by $Q_1(0)\setminus\left(Y_{\sigma_1}^1 \cup Y_{\sigma_1}^2\right)$. Let

$$V_1 = \{i\in\{1,2,\dots N_1\} : P_i \cap Z_1 \neq \emptyset\}, \quad V_2 = \{i\in\{1,2,\dots N_1\} : P_i \cap Z_2 \neq \emptyset\}, \tag{85}$$

note that $V_1 \cap V_2 \neq \emptyset$. Now by (83) and (84) we know

$$\sum_{i\in V_1}\int_{\partial P_i} F_1(z)\, dH^1 z \leq c\epsilon^{-\frac{1}{q}}\epsilon^{\frac{\alpha}{3q}-\tau}. \tag{86}$$

Now each parallelogram P_i is composed of two triangles, denote them τ_i^1, τ_i^2. See figure 4.

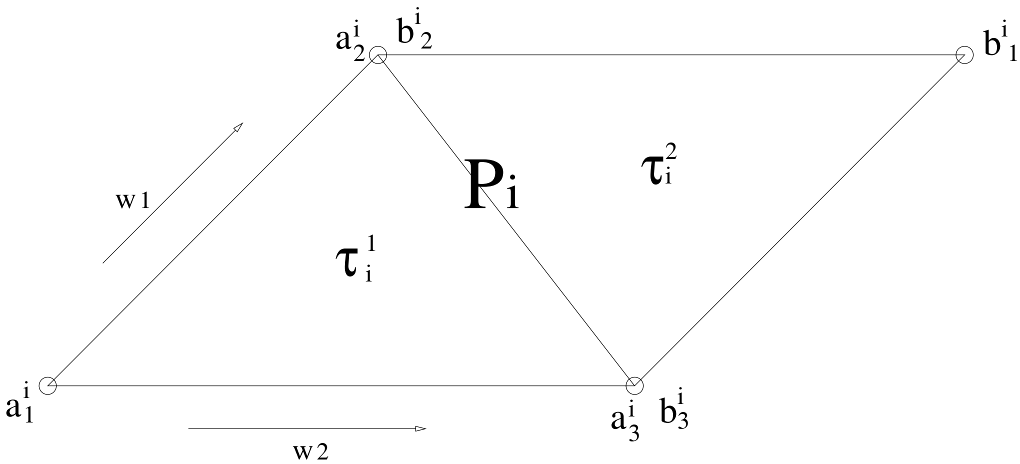

FIGURE 4

Let $\{a_1^i, a_2^i, a_3^i\}$ denote the corners of the τ_i^1 where $[a_1^i, a_2^i] \subset \partial P_i$ and $[a_1^i, a_3^i] \subset \partial P_i$ and let $\{b_1^i, b_2^i, b_3^i\}$ denote the corners of τ_i^2 where $[b_1^i, b_2^i] \subset \partial P_i$ and $[b_1^i, b_3^i] \subset \partial P_i$. Now if $i \in V_1$ then $P_i \subset Q_{\mathcal{C}_1\varphi\epsilon^{\frac{1}{q}}}(c_{p(i)})$ for some $p(i) \notin B_1 \cup B_2$ and $F_1(x) \geq \left|Du(x) - A_{p(i)}\right|^p$ for all $x \in P_i$. See figure 5.

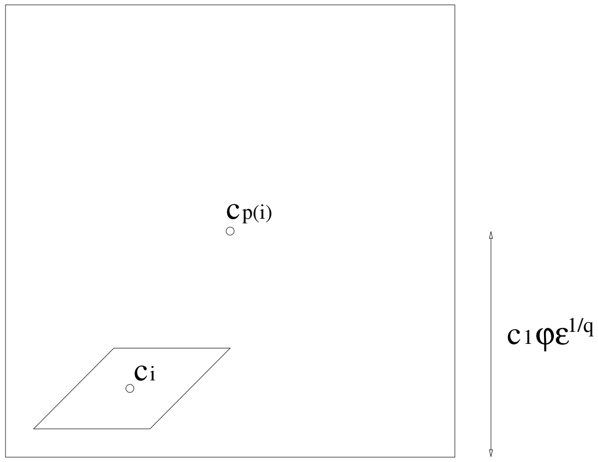

FIGURE 5

Now

$$\begin{aligned}\left|\left(u\left(a_1^i\right)-u\left(a_2^i\right)\right)-A_{p(i)}\left(a_1^i-a_2^i\right)\right| &\leq \int_{\left\{\left[a_1^i,a_2^i\right]:\left|Du(z)-A_{p(i)}\right|\leq\epsilon^{\frac{\alpha}{3q}}\right\}}\left|Du\left(z\right)-A_{p(i)}\right|dL^1z \\ &\quad+\int_{\left\{\left[a_1^i,a_2^i\right]:\left|Du(z)-A_{p(i)}\right|>\epsilon^{\frac{\alpha}{3q}}\right\}}\left|Du\left(z\right)-A_{p(i)}\right|dL^1z \\ &\leq \left|a_1^i-a_2^i\right|\epsilon^{\frac{\alpha}{3q}}+\epsilon^{\frac{(1-p)\alpha}{3q}}\int_{a_1^i}^{a_2^i}\left|Du\left(z\right)-A_{p(i)}\right|^p dL^1z \\ &\leq \left|a_1^i-a_2^i\right|\epsilon^{\frac{\alpha}{3q}}+\epsilon^{\frac{(1-p)\alpha}{3q}}\int_{a_1^i}^{a_2^i}F_1\left(z\right)dH^1z.\end{aligned}$$

And in exactly the same way

$$\left|\left(u\left(a_1^i\right)-u\left(a_3^i\right)\right)-A_{p(i)}\left(a_1^i-a_3^i\right)\right|\leq\left|a_1^i-a_3^i\right|\epsilon^{\frac{\alpha}{3q}}+\epsilon^{\frac{(1-p)\alpha}{3q}}\int_{a_1^i}^{a_3^i}F_1\left(z\right)dH^1z. \tag{87}$$

Which implies

$$\left|D\tilde{u}_{\lfloor\tau_1^i}-A_{p(i)}\right|\leq c\epsilon^{\frac{\alpha}{3q}}+c\epsilon^{-\frac{1}{q}}\epsilon^{\frac{(1-p)\alpha}{3q}}\int_{\partial P_i}F_1\left(z\right)dH^1z,$$

exactly the same inequality holds for τ_2^i.

So

$$\begin{aligned}\sum_{i\in V_1}\sum_{q=1}^{2}\left|D\tilde{u}_{\lfloor\tau_q^i}-A_{p(i)}\right|\epsilon^{\frac{2}{q}} &\leq 2\mathrm{Card}\left(V_1\right)\epsilon^{\frac{2}{q}}\epsilon^{\frac{\alpha}{3q}}+\sum_{i\in V_1}c\epsilon^{\frac{1}{q}}\epsilon^{\frac{(1-p)\alpha}{3q}}\int_{\partial P_i}F_1\left(z\right)dH^1z \\ &\overset{(86)}{\leq} c\epsilon^{\frac{(1-p)\alpha}{3q}}\epsilon^{\frac{\alpha}{3q}-\tau}.\end{aligned} \tag{88}$$

Now for $i\in V_2$ we know $P_i\subset Q_{\mathcal{C}_1\epsilon^{\frac{1}{q}}}\left(c_{p(i)}\right)$ for some $p\left(i\right)\in B_2$ so (see (81))

$$\int_{Q_{\mathcal{C}_1\varphi}(0)}\left|Dv_i\left(z\right)-A_{p(i)}\right|^p dL^2z\leq\mathcal{C}_2\epsilon^{\frac{\tau}{ep-1}}.$$

Now let $\tau = \frac{\alpha(p-1)}{q}$ so $\epsilon^{\frac{\tau}{ep-1}} = \epsilon^{\frac{\alpha(p-1)}{q(ep-1)}}$, so since v_i is Lipschitz

$$\left(\int_{Q_{\mathcal{C}_1\varphi}(0)} \left|Dv_i(z) - A_{p(i)}\right|^3 dL^2 z\right)^{\frac{1}{3}} \leq c\epsilon^{\frac{\alpha(p-1)}{3q(ep-1)}}.$$

Let $l_{A_{p(i)}}$ be the affine function with $l_{A_{p(i)}}(0) = v_i(0)$, and $Dl_{A_{p(i)}} = A_{p(i)}$. Let $g := v_i - l_{A_{p(i)}}$, so $g(0) = 0$ and so by Morrey's inequality ([10] Section 4.5.3, Theorem 3) we have

$$\begin{aligned}\sup_{x \in Q_{\mathcal{C}_1\varphi}(0)} |g(x)| &\leq c\left(\int_{Q_{\mathcal{C}_1\varphi}(0)} |Dg(z)|^3 dL^2 z\right)^{\frac{1}{3}} \\ &\leq c\epsilon^{\frac{\alpha(p-1)}{3q(ep-1)}}.\end{aligned}$$

Now recall $u(x) = v_i\left(\frac{x - c_{p(i)}}{\epsilon^{\frac{1}{q}}}\right)\epsilon^{\frac{1}{q}}$ for $x \in P_1$, so

$$\sup\left\{\left|u\left(\epsilon^{\frac{1}{q}}z + c_{p(i)}\right) - l_{A_{p(i)}}\left(\epsilon^{\frac{1}{q}}z\right)\right| : z \in Q_{\mathcal{C}_1\varphi}(0)\right\} \leq c\epsilon^{\frac{1}{q}}\epsilon^{\frac{\alpha(p-1)}{3q(ep-1)}}. \tag{89}$$

Now take triangle τ_i^1, note that

$$\begin{aligned}\left|D\tilde{u}_{\lfloor\tau_i^1}\left(a_2^i - a_1^i\right) - A_{p(i)}\left(a_2^i - a_1^i\right)\right| &= \left|\left(u\left(a_2^i\right) - u\left(a_1^i\right)\right) - \left(l_{A_{p(i)}}\left(a_2^i\right) - l_{A_{p(i)}}\left(a_1^i\right)\right)\right| \\ &\leq c\epsilon^{\frac{1}{q}}\epsilon^{\frac{\alpha(p-1)}{3q(ep-1)}}.\end{aligned}$$

Thus

$$\left|D\tilde{u}_{\lfloor\tau_i^1}\left(\frac{a_2^i - a_1^i}{\left|a_2^i - a_1^i\right|}\right) - A_{p(i)}\left(\frac{a_2^i - a_1^i}{\left|a_2^i - a_1^i\right|}\right)\right| \leq c\epsilon^{\frac{\alpha(p-1)}{3q(ep-1)}}.$$

In exactly the same way

$$\left|D\tilde{u}_{\lfloor\tau_i^1}\left(\frac{a_3^i - a_1^i}{\left|a_3^i - a_1^i\right|}\right) - A_{p(i)}\left(\frac{a_3^i - a_1^i}{\left|a_3^i - a_1^i\right|}\right)\right| \leq c\epsilon^{\frac{\alpha(p-1)}{3q(ep-1)}}.$$

Thus $\left|D\tilde{u}_{\lfloor\tau_i^1} - A_{p(i)}\right| \leq c\epsilon^{\frac{\alpha(p-1)}{3q(ep-1)}}$. Let $K_\epsilon := N_{c\epsilon^{\frac{\alpha(p-1)}{3q(ep-1)}}}(K)$. So we have shown

$$D\tilde{u}_{\lfloor\tau_i^w} \in K_\epsilon \text{ for every } i \in V_2, w = 1, 2. \tag{90}$$

Now note that $Q_1(0) \setminus (Z_1 \cup Z_2) = Q_1(0) \setminus \left(\bigcup_{i \in G \cup B_2} Q_{\frac{\mathcal{C}_1\varphi}{2}\epsilon^{\frac{1}{q}}}(c_i)\right)$ and note

$$\begin{aligned}L^2\left(Q_1(0) \setminus \left(\bigcup_{i \in G \cup B_2} Q_{\frac{\mathcal{C}_1\varphi}{2}\epsilon^{\frac{1}{q}}}(c_i)\right)\right) &\leq c\mathrm{Card}(B_1)\epsilon^{\frac{2}{q}} \\ &\overset{(79)}{\leq} c\epsilon^{\frac{\alpha}{3q}}.\end{aligned}$$

So as

$$\bigcup_{i \in \{1,2,\dots N_1\} \setminus (V_1 \cup V_2)} P_i \overset{(85)}{\subset} Q_1(0) \setminus \left(\bigcup_{i \in G \cup B_2} Q_{\frac{\mathcal{C}_1\varphi}{2}\epsilon^{\frac{1}{q}}}(c_i)\right)$$

so

$$\sum_{i \in \{1,2,\dots N_1\} \setminus (V_1 \cup V_2)} L^2(P_i) \leq c\epsilon^{\frac{\alpha}{3q}}. \tag{91}$$

Hence

$$\begin{aligned}
\int_{Q_1(0)} d\left(D\tilde{u}\left(z\right),K_\epsilon\right)dL^2z &\leq c\sum_{i=1}^{N_1}\sum_{w=1}^{2} d\left(D\tilde{u}_{\lfloor\tau_i^w}, K_\epsilon\right)\epsilon^{\frac{2}{q}} + c\epsilon^{\frac{1}{q}} \\
&\leq c\sum_{i\in V_1}\sum_{w=1}^{2} d\left(D\tilde{u}_{\lfloor\tau_i^w}, K_\epsilon\right)\epsilon^{\frac{2}{q}} + c\sum_{i\in V_2}\sum_{w=1}^{2} d\left(D\tilde{u}_{\lfloor\tau_i^w}, K_\epsilon\right)\epsilon^{\frac{2}{q}} \\
&\quad + \sum_{i\in\{1,2,\dots N_1\}\setminus(V_1\cup V_2)} cL^2\left(P_i\right) + c\epsilon^{\frac{1}{q}} \\
&\overset{(88),(90),(91)}{\leq} c\epsilon^{\frac{(1-p)\alpha}{3q}-\tau}\epsilon^{\frac{\alpha}{3q}} \\
&= c\epsilon^{\frac{4(1-p)\alpha}{3q}}\epsilon^{\frac{\alpha}{3q}} \\
&\leq c\epsilon^{-\delta}\epsilon^{\frac{\alpha}{3q}}.
\end{aligned}$$

Now in the case where $e=1$, $K_\epsilon = N_{c\epsilon^{\frac{\alpha}{3q}}}\left(K\right)$. So

$$\begin{aligned}
\int_{Q_1(0)} d\left(D\tilde{u}\left(z\right),K\right)dL^2z &\leq \int_{Q_1(0)} d\left(D\tilde{u}\left(z\right),K_\epsilon\right)dL^2z + c\epsilon^{\frac{\alpha}{3q}} \\
&\leq c\epsilon^{-\delta}\epsilon^{\frac{\alpha}{3q}}.
\end{aligned}$$

Now for $e>1$, since $e\leq 4$ and $p\leq 2$, and recall $\frac{p-1}{q} = \frac{\delta}{2\alpha}$ so

$$\begin{aligned}
\epsilon^{\frac{p-1}{3q(ep-1)}} &= \epsilon^{\frac{\delta}{6\alpha(ep-1)}} \\
&\leq \epsilon^{\frac{\delta}{84}} \\
&\leq \epsilon^{\frac{\delta}{84q}}.
\end{aligned}$$

Hence $K_\epsilon \subset N_{\epsilon^{\frac{\delta}{84q}}}\left(K\right)$. So

$$\int_{Q_1(0)} d\left(D\tilde{u}\left(z\right), N_{\epsilon^{\frac{\delta}{84q}}}\left(K\right)\right)dL^2z \leq \mathcal{C}_3\epsilon^{-\delta}\epsilon^{\frac{\alpha}{3q}}. \quad \square$$

Proof of Theorem 2.

To simplify details we will take $\Omega = Q_1(0)$. It will be clear that the proof works for any bounded Lipschitz domain. Suppose

$$\inf_{v\in B_F^h} \mathcal{F}_h\left(v\right) \geq \mathcal{A}h^{\frac{1}{3}}. \tag{92}$$

Let $q\geq 1$. If for some ϵ, there exists $u\in A_F^\zeta$ such that

$$\begin{aligned}
I_\epsilon^q\left(u\right) &\leq \epsilon^{\frac{1}{3q}+\delta} \\
&= \epsilon^{\frac{1+3q\delta}{3q}}
\end{aligned}$$

let $\alpha = 1+3q\delta$. Now for $q>1$ by Theorem 1 we have an $(p,q,4)$ Liouville theorem, so $h=\epsilon^{\frac{1}{q}}$ by Proposition 3 we have

$$\begin{aligned}
\mathcal{F}_h\left(\tilde{u}\right) &\leq \mathcal{C}_3 h^{\frac{1}{3}+q\delta-\frac{\delta}{2}} \\
&\leq \mathcal{C}_3 h^{\frac{1}{3}+\frac{\delta}{2}}
\end{aligned}$$

which contradicts (92) for small enough h (depending on δ, $\mathcal{A}$, q, σ and ζ). So we have established (8).

For the case $q=1$, suppose $\int_{v\in B_F^h}\mathcal{F}_0\left(v\right)\geq \mathcal{A}h^{\frac{1}{3}}$. Since from [6] we have an $(p,1,1)$ Liouville theorem. So let $h=\epsilon$, by Proposition 3 for $h=\epsilon$ we have $\mathcal{F}_0\left(\tilde{u}\right)\leq \mathcal{C}_3h^{\frac{1}{3}+\frac{\delta}{2}}$, contradiction for small enough h. So we have shown (9). $\square$

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