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BOUNDS ON THE SPEED AND ON REGENERATION TIMES FOR CERTAIN PROCESSES ON REGULAR TREES

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Abstract. We develop a technique that provides a lower bound on the speed of transient random walk in a random environment on regular trees. A refinement of this technique yields upper bounds on the first regeneration level and regeneration time. In particular, a lower and upper bound on the covariance in the annealed invariance principle follows. Our methods are general and also apply in the case of once-reinforced random walk. Durrett, Kesten and Limic [11] prove an upper bound of the form $b/(b+\delta)$ for the speed on the b-ary tree, where δ is the reinforcement parameter. For $\delta > 1$ we provide a lower bound of the form $\gamma^2\, b/(b+\delta)$, where γ is the survival probability of an associated branching process.

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1. Introduction

Random processess with long memory have gained considerable attention in the recent past. Two emblematic examples of such processes are random walks in a random environment and reinforced processes. Although considerable progress has been achieved, there are many basic questions that remain open. We refer to the overviews by Sznitman [25] and Zeitouni [27],[28] for random walk in a random environment on $\mathbb{Z}^d$, and by Pemantle [20] for reinforced processes on $\mathbb{Z}^d$ and on trees.

In this article we look at certain transient processes on regular trees, more precisely at random walk in a random environment, and at once-reinforced random walk. An important question is to obtain an explicit expression for the speed (if at all it exists), or at least to get good estimates. This is in general a hard question, even for Markov chains as the biased random walk on a general tree, i.e. a graph without cycles. For this model there is in general no explicit expression for the speed, and often only an upper bound is at hand. It is in general hard to find a lower bound, and we refer to Chen [3] for several examples. We also point out to random walks on general graphs (Virág [26]) where basically no lower bound on the speed is available.

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For random walk in a random environment, the speed is explicitly known only in one-dimensional models. On $\mathbb{Z}^d$, $d \geq 2$, not much is known about the speed, and even worse, if $d \geq 3$, it is still open if a law of large numbers with constant speed holds, see [25],[27],[28]. On regular trees however, a law of large numbers holds, see [13], and transience implies that the speed is positive. This follows from Theorem 1.1 in Aidékon [1] that treats the more general setting of Galton-Watson trees. One of our goals is to find a *lower* bound on the speed for random walks in a random environment on regular trees. Our approach is general and we apply it to another class of processes with long memory: once edge-reinforced random walk. Once edge-reinforced random walk on regular trees is transient, and has positive speed, see Theorem 1 and 2 in Durrett, Kesten and Limic [11]. They propose an upper bound on the speed, but no lower bound that is always positive is at hand. With similar techniques than in the setting of random walk in random environment, we derive a lower bound.

In order to provide a lower bound on the speed, it is instrumental to find a lower bound for the escape probability from the root, as well as an upper bound for the expected number of returns to the root. Both these bounds are obtained with the help of an auxiliary branching process that already appeared in Collevecchio [5]. In particular the escape probability is bounded from below by the survival probability of the branching process, see the Propositions 2.6 and 2.13. For once-reinforced random walk, the branching process can be constructed in such a way that its survival probability is always positive, whereas for random walk in random environment we need additional assumptions.

By a refinement of our methods, we are moreover able to derive a *common* explicit upper bound on all the moments of a first regeneration time τ_1. These bounds are general and hold for random walk in a random environment as well as for once edge-reinforced random walk, see Theorem 3.7. In words, this first regeneration time is the first time the height of the walk reaches a new maximum, and from then on never backtracks below this maximum. Regeneration times enjoy a wide-spread use in different settings, and we refer for instance to Lyons, Pemantle and Peres [17] for biased random walk on a Galton-Watson tree, to Durrett, Kesten and Limic [11] for once-reinforced random walk on a regular tree, and to Sznitman [25] for random walk in a random environment on $\mathbb{Z}^d$.
The main step is to derive an explicit upper exponential tail on the first regeneration level ℓ_1, defined as $\ell_1 = |X_{\tau_1}|$, where $|\cdot|$ denotes the height of a vertex, see Theorem 3.5. We inspire ourselves from Collevecchio [6], where a similar technique was introduced, although in the setting of the vertex-reinforced jump process. Let us mention that a detailed analysis of the tail behaviour of the first regeneration time is presented in Proposition 2.1 and 2.2 in Aidékon [2], revealing an exponential and a subexponential regime on regular trees. We emphasize that we obtain *explicit* upper bounds on all moments of the first regeneration time under certain assumptions, in contrast to [2], where only the finiteness of the moments follows. In particular, these bounds on the first regneration level resp. regeneration time imply a lower and an upper bound on the covariance of the Brownian motion that appears as the limiting object in an annealed invariance principle, see Theorem 3.8 and Proposition 3.9.

This article is organised as follows. In Section 2, we provide a lower bound on the speed for random walk in a random environment and for once edge-reinforced random walk, and in Section 3 we derive moment bounds on the first regeneration time that are completely general and hold for random walk in a random environment and for once edge-reinforced random walk.

2. On the speed

Let us start by introducing some notation. Consider the b-ary regular tree $\mathcal{G}_b$ with root ρ. We assume that the root ρ has a parent $\overleftarrow{\rho}$. Hence each vertex in the tree is connected to $b+1$ vertices, except for $\overleftarrow{\rho}$, that is only connected to ρ. For any vertex ν, denote by $|\nu|$ its distance to the root, i.e. the number of edges on the unique self-avoiding path connecting ν and ρ. Level i is the set of vertices ν such that $|\nu| = i$, with the exception that $|\overleftarrow{\rho}| = -1$. For $\nu \neq \overleftarrow{\rho}$, define $\overleftarrow{\nu}$, called the parent of ν, to be the unique vertex at level $|\nu| - 1$ connected to ν. We say that ν is a child of $\overleftarrow{\nu}$. We say that a vertex ν_0 is a descendant of the vertex ν if the latter lies on the unique self-avoiding path connecting ν_0 to ρ, and $\nu_0 \neq \nu$. In this case, ν is said to be an ancestor of ν_0. For any vertex μ, let Λ_μ be the subtree of $\mathcal{G}_b$ consisting of μ, its descendants and the edges connecting them, i.e. the b-ary subtree rooted at μ. Let $\overleftarrow{\Lambda}_\mu$ be the smallest subtree of $\mathcal{G}_b$ containing Λ_μ and the vertex $\overleftarrow{\mu}$.

2.1 Random Walk in Random Environment Let us define the random environment. To each vertex ν, different from $\overleftarrow{\rho}$, we assign a b-dimensional random vector with positive entries

$$\mathbf{A}_\nu \stackrel{def}{=} (A_\nu^{(1)}, A_\nu^{(2)}, \dots A_\nu^{(b)}).$$

We assume that these vectors are i.i.d. under the measure $\mathbb{P}$. Moreover, following Lyons and Pemantle [16], we assume that the coordinates are identically distributed. The random environment ω is defined by $\omega(\overleftarrow{\rho}, \rho) = 1$ and for any vertex $\nu \neq \overleftarrow{\rho}$,

$$\omega(\nu, \overrightarrow{\nu}^{(i)}) = \frac{A_\nu^{(i)}}{1 + \sum_j A_\nu^{(j)}}; \quad \omega(\nu, \overleftarrow{\nu}) = \frac{1}{1 + \sum_j A_\nu^{(j)}}. \tag{2.1}$$

For a vertex ν we define the Markov chain $\{X_n, n \geq 0\}$ started at ν by

$$\begin{aligned} \mathbf{P}_{\nu,\omega}(X_0 = \nu) &= 1 \\ \mathbf{P}_{\nu,\omega}(X_{n+1} = \mu_1 | X_n = \mu_0) &= \omega(\mu_0, \mu_1), \end{aligned}$$

for any pair of neighbors μ_0, μ_1. We introduce further the annealed measure as the semi-direct product $\mathbf{P}_\nu = \mathbb{P} \times \mathbf{P}_{\nu,\omega}$. We write $\mathbf{P}_\omega$ and $\mathbf{P}$ for $\mathbf{P}_{\rho,\omega}$ resp. $\mathbf{P}_\rho$. We also write A and $\mathbf{A} = (A^{(1)}, \dots, A^{(b)})$ for a generic copy of $A_\nu^{(i)}$, $1 \leq i \leq b$, respectively for a generic copy of $\mathbf{A}_\nu = (A_\nu^{(1)}, \dots, A_\nu^{(b)})$. We introduce the hitting times of a vertex ν respectively of a level i

$$T(\nu) \stackrel{def}{=} \inf\{k \geq 0\colon X_k = \nu\} \qquad \text{and} \qquad T_i \stackrel{def}{=} \inf\{k \geq 0\colon |X_k| = i\}. \tag{2.2}$$

We further introduce the respective return times

$$D \stackrel{def}{=} \inf\{n \geq 1\colon X_n = \overleftarrow{X_0}\}, \qquad D(\nu) \stackrel{def}{=} \inf\{n \geq 1\colon X_{n-1} = \nu,\, X_n = \overleftarrow{\nu}\}, \tag{2.3}$$

and the annealed return probability

$$\beta \stackrel{def}{=} \mathbf{P}(D < \infty). \tag{2.4}$$

To each ordered pair of neighbors $\nu, \mu \in$Vert$(\mathcal{G}_b)$ assign a collection of independent exponentials $h_k(\nu, \mu)$, $k \geq 0$, each with mean one. We assume that all these collections are independent. Using these exponentials, we now provide a construction of random walk in random environment on an arbitrary subtree (see [22] for a similar construction for reinforced processes).

Definition 2.1. (Extension $\mathbf{Y}^{\mathcal{C}}$) Fix a subtree $\mathcal{C}$ of $\mathcal{G}_b$. The extension $\mathbf{Y}^{\mathcal{C}}$ of $\mathbf{X}$ on the subtree $\mathcal{C}$ is defined as follows. Fix a starting point η in $\mathcal{C}$, i.e. $Y_0^{\mathcal{C}} = \eta$. We define $\mathbf{Y}^{\mathcal{C}}$ iteratively in the following way. Let $s_1(\nu)$ be the first time $\mathbf{Y}^{\mathcal{C}}$ reaches some vertex ν. Define $N_\nu^{\mathcal{C}}$ to be the set of neighbors of ν in $\mathcal{C}$. The first jump after $s_1(\nu)$ is towards the neighbor $\mu \in N_\nu^{\mathcal{C}}$ for which the following minimum

$$\min_{\eta \in N_\nu^{\mathcal{C}}} \frac{h_1(\nu, \eta)}{\omega(\nu, \eta)} \tag{2.5}$$

is a.s. attained. We define $s_k(\nu)$, $k \geq 2$, inductively via

$$s_k \stackrel{def}{=} \inf \left\{ n > s_{k-1} \colon Y_n^{\mathcal{C}} = \nu \right\}, \text{ and}$$

$$j_k(\nu, \mu) \stackrel{def}{=} 1 + \text{ number of times } \mathbf{Y}^{\mathcal{C}} \text{ jumped from } \nu \text{ to its neighbor } \mu \text{ by time } s_k.$$

The first jump after s_k is towards the neighbor μ for which the following minimum

$$\min_{\mu \in N_\nu^{\mathcal{C}}} \frac{h_{j_k}(\nu, \mu)}{\omega(\nu, \mu)} \tag{2.6}$$

is a.s. attained. With a slight abuse of notation, we denote the quenched and annealed law of the extension $\mathbf{Y}^{\mathcal{C}}$ again by $\mathbf{P}_{\cdot,\omega}$ resp. $\mathbf{P}_{\cdot}$.

Remark 2.2. The extension processes will play a crucial role in our proofs. They are coupled to the original process $\mathbf{X}$ in the following sense. Let $\mathbf{Y}^{\mathcal{C}}$ be the extension of $\mathbf{X}$ on $\mathcal{C}$, started at a vertex ν in $\mathcal{C}$. Denote with $\theta_\cdot$ the canonical time shift, and suppose that $\mathbf{X}$ hits ν. Since both processes are generated by the same exponential variables, it follows that $\mathbf{Y}^{\mathcal{C}}$ coincides with the process $\mathbf{X} \circ \theta_{T(\nu)}$, of course only observed on the subtree $\mathcal{C}$, which is called *restriction* process. For a rigorous definition of restriction process see [4] or [9]. Extension processes were used in [6] to prove the strong law of large numbers for vertex jump-reinforced processes. $\square$

A child $\nu^{(j)}$ of ν is called a *first child* if it is a.s. the minimiser of

$$\min_{1 \leq i \leq b} \frac{h_1(\nu, \nu^{(i)})}{\omega(\nu, \nu^{(i)})} \quad a.s. \tag{2.7}$$

Let us now turn to the lower bound on the speed. Lyons and Pemantle [16] (see also Menshikov and Petritis [18]) established the following recurrence-transience dichotomy:

$$\mathbf{X} \text{ is transient if } \inf_{0 \leq t \leq 1} \mathbb{E}[A^t] > \tfrac{1}{b}, \text{ and recurrent otherwise.} \tag{2.8}$$

Our standing assumption is that the walk is transient. Gross [13] proves a strong law of large numbers

$$v \stackrel{def}{=} \lim_{n \to \infty} \frac{|X_n|}{n} \geq 0 \quad \mathbf{P} - \text{a.s.} \tag{2.9}$$

The natural question to ask now is in which cases v is positive. This question was answered recently in Aidékon [1] in the more general setting of Galton-Watson trees. In our setting, on regular trees, it turns out that v is always positive, see Theorem 1.1 in [1]. We will now derive a lower bound on the speed v. For $n \geq 1$, we define

$$L(\nu, n) \stackrel{def}{=} \sum_{j=1}^{n} \mathbb{1}_{\{X_j = \nu\}}, \quad \text{and} \quad L(\nu) \stackrel{def}{=} \sum_{j=1}^{\infty} \mathbb{1}_{\{X_j = \nu\}}, \tag{2.10}$$

the number of visits to ν by time n, resp. the total number of visits. Under transience, it is well-known that $v = \lim_{n \to \infty} |X_n|/n$ exists. Here is the main result of this subsection.

Proposition 2.3. *Under transience, it holds that*

$$v \geq \frac{1-\beta}{\mathbf{E}[L(\rho)]} > 0 \quad \mathbf{P} - a.s. \tag{2.11}$$

Before proving Proposition 2.3, we provide first a lemma. Let

$$\Pi_k = \sum_{\nu\colon |\nu|=k} \mathbb{1}_{\{T(\nu)<\infty\}} \tag{2.12}$$

be the number of vertices visited at level k. Recall β in (2.4). We have

Lemma 2.4. *Assume transience, i.e. $\beta < 1$. Then Π_k is stochastically dominated by a geometric random variable with parameter $1-\beta$.*

Proof. One vertex at level k is visited for sure. Call this vertex σ_1. Notice that, after $T(\sigma_1)$, a necessary condition to visit a further vertex at level k is that the walk returns to the parent of σ_1. To obtain an upper bound for Π_k, we adopt the following strategy. If the walk returns to the parent of σ_1, we consider the extension $Y^{(\sigma_1)}$ of $\mathbf{X}$ to the subtree obtained by cutting the subtree Λ_{σ_1}. This ensures that the second visit at level k will be at a new vertex σ_2, different from σ_1. We repeat this procedure iteratively, and it clearly yields an upper bound on the number of vertices σ_i visited at level k. Each time a new vertex σ_i is visited, there is a chance of escape to infinity with annealed probability $1-\beta > 0$, because of stationarity. Since all subtrees Λ_{σ_i} are disjoint, the trials of escape are independent. It follows that Π_k is dominated by a geometric with parameter $1-\beta$. This ends the proof. $\square$

Proof of Proposition 2.3. Notice that

$$\lim_{n\to\infty} \frac{T_n}{n} = 1/v \qquad \mathbf{P} \text{ a.s.} \tag{2.13}$$

Label the vertices at level k by $\nu_{k,1}, \nu_{k,2}, \ldots, \nu_{k,b^k}$. We have that for $n \geq 1$,

$$\mathbf{E}[T_n] \leq 1 + \mathbf{E}[L(\overleftarrow{\rho})] + \sum_{k=0}^{n-1} \sum_{j=1}^{b^k} \mathbf{E}\big[L(\nu_{k,j}) \mathbb{1}_{\{T(\nu_{k,j})<\infty\}}\big]. \tag{2.14}$$

Fix a vertex ν, and define $\widetilde{L}(\nu)$ to be the total time spent in the vertex ν by the extension of $\mathbf{X}$ to $\overleftarrow{\Lambda}_\nu$ started at ν. Then $L(\nu) \leq \widetilde{L}(\nu)$, and the law of $\widetilde{L}(\nu)$ under $\mathbf{P}_\nu$ is equal to the law of $L(\nu)$ under $\mathbf{P}$. Moreover the random variables $\widetilde{L}(\nu)$ and $\mathbb{1}_{\{T(\nu)<\infty\}}$ are independent under the annealed measure. We use independence, and then stationarity, and obtain that the sum on the right-hand side of (2.18) is smaller than

$$\sum_{k=0}^{n-1} \sum_{j=1}^{b^k} \mathbf{E}[\widetilde{L}(\nu_{k,j})]\, \mathbf{P}\big[T(\nu_{k,j}) < \infty\big] = \mathbf{E}[L(\rho)] \sum_{k=0}^{n-1} \mathbf{E}[\Pi_k] \leq \mathbf{E}[L(\rho)] \frac{n}{1-\beta}, \tag{2.15}$$

where in the last step we used Lemma 2.4. Using (2.14) and (2.15), and by Fatou's lemma, we obtain that $\mathbf{P}$-a.s.,

$$\lim_{n\to\infty} T_n/n \leq \liminf_{n\to\infty} \mathbf{E}[T_n/n] = \mathbf{E}[L(\rho)](1-\beta)^{-1}. \tag{2.16}$$

The claim of the theorem follows now from (2.13). $\square$

Our main task is now to derive upper bounds on β and on the expectation of $L(\rho)$.

2.1.1 Estimates on the return probability β. In the last section, we provided a lower bound in terms of the annealed return probability β. In this section, we will derive an upper bound on β in terms of the extinction probability α of a certain branching process, in the spirit of Collevecchio [5]. This allows to obtain an explicit lower bound on the speed.
Let us start by constructing the branching process.

Definition 2.5. *(Color scheme)* Fix an integer $\psi \geq 1$, and denote with $\mathbf{Y}(\nu, \mu)$ the extension of $\mathbf{X}$ to the unique ray connecting the vertices μ and ν. We introduce the following color scheme. A vertex ν at level ψ is colored if and only if the $\mathbf{Y}(\overleftarrow{\rho}, \nu)$, started at ρ, hits ν before $\overleftarrow{\rho}$. A vertex ν at level $k\psi$, $k \geq 2$, is colored if and only if

- its ancestor at level $(k-1)\psi$, say μ, is colored, and
- $Y(\overleftarrow{\mu}, \nu)$, started at μ, hits ν before $\overleftarrow{\mu}$.

All the other vertices are uncolored, and only vertices that are at a level $k\psi$, $k \geq 1$, can be colored.

Under the *annealed* measure, the number of colored vertices form a homogeneous branching process, since the offspring is each time determined by disjoint parts of the environment. We denote this branching process with $\mathbf{Z}_\psi$. We formulate the following

Proposition 2.6. *Denote with α_ψ the extinction probability of Z_ψ. Then $\beta \leq \alpha_\psi$. If moreover $\mathbb{E}[A^{-1}] < b$, then there is an integer $\psi \geq 1$ such that $\alpha_\psi < 1$.*

Proof. Let us show that $\beta \leq \alpha_\psi$ in the case $\alpha_\psi < 1$ (otherwise there is nothing to prove). Assume that $\mathbf{Z}_\psi$ survives. Choose vertices μ and ν as in definition 2.5. By remark 2.2, the processes $Y(\overleftarrow{\mu}, \nu)$ and $\mathbf{X}$ coincide, from the time the latter hits μ until its last visit to the path connecting $\overleftarrow{\mu}$ to ν. It follows that, if $\mathbf{X}$ hits ν before μ, then so does $Y(\overleftarrow{\mu}, \nu)$. It follows that all vertices $X_{T_{k\psi}}$, $k \geq 1$, are colored. In particular, if the branching process survives, then each level $k\psi$, $k \geq 1$, is hit before returning to the parent of the root. Hence $\{Z_\psi \text{ survives}\} \subseteq \{D = \infty\}$, and $\beta \leq \alpha_\psi$ follows. Let us now show that if $\mathbb{E}[A^{-1}] < b$, then we can find $\psi \geq 1$ such that $\mathbf{Z}_\psi$ is supercritical. We choose a vertex μ, and then a vertex ν at level $|\mu| + \psi$. Then the extension $\mathbf{Y}(\overleftarrow{\mu}, \nu)$, started at μ, hits ν before $\overleftarrow{\mu}$ with (annealed) probability

$$\mathbb{E}\big[\big(\sum_{r=1}^{\psi+1} \prod_{j=1}^{r-1} A_j^{-1}\big)^{-1}\big], \tag{2.17}$$

where $A_j, 1 \leq j \leq \psi$, is an enumeration of the variables A along the ray connecting μ to ν. By Jensen's inequality,

$$\mathbb{E}\big[\big(\sum_{r=1}^{\psi+1} \prod_{j=1}^{r-1} A_j^{-1}\big)^{-1}\big] \geq \mathbb{E}\big[\sum_{r=1}^{\psi+1} \prod_{j=1}^{r-1} A_j^{-1}\big]^{-1}. \tag{2.18}$$

By independence, we find for large ψ,

$$\mathbb{E}\big[\sum_{r=1}^{\psi+1} \prod_{j=1}^{r-1} A_j^{-1}\big] = \sum_{r=1}^{\psi+1} \mathbb{E}[A^{-1}]^{r-1} = \frac{1 - \mathbb{E}[A^{-1}]^{\psi+1}}{1 - \mathbb{E}[A^{-1}]}. \tag{2.19}$$

By the assumption $\mathbb{E}[A^{-1}] < b$, we find that

$$\lim_{\psi\to\infty} b^{-\psi}\,\mathbb{E}\Big[\sum_{r=1}^{\psi+1}\prod_{j=1}^{r-1} A_j^{-1}\Big] = 0\,. \tag{2.20}$$

Hence, if we choose ψ large enough, then we can make sure that

$$b^{\psi}\,\mathbb{E}\Big[\Big(\sum_{r=1}^{\psi+1}\prod_{j=1}^{r-1} A_j^{-1}\Big)^{-1}\Big] > 1\,. \tag{2.21}$$

Notice that the left-hand side of the last display is the expected offspring of the branching process Z_ψ, so that we can choose ψ s.t. $\mathbf{Z}_\psi$ is supercritical. This finishes the proof of the proposition. □

Definition 2.7. We denote with $\mathbf{p} := \{p_k,\, k \in \{0, 1, \dots, b^\psi\}\}$ the offspring distribution of the branching process $\mathbf{Z}_\psi$. The mean offspring is

$$m_\psi \stackrel{def}{=} \sum_{k=0}^{b^\psi} k p_k. \tag{2.22}$$

Proposition 2.6 implies that if $\mathbb{E}[A^{-1}] < b$, then there is $\psi \geq 1$ such that $m_\psi > 1$.

2.1.2 An explicit upper bound on the expectation of $L(\rho)$. Our standing assumption in the remaining subsections is that

$$\text{we can find } \psi \geq 1 \text{ such that } \alpha_\psi < 1, \tag{2.23}$$

where we recall α_ψ in Proposition 2.6. Condition (2.23) is in particular satisfied if $\mathbb{E}[A^{-1}] < b$. For $p \geq 1, n \geq 1$, we introduce the function

$$\theta(p,n) \stackrel{def}{=} \begin{cases} c_p\, b\, \mathbb{E}[(1+\frac{1}{\sum_{i=1}^b A^{(i)}})^p]\, n^{p-1}\, \frac{\mathbb{E}[A^{-p}]^n - 1}{\mathbb{E}[A^{-p}] - 1}, & \text{if } n \geq 2, \\ c_p\, \mathbb{E}[(1+\frac{1}{\sum_{i=1}^b A^{(i)}})^p], & \text{if } n = 1, \end{cases} \tag{2.24}$$

where the r.h.s. is infinite if $\mathbb{E}[A^{-p}] = \infty$, and the constants c_p are introduced in Lemma 4.1 in the Appendix. We have the following

Proposition 2.8. *If $\mathbb{E}[A^{-p-\varepsilon}] < \infty$ for some $p \geq 1$ and some $\varepsilon > 0$, then for all $n \geq 1$, $\theta(p+\varepsilon, n) < \infty$, and*

$$\mathbf{E}[L(\rho)^p] \leq \theta(p+\varepsilon, 1)^{1/q} + \sum_{n=2}^{\infty} \theta(p+\varepsilon, n)^{1/q} \alpha_\psi^{b^{n-2}/q'} \Big(\sum_{i=1}^{b} (-1)^{i-1}\binom{b}{i}\alpha_\psi^{b^{n-2}(i-1)}\Big)^{1/q'}, \tag{2.25}$$

where $q = 1+\varepsilon/p$, and $q' = 1+p/\varepsilon$ is the dual of q.

Before proving Proposition 2.8, we formulate an auxiliary result. We first introduce some notation. Fix $n \geq 2$. Choose b distinct vertices ν_i, $1 \leq i \leq b$, at level n, with different ancestors at level one. More precisely, we choose ν_i with ancestor $\overrightarrow{\rho_i}$ at level one, and call this set of vertices $\mathcal{A}_n$. We label the vertices on the ray connecting $\overrightarrow{\rho_i}$ to ν_i by $\sigma_j^{(i)}$, $1 \leq j \leq n$, with $\sigma_1^{(i)} = \overrightarrow{\rho}_i$ and $\sigma_n^{(i)} = \nu_i$. Denote with Γ_n the subtree composed by the root ρ, its parent $\overleftarrow{\rho}$, the vertices $\sigma_j^{(i)}, 1 \leq j \leq n, 1 \leq i \leq b$, and the edges connecting them. For $n = 1$, Γ_1 is simply the subtree composed by the root and its children, with the edges connecting them, and $\mathcal{A}_1$ is the set of children of the root. We denote with $\mathbf{Y}$ the extension of $\mathbf{X}$ to Γ_n, and we introduce $\widetilde{T}_{\mathcal{A}_n} = \inf\{n \geq 0\colon Y_n \in \mathcal{A}_n\}$, and $\widetilde{T}(\rho) \stackrel{def}{=} \inf\{n \geq 1\colon Y_n = \rho\}$. We further define

$$\widetilde{L}(\rho, \widetilde{T}_{\mathcal{A}_n}) \stackrel{def}{=} \sum_{i=0}^{\infty} \mathbb{1}_{\{Y_i=\rho,\, i<\widetilde{T}_{\mathcal{A}_n}\}}\,.$$

Recall $\theta(p,n)$ in (2.24). We have the following

Proposition 2.9. *If* $\mathbb{E}[A^{-p}] < \infty$ *for some* $p \geq 1$, *then*

$$\mathbf{E}\big[\widetilde{L}(\rho, \widetilde{T}_{\mathcal{A}_n})^p\big] \leq \theta(p,n) < \infty\,. \tag{2.26}$$

Proof of Proposition 2.9. Fix $n \geq 2$. To escape from the root, the walk $\mathbf{Y}$ has to jump to one of the children of the root, and then hit the set $\mathcal{A}_n$ before returning to the root. Hence

$$q_\omega \stackrel{def}{=} \mathbf{P}_\omega(\widetilde{T}_{\mathcal{A}_n} < \widetilde{T}(\rho)) = \sum_{i=1}^{b} \omega(\rho, \rho^{(i)})\, p_{i,\omega}, \quad \text{where} \quad p_{i,\omega} = \Big(\sum_{j=1}^{n}\prod_{k=1}^{r-1}\frac{\omega(\sigma_k^{(i)}, \sigma_{k-1}^{(i)})}{\omega(\sigma_k^{(i)}, \sigma_{k+1}^{(i)})}\Big)^{-1}. \tag{2.27}$$

It follows that under the quenched measure, $\widetilde{L}(\rho, \widetilde{T}_{\mathcal{A}_n})$ is a geometric variable with parameter q_ω. Hence, with the help of Lemma 4.1 in the Appendix, we find that

$$\mathbf{E}[\widetilde{L}(\rho, \widetilde{T}_{\mathcal{A}_n})^p] \leq c_p \mathbb{E}[q_\omega^{-p}]\,. \tag{2.28}$$

It follows from (2.27), and by independence, that

$$\mathbb{E}[q_\omega^{-p}] \leq \mathbb{E}[(\min_i p_{i,\omega})^{-p}\,(1-\omega(\rho, \overleftarrow{\rho}))^{-p}] = \mathbb{E}[(\min_i p_{i,\omega})^{-p}]\,\mathbb{E}[(1-\omega(\rho, \overleftarrow{\rho}))^{-p}]. \tag{2.29}$$

We use that

$$\mathbb{E}[(\min_i p_{i,\omega})^{-p}] = \mathbb{E}[\max_i p_{i,\omega}^{-p}] \leq \mathbb{E}[\Sigma_i p_{i,\omega}^{-p}] = b\,\mathbb{E}[p_{1,\omega}^{-p}], \tag{2.30}$$

and we find by (2.27), by Jensen's inequality and by independence that

$$\mathbb{E}[p_{1,\omega}^{-p}] \leq n^{p-1}\sum_{j=1}^{n}\mathbb{E}[A^{-p}]^j = n^{p-1}\frac{\mathbb{E}[A^{-p}]^n - 1}{\mathbb{E}[A^{-p}] - 1}. \tag{2.31}$$

Now observe that

$$\mathbb{E}[(1-\omega(\rho, \overleftarrow{\rho}))^{-p}] = \mathbb{E}[(1+\tfrac{1}{\sum_i A_i})^p], \tag{2.32}$$

and by collecting the results from (2.28) to (2.32), the claim of the Proposition follows for $n \geq 2$. For $n = 1$, a similar (and simpler) argument shows the claim. This finishes the proof of the Proposition. □

Proof of Proposition 2.8. In the course of this proof, we denote with $\mathbf{Y}^{(\nu)}$ the extension of $\mathbf{X}$ to $\overleftarrow{\Lambda}_\nu$, and let

$$D^{(\nu)} \stackrel{def}{=} \inf\{n \geq 1\colon Y_n^{(\nu)} = \overleftarrow{\nu}\}, \quad \text{and} \quad C(\nu) = \{D^{(\nu)} = \infty\}. \tag{2.33}$$

Suppose that $|\nu| \geq 1$ and $C(\nu)$ holds. Then if the process visits ν it will never return to $\overleftarrow{\nu}$, and in particular it will not increase the local time spent at the root ρ. Define

$$\begin{aligned} d = \inf\{k \geq 1\colon &\text{there are } b \text{ distinct vertices } \nu_1, \dots, \nu_b \text{ at level } k \text{ with different} \\ &\text{ancestors at level 1 s.t. } C(\nu_i) \text{ holds for all } 1 \leq i \leq b\}. \end{aligned} \tag{2.34}$$

On $\{d = n\}$, we choose b distinct vertices $\nu_1, \ldots, \nu_b$ at level n with different ancestors at level 1 s.t. $C(\nu_i)$ holds for all $1 \le i \le b$, and in the notation used in Propostion 2.9, we denote this set of vertices with $\mathcal{A}_n$. Notice that

$$L(\rho)\, \mathbb{1}_{\{d=n\}} \le \widetilde{L}(\rho, \widetilde{T}_{\mathcal{A}_n})\, \mathbb{1}_{\{d=n\}}\,. \tag{2.35}$$

With the help of (2.35), we infer that for q, q' as in the proposition,

$$\mathbf{E}[L(\rho)^p] \le \sum_{n=1}^{\infty} \mathbf{E}[\widetilde{L}(\rho, \widetilde{T}_{\mathcal{A}_n})^p, d = n] \le \sum_{n=1}^{\infty} \mathbf{E}[\widetilde{L}(\rho, \widetilde{T}_{\mathcal{A}_n})^{pq}]^{1/q}\, \mathbf{P}[d = n]^{1/q'}, \tag{2.36}$$

where in the last inequality we used Cauchy-Schwarz's inequality. Let us now estimate $\mathbf{P}(d = n)$. The events $C(\nu)_{|\nu|=n}$ are determined by disjoint parts of the environment, and are thus independent and identically distributed under the annealed measure. Fix $n \ge 2$. At level $n-1$, there are b families of b^{n-2} vertices each that have different ancestors at level one. If $\{d = n\}$ holds, then the event $C(\cdot)^c$ holds for all b^{n-2} vertices in at least one of these families of vertices at level $n-1$. With $\mathbf{P}(C(\cdot)) = 1 - \beta$, it follows

$$\mathbf{P}(d = n) \le 1 - (1 - \mathbf{P}(C^c)^{b^{n-2}})^b = 1 - (1 - \beta^{b^{n-2}})^b\,,$$

and with Proposition 2.6, it follows that

$$\mathbf{P}(d = n) \le 1 - (1 - \alpha_\psi^{b^{n-2}})^b = \alpha_\psi^{b^{n-2}} \sum_{i=1}^{b} (-1)^{i-1} \binom{b}{i} \alpha_\psi^{b^{n-2}(i-1)}\,. \tag{2.37}$$

Together with the trivial bound $\mathbf{P}(d = 1) \le 1$, this finishes the proof of the proposition. $\square$

2.1.3 An explicit lower bound on the speed and an example. Recall α_ψ in Proposition 2.6. The propositions 2.3, 2.6 and 2.8 (applied with $p = \varepsilon = 1$) imply the following

Theorem 2.10. *Assume (2.23), and that $\mathbb{E}[A^{-2}] < \infty$. Then it holds $\mathbf{P}$-a.s. that*

$$v \ge \frac{1-\alpha_\psi}{\mathbf{E}[L(\rho)]} \ge \frac{1-\alpha_\psi}{\theta(2,1)^{1/2} + \sum_{n=2}^{\infty} \theta(2,n)^{1/2}\, \alpha_\psi^{b^{n-2}/2} \left(\sum_{i=1}^{b} (-1)^{i-1} \binom{b}{i} \alpha_\psi^{b^{n-2}(i-1)}\right)^{1/2}} > 0\,.$$

An example. Let us now provide an explicit example on the regular binary tree (i.e. $b = 2$). We choose $A_1 = A_2$, and we write A for a copy of A_1 resp. A_2. We choose the following distribution

$$\mathbb{P}[A = 3/10] = \kappa, \quad \mathbb{P}[A = 7/2] = 1 - \kappa, \quad \kappa \in (0, 1/2].$$

We compute m_1, which is given by the left-hand side of (2.21) with ψ replaced by one, and find that for all $\kappa \in (0, 1/2]$,

$$m_1 = 2\,\mathbb{E}[\tfrac{A}{1+A}] = (182 - 128\kappa)/117 > 1, \tag{2.38}$$

so that $\alpha_1 < 1$. The extinction probability α_1 is given by the smallest solution of $x = p_0 + p_1 x + p_2 x^2$, hence $\alpha_1 = p_0/p_2$. Let us compute now the offspring distribution $\mathbf{p}$. We denote the site environment corresponding to the events $\{A = 3/10\}$ and $\{A = 7/2\}$ by ω_1 resp. ω_2. It follows that

$$\begin{aligned} \omega_1(\rho, \overleftarrow{\rho}) &= 5/8, \qquad \omega_1(\rho, \rho^{(1)}) = \omega_1(\rho, \rho^{(2)}) = 3/16, \\ \omega_2(\rho, \overleftarrow{\rho}) &= 1/8, \qquad \omega_2(\rho, \rho^{(1)}) = \omega_2(\rho, \rho^{(2)}) = 7/16. \end{aligned} \tag{2.39}$$

We obtain that

$$\begin{aligned} p_0 &= \mathbb{E}[\omega(\rho,\overleftarrow{\rho})] = 1/8+\kappa/2,\\ p_1 &= \mathbb{E}[\omega(\rho,\rho^{(1)})\tfrac{\omega(\rho,\overleftarrow{\rho})}{\omega(\rho,\overleftarrow{\rho})+\omega(\rho,\rho^{(2)})}] + \mathbb{E}[\omega(\rho,\rho^{(2)})\tfrac{\omega(\rho,\overleftarrow{\rho})}{\omega(\rho,\overleftarrow{\rho})+\omega(\rho,\rho^{(1)})}] = 7/36+11\kappa/117, \qquad (2.40)\\ p_2 &= 1-p_0-p_1 = 49/72-139\kappa/234. \end{aligned}$$

Hence $\alpha_1 = (117+468\kappa)/(637-556\kappa)$. Further we find that $\mathbb{E}[A^{-2}]-1 = 4864\kappa/441-45/49$, and that $\mathbb{E}[(1+\frac{1}{2A})^2] = 64/49+2560\kappa/441$. Let us choose now $\kappa = 1/30$. We obtain from Theorem 2.10 and the above computations that

$$v \geq 0.1229 \quad \text{if} \quad \kappa = 1/30\,.$$

2.2 Once edge-reinforced random walk

Durrett, Kesten and Limic [11] prove transience and provide a law of large numbers with positive speed for once edge-reinforced random walk on a regular tree. However their methods do not give a lower bound for the speed that is always positive. Collevecchio [5] proves transience for this process defined on supercritical Galton–Watson trees. The same was proved, independently and with different methods by Dai [7]. In this section, we provide a lower bound on the speed by using a refinement of the methods from [5].

Let us first define the process. Fix $\delta > 0$, and denote with $\{\nu,\mu\}$ the edge connecting the neighboring vertices ν and μ. Once δ-edge-reinforced random walk (ORRW(δ) or simply ORRW) $\mathbf{X} = \{X_k, k \geq 0\}$ is a discrete-time process on the regular b-ary tree $\mathcal{G}_b$, and is defined as follows. Each edge has initial weight one, i.e. $W(\{\nu,\mu\},0) = 1$, with the exception of the edge $\{\overleftarrow{\rho},\rho\}$, which has weight δ, i.e. $W(\{\overleftarrow{\rho},\rho\},0) = \delta$. This exception helps to simplify our exposition. This initial weight configuration is called *initially fair*. For $n \geq 1$, we update the weight W of the edges according to the following rule:

$$W(\{\nu,\mu\},n) = \begin{cases} \delta, & \text{if } \{X_{k-1},X_k\} = \{\nu,\mu\} \text{ for some } 1 \leq k \leq n,\\ 1, & \text{otherwise.} \end{cases} \qquad (2.41)$$

ORRW starts from ρ, i.e. $X_0 = \rho$, and we define inductively $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, and the transition probabilities

$$\mathbf{P}(X_{n+1} = \mu \mid \mathcal{F}_n) = \frac{W(\{X_n,\mu\},n)}{\sum_{\nu:\nu\sim X_n} W(\{X_n,\nu\},n)}, \qquad (2.42)$$

if μ is a neighbor of X_n, and zero otherwise. The canonical law of this process is denoted with $\mathbf{P}$. Later on, we will also use the following initial weights, where not only the edge $\{\overleftarrow{\rho},\rho\}$ has weight δ, but a connected collection of edges containing the edge $\{\overleftarrow{\rho},\rho\}$, i.e. if some edge has weight δ, then each edge on the path connecting this edge to the root has weight δ. We denote with $\mathbb{W}$ the set of such initial weight configurations. Of course, $\mathbb{W}$ contains the initially fair weights, that we denote from now on with w_0. For $w \in \mathbb{W}$ let $w(\{\nu,\mu\})$ be the weight that w assigns to the edge $\{\nu,\mu\}$. For any weight configuration $w \in \mathbb{W}$, define $W_w(\{\nu,\mu\},0) = w(\{\nu,\mu\})$, and for $n \geq 1$,

$$W_w(\{\nu,\mu\},n) = \begin{cases} \delta, & \text{if } \{X_{k-1},X_k\} = \{\nu,\mu\} \text{ for some } 1 \leq k \leq n,\\ w(\{\nu,\mu\}), & \text{otherwise.} \end{cases}$$

The transition probabilities are defined similarly as in (2.42), with $W(\cdot,n)$ replaced by $W_w(\cdot,n)$. The canonical law of ORRW started at ρ and in the initial weight configuration $w \in \mathbb{W}$ is denoted with $\mathbf{P}_w$ (clearly $\mathbf{P} = \mathbf{P}_{w_0}$). Recall the exponential random variables $h_k(\cdot,\cdot)$, $k \geq 1$, with mean one, used in definition 2.1 and fix a subtree $\mathcal{C}$ of $\mathcal{G}_b$.

Definition 2.11. [Extension $\mathbf{Y}^{\mathcal{C}}$ on the subtree $\mathcal{C}$] The extension $\mathbf{Y}^{\mathcal{C}}$ of $\mathbf{X}$ on the subtree $\mathcal{C}$ is defined as follows. Fix a starting point η in $\mathcal{C}$, i.e. $Y_0^{\mathcal{C}} = \eta$ and an initial weight configuration $w \in \mathbb{W}$. We define $\mathbf{Y}^{\mathcal{C}}$ iteratively in the following way. Let $s_1(\nu)$ be the first time $\mathbf{Y}^{\mathcal{C}}$ reaches some vertex ν. Define $N_\nu^{\mathcal{C}}$ to be the set of neighbors of ν in $\mathcal{C}$. The first jump after $s_1(\nu)$ is towards the neighbor $\mu \in N_\nu^{\mathcal{C}}$ for which the following minimum

$$\min_{\mu \in N_\nu^{\mathcal{C}}} \frac{h_1(\nu,\mu)}{W_w(\{\nu,\mu\}, s_1(\nu))} \tag{2.43}$$

is a.s. attained. We define $s_k(\nu)$, $k \geq 2$, inductively via

$$s_k(\nu) \stackrel{def}{=} \inf\left\{n > s_{k-1} \colon Y_n^{\mathcal{C}} = \nu\right\}, \text{ and}$$

$$j_k(\nu,\mu) \stackrel{def}{=} 1 + \text{ number of times } \mathbf{Y}^{\mathcal{C}} \text{ jumped from } \nu \text{ to its neighbor } \mu \text{ by time } s_k.$$

The first jump after $s_k(\nu)$ is towards the neighbor μ for which the following minimum

$$\min_{\mu \in N_\nu^{\mathcal{C}}} \frac{h_{j_k}(\nu,\mu)}{W_w(\{\nu,\mu\}, s_1(\nu))} \tag{2.44}$$

is a.s. attained.

The comments in remark 2.2 also apply here. We now introduce a similar color scheme as in definition 2.5.

Definition 2.12. Fix an integer $\psi \geq 1$, and denote with $\mathbf{Y}(\overleftarrow{\mu}, \nu)$, for a descendant ν of μ, the extension of ORRW on the ray connecting $\overleftarrow{\mu}$ to ν, started at μ, in the following initial weight configuration. The edge $\{\overleftarrow{\mu}, \mu\}$ has weight δ and all the other edges in the path connecting μ to ν have initial weight 1. A vertex ν at level ψ is colored if and only if $\mathbf{Y}(\overleftarrow{\rho}, \nu)$ hits ν before $\overleftarrow{\rho}$. A vertex ν at level $k\psi$, $k \geq 2$, is colored if and only if

- its ancestor at level $(k-1)\psi$, say μ, is colored, and
- $Y(\overleftarrow{\mu}, \nu)$ hits ν before $\overleftarrow{\mu}$.

All the other vertices are uncolored, and only vertices that are at a level $k\psi$, $k \geq 1$, can be colored.

This color scheme constitutes again a homogeneous branching process, with extinction probability α_ψ. Notice that for every $b \geq 2$, and every $\delta > 0$, we can always find an integer $\psi \geq 1$ such that

$$b^\psi \prod_{j=1}^{\psi} \frac{j}{j+\delta} > 1. \tag{2.45}$$

We define D in the same way as in (2.3), and also $\beta_w = \mathbf{P}_w(D = \infty)$, and we write $\beta = \beta_{w_0}$. Recall $\mathbb{W}$ below (2.42). We have the following

Proposition 2.13. *If ψ is such that* (2.45) *holds, then $\alpha_\psi < 1$. If $\delta > 1$, then for every $w \in \mathbb{W}$, it holds that $\beta_w \leq \alpha_\psi$.*

Proof. The probability that $\mathbf{Y}(\overleftarrow{\rho}, \nu)$, started at ρ, in the initially fair weight configuration w_0, hits level ψ before it hits $\overleftarrow{\rho}$ is equal to (see Lemma 1 in [5])

$$\prod_{j=1}^{\psi} \frac{j}{j+\delta}. \tag{2.46}$$

Hence the mean of the offspring distribution of the colored process is equal to $b^{\psi} \prod_{j=1}^{\psi} \frac{j}{j+\delta}$, which is larger than one by our choice of ψ. This shows that $\alpha_{\psi} < 1$. Now choose an initial weight configuration $w \in \mathbb{W}$. If $\delta > 1$, we can couple the extension $\mathbf{Y}(\overleftarrow{\rho}, \nu)$, started at ρ, in the initially fair weight configuration w_0, to the extension $\widetilde{\mathbf{Y}}(\overleftarrow{\rho}, \nu)$, started at ρ, in the weight configuration $w \in \mathbb{W}$, in such a way that $|\widetilde{\mathbf{Y}}| \geq |\mathbf{Y}|$. To do this, we choose a family of independent variables $(E_n^{\uparrow}, E_n^{\downarrow})_{n \geq 1}$, with i.i.d. exponential entries with mean 1. At each time point n, the vector $(E_n^{\uparrow}, E_n^{\downarrow})$ is attached *both* to the positions Y_n and $\widetilde{Y}_n$, with $E_n^{\uparrow}$ attached to the edge connecting Y_n and $\widetilde{Y}_n$ to the vertex ν at level $|Y_n| + 1$ resp. $\widetilde{\nu}$ at level $|\widetilde{Y}_n| + 1$, and $E_n^{\downarrow}$ attached to the edge connecting Y_n and $\widetilde{Y}_n$ to the vertex μ at level $|Y_n| - 1$ resp. $\widetilde{\mu}$ at level $|\widetilde{Y}_n| - 1$. The jump of $\mathbf{Y}$ at time $n+1$ is to the vertex ν or μ for which the minimum

$$\min\{ \frac{E_n^{\uparrow}}{W_{w_0}(\{Y_n, \nu\}, n)}, \frac{E_n^{\downarrow}}{W_{w_0}(\{Y_n, \mu\}, n)} \} \tag{2.47}$$

is a.s. attained, and similarly for $\widetilde{\mathbf{Y}}$, where we replace the weights W_{w_0} by W_w, and the vertices ν, μ by $\widetilde{\nu}$, $\widetilde{\mu}$. Notice that in this way the extensions $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$ have the same distribution as in the definition 2.11. Let

$$r = \inf\{n \geq 1 \colon |Y_n| \neq |\widetilde{Y}_n|\}$$

be the first splitting time, and for ease of notation, let e_0, e_1 be the two edges incident to $Y_{r-1} = \widetilde{Y}_{r-1}$, where e_1 connects Y_{r-1} to its child on the path, and e_0 connects Y_{r-1} to its parent $\overleftarrow{Y}_{r-1}$. Clearly $W_w(e_0, r-1) = W_{w_0}(e_0, r-1) = \delta$, since the edge e_0 is crossed by both processes. Also, by construction, $W_w(e, r-1) \geq W_{w_0}(e, r-1)$ for any edge e lying on the path connecting $\overleftarrow{\rho}$ to ν. If we would have $W_w(e_1, r-1) = W_{w_0}(e_1, r-1)$, then, by the construction of the coupling in (2.47), $Y_r = \widetilde{Y}_r$, a contradiction. Hence $W_w(e_1, r-1) = \delta$ and $W_{w_0}(e_1, r-1) = 1$. It follows again from (2.47) that the only way $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$ can split is that $|\widetilde{Y}_r| = |Y_r| + 2$. Define

$$s = \inf\{n > r \colon |Y_n| = |\widetilde{Y}_n|\}.$$

For any edge e lying on the path connecting $\overleftarrow{\rho}$ to ν, we have that $W_w(e, s) \geq W_{w_0}(e, s)$, and we can reiterate the previous argument to prove that $|\widetilde{\mathbf{Y}}| \geq |\mathbf{Y}|$. Consider the coloring process, defined in the same way as above (2.45), but on the weight configuration w. It follows that, if the coloring process associated to $\mathbf{Y}$ survives, then as $|\widetilde{\mathbf{Y}}| \geq |\mathbf{Y}|$, the coloring process associated to $\widetilde{\mathbf{Y}}$ survives. But on this last event, $D = \infty$. Hence $\beta_w = \mathbf{P}_w(D < \infty) \leq \alpha_{\psi}$. $\square$

The random variable $L(\cdot)$ is defined in the same way as in (2.10). We have the following

Proposition 2.14. *If $\delta > 1$, under $\mathbf{P}_{w_0}$, the random variable $L(\rho)$ is stochastically dominated by a geometric variable with parameter $(1 - \alpha_{\psi})\, b/(b+\delta)$.*

Proof. Recall that $\mathbf{X}$ starts from ρ in the initially fair weight configuration w_0. With probability $b/(b+\delta)$ the first jump will be towards one of the children of ρ. Then, started at this child of ρ, with probability $1-\beta$, the process will never return to ρ. Whenever it returns to ρ, it starts on some random weight configuration $w \in \mathbb{W}$, depending on the past of the path. Under $\mathbf{P}_w$, the probability that ORRW jumps to one of the children of ρ is greater than $b/(b+\delta)$. To see this, recall that the edge $\{\overleftarrow{\rho}, \rho\}$ has weight δ, and we change all the weights on the edges connecting ρ to its children to one. Since $\delta > 1$, this decreases the probability to jump to level one, and we obtain the lower bound for this probability. Under $\mathbf{P}_w$, ORRW, started at a child ν of ρ, has probability larger than $1 - \beta_{\overline{w}}$ of never returning to ρ, where $\overline{w}$ is the weight configuration induced by w on $\overleftarrow{\Lambda}_\nu$. With the help of Proposition 2.13, we find that, for any $w \in \mathbb{W}$, the escape probability from ρ is at least $(1-\alpha_\psi)b/(b+\delta)$, and it follows that the number of returns to ρ is stochastically dominated by a geometric variable with parameter $(1-\alpha_\psi)b/(b+\delta)$. □

We recall from [11] that a law of large numbers with positive speed holds, i.e. $\mathbf{P}$-a.s., $v = \lim_{n\to\infty} |X_n|/n > 0$. Further it is shown that $v \le b/(b+\delta)$, but no lower bound is available. We are now ready to provide a lower bound for the speed that is always positive.

Theorem 2.15. *If $\delta > 1$, choose $\psi \ge 1$ such that* (2.45) *holds. Then the speed v satisfies*

$$v \ge \frac{1-\beta}{\mathbf{E}[L(\rho)]} \ge (1-\alpha_\psi)^2 \frac{b}{b+\delta} > 0. \tag{2.48}$$

Remark 2.16. Notice that in the case of $\delta < b$ we can compare $|\mathbf{X}|$ with a simple random walk on the non-negative integers with drift equal to $(b-\delta)/(b+\delta) > 0$. It follows that for $\delta < b$ we have $v \ge (b-\delta)/(b+\delta)$. In this case, we find that the lower bound in (2.48) is larger than $(b-\delta)/(b+\delta)$ if and only if $\alpha_\psi < 1 - \sqrt{1-\delta/b}$. The challenging case is $\delta \ge b$, which is covered by Theorem 2.15.

Proof of Theorem 2.15. Define the random variable Π_k in the same way as in (2.12). Observe that the same result as Lemma 2.4 in the previous section holds, with exactly the same proof. By straightforward modifications, we further see that Proposition 2.3 holds in the setting of once edge-reinforced random walk. The first inequality follows. The second and third inequality then follow directly from Propositions 2.14 and 2.13. □

Next we show monotonicity of the lower bound on the speed in (2.48).

Proposition 2.17. *Choose $\delta_2 > \delta_1 \ge 1$. Then for every $\psi \ge 1$, $\alpha_\psi(\delta_1) \le \alpha_\psi(\delta_2)$, and in particular the lower bound in* (2.48) *is decreasing in δ for $\delta > 1$.*

Proof. Denote with $\mathbf{Y}^{(1)}$ and $\mathbf{Y}^{(2)}$ the extensions on rays $[\overleftarrow{\rho}, \infty)$ corresponding to ORRW(δ_1) resp. ORRW(δ_2), started at ρ, in the initially fair weight configuration $w_0^{(\delta_1)}$ resp. $w_0^{(\delta_2)}$. Using the same coupling as in (2.47), we can show that $|\mathbf{Y}^{(1)}| \ge |\mathbf{Y}^{(2)}|$. To see this, call r to be the first time the two processes split, and let e_0 and e_1 be as in in proof of Proposition 2.13. Next we show that none of the processes traversed edge e_1 by time $r-1$. In fact, as the two processes coincide up to time $r-1$, if one of them traversed e_1, also the other did. On the other hand, both of them traversed e_0 by time $r-1$, in order to reach $Y_{r-1}^{(1)} = Y_{r-1}^{(2)}$. Hence $\mathbf{P}(|Y_r^{(1)}| = |Y_{r-1}^{(1)}| + 1) = 1/2 = \mathbf{P}(|Y_r^{(2)}| = |Y_{r-1}^{(2)}| + 1)$. By construction of the coupling, this would imply that $Y_r^{(1)} = Y_r^{(2)}$, which contradicts the definition of r. As none of the processes traversed edge e_1 by time $r-1$, while both traversed e_0, using the fact $\delta_2 > \delta_1$ we infer that $|Y_r^{(1)}| > |Y_r^{(2)}|$. Denote with t the first time, after r, when the two processes meet, and let r_1 be the first time after t, when the two processes split again. As $|Y_k^{(1)}| \ge |Y_k^{(2)}|$ for all $k \le r_1 - 1$, we have that there is no edge reinforced by $Y_k^{(2)}$ which has not been reinforced by $Y_k^{(1)}$, $k \leq r_1 - 1$. This, together with the fact that $\delta_1 > \delta_2 > 1$, and the construction of the coupling, implies that $\mathbf{P}(|Y_{r_1}^{(1)}| = |Y_{r_1-1}^{(1)}| + 1) \geq \mathbf{P}(|Y_{r_1}^{(2)}| = |Y_{r_1-1}^{(2)}| + 1)$. By construction of the coupling, we have that $|Y_{r_1}^{(1)}| > |Y_{r_1}^{(2)}|$. By reiterating this argument, we get $|\mathbf{Y}^{(1)}| \geq |\mathbf{Y}^{(2)}|$. This implies that for every ψ, $\alpha_\psi(\delta_1) \leq \alpha_\psi(\delta_2)$, and it follows that the lower bound in (2.48) is decreasing in δ. $\square$

3. Moment bounds on the first regeneration time

In addition to providing an explicit lower bound on the speed, our methods can be extended to give an explicit upper bound on the tail of a certain regeneration level. We present a unified approach that applies both for random walk in a random environment and once edge-reinforced random walk. Hence, in what follows, $\mathbf{X}$ denotes either one of these processes. We start by defining the regeneration times.

Definition 3.1. We define the first regeneration level as follows

$$\ell_1 \stackrel{def}{=} \inf\{k \geq 1 \colon D(X_{T_k}) = \infty\},$$

and iteratively

$$\ell_n \stackrel{def}{=} \inf\{k > \ell_{n-1} \colon D(X_{T_k}) = \infty\},$$

where $D(\cdot)$ is defined in (2.3) and we use the convention $\inf \varnothing = \infty$. The regeneration times are defined as $\tau_n = T_{\ell_n}, n \geq 1$, on the event $\{\ell_n < \infty\}$.

In other words, a regeneration time occurs when the walk hits a level for the first time and then never backtracks to the previous level. Clearly, these are not stopping times. It is easy to see that under transience, it holds that for all $n \geq 1$, $\tau_n < \infty$ $\mathbf{P}$-a.s. It is also known that in the setting of random walks in random environment, the first regeneration level ℓ_1 has exponential moments under the conditioned measure $\mathbf{P}(\cdot|D = \infty)$. This is for instance proved in in Lemma 4.2 in [10] for biased random walks on Galton-Watson trees, and can be directly adapted to our setting. For once edge-reinforced random walk with $\delta > 1$, we know that ℓ_1 has all moments finite under $\mathbf{P}(\cdot|D = \infty)$, see Lemma 7 in [11] (this statement is actually proved for certain cut levels, but notice that our regeneration level is smaller than the cut level in [11]).
We now present a unified approach that applies to both settings, and that provides explicit estimates for the tail of ℓ_1 and for the moments of τ_1.

3.1 The tail of the first regeneration level We assume that we can choose ψ such that (2.23) is fulfilled for random walk in a random environment resp. once edge-reinforced random walk. Recall that for ORRW(δ), this is always possible, see (2.45) and Proposition 2.13.
We will find *explicit* exponential tails on ℓ_1. These tail estimates on ℓ_1 are obtained by refining the color scheme from definitions 2.5 resp. 2.12.

Definition 3.2. Let ν be a vertex at level $k\psi$, $k \geq 1$. Let Θ_ν be the set of vertices μ in Λ_ν which are first children and whose distance from ν is a multiple of $\zeta\psi$. Let Σ_ν be the set of vertices μ in Λ_ν such that

- μ is colored (in particular $|\mu|$ - $|\nu|$ is a multiple of ψ),
- all ancestors of μ in Λ_ν do not belong to Θ_ν.

In other words, Σ_ν is the set of colored vertices in Λ_ν minus the colored vertices that are elements of subtrees generated by vertices μ that are first children and $|\mu| - |\nu| = k\zeta\psi$, $k \geq 1$. Further let

$$B(\nu) \stackrel{def}{=} \{\Sigma_\nu \text{ is infinite}\}, \quad \text{and} \quad B_0 \stackrel{def}{=} B(\rho), \;\; B_i \stackrel{def}{=} B(X_{T_{\psi\zeta i}}), \;\; i \geq 1. \tag{3.1}$$

In a first step, we introduce an auxiliary branching process and use it to derive an explicit lower bound on the probability of B_0, see Lemma 3.3. In a second step, in Lemma 3.4, we then show that the events B_i are independent. In [6], section 3, the counterpart of these lemmata for vertex-reinforced jump processes are stated and proved in a similar way.
For any pair of distributions f_1 and f_2, denote by $f_1 \overline{*} f_2$ the distribution of $\sum_{k=1}^V M_k$, where

- V has distribution f_1, and
- $\{M_k,\, k \in \mathbb{N}\}$ is a sequence of i.i.d. random variables, independent of V, each with distribution f_2.

We set $\mathbf{p}^{(1)} := \mathbf{p}$, and define, by recursion, $\mathbf{p}^{(j)} := \mathbf{p}^{(j-1)} \overline{*} \mathbf{p}$ for $j \geq 2$. The distribution $\mathbf{p}^{(j)}$ describes the number of elements, at time j, in a population which evolves like a branching process generated by one ancestor and with offspring distribution $\mathbf{p}$. Let $q_0 = p_0 + p_1$, and for $k \in \{1, \ldots, b^\psi - 1\}$, set $q_k = p_{k+1}$. Set $\mathbf{q}$ to be the distribution which assigns to $i \in \{0, \ldots, b^\psi - 1\}$ probability q_i. For $j \geq 2$, let $\mathbf{q}^{(j)} := \mathbf{p}^{(j-1)} \overline{*} \mathbf{q}$. Denote by $q_i^{(j)}$ the weight that the distribution $\mathbf{q}^{(j)}$ assigns to $i \in \{0, \ldots, (b^\psi - 1)b^{(j-1)\psi}\}$. The mean of $\mathbf{q}^{(j)}$ is $m_\psi^{j-1}(m_\psi - 1)$. From now on, ζ denotes the smallest positive integer such that

$$m_\psi^{\zeta-1}(m_\psi - 1) > 1. \tag{3.2}$$

(This is possible since we chose ψ such that $m_\psi > 1$.) Define γ to be the smallest positive solution of the equation

$$x = \sum_{k=0}^{\vartheta} x^k q_k^{(\zeta)}, \quad \text{where } \vartheta = b^{(\zeta-1)\psi}(b^\psi - 1). \tag{3.3}$$

Lemma 3.3. *Assume* (3.2). *We have that for $i \geq 0$, $\mathbf{P}(B_i) = \mathbf{P}(B_0) \geq 1 - \gamma > 0$.*

Proof. Fix i and notice that by stationarity, $\mathbf{P}(B_i) = \mathbf{P}(B_0)$. From the definition of Σ_ρ, it follows that the offspring distribution of colored vertices at level $\zeta\psi$ in Σ_ρ is obtained as follows. The number of vertices at level $(\zeta - 1)\psi$ has law $\mathbf{p}^{((\zeta-1)\psi)}$. Each vertex at level $\zeta\psi$ has a number of colored offspring distributed as $\mathbf{p} = \mathbf{p}^{(1)}$. If from each of these offspring we delete the first child, the number of the remaining colored offspring is distributed as $\mathbf{q}$. Hence the offspring distribution modeling Σ_ν is given by $\mathbf{q}^{(\zeta)} = \mathbf{p}^{(\zeta-1)} \overline{*} \mathbf{q}$. Then, from the basic theory of branching processes we know that the extinction probability equals the smallest positive solution of the equation (3.3). In virtue of (3.2) we have that $\gamma < 1$. □

Lemma 3.4. *The events B_i, $i \geq 1$, are independent under* $\mathbf{P}$.

Proof. Choose integers $0 < i_1 < i_2 < \ldots < i_k$. It is enough to prove that

$$\mathbf{P}\Big(\bigcap_{j=1}^k B_{i_j}\Big) = \prod_{j=1}^k \mathbf{P}(B_{i_j}). \tag{3.4}$$

We proceed by backward recursion. We use the notation introduced in definition 2.1. The set $B(\nu)$ belongs to the sigma-algebra generated by $\{h_i(\eta,\mu)\colon\ \eta,\mu \in \mathrm{Vert}(\Lambda_\nu) \text{ and } i \geq 1\}$. Notice that each X_{T_i}, $i \geq 1$, is a first child. Hence the set $\cap_{j=1}^{k-1} B_{i_j} \cap \{X_{T_{\psi\zeta i_k}} = \nu\}$ belongs to $\{h_i(\eta,\mu)\colon\ \eta \notin \mathrm{Vert}(\Lambda_\nu)\}$. As the two events belong to disjoint collections of independent exponential variables, they are independent. We have

$$\mathbf{P}(\bigcap_{j=1}^{k} B_{i_j}) = \sum_\nu \mathbf{P}\big(B_{i_k} \cap \bigcap_{j=1}^{k-1} B_{i_j} \cap \{X_{T_{\psi\zeta i_k}} = \nu\}\big) = \sum_\nu \mathbf{P}(B(\nu))\ \mathbf{P}\big(\bigcap_{j=1}^{k-1} B_{i_j} \cap \{X_{T_{\psi\zeta i_k}} = \nu\}\big).$$

From stationarity, it follows that $\mathbf{P}(B(\nu)) = \mathbf{P}(B_0)$, and from the independence of $B(\nu)$ and $\{X_{T_{i\psi\zeta}} = \nu\}$, we infer that for an arbitrary vertex ν, and each $i \geq 1$,

$$\mathbf{P}(B(\nu)) = \mathbf{P}(B_i)\,. \tag{3.5}$$

Now the right-hand side of (3.1) equals

$$\mathbf{P}(B_0) \sum_\nu \mathbf{P}\big(\bigcap_{j=1}^{k-1} B_{i_j} \cap \{X_{T_{\psi\zeta i_k}} = \nu\}\big) = \mathbf{P}(B_{i_k})\mathbf{P}\big(\bigcap_{j=1}^{k-1} B_{i_j}\big)\,. \tag{3.6}$$

(3.4) follows now by iteration. □

Theorem 3.5. *Assume (2.23). For $n \geq 1$, we have that*

$$\mathbf{P}(\ell_1 \geq n\psi\zeta) \leq \gamma^{n-1}, \tag{3.7}$$

where γ is defined in (3.3).

Proof. Notice that on the event B_i, the colored process survives in the subtree $\Lambda_{X_{T_{i\psi\zeta}}}$. It follows that $B_i \subseteq \{\text{level } i\psi\zeta \text{ is a regeneration level}\}$. Hence

$$\{\ell_1 \geq n\psi\zeta\} \subseteq \bigcap_{i=1}^{n-1} B_i^c\,,$$

and the Theorem now follows from the Lemmata 3.3 and 3.4. □

3.2 Moment bounds for the first regeneration time

Recall the first regeneration time in Definition 3.1, and define

$$\Pi = \sum_{\nu \in \Lambda} \mathbb{1}_{\{T(\nu) \leq \tau_1\}}$$

to be the number of distinct vertices visited by time τ_1. We denote with $M(n,q)$ the n-th moment of a geometric variable with parameter q. We have the following explicit bound on the moments of Π, which implies an explicit bound on the moments of τ_1, see Theorem 3.7 below.

Proposition 3.6. *Assume (2.23). For $p \geq 1$, it holds that*

$$\mathbf{E}[\Pi^p] \leq \gamma^{-1/2}\Big(1 - \gamma^{\frac{1}{2\psi\zeta}}\Big)^{-1}\Big(M\big(p, 1 - \gamma^{\frac{1}{2\psi\zeta}}\big) - 1\Big)\, M^{1/2}(2p, 1-\beta)\,. \tag{3.8}$$

Proof. Recall $\Pi_k = \sum_{\nu:\, |\nu|=k} \mathbb{1}_{\{T(\nu)<\infty\}}$, $k \geq -1$, which is the number of vertices visited at level k, and observe that

$$\Pi = \sum_{n=1}^{\infty} \sum_{\nu} \mathbb{1}_{\{T(\nu)\leq T_n\}}\, \mathbb{1}_{\{\ell_1=n\}} \leq \sum_{n=1}^{\infty} \sum_{\nu:|\nu|<n} \mathbb{1}_{\{T(\nu)<\infty\}}\, \mathbb{1}_{\{\ell_1=n\}} = \sum_{n=1}^{\infty} \sum_{k=-1}^{n-1} \Pi_k\, \mathbb{1}_{\{\ell_1=n\}}. \tag{3.9}$$

We use Jensen's inequality, and obtain that

$$\mathbf{E}[\Pi^p] \leq \sum_{n=1}^{\infty} \mathbf{E}[\big(\sum_{k=-1}^{n-1} \Pi_k\big)^p \mathbb{1}_{\{\ell_1=n\}}] \overset{(Jensen)}{\leq} \sum_{n=1}^{\infty} (n+1)^{p-1} \sum_{k=-1}^{n-1} \mathbf{E}[\Pi_k^p \mathbb{1}_{\{\ell_1=n\}}]. \tag{3.10}$$

First notice that Lemma 2.4, proved for random walk in a random environment, holds also for once edge-reinforced random walk with the same proof. We first use Cauchy-Schwarz's inequality, and then Lemma 2.4 together with Lemma 4.1 from the Appendix to obtain that the right-hand side of the last display is smaller than

$$\sum_{n=1}^{\infty} (n+1)^{p-1} \sum_{k=-1}^{n-1} \mathbf{E}[\Pi_k^{2p}]^{1/2}\, \mathbf{P}(\ell_1 = n)^{1/2} \leq M^{1/2}(2p, 1-\beta) \sum_{n=1}^{\infty} (n+1)^p\, \mathbf{P}(\ell_1 \geq n)^{1/2}. \tag{3.11}$$

Finally, with Theorem 3.5, we obtain that

$$\sum_{n=1}^{\infty} (n+1)^p\, \mathbf{P}(\ell_1 \geq n)^{1/2} \leq \gamma^{-\frac{1}{2}} \sum_{n=2}^{\infty} n^p\, \gamma^{\frac{n-1}{2\psi\zeta}} = \gamma^{-1/2} \Big(1 - \gamma^{\frac{1}{2\psi\zeta}}\Big)^{-1} \Big(M(p, 1-\gamma^{\frac{1}{2\psi\zeta}}) - 1\Big). \tag{3.12}$$

The claim (3.8) now follows by collecting the results in (3.10) to (3.12). $\square$

We are now ready to state the main result of this subsection.

Theorem 3.7. *Assume (2.23) and that $\mathbb{E}[A^{-p-\varepsilon}] < \infty$ for some $p \geq 1$ and $\varepsilon > 0$. It holds that*

$$\mathbf{E}[\tau_1^p] \leq \tfrac{\pi^2}{6}\, \mathbf{E}[L(\rho)^{p+\varepsilon}]^{\frac{1}{q}}\, \mathbf{E}[\Pi^{2(p-1)q'}]^{\frac{1}{2q'}}\, \mathbf{E}[\Pi^{4q'}]^{\frac{1}{2q'}} < \infty, \tag{3.13}$$

where $q = 1 + \varepsilon/p$, and $q' = 1 + p/\varepsilon$ is the dual of q.

Proof. By Jensen's inequality, we find

$$\mathbf{E}[\tau_1^p] = \mathbf{E}[\big(\sum_{i=1}^{\Pi} L(\sigma_i)\big)^p] \leq \mathbf{E}[\Pi^{p-1} \sum_{i=1}^{\Pi} L(\sigma_i)^p] = \sum_{i=1}^{\infty} \mathbf{E}[\Pi^{p-1} L(\sigma_i)^p \mathbb{1}_{\{\Pi \geq i\}}]. \tag{3.14}$$

By Hölder's inequality, and by stationarity, the right-hand side of the last display is smaller than

$$\mathbf{E}[L(\rho)^{p+\varepsilon}]^{\frac{1}{q}}\, \mathbf{E}[\Pi^{2(p-1)q'}]^{\frac{1}{2q'}} \sum_{i=1}^{\infty} \mathbf{P}(\Pi \geq i)^{\frac{1}{2q'}} \tag{3.15}$$

By Chebychev's inequality, we find that

$$\sum_{i=1}^{\infty} \mathbf{P}(\Pi \geq i)^{\frac{1}{2q'}} \leq \sum_{i=1}^{\infty} i^{-2} \mathbf{E}[\Pi^{4q'}]^{\frac{1}{2q'}} = \tfrac{\pi^2}{6} \mathbf{E}[\Pi^{4q'}]^{\frac{1}{2q'}}. \tag{3.16}$$

Putting (3.14),(3.15) and (3.16) together, we obtain the claim. $\square$

3.3 An invariance principle and bounds on the covariance

For ORRW, an invariance principle is known, see Theorem 3 in Durrett, Kesten and Limic [11]. For RWRE, an annealed invariance principle easily follows from the results of Aidékon [2]. We further refer to Peres and Zeitouni [21] for a quenched invaraince principle for biased random walks on Galton-Watson trees. Define

$$B^n_\cdot = \frac{1}{\sqrt{n}}(|X_{[\cdot n]}| - [\cdot n]v), \ \ \beta^n_t = B^n_t + (nt - [nt])(B^n_{t+1} - B^n_t), \ \ n \geq 1,$$

i.e. β is the polygonal interpolation of $k/n \to B^n_{k/n}$, $k \geq 0$. We endow the space $C(\mathbb{R}_+, \mathbb{R})$ of continuous functions with the topology of uniform convergence on compacts, and with its Borel σ-algebra.

Proposition 3.8. *The $C(\mathbb{R}_+, \mathbb{R})$-valued random variable $\beta^n_\cdot$ converge under $\mathbf{P}$ in law to a Brownian motion $B.$ with covariance*

$$K = \mathbf{E}[(\ell_1 - v\tau_1)^2|D = \infty] \ \mathbf{E}[\tau_1|D = \infty]^{-1}.$$

Proof. For ORRW, we refer to Theorem 3 in [11]. For RWRE, observe that the second moment of τ_1, und thus of ℓ_1, is finite, as follows from Propositions 2.1 and 2.2 in Aidékon [2]. Since $\mathbf{P}[D = \infty] = 1 - \beta > 0$, also $\mathbf{E}[\ell_1^2|D = \infty] \leq \mathbf{E}[\tau_1^2|D = \infty] < \infty$. Further it is well-known that

$$\begin{aligned} &(\tau_{i+1} - \tau_i, \ell_{i+1} - \ell_i)_{i\geq 1} \text{ is an i.i.d. sequence under } \mathbf{P}, \text{ and for } i \geq 1, \\ &(\tau_{i+1} - \tau_i, \ell_{i+1} - \ell_i) \text{ has same law under } \mathbf{P} \text{ as } (\tau_1, \ell_1) \text{ under } \mathbf{P}(\,\cdot \mid D = \infty), \end{aligned} \tag{3.17}$$

see [13] (see also [17] for a similar statement for biased random walks on Galton-Watson trees). With the help of this i.i.d. structure, the proof of the invariance principle is now quite standard, see for instance Theorem 3 in Durrett, Kesten and Limic [11] and also Theorem 3.3 in Shen [24]. □

With the help of Theorem 3.5 and Theorem 3.7, we obtain explicit bounds on the covariance K via the following proposition. For RWRE (resp. ORRW) denote with w the right-hand side in inequality (2.10) (resp. (2.48)) , so that $v \geq w$. Let a be the smallest even integer larger or equal to $[3/w] + 1$. As $w \leq 1$, we have $a \geq 4$.

Proposition 3.9. *In the case of RWRE, we assume that (2.23) holds and that $\mathbb{E}[A^{-2-\varepsilon}] < \infty$ for some $\varepsilon > 0$. In the case of ORRW we choose ψ satisfying (2.45). Then we have the following common upper bound on the covariance K*

$$K \leq (1 - \alpha_\psi)^{-1}(\mathbf{E}[\ell_1^2] + \mathbf{E}[\tau_1^2]) \quad \textit{for RWRE and ORRW,} \tag{3.18}$$

and the following lower bound

$$\begin{aligned} &K \geq b\,(1 - \alpha_\psi)\,\mathbf{E}[\tau_1]^{-1}\,\mathbb{E}[\omega(\rho,\ \overrightarrow{\rho}_1)^{\frac{a}{2}}]\mathbb{E}[\omega(\overrightarrow{\rho}_1,\ \rho)^{\frac{a}{2}-1}\big(1 - \omega(\overrightarrow{\rho}_1,\ \rho)\big)] \quad \textit{for RWRE,} \\ &K \geq (1 - \alpha_\psi)\,\mathbf{E}[\tau_1]^{-1}\left(\frac{b}{b+\delta}\right)^2\left(\frac{\delta}{b+\delta}\right)^{a/2-1}\left(\frac{\delta}{b-1+2\delta}\right)^{a/2-1} \quad \textit{for ORRW.} \end{aligned} \tag{3.19}$$

Proof of Proposition 3.9. We start with the upper bound. We use the trivial bound $(a - b)^2 \leq a^2 + b^2$, $a, b \geq 0$, and $v \leq 1$ to obtain that

$$K \leq \mathbf{E}[\ell_1^2|D = \infty] + \mathbf{E}[\tau_1^2|D = \infty] \leq (1 - \beta)^{-1}(\mathbf{E}[\ell_1^2] + \mathbf{E}[\tau_1^2]). \tag{3.20}$$

The upper bound (3.18) follows from Proposition 2.6. Let us now turn to the lower bound (3.19) for random walk in random environment. We use the following approach

$$\mathbf{E}[(\ell_1 - v\tau_1)^2|D=\infty] \geq \mathbf{E}[(\ell_1 - v\tau_1)^2 \mathbb{1}_{\{v\tau_1 \geq \ell_1+1\}}|D=\infty] \geq \mathbf{P}[v\tau_1 \geq \ell_1+1|D=\infty], \quad (3.21)$$

where the last inequality comes from the fact that on the event $\{v\tau_1 \geq \ell_1 + 1\}$ we have $(\ell_1 - v\tau_1)^2 \geq 1$. Hence

$$K \geq \mathbf{P}(v\tau_1 \geq \ell_1+1|D=\infty)\ \mathbf{E}[\tau_1|D=\infty]^{-1} \geq \mathbf{P}(v\tau_1 \geq \ell_1+1, D=\infty)\mathbf{E}[\tau_1]^{-1}. \quad (3.22)$$

Next we find a suitable subset of $\{v\tau_1 \geq \ell_1+1\}$ whose probability is easy to compute. Consider the event

$$C \stackrel{def}{=} \{T_2 = a,\ D(X_{T_2}) = \infty,\ \cup_{i=1}^{b}\{X_j \in \{\rho, \overrightarrow{\rho}_i\},\ \forall j \leq T_2 - 1\}\}.$$

If this event holds then the walk, started at the root ρ, visits level two first at time a and, after this time, never goes back to level 1.Moreover before time T_2, the process $\mathbf{X}$ visits only the vertices ρ and $\overrightarrow{\rho_i}$ for some i, and hence it does not return to $\overleftarrow{\rho}$. As $a \geq 4$, it jumps at least once from $\overrightarrow{\rho_1}$ to ρ, so that level one cannot be a cut level and $\ell_1 = 2$. As $a \geq [3/w]+1 \geq [3/v]+1$, we have

$$C \subset \{\ell_1 = 2, \tau_1 \geq [3/v]+1,\ D=\infty\}.$$

On the event $\{\ell_1 = 2, \tau_1 \geq [3/v]+1\}$ we have that $v\tau_1 \geq 3$, hence $v\tau_1 - \ell_1 \geq 1$. In other words,

$$C \subset \{v\tau_1 \geq \ell_1+1,\ D=\infty\}. \quad (3.23)$$

We first focus on the RWRE case. Let us now compute the probability of the event C. The Markov property implies that

$$\mathbf{P}_\omega(C) = \sum_{i=1}^{b} \omega(\rho, \overrightarrow{\rho_i})^{\frac{a}{2}} \omega(\overrightarrow{\rho_i}, \rho)^{\frac{a}{2}-1}\big(1-\omega(\overrightarrow{\rho_i},\ \rho)\big)\ \mathbf{E}_\omega[\mathbf{P}_{X_{T_2},\omega}(D=\infty)].$$

The random variables $\omega(\rho, \overrightarrow{\rho_i})$, $\omega(\overrightarrow{\rho_i}, \rho)\big(1-\omega(\overrightarrow{\rho_i},\ \rho)\big)$ and $\mathbf{E}_\omega[\mathbf{P}_{X_{T_2},\omega}(D=\infty)]$ are independent, since they are measurable w.r.t. disjoint parts of the environment. We use in addition stationarity to find that

$$\mathbf{P}(C) = b\,\mathbb{E}[\omega(\rho, \overrightarrow{\rho_1})^{\frac{a}{2}}]\,\mathbb{E}[\omega(\overrightarrow{\rho_1}, \rho)^{\frac{a}{2}-1}(1-\omega(\overrightarrow{\rho_1}, \rho))]\,\mathbf{E}[\mathbf{P}_{X_{T_2},\omega}(D=\infty)].$$

Again, by independence and stationarity,

$$\begin{aligned}\mathbf{E}[P_{X_{T_2},\omega}(D=\infty)] &= \sum_{\nu} \mathbf{E}[\mathbf{P}_{\nu,\omega}(D=\infty),\ X_{T_2}=\nu] \\ &= \sum_{\nu} \mathbf{P}_\nu(D=\infty)\,\mathbf{P}(X_{T_2}=\nu) = \mathbf{P}(D=\infty) = 1-\beta.\end{aligned}$$

It follows that

$$\mathbf{P}(v\tau_1 - \ell_1 \geq 1, D=\infty) \geq \mathbf{P}(C) = b\,\mathbb{E}[\omega(\rho, \overrightarrow{\rho_1})^{\frac{a}{2}}]\mathbb{E}[\omega(\overrightarrow{\rho_1}, \rho)^{\frac{a}{2}-1}\big(1-\omega(\overrightarrow{\rho_1}, \rho)\big)](1-\beta). \quad (3.24)$$

The lower bound (3.19) for RWRE now follows from (3.22), (3.24) and Proposition 2.6. Let us now turn to the proof of the lower bound (3.19) for ORRW. We follow the same strategy as above, and we see that (3.22) and (3.23) hold. It remains to compute the probability of the event C:

$$\mathbf{P}(C) = \Big(\frac{b}{b+\delta}\Big)^2 \Big(\frac{\delta}{b+\delta}\Big)^{a/2-1} \Big(\frac{\delta}{b-1+2\delta}\Big)^{a/2-1}.$$

By proceeding as in (3.24) and above, and with the help of Proposition 2.13, the proof of (3.19) is completed. □

4. Appendix

Lemma 4.1. *Let $M(n,q)$ denote the n-th moment of a geometric random variable with parameter q. Then for $n \geq 1$, $M(n,q) \leq c_n\, q^{-n}$, for some constant c_n that only depends on n.*

Proof. We define $g(q,n) \stackrel{def}{=} \sum_{k=1}^{\infty} k^n (1-q)^{k-1}$, and notice that $M_n^{(q)} = \sum_{k=1}^{\infty} k^n q(1-q)^{k-1} = qg(q,n)$. Since $0<q<1$, it is enough to show that there are coefficients $a_\cdot^{(n)}$ such that

$$g(q,n) = \frac{\sum_{i=1}^{n} a_i^{(n)} q^{n-i}}{q^{n+1}} = \sum_{i=1}^{n} a_i^{(n)} q^{-i-1}. \tag{4.1}$$

We prove (4.1) by induction. As $g(q,1) = 1/q^2$, (4.1) holds for $n=1$. Suppose now (4.1) holds for $n-1$. We have

$$\begin{aligned} g(q,n) - g(q,n-1) &= \sum_{k=1}^{\infty} k^{n-1}(k-1)(1-q)^{k-1} \\ &= (1-q)\frac{\mathrm{d}}{\mathrm{d}(1-q)} \sum_{k=1}^{\infty} k^{n-1}(1-q)^{k-1} = (1-q)\frac{\mathrm{d}}{\mathrm{d}(1-q)} g(q,n-1), \end{aligned} \tag{4.2}$$

where $\frac{\mathrm{d}}{\mathrm{d}x}$ denotes the derivative with respect x. By the induction hypothesis,

$$\frac{\mathrm{d}}{\mathrm{d}(1-q)} g(q,n-1) = \sum_{i=1}^{n-1} (i+1) a_i^{(n-1)} q^{-i-2}, \tag{4.3}$$

and hence, using (4.1) to (4.3),

$$g(q,n) = n\, a_{n-1}^{(n-1)} q^{-n-1} + \sum_{i=2}^{n-1} i(a_{i-1}^{(n-1)} - a_i^{(n-1)}) q^{-i-1} - a_1^{(n-1)} q^{-2}. \tag{4.4}$$

This shows (4.1), and the proof is finished. □

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