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by

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Abstract

We present new bell inequalities for arbitrary dimensional bipartite quantum systems. The maximal violation of the inequalities is computed. The Bell inequality is capable of detecting quantum entanglement of both pure and mixed quantum states more effectively.

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The problem related to the local realism and quantum mechanics was first highlighted by the paradox of Einstein, Podolsky and Rosen [1]. Later, Bell proposed a remarkable inequality that should be obeyed by any local realistic theory [2]. From then on investigation of Bell theorem for a general quantum system has been regarded as one of the most important challenges in quantum mechanics and quantum information science [3–8].

Since Bell's work there have appeared many important further results such as Clauser-Horne-Shimony-Holt (CHSH) [9] and Mermin-Ardehali-Belinskii-Klyshko (MABK) inequalities [10]. In 1991 Gisin presented a theorem saying that any pure entangled states of two spin-1/2 particles (qubits) violate the CHSH inequality [11]. Soon after a more complete and simpler proof of this theorem had been given for two arbitrary spin-j particles systems [12]. Since then there have been many results on studying the Gisin's theorem for general bipartite systems in terms of various kinds of Bell inequalities. In [13] the authors showed analytically that all pure entangled two-qudit states violate the an inequality. A Bell inequality for two d-dimensional particles was also given in [14].

In this paper, we present new Bell inequalities for bipartite quantum states. The maximal violation of the Bell inequalities is computed analytically for pure states. The inequality and it's maximal violation are also shown to be valid for all bipartite even dimensional bipartite mixed states and some odd dimensional bipartite mixed states. It is shown that some entangled quantum states that are unrecognizable by using the Bell inequality given in [Phys. Lett. A 162, 15(1992)] can be detected by the new Bell inequalities.

We consider bipartite states on $N \times N$ systems. For even N, let Γ_x, Γ_y and Γ_z be block-diagonal matrices, in which each block is an ordinary Pauli matrix, σ_x, σ_y and σ_z repectively, as described in [12] for Γ_x and Γ_z. When N is odd, we set the elements of the kth row and the kth column in Γ_x, Γ_y and Γ_z to be zero. The rest elements of Γ_x, Γ_y and Γ_z are the block-diagonal matrices as for the even N case. Let $\Pi(k)$ be an $N \times N$ matrix whose only nonvanishing entry is $(\Pi(k))_{kk} = 1$, $k \in 1, 2, \cdots, N$ for odd N and be a null matrix for even N. We define observables

$$A = \vec{a} \cdot \vec{\Gamma} + \Pi(k) = a_x\Gamma_x + a_y\Gamma_y + a_z\Gamma_z + \Pi(k) \tag{1}$$

and

$$B = \vec{b} \cdot \vec{\Gamma} + \Pi(k) = b_x\Gamma_x + b_y\Gamma_y + b_z\Gamma_z + \Pi(k), \tag{2}$$

where $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$ are unit vectors.

We define the Bell operator as

$$\mathcal{B} = A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2, \tag{3}$$

where

$$\begin{aligned} A_i &= \vec{a}_i \cdot \vec{\Gamma} + \Pi(k) = a_i^x\Gamma_x + a_i^y\Gamma_y + a_i^z\Gamma_z + \Pi(k), \\ B_i &= \vec{b}_j \cdot \vec{\Gamma} + \Pi(k) = b_j^x\Gamma_x + b_j^y\Gamma_y + b_j^z\Gamma_z + \Pi(k). \end{aligned}$$

Theorem: If there exists local hidden variable model to describe the system, the inequality

$$|\langle \mathcal{B} \rangle| \leq 2 \tag{4}$$

must hold for any $\vec{a}_i$, $\vec{b}_i$, $i = 1, 2$, and all $k \in 1, 2, \cdots, N$.

The Proof of this theorem is straightforward. Note that for any 3-dimensional unit vectors $\vec{a}$ and $\vec{b}$, the eigenvalues of A and B are either 1 or -1. Then as discussed for two-qubit case, if there exists local hidden variable model to describe the system, one has

$$\begin{aligned} |\langle \mathcal{B} \rangle| &= |\langle A_1 \otimes B_1 + A_1 \otimes B_2 + A_2 \otimes B_1 - A_2 \otimes B_2 \rangle| \\ &= |\langle A_1 \otimes (B_1 + B_2) \rangle + \langle A_2 \otimes (B_1 - B_2) \rangle| \\ &\leq |\langle A_1 \rangle||\langle (B_1 + B_2) \rangle| + |\langle A_2 \rangle||\langle (B_1 - B_2) \rangle| \leq 2. \end{aligned}$$

Now we compute the maximal violation of the Bell inequality.

Proposition 1: For any bipartite pure state $|\psi\rangle$ with even N, the maximal violation of the Bell inequality (4) is given by

$$\max\langle\psi|\mathcal{B}|\psi\rangle = 2\sqrt{\tau_1+\tau_2}, \tag{5}$$

where τ_1 and τ_2 are the two largest eigenvalues of the matrix $R^T R$, R is the matrix with entries $R_{\alpha\beta} = \langle\psi|\Gamma_\alpha \otimes \Gamma_\beta|\psi\rangle$, α, $\beta = x, y, z$.

[Proof] If N is even, we have the maximal violation of the Bell inequalities (4),

$$\begin{aligned}
\max\langle\psi|\mathcal{B}|\psi\rangle &= \max_{\vec{a}_1,\vec{a}_2,\vec{b}_1,\vec{b}_2}[\langle\psi|\sum_{\alpha=x,y,z} a_1^\alpha\Gamma_\alpha \otimes \sum_{\beta=x,y,z}(b_1^\beta + b_2^\beta)\Gamma_\beta|\psi\rangle \\
&\quad +\langle\psi|\sum_{\alpha=x,y,z} a_2^\alpha\Gamma_\alpha \otimes \sum_{\beta=x,y,z}(b_1^\beta - b_2^\beta)\Gamma_\beta|\psi\rangle] \\
&= \max_{\vec{a}_1,\vec{a}_2,\vec{b}_1,\vec{b}_2}[\vec{a}_1 \cdot R(\vec{b}_1+\vec{b}_2) + \vec{a}_2 \cdot R(\vec{b}_1-\vec{b}_2)] \\
&= \max_{\vec{b}_1,\vec{b}_2}[||R(\vec{b}_1+\vec{b}_2)|| + ||R(\vec{b}_1-\vec{b}_2)||] \\
&= \max_{\theta,\vec{c}\perp\vec{c}'} 2[\cos\theta||R\vec{c}|| + \sin\theta||R\vec{c}'||] \\
&= \max_{\vec{c}\perp\vec{c}'} 2\sqrt{||R\vec{c}||^2 + ||R\vec{c}'||^2} = 2\sqrt{\tau_1+\tau_2},
\end{aligned}$$

where $\vec{a}_i = (a_i^x, a_i^y, a_i^z)$, $\vec{b}_j = (b_j^x, b_j^y, b_j^z)$, $i, j = 1, 2$. □

Proposition 2: For any bipartite pure state $|\Psi\rangle$ in the Schmidt bi-orthogonal form,

$$|\Psi\rangle = \sum_{i=1}^{N} c_i|ii\rangle, \quad c_i \in I\!R, \quad \sum_i c_i^2 = 1 \tag{6}$$

with odd N, the maximal violation of the Bell inequality (4) is given by

$$\max\langle\Psi|\mathcal{B}|\Psi\rangle = 2\sqrt{\tau_1+\tau_2} + 2\langle\Psi|\Pi(k)\otimes\Pi(k)|\Psi\rangle, \tag{7}$$

where τ_1 and τ_2 are defined in Proposition 1.

[Proof] For odd N any $k \in \{1, 2, \cdots, N\}$, similarly we have

$$\begin{aligned}
\max\langle\Psi|B|\Psi\rangle &= \max_{\vec{a}_1,\vec{a}_2,\vec{b}_1,\vec{b}_2}[\langle\Psi|(\sum_{\alpha=x,y,z} a_1^\alpha\Gamma_\alpha + \Pi(k)) \otimes (\sum_{\beta=x,y,z}(b_1^\beta + b_2^\beta)\Gamma_\beta + 2\Pi(k))|\Psi\rangle \\
&\quad +\langle\Psi|(\sum_{\alpha=x,y,z} a_2^\alpha\Gamma_\alpha + \Pi(k)) \otimes (\sum_{\beta=x,y,z}(b_1^\beta - b_2^\beta)\Gamma_\beta)|\Psi\rangle] \\
&= \max_{\vec{a}_1,\vec{a}_2,\vec{b}_1,\vec{b}_2}[\vec{a}_1 \cdot R(\vec{b}_1+\vec{b}_2) + \vec{a}_2 \cdot R(\vec{b}_1-\vec{b}_2)] + 2\langle\Psi|\Pi(k)\otimes\Pi(k)|\Psi\rangle \\
&= \max_{\vec{b}_1,\vec{b}_2}[||R(\vec{b}_1+\vec{b}_2)|| + ||R(\vec{b}_1-\vec{b}_2)||] + 2\langle\Psi|\Pi(k)\otimes\Pi(k)|\Psi\rangle \\
&= \max_{\theta,\vec{c}\perp\vec{c}'} 2[\cos\theta||R\vec{c}|| + \sin\theta||R\vec{c}'||] + 2\langle\Psi|\Pi(k)\otimes\Pi(k)|\Psi\rangle \\
&= \max_{\vec{c}\perp\vec{c}'} 2\sqrt{||R\vec{c}||^2 + ||R\vec{c}'||^2} + 2\langle\Psi|\Pi(k)\otimes\Pi(k)|\Psi\rangle \\
&= 2\sqrt{\tau_1+\tau_2} + 2\langle\Psi|\Pi(k)\otimes\Pi(k)|\Psi\rangle.
\end{aligned}$$

$\square$

Remark: For even N, the formula (5) is also valid for any bipartite mixed quantum states ρ. One only needs to redefine $R_{\alpha\beta} = Tr[\rho\Gamma_\alpha \otimes \Gamma_\beta]$ for $\alpha, \beta = x, y, z$. The formula (7) doesn't fit for general quantum states with odd N. However, for some quantum mixed states their maximal violation of the Bell inequality (4) can be still computed by the formula (see the example 2 below).

Moreover the Bell inequality in [12] is a special case of (4) in the sense that it can be obtained by setting a_y, b_y in (1) and (2) to be zero, and $k = N$ in our Bell operator (3). For $k = N$, the maximal violation of (4) for an arbitrary bipartite quantum state (6) is the same as the violation values given in [12]. Which means that the parameters a_y, b_y do not contribute to the maximal violation in this case. However, even in this case the formula (5) and (7) have their own advantages. On one hand, one can compute the maximal violation without choosing proper Bell operator as is needed in [12]. On the other hand, for odd N, by adjusting k more entangled quantum states can be detected. In the following we give two examples to show these properties.

Example 1: Consider a 3×3 pure state with Schmidt decomposition $|\psi\rangle = (|11\rangle + |33\rangle)/\sqrt{2}$. Using the Bell operator given in [12] one gets the maximal violation 2, which fails to detect the entanglement. Now taking $k = 2$ we obtain the maximal violation of our Bell inequality (3) $2\sqrt{2}$, which means that $|\psi\rangle$ is entangled.

Our Bell inequality valids also for all mixed states with even N and for some mixed states with odd N. Therefore it can be used to detect experimentally the entanglement of mixed states.

Example 2: Consider that the maximally entangled state $|\psi_+\rangle = \sum_{i=1}^{N} \frac{1}{\sqrt{N}} |ii\rangle$ mixed with noise:

$$\rho(x) = \frac{x}{N^2} I + (1 - x)|\psi_+\rangle\langle\psi_+|. \tag{8}$$

For even N, the maximal violation of $\rho(x)$ is $2\sqrt{2}(1 - x)$. Therefor, Bell inequality (4) detect entanglement of $\rho(x)$ for $0 \leq x < 0.292893$. If N is odd, first note that for any $k \in \{1, 2, \cdots, N\}$ and $\alpha \in \{x, y, z\}$, $(\Gamma_\alpha)_{kk} = 0$. Thus we have

$$\mathrm{Tr}[\rho(x)(\Gamma_\alpha \otimes \Pi(k))] = \mathrm{Tr}[\rho(x)(\Pi(k) \otimes \Gamma_\alpha)] = 0. \tag{9}$$

Taking into account (9) we have the maximal violation

$$\begin{aligned}\max \mathrm{Tr}[\rho(x)B] &= \max_{\vec{a}_1,\vec{a}_2,\vec{b}_1,\vec{b}_2}\Big\{\mathrm{Tr}[\rho(x)(\sum_{\alpha=x,y,z} a_1^\alpha\Gamma_\alpha+\Pi(k))\otimes(\sum_{\beta=x,y,z}(b_1^\beta+b_2^\beta)\Gamma_\beta+2\Pi(k))]\\ &\quad +\mathrm{Tr}[\rho(x)(\sum_{\alpha=x,y,z} a_2^\alpha\Gamma_\alpha+\Pi(k))\otimes(\sum_{\beta=x,y,z}(b_1^\beta-b_2^\beta)\Gamma_\beta)]\Big\}\\ &= \max_{\vec{a}_1,\vec{a}_2,\vec{b}_1,\vec{b}_2}[\vec{a}_1\cdot R(\vec{b}_1+\vec{b}_2)+\vec{a}_2\cdot R(\vec{b}_1-\vec{b}_2)]+2\mathrm{Tr}[\rho(x)\Pi(k)\otimes\Pi(k)]\\ &= \max_{\vec{c}\perp\vec{c}'} 2\sqrt{||R\vec{c}||^2+||R\vec{c}'||^2}+2\mathrm{Tr}[\rho(x)\Pi(k)\otimes\Pi(k)]\\ &= 2\sqrt{\tau_1+\tau_2}+2\mathrm{Tr}[\rho(x)\Pi(k)\otimes\Pi(k)],\end{aligned}$$

where $R_{\alpha\beta}=Tr[\rho\Gamma_\alpha\otimes\Gamma_\beta]$. For $N=3$, the maximal violation of $\rho(x)$ is $\frac{2}{25}(5-4x)+\frac{8\sqrt{2}}{5}(1-x)$. Hence the Bell inequality (4) can detect then the entanglement of $\rho(x)$ for $0\le x<0.2566$ in this case.

We have studied bipartite Bell inequality by constructing new Bell operator which includes the Gisin's Bell inequalities in [12] as a special case. The maximal violation of these Bell inequalities for pure states in Schmidt forms has been obtained. The formulae of maximal violation valid also for all pure and mixed quantum states in even dimensional bipartite systems and for some mixed states in odd dimensional bipartite ones. The new Bell inequality has been shown to be capable of detecting quantum entanglement more effectively.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935).

[2] J. S. Bell, Physics 1, 195 (1964).

[3] N. D. Mermin, Phys. Rev. D 22, 356 (1980); A. Gard and N. D. Mermin, Found. Phys. 14, 1 (1984).

[4] D. Kaszlikowski, P Gnacinski, M. Zukowski, W. Miklaszewski, and A. Zeilinger, Phys. Rev. Lett. 85, 4418 (2000).

[5] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. 88, 040404 (2002).

[6] N. D. Mermin, Phys. Rev. Lett. 65, 1838 (1990).

[7] M. Ardehali, Phys. Rev. A 46, 5375 (1992).

[8] N. J. Cerf, S. Massar, and S. Pironio, Phys. Rev. Lett. 89, 080402 (2002).

[9] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).

[10] N.D. Mermin, Phys. Rev. Lett. 65, 1838 (1990); S.M. Roy and V. Singh, ibid. 67, 2761 (1991); M. Ardehali, Phys. Rev. A 46, 5375 (1992); A.V. Belinskii and D.N. Klyshko, Phys. Usp. 36, 653 (1993); N. Gisin and H. Bechmann-Pasquinucci, Phys. Lett. A 246, 1 (1998).

[11] N. Gisin, Phys. Lett. A, 154, 201(1992).

[12] N. Gisin and A. Peres, Phys. Lett. A, 162, 15(1992).

[13] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. 88, 040404 (2002); A. Acín, T. Durt, N. Gisin, and J. I. Latorre, Phys. Rev. A 65, 052325 (2002).

[14] D. Kaszlikowski, L. C. Kwek, J. L. Chen, M. Zukowski, and C. H. Oh, Phys. Rev. A 65, 032118 (2002).