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**A counter example to weak maximum principles for locally vanishing elliptic operators**

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by


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# A counter example to weak maximum principles for locally vanishing elliptic operators

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**Abstract**

For the validity of the weak maximum principle for classical solutions of elliptic partial differential equations it is sufficient that the coefficient matrix $a^{ij}(x)$ is non-negative. In this note we consider maximum principles for weak solutions of elliptic partial differential equations in divergence form with bounded coefficients $a^{ij}$. We demonstrate that the assumption that the coefficient matrix $a^{ij}(x)$ is positive almost everywhere is essential and cannot be weakened. To this end we give a counter example originating from geometrically linear elasticity.

**Note**

After submission of this paper we learned that there are much simpler examples which demonstrate that the positivity of the coefficient matrix $a^{ij}(x)$ is essential and cannot be weakened. In two space dimensions, let the coefficient matrix be given by

$$a^{ij}(x_1, x_2) = \begin{pmatrix} x_2^2 & -x_1x_2 \\ -x_1x_2 & x_1^2 \end{pmatrix}.$$

Then any $u$ that is some $C^2$-function of $x_1^2 + x_2^2$ is a solution of Equation (1.3) below with $f \equiv 0$. Hence $u$ does not need to satisfy $\sup_\Omega u = \sup_{\partial\Omega} u$. We remark that this example works already for classical solutions of equations in divergence form.

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# 1 Introduction

The weak maximum principle for classical solutions of second order linear elliptic partial differential equations also holds true if the coefficient matrix is non-negative, see [4, Section 3.1] and the short review below. In this article we show that the corresponding weak maximum principle for weak (sub-)solutions of elliptic partial differential equations in divergence form does *not* allow a generalisation to non-negative coefficient matrices, i.e., the strict ellipticity condition on $a^{ij}(x)$ is essential. We do so by constructing a counter example which arises in the context of some non-convex variational problems describing microstructures in crystals.

First we fix the notations and briefly review some weak maximum principles related to our result. Our counter example involves the highest order term of general second-order linear elliptic partial differential equations only. Therefore we restrict the presentation of the weak maximum principles to the highest order term in second-order linear elliptic partial differential equations. We refer to [4, 5, 7, 8] for the treatment of more general cases.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $f \in L^p(\Omega)$ and $a^{ij} \in L^\infty(\Omega)$, $1 \le i, j \le n$ such that $a^{ij} = a^{ji}$ and uniformly for almost every $x \in \Omega$

$$\lambda \, \mathrm{Id} \le (a^{ij}(x))_{1 \le i,j \le n} \le \Lambda \, \mathrm{Id}, \tag{1.1}$$

where Id denotes the identity matrix in $\mathbb{R}^{n \times n}$ and $0 \le \lambda \le \Lambda$ are two real numbers. In case of $\lambda > 0$, this ensures the strict ellipticity of the partial differential equation

$$-a^{ij}(x)\partial_{ij}u(x) = f(x) \quad \text{in } \Omega \tag{1.2}$$

as well as of the related partial differential equation in divergence form

$$-\partial_i\big(a^{ij}(x)\partial_j u(x)\big) = f(x) \quad \text{in } \Omega. \tag{1.3}$$

Here and in the following, the summation convention of repeated indices is applied. If the coefficient matrix is simply given to be positive for almost every $x \in \Omega$, we say that the above partial differential equations are elliptic.

We suppose tacitly in the following that the divergence theorem holds for $\Omega$. This enables us to speak about the weak formulation of the left hand side of (1.3).

Next we recall the weak maximum principle for classical solutions of (1.2).

**Theorem 1.1.** *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be such that*

$$a^{ij}(x)\partial_{ij}u(x) \ge 0 \ (\le 0) \quad \text{in } \Omega$$

*and the coefficient matrix fulfils (1.1) with $\lambda \ge 0$. Then the maximum (minimum) of $u$ in $\overline{\Omega}$ is achieved on $\partial\Omega$, that is*

$$\sup_{\Omega} u = \sup_{\partial\Omega} u \quad (\inf_{\Omega} u = \inf_{\partial\Omega} u).$$

*Proof.* See, e.g., [4, Section 3.1]. □

We emphasise that this theorem holds true for non-negative coefficient matrices. Moreover, if additionally $a^{ij}(x) \in C^1(\Omega)$, Theorem 1.1 can be written in divergence form and it holds true for non-negative coefficient matrices, too, see [4, Section 3.6].

Next we recall the weak maximum principle for weak solutions of second order elliptic partial differential equations in divergence form. As before we restrict ourselves to the highest order term and assume $a^{ij} \in L^\infty(\Omega)$.

**Theorem 1.2.** *Let $u \in H^{1,2}(\Omega)$ be such that*

$$\partial_i \left( a^{ij}(x) \partial_j u(x) \right) \geq 0 \ (\leq 0) \quad \text{in } \Omega \tag{1.4}$$

*and the coefficient matrix is positive for almost every $x \in \Omega$. Then*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ \quad (\inf_\Omega u \geq \inf_{\partial\Omega} u^-),$$

*where $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$.*

*Proof.* See [9] and, e.g., [4, Section 8.1]. □

In Theorem 2.1 below we show that the above theorem does *not* generalise to non-negative coefficients. As a corollary to our result in Theorem 2.1 we obtain that also the maximum principles for weak solutions of elliptic equations in divergence form with a non-zero right hand side do not generalise to non-negative coefficient matrices. To give an example, we recall a recent theorem by Li and Wang [6], which gives an Alexandrov-Bakelman-Pucci maximum principle for elliptic equations in divergence form. To recall this theorem, we first fix some more notions and notations.

As $\Omega$ is bounded, we may assume throughout that there exists an open ball $B_d(0)$ in $\mathbb{R}^n$ with radius $d > 0$ centred at the origin such that

$$\Omega \subset B_d(0).$$

A *weak subsolution* of (1.3) is a function $u \in H^{1,2}(\Omega)$ for which

$$\int_\Omega a^{ij}(x) \partial_j u(x) \partial_i v(x) \, \mathrm{d}x \leq \int_\Omega f(x) v(x) \, \mathrm{d}x \tag{1.5}$$

holds for all $v \in C_0^1(\Omega)$.

Let $Y_\psi$ be the solution of the obstacle problem

$$\text{Find } u \in K_\psi \text{ such that } J(u) = \inf_{w \in K_\psi} J(w),$$

where

$$J(w) := \int_{B_{2d}(0)} a^{ij} \partial_i w \partial_j w \, \mathrm{d}x$$

for $w \in H_0^{1,2}(B_{2d}(0))$ and the set $K_\psi$ is defined by

$$K_\psi := \{w \in H_0^{1,2}(B_{2d}(0)) \mid w \geq \psi \text{ a.e. in } \Omega\}$$

with arbitrary $\psi \in H_0^{1,2}(\Omega)$, $\psi \leq 0$ on $\partial\Omega$.

As is shown in [6, Lemma 1.2], there exists a unique function $Y_\psi \in K_\psi$ with the property

$$J(Y_\psi) = \inf_{w \in K_\psi} J(w). \tag{1.6}$$

**Theorem 1.3.** *Let (1.1) hold for $0 < \lambda \leq \Lambda$, let $f \in L^p(\Omega)$ with $p > \frac{n}{2}$ if $n \geq 4$ or $p = 2$ if $n = 1, 2, 3$, and let $u \in H^{1,2}(\Omega)$ be a weak subsolution of (1.3). Then there exists a constant $C$ depending only on $\lambda$, $n$ and $p$ such that*

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + Cd^{2-\frac{n}{p}} \left( \int_{\{\psi = Y_\psi\} \cap \Omega} (f^+)^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$

*where $\psi := u - \sup_{\partial\Omega} u^+$ and $Y_\psi$ is the function in* (1.6).

*Proof.* See [6][Theorem 1.3]. □

Furthermore we mention the estimate, see [6, Theorem 3.1],

$$\sup_\Omega u \leq \sup_{\partial\Omega} u^+ + Cr^{2-\frac{n}{p}} \|f\|_{L^p(\Omega)}$$

with $|\Omega| = \omega_n r^n$ and $\omega_n$ denoting the volume of the unit ball in $\mathbb{R}^n$, which is similar to Theorem 1.3 but with a different right hand side. Our counter example applies for this altered formulation, too. That is, we deduce that those theorems cannot be extended to non-negative coefficients, see Corollary 2.2.

# 2 Non-negative coefficient matrices

We shall prove by a counter example in space dimensions $n \leq 3$ that the assumption on the positivity of $a^{ij}$ in Theorem 1.2 is essential and may not be replaced by $a^{ij}$ being non-negative. We expect that our result holds for arbitrary space dimensions $n$. In particular, Theorem 1.2 is violated when the set

$$\left\{x \in \Omega \;\middle|\; a^{ij}(x) = 0\right\}$$

has positive measure.

For this counter example we take resort to specific elastic energy functionals and make use of the explicit formulas for the relaxation of those functionals found in [3] for space dimensions $n = 2, 3$. For $n = 1$ we newly derive the explicit formula of the relaxed elastic energy in the appendix. Subsequently we introduce our example and illuminate the physical background.

The steady state of elastically stressed solids can be characterised as the minimiser of an elastic energy functional in the vector-valued variable $u \in$

$H^{1,2}(\Omega;\mathbb{R}^n)$ that represents the deformation of the material with respect to the chosen reference state. Due to the inherent rotational symmetry (frame indifference), the Euler-Lagrange equation leads to an elliptic partial differential equation in the (linearised) strain

$$\varepsilon = \varepsilon(u) := \frac{1}{2}\left(\nabla u + \nabla^t u\right),$$

which defines a symmetric $n \times n$-matrix. Let $A : B := \mathrm{tr}(A^t B)$ denote the scalar product between symmetric $n \times n$ matrices $A$, $B$. For the construction of the counter example we consider a solid consisting of two homogeneous constituents with different mechanical properties, as studied for instance in [1, 3]. For constituent $k \in \{1, 2\}$, the elastic energy density is given by

$$W_k(\tilde{\varepsilon}) := \frac{1}{2}\alpha_k\left(\tilde{\varepsilon} - \varepsilon_k^T\right) : \left(\tilde{\varepsilon} - \varepsilon_k^T\right) + w_k, \quad \tilde{\varepsilon} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}, \tag{2.1}$$

and $\alpha_k$, $\varepsilon_k^T$, $w_k \geq 0$ are material parameters which denote the elasticity tensor, the transformation strain tensor and the value of the local minima, respectively.

The material can form microstructure which is described by a function $\tilde{d} : \Omega \to \{0, 1\}$ which gives the spatial distribution of the two phases. In consequence, the energy on the micro scale reads

$$W(\tilde{d}, \tilde{\varepsilon}) := \tilde{d} W_1(\tilde{\varepsilon}) + \left(1 - \tilde{d}\right) W_2(\tilde{\varepsilon}). \tag{2.2}$$

As introduced and explained in [3], the relaxation of the effective elastic energy functional for $d \in [0, 1]$ and $\varepsilon \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$ is given by

$$\widehat{W}(d, \varepsilon) := \inf_{\substack{\langle \tilde{d}\rangle = d \\ \tilde{d}\in\{0,1\}}} \inf_{\tilde{u}_{|\partial\Omega} = \varepsilon x} \fint_\Omega W\left(\tilde{d}, \varepsilon(\tilde{u})\right) \mathrm{d}x, \tag{2.3}$$

where $\langle \tilde{d}\rangle := \fint_\Omega \tilde{d}(x)\,\mathrm{d}x := \frac{1}{|\Omega|}\int_\Omega \tilde{d}(x)\,\mathrm{d}x$. The infimum over $\tilde{d}$ is the result of homogenisation subject to the constraint that the volume fraction of the selected phase is preset by $d$. This infimum is taken over functions $\tilde{d}$ with bounded variation in $\Omega$ and values 0 or 1 a.e.,

$$\tilde{d} \in BV(\Omega;\, \{0, 1\}), \tag{2.4}$$

ensuring $\widehat{W} \geq 0$. Due to (2.4), the definition (2.3) is meaningful only for $d \in [0, 1]$. The second infimum in (2.3) is taken over functions $\tilde{u} \in H^{1,2}(\Omega;\mathbb{R}^n)$, where the condition $\tilde{u}_{|\partial\Omega} = \varepsilon x$ has to be read as $\tilde{u}(x) = \varepsilon x$ for a.e. $x \in \partial\Omega$.

The minimisation problem (2.3) gives rise to an elliptic partial differential equation in divergence form (1.3) with coefficients in $L^\infty(\Omega)$, see Eqn. (2.5) below, where due to (2.4) the ellipticity condition (1.1) holds with $\lambda = 0$ on a set of positive measure. We will show that the extension of Theorem 1.2 to $0 \leq \lambda$ contradicts the results in [3], i.e. the assumption $a^{ij}$ being positive almost everywhere in Theorem 1.2 is essential.

**Theorem 2.1.** *Let $n \le 3$ and let $a^{ij}$ be non-negative. If $u \in H^{1,2}(\Omega)$ is such that* (1.4) *is satisfied, then the conclusion of Theorem 1.2 does* not *hold true in general.*

*Proof.* Set $\tilde{d}_1 \equiv \tilde{d}$, $\tilde{d}_2 \equiv (1-\tilde{d})$. Then (2.1)–(2.3) yield for given $d_0 \in [0,1]$, $\varepsilon_0 \in \mathbb{R}^{n\times n}_{\text{sym}}$

$$\begin{aligned}\widehat{W}(d_0,\varepsilon_0) &:= \inf_{\substack{\langle \tilde{d}\rangle = d_0 \\ \tilde{d}\in\{0,1\}}} \inf_{\tilde{u}_{|\partial\Omega}=\varepsilon_0 x} ⨍_\Omega \tilde{d}_1 W_1(\varepsilon(\tilde{u})) + \tilde{d}_2 W_2(\varepsilon(\tilde{u}))\,\mathrm{d}x \\ &= \inf_{\substack{\langle \tilde{d}\rangle = d_0 \\ \tilde{d}\in\{0,1\}}} \inf_{\tilde{u}_{|\partial\Omega}=\varepsilon_0 x} ⨍_\Omega \sum_{k=1}^2 \tilde{d}_k \frac{\alpha_k}{2}\left(\varepsilon(\tilde{u})-\varepsilon_k^T\right):\left(\varepsilon(\tilde{u})-\varepsilon_k^T\right)\,\mathrm{d}x.\end{aligned}$$

We look at the minimisation problem in $\tilde{u}$, i.e., for fixed $\tilde{d} \in BV(\Omega;\{0,1\})$ with $\langle \tilde{d}\rangle = d_0$, we consider

$$E_{\tilde{d}}(\tilde{u}) := ⨍_\Omega \sum_{k=1}^2 \tilde{d}_k \frac{\alpha_k}{2}\left(\varepsilon(\tilde{u})-\varepsilon_k^T\right):\left(\varepsilon(\tilde{u})-\varepsilon_k^T\right)\,\mathrm{d}x,$$

which we need to minimise over all $\tilde{u} \in H^{1,2}(\Omega;\mathbb{R}^n)$ with $\tilde{u}(x) = \varepsilon_0 x$ for a.e. $x \in \partial\Omega$. A necessary condition for the optimal $u = u_{\text{opt}}$ is that the first variation equals zero,

$$\int_\Omega \sum_{k=1}^2 \tilde{d}_k \alpha_k \left(\varepsilon(u)-\varepsilon_k^T\right) : \varepsilon(\zeta)\,\mathrm{d}x = 0 \quad \text{for all } \zeta \in H^{1,2}(\Omega;\mathbb{R}^n). \tag{2.5}$$

Next we would like to integrate (2.5) by parts in order to obtain a partial differential equation of the form (1.3). Since $\tilde{d}_k \in BV(\Omega;\{0,1\})$ yields additional terms, we consider a regularized problem. Let $\delta > 0$. According to [10, Theorem 5.3.3] there exists a function $\tilde{d}^\delta \in C^\infty(\Omega)$ such that $0 \le \tilde{d}^\delta \le 1$ and

$$\lim_{\delta\searrow 0}\int_\Omega \left|\tilde{d}-\tilde{d}^\delta\right|\,\mathrm{d}x = 0, \qquad \lim_{\delta\searrow 0}\left\|\nabla\tilde{d}^\delta\right\| = \left\|\nabla\tilde{d}\right\|, \tag{2.6}$$

where $\left\|\nabla\tilde{d}\right\|$ denotes the total variation of $u \in BV(\Omega;\{0,1\})$. With $\tilde{d}_1^\delta := \tilde{d}^\delta$, $\tilde{d}_2^\delta := 1-\tilde{d}^\delta$, Eqn. (2.5) becomes

$$\int_\Omega \sum_{k=1}^2 \tilde{d}_k^\delta \alpha_k \left(\varepsilon(u)-\varepsilon_k^T\right) : \varepsilon(\zeta)\,\mathrm{d}x = 0 \quad \text{for all } \zeta \in H^{1,2}(\Omega;\mathbb{R}^n). \tag{2.7}$$

The corresponding Euler-Lagrange equations are

$$\begin{aligned}\operatorname{div}\left(\sum_{k=1}^2 \tilde{d}_k^\delta \alpha_k \varepsilon(u)\right) &= \sum_{k=1}^2 \alpha_k \varepsilon_k^T \nabla \tilde{d}_k^\delta \quad \text{in } \Omega, \\ u(x) &= \varepsilon_0 x, \qquad\qquad x \in \partial\Omega,\end{aligned} \tag{2.8}$$

which form a system of elliptic partial differential equations in divergence form with coefficients in $L^\infty(\Omega)$ that vanish locally.

Next we want to apply Theorem 1.2 to this system of partial differential equations. To this end, we choose $\varepsilon_1^T = \varepsilon_2^T = 0$ and make the assumptions $\alpha_k \in \mathbb{R}_{>0}$, $k = 1, 2$, and $(\nabla u)^t = \nabla u$. This ensures that the system (2.8) decouples and that the right hand side is zero. Then (2.8) reads

$$\left.\begin{aligned} \operatorname{div}\left(\sum_{k=1}^{2} \tilde{d}_k^\delta \alpha_k \nabla u_\ell\right) &= 0 \qquad \text{in } \Omega, \\ u_\ell(x) &= (\varepsilon_0 x)_\ell\,, \quad x \in \partial\Omega, \end{aligned}\right\} \quad \ell = 1, \dots, n. \tag{2.9}$$

For each $\ell$, this corresponds to (1.3) with $f = 0$ and $a^{ij}(x) = -\sum_{k=1}^2 \tilde{d}_k^\delta(x)\alpha_k\delta^{ij}$, where $\delta^{ij}$ denotes the Kronecker Delta.

If Theorem 1.2 extended to non-negative coefficients, it would hold $u_\ell(x) = (\varepsilon_0 x)_\ell$ in $\overline{\Omega}$ for a weak (sub-)solution $u_\ell \in H^{1,2}(\Omega)$ of (2.9). Indeed, the solution to (2.9) would be unique, and $u_\ell(x) = (\varepsilon_0 x)_\ell$ solves the partial differential equation $(2.9)_1$[1] and satisfies the boundary conditions $(2.9)_2$. Then also (2.7) with $\varepsilon_k^T = 0$, $\alpha_k \in \mathbb{R}_{>0}$ for $k = 1, 2$ and with $(\nabla u)^t = \nabla u$ would be fulfiled with $\varepsilon(u_{\text{opt}}) = \nabla u_{\text{opt}} = \varepsilon_0$ in $\overline{\Omega}$ (as follows by an integration by parts). So we would obtain

$$\begin{aligned} E_{\tilde{d}^\delta}(u) &= \fint_\Omega \sum_{k=1}^{2} \tilde{d}_k^\delta \frac{\alpha_k}{2} \varepsilon_0 : \varepsilon_0 \,\mathrm{d}x \\ &= \sum_{k=1}^{2} \frac{\alpha_k}{2} \varepsilon_0 : \varepsilon_0 \fint_\Omega \tilde{d}_k^\delta \,\mathrm{d}x \\ &\longrightarrow d_0 W_1(\varepsilon_0) + (1 - d_0) W_2(\varepsilon_0) = E_{\tilde{d}}(u) \qquad \text{for } \delta \searrow 0, \end{aligned}$$

since, by (2.6), $\fint_\Omega \tilde{d}^\delta(x)\,\mathrm{d}x \longrightarrow \fint_\Omega \tilde{d}(x)\,\mathrm{d}x = d_0$ as $\delta \searrow 0$. Consequently,

$$\widehat{W}(d_0, \varepsilon_0) = d_0 W_1(\varepsilon_0) + (1 - d_0) W_2(\varepsilon_0). \tag{2.10}$$

It turns out that the identity (2.10) is in contradiction to the explicit formulas of $\widehat{W}(d_0, \varepsilon_0)$ for $n = 2, 3$ in [3] and for $n = 1$ derived in the appendix. For illustration, if $n = 2$, we have

$$\widehat{W}(d_0, \varepsilon_0) = d_0 W_1(\varepsilon_1^*) + (1 - d_0) W_2(\varepsilon_2^*) + \beta^* d_0(1 - d_0) \det(\varepsilon_2^* - \varepsilon_1^*), \tag{2.11}$$

where $\beta^* \in \mathbb{R}$ and $\varepsilon_k^* \in \mathbb{R}^{n\times n}_{\text{sym}}$ depend in an involved nonlinear way on $d_0$, $\varepsilon_0$ and $\alpha_k$, see the appendix, in particular (2.12), for details. The formulas (2.10) and (2.11) coincide if and only if $\varepsilon_k^* = \varepsilon_0$, $k = 1, 2$. From $\varepsilon_1^* = \varepsilon_2^*$ we learn, cf. (2.13),

$$(\alpha_2 - \alpha_1)\varepsilon_0 = \left(\alpha_2 \varepsilon_2^T - \alpha_1 \varepsilon_1^T\right) = 0$$

[1] A note after submission of this paper: with the definition of $\tilde{\delta}_k^\delta$ as in the text, $u_l(x) = (\varepsilon_0 x)_l$ is *not* a solution of (2.9) as wrongly stated.

for all $\varepsilon_0 \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, which yields $\alpha_1 = \alpha_2$. Thus (2.10) and (2.11) are different in general.

Similarly, if $n = 1$, we obtain equality of (2.10) (with $d_0$ replaced by $d$ and $\varepsilon_0$ replaced by $e$) and (2.16) in the appendix if and only if $e^*_k = e$, $k = 1, 2$, which yields the constraint $\alpha_1 = \alpha_2$. So the two formulas for $\widehat{W}$ are different in general.

The same conclusion can be drawn for $n = 3$, taking resort to the explicit formula of $\widehat{W}(d_0, \varepsilon_0)$ in [3]. Hence the conclusion of Theorem 1.2 does not hold in general when $a^{ij}(x)$ vanishes on a set with positive measure. □

**Corollary 2.2.** Let $n \le 3$ and let (1.1) hold for $0 \le \lambda \le \Lambda$. If $u \in H^{1,2}(\Omega)$ is a weak subsolution of (1.3) for $f \in L^2(\Omega)$, the conclusion of Theorem 1.3 does *not* hold true.

*Proof.* With $f = 0$, the same proof as of Theorem 2.1 applies. □

# Appendix

In this appendix we derive the explicit formula for $\widehat{W}(d, \varepsilon)$ if $n = 1$. We start from the explicit formula (2.11) in two dimensions and consider situations where the input data such as the applied macroscopic strain or the eigenstrains can be projected to a one-dimensional manifold.

First we recall some notations and refer to [2] for further details. The scalar $\beta^* \in [0, \gamma^*]$ determines certain classes of microstructures in 2D, where the constant $\gamma^* > 0$ depends on the elastic moduli $\alpha_k$ only. We set

$$\begin{aligned} \varepsilon^*_1(d,\varepsilon) &= \alpha^{-1}(\beta^*, d)[(\alpha_2 - \beta^* Q)\varepsilon - (1-d)\left(\alpha_2\varepsilon_2^T - \alpha_1\varepsilon_1^T\right)], \\ \varepsilon^*_2(d,\varepsilon) &= \alpha^{-1}(\beta^*, d)[(\alpha_1 - \beta^* Q)\varepsilon + d\left(\alpha_2\varepsilon_2^T - \alpha_1\varepsilon_1^T\right)], \end{aligned} \tag{2.12}$$

where $\alpha(\beta^*, d) := (1-d)\alpha_1 + d\alpha_2 - \beta^* Q$ with $Q\varepsilon := \varepsilon - \mathrm{tr}(\varepsilon)\mathrm{Id}$, $\varepsilon \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$. The function $\varphi(\beta^*, d, \varepsilon) := -\det(\varepsilon^*_2 - \varepsilon^*_1)$ plays an important role, compare also with (2.11). As explained thoroughly in [2], Regime 0, where there is no dependence on the microstructure, is characterised by $\varphi \equiv 0$; Regime I, where there are two optimal laminates of rank I, is defined by $\varphi(0, d, \varepsilon) > 0$; Regime III, where two optimal microstructures of rank II exist, is defined by $\varphi(\gamma^*, d, \varepsilon) < 0$; and finally Regime II, where there is a unique microstructure of rank I, is characterised by $\varphi(0, d, \varepsilon) \le 0$ and $\varphi(\gamma^*, d, \varepsilon) \ge 0$. In particular, in Regime II, there is a unique solution $\beta_{II}$ of $\varphi(\cdot, d, \varepsilon) = 0$.

To derive the formula in one space dimension, we pick data constant in the $y$-direction such that $\varepsilon = \mathrm{diag}(e, 0)$, $\varepsilon^T_1 = \mathrm{diag}\left(e^T_1, 0\right)$, $\varepsilon_2 = \mathrm{diag}\left(e^T_2, 0\right)$ with $e, e^T_1, e^T_2 \in \mathbb{R}$. Moreover, we require that the stresses $\alpha_k\varepsilon$ are diagonal matrices of the form $\mathrm{diag}(\cdot, 0)$. This allows us to assume that $C_{1,12} = C_{2,12} = 0$, where we use a scaled version of the Voigt notation, see e.g. [1], and restrict ourselves to a cubic material.

Under these assumptions, we will verify below that $\varepsilon^*_1$, $\varepsilon^*_2$ are also contained in the selected one-dimensional submanifold of $\mathbb{R}^2$.

Using this fact we first prove that Regime II is of no relevance in 1D. Indeed, in order to compute the value of $\beta^*$ in Regime II in 1D, cf. [2], we observe that

$$\begin{aligned}&(\varepsilon_1^* - \varepsilon_2^*)(d,\varepsilon) = 0\\ &\qquad \Leftrightarrow (\alpha_2 - \beta^* Q)\varepsilon - (1-d)\left(\alpha_2\varepsilon_2^T - \alpha_1\varepsilon_1^T\right) = (\alpha_1 - \beta^* Q)\varepsilon + d\left(\alpha_2\varepsilon_2^T - \alpha_1\varepsilon_1^T\right),\end{aligned}$$

which is equivalent to

$$(\alpha_2 - \alpha_1)\varepsilon = \left(\alpha_2\varepsilon_2^T - \alpha_1\varepsilon_1^T\right). \tag{2.13}$$

For $\alpha_1 = \alpha_2$ we then obtain $\varphi \equiv 0$, i.e. we are in Regime 0. Otherwise we obtain

$$\varepsilon = (\alpha_2 - \alpha_1)^{-1}\left(\alpha_2\varepsilon_2^T - \alpha_1\varepsilon_1^T\right). \tag{2.14}$$

By formula (2.14) only $\varepsilon$ is determined; the other two parameters $\beta^*$ and $d$ for Regime II are free. Hence there is no unique $\beta_{II}$ solving $\varphi(\cdot, d, \varepsilon) = 0$, i.e., Regime II does not occur. Furthermore, since rank-II-laminates cannot occur in 1D, Regime III does not play a role in this case. Hence $\gamma^* = 0$. Thus $\beta^*$ is identically zero in one dimension.

With $\beta^* = 0$, taking resort to the Voigt notation (see the computations in [2]), we show that for $k = 1, 2$

$$\varepsilon_k^*(d,\varepsilon) = \mathrm{diag}(e_k^*, 0). \tag{2.15}$$

Indeed,

$$\alpha_k\varepsilon = \begin{pmatrix} C_{k,11} & 0 & 0\\ 0 & C_{k,11} & 0\\ 0 & 0 & 2C_{k,44}\end{pmatrix}\begin{pmatrix} e\\ 0\\ 0\end{pmatrix} = \begin{pmatrix} C_{k,11}e\\ 0\\ 0\end{pmatrix}.$$

Thus

$$\begin{aligned}\alpha(0,d) &= (1-d)\alpha_1 + d\alpha_2\\ &= \mathrm{diag}(dC_{1,11} + (1-d)C_{2,11}, dC_{1,11} + (1-d)C_{2,11}, 2dC_{1,44} + 2(1-d)C_{2,44})\end{aligned}$$

and $\alpha_2\varepsilon_2^T - \alpha_1\varepsilon_1^T = \left(C_{2,11}e_2^T - C_{1,11}e_1^T, 0, 0\right)$. Hence, by (2.12), we obtain (2.15). Finally (2.11) now yields the formula in 1D

$$\widehat{W}(d,e) = dW_1(e_1^*) + (1-d)W_2(e_2^*) \tag{2.16}$$

for all $d \in [0,1]$ and $e \in \mathbb{R}$.

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