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The tragedy of the commons in a
multi-population complementarity game

by

Jürgen Jost, and Wei Li

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Wei Li

Complexity Science Center, Hua-Zhong Normal University,
Wuhan 430079, P.R. China

Jürgen Jost

Max Planck Institute for Mathematics in the Sciences,
Inselstr.22, 04103 Leipzig, Germany

We study a complementarity game with multiple populations whose members' offered contributions are put together towards some common aim. When the sum of the players' offers reaches or exceeds some threshold K, they each receive K minus their own offers. Else, they all receive nothing. Each player tries to offer as little as possible, hoping that the sum of the contributions still reaches K, however. The game is symmetric at the individual level, but has many equilibria that are more or less favorable to the members of certain populations. In particular, it is possible that the members of one or several populations do not contribute anything, a behavior called defecting, while the others still contribute enough to reach the threshold. Which of these equilibria then is attained is decided by the dynamics at the population level that in turn depends on the strategic options the players possess. We find that defecting occurs when more than 3 populations participate in the game, even when the strategy scheme employed is very simple, if certain conditions for the system parameters are satisfied. The results are obtained through systematic simulations.

I.   INTRODUCTION

We consider a complementarity game with several populations that need to cooperate to achieve some common goal. In each round, by random assignment, groups of I individuals are formed, one from each population. Here, I of course is the number of participating populations. Each agent i offers to contribute k_i units.[?] In order to reach the common goal, at least K units have to be provided. The question then is which of the partners should contribute how much to reach that level K. Thus, the contributions should satisfy

$$\sum_i k_i \geq K. \tag{1}$$

The pay-off for agent i then is

$$K - k_i. \tag{2}$$

The pay-off is 0 when (1) is not satisfied. Thus, in order to maximize her/his pay-off, each agent wants to contribute as little as possible, provided however, that the joint contribution of the two partners satisfies (1). In particular, in this game, any collection of values k_i with

$$\sum_i k_i = K \tag{3}$$

yield a Nash equilibrium. This means that neither agent can gain by unilaterally deviating from it when the other player does not change her/his contribution. Increasing the own contribution would simply be wasteful, as it lowers the pay-off (2), and decreasing it would make the entire endeavor fail, by violating (1), and the pay-off would shrink to 0.

Thus, there are multiple Nash equilibria. The question then is which one of them is attained. The fairest equilibrium seems to be

$$k_i = \frac{K}{I}, \tag{4}$$

that is, where the partners contribute equally.[?] There is nothing that guarantees, however, that such a fair situation will be achieved in groups of selfish players. When the players can negotiate, the outcome will depend on their respective negotiating strength, their bargaining position. Of course, assuming such a bargaining position will already bring additional elements into the game. When, for instance, one of the players is in a position to sustain higher losses, then their pay-offs are no longer really equal, because even though the numbers are the same, these

numbers matter less to some than to other players.

In any case, in a single-shot game without additional elements that affect the players differently, there is no way to decide between the different possible equilibria. This might change when the game is played repeatedly. Everything else being equal, one could, for instance, guess that in the long run, things will average out, and the symmetric equilibrium (4) might be achieved. Of course, playing the game repeatedly enlarges the strategy space, in the sense that a player in a given round will choose an action that may also affect her/his future pay-offs. She/he may therefore try to induce the opponents to make unfavorable, i.e., higher contribution in subsequent rounds, for instance by playing low her/himself, even though that might cause failure in the present round. Of course, when the opponents are doing the same, neither would gain any pay-off, and the question then might be which of them can hold out longer. On the other hand, from observing the opponents over a couple of rounds, a player may also try to gain some insights into the opponents' behavior and exploit observed regularities in that behavior. The opponents, however, again will do the same. Thus, the players will try to mutually adapt to each other. This will then trigger an interesting dynamics when the number of rounds played grows. The players may utilize rather complex strategies then, for example for making predictions of the opponents' future behavior on the basis of observations of many rounds. In simulations, one may then break the symmetry by allowing the players access to different strategy types. In that way, one can investigate what type of strategy is better than others.

We can then also set up an evolutionary process on the basis of some evaluation criterion. When we have populations of players, it is natural to let each player not always play the same opponent, but rather pair her/him with randomly selected opponents in each round. In that way, the population gets sampled, and many effects will get averaged out so that the essential structure of the game will emerge. Again, each player is allowed to play a fixed number of rounds, the same for everybody so that we can compare their accumulated pay-offs. On the basis of that evaluation, the evolutionary step then produces the next generation of players. The more successful ones have more offspring than the others, and the poor performers may not get any offspring at all.

There are some sources of randomness involved. The members of the populations are randomly initialized. For instance, in the simplest case, each agent a just can play one simple number k_a which might be randomly drawn from the integers between 0 and K, independently and identically for each agent. Also, the pairing of the agents from the different population in each round is done randomly. Finally, the evolutionary step contains some random selection and mutation scheme. In [3], we have systematically investigated the effects of the various sources of randomness. Therefore, here we shall not address this issue in detail.

Concerning the population, we have two options. We could either work with one single population in which I players get randomly grouped in each round, a game being successful when the sum of their contributions is at least K, see (1), and receiving pay-offs according to (2). Or we could work with distinct populations so that in each round, groups of I players, one from each population, are randomly chosen to play the game.

In the first case, we could compose the – single – population of different strategy types and see how they perform and how the population evolves. In the second case, we could also equip the populations with access to different strategy spaces and study how those perform at the population level. In that case, we not only have an – indirect – competition for accumulated pay-off within each population, which decides about the evolutionary update, but also a competition between the populations for achieving a favorable equilibrium value. Thus, we can see the interaction of competitions at two different scales, individuals struggling inside their own population to be more successful than others, and populations or strategy types competing with each other at a cumulative scale.

In particular, we can investigate in a quantitative manner a version of the tragedy of the commons. This is concerned with situations where individual contributions are needed for a collective benefit, but individuals can gain an advantage by participating in the benefit without contributing. Others then need to contribute if the benefit is to be sustained. Typically, however, in the end nobody will contribute and therefore also nobody will benefit. In our scenario, when some agents will stop contributing, it is still advantageous for the other ones to contribute more and fill the gap, as long as at least two agents contribute. In turn, when only two contribute, neither of them can gain an advantage by stopping her/his contributions. Thus, our question here is under which circumstances some agents can discover that by contributing less, they will force the other ones to contribute more. The problem for the agents is that when at an equilibrium, one of them will suddenly decrease or terminate her/his contributions, then all of them will loose, including the defector. Therefore, the tendency to defect needs to start already before the equilibrium is reached, and in turn it will then affect that equilibrium.

Also, the fact that in a completely symmetric situation, some agents might start to contribute less or stop contributing, is a phenomenon of symmetry breaking due to random fluctuations which can be investigated here in a simple setting. In particular, we shall see that whereas the equilibrium reached for two or even three populations tends to be symmetric, this is no longer the case for more populations. Thus, in our scenario, the tragedy of the commons results only when there are sufficiently many participants.

II. DEFECTING

As explained in the introduction, in each instance of the game, we form a group of I players, one from each of the I different populations. To start with, we let each population have the same number N of agents. Therefore, in each round, we can let N such disjoint groups play simultaneously. This ensures that every player plays the same number of times. We then fix a generation time T. That is, after playing T rounds, the sum of the pay-offs over these round is computed for each agent. In each population, we then form a new generation by some evolutionary scheme, that is, the expected number of offspring in the next generation is a positive function of an agent's accumulated pay-off. The population size N will be kept constant across generations. Thus, successful agents could have more than one offspring whereas less successful might not have any. Thus, the populations will evolve by rewarding the more successful agents at the expense of the others within each population.

Here, we consider the simplest case where all populations have access to the same strategy space. The question we want to address then is whether the fair equilibrium (4) is attained or not. Our previous investigations [1, 2] indicate that the equilibrium (4) is stable for two populations.[?] Here, we investigate the situation with 3 and more populations.

After sufficiently many generations, the game dynamics will converge to some equilibrium where the players of each fixed population always contribute the same offer k_i such that

$$\sum_i k_i \geq K \tag{5}$$

(because of some random fluctuations in the course of the dynamics, usually the sum is slightly larger than K, that is, the populations use some safety margin in their offers). Whenever the populations settle at such an equilibrium, no single player can gain any advantage from deviating from it (except for narrowing down the safety margin, but we shall ignore that issue in our discussion). The equilibrium is a Nash equilibrium (see e.g. [4] for game theoretic concepts utilized in our discussion). At the equilibrium, however, the contributions k_i coming from the different populations need not all be equal. That is, inside each population, the agents are homogeneous, but the populations themselves may be different from each other.

We find that, when we have 4 or more populations, the members of some populations discover the possibility of defecting. This means that the final equilibrium reached has at least one of the $k_i = 0$. Since the populations are symmetric to each other as regards their sizes and strategy spaces, which of them discovers the possibility of defecting is solely determined by random fluctuations in the individual conditions or the dynamics. What we are interested therefore is not which population defects, but how many of them do.

For the resulting equilibrium, we find a minimal breaking of symmetry. That is, after relabelling, there will be I_1 populations with $k_i = \frac{K}{I_1}$ for $1 \leq i \leq I_1$ and $I - I_1$ populations with $k_j = 0$ for $I_1 + 1 \leq j \leq I$. That is, $I - I_1$ populations defect completely, whereas the other ones contribute equally.

III. SIMULATIONS

As already mentioned in the previous sections, here we utilize a very simple strategy to study the defecting behavior when more than two populations participate in the game. There are at least two advantages by starting from a simple strategy. First, we can identify the critical factors which may lead to defecting, if available, more efficiently than we can when faced with a case of a more complex strategy. Second, simple strategies usually may allow for some analytical understanding, which is of course more appealing. In this simple strategy, initially all the members of each population choose random numbers uniformly distributed between 0 and K. After certain rounds of interactions, say T, their payoffs will be compared and the more successful players will be more represented in the next generation, namely having more offsprings. It is very probable that some poorly performed players may not have any offsprings at certain and therefore will be eliminated right away. To maintain the diversity of parents pool, we allow random mutations to occurs, but only with a tiny probability. In such a simple scheme one would expect that the convergence to the equilibrium is efficient when the population size is modest.

In our simulations, the random mutation rate is kept fixed to be 1 percent as usual. But other parameters such as the generation time T, the population size N, the maximum offer K and the number of populations I are not fixed. By varying those parameters we simply intend to investigate how the defecting, once emerges, depends on certain parameters. Or, how will the interplay between the parameters may finally culture the selfish defecting behavior? Which parameters are playing the leading role?

We have already known that the defecting does not appear in the two-population version of our game. That is, each population has to make certain contributions in order to reach a favorable goal. It is not possible for defecting to

occur under totally symmetric situations. We have checked various possibilities of parameters' combination as I, the number of populations, is increased up to 3, of course, under symmetric conditions. But the 3-population game is very similar to the 2-population one. That is, there is certainly no chance for defecting. In most cases, the equilibrium ends up where each population shares the equal contribution.

If I is increased up to 4, some interesting phenomena show up. Namely, we find that defecting, that is at least one population contributes nothing, may emerge once certain critical condition, regarding T, N and K, is satisfied. The first requirement is enforced upon the generation time T, which should be as small as possible (better be 1 which corresponds to immediate comparison of interaction outcomes). If T is too large, the defecting tendency will be suppressed as the accumulated pay-offs of the players who intend to defect will be smaller than the counterparts of those who contribute. Consequently the defecting tendency will be less rewarded and eventually gets eliminated before the equilibrium is reached. This also indicates that the tendency to defect needs to start from the beginning and only takes effects if it is not eliminated too early. The second requirement is related to both N and K. We find at a certain K, there is a critical threshold of N for the occurrence of defecting. Or equivalently for a fixed N, there is a critical threshold of K. This condition is quite natural as N/K is proportional to the initial number of defectors (those who offers 0) within a certain population. This value has to be large enough so that the defecting behavior can be spread within the system and eventually dominates, which generates a uniform population whose members only offers 0. This condition is reasonable as well if one makes a connection between the epidemic spreading and the spreading of defecting behavior in our game.

In Fig. 1 we show one simulation of the 4-population game. We see that at the start of the game the situation is nearly symmetric, except for some fluctuations. But after some rounds of interactions, one population starts to offer less than the rest of populations do, probably due to some random fluctuations, though rather small. Apparently that small chance grows very fast and eventually the whole population contributes nothing whereas the rest populations contribute almost equally. The 5-population game is no more different than the 4-population one except that 2 populations may be defectors and the rest three are contributors who could offer differently due to fluctuations accumulated.

IV. CONCLUSION

We have investigated a game where several players need to contribute enough so that the sum of the contributions reaches or exceeds some given threshold. Each player thus hopes that she/he can get away with a small contribution while the other players contribute so much that the threshold is still reached. This is the same situation for every player. Thus, the game is completely symmetric. In particular, it possesses a symmetric Nash equilibrium where all contribute equally. However, this is not the only equilibrium. Any state where the sum of the contributions is equal to the threshold is an equilibrium. This includes states where some players do not contribute anything while the others then have to make up the deficit. This is a version of the tragedy of the commons. The question addressed in this contribution then is under which circumstances defecting sets in an evolutionary population game among equal agents. That is, when do the members of some agent populations by chance, that is, as a result of random fluctuations in the evolution of the game, discover that they can contribute less and in turn force the others to contribute more? We have found that under rather general conditions, such a defecting behavior is found for 4 or more players. That is, our evolutionary population game produces a phase transition towards the tragedy of the commons.

[1] J. Jost and W. Li, Individual strategies in complementarity games and population dynamics, Physica A **345**, 245-266 (2005).
[2] J. Jost and W. Li, Learning, evolution and population dynamics, Adv.Complex Syst.11, 901–926 (2008)
[3] J. Jost and W. Li, Randomness, heterogeneity and population dynamics
[4] J.Weibull, Evolutionary game theory, MIT Press, 1995
[] All numbers will be nonnegative integers $\leq K$.
[] Of course, $\frac{K}{I}$ need not be an integer, so that, more precisely, k_i should be smallest integer $\geq \frac{K}{I}$, but we shall choose K and ignore this trivial technical point.
[] The game considered in these references uses two players (called "buyers" and "sellers") that have to make offers b, s between 0 and some maximum K. When $b \geq s$, the first player gets $K - b$, the second one s, else they get nothing. Putting $k_1 = b$, $k_2 = K - s$, that game is converted into the one considered in the present paper.
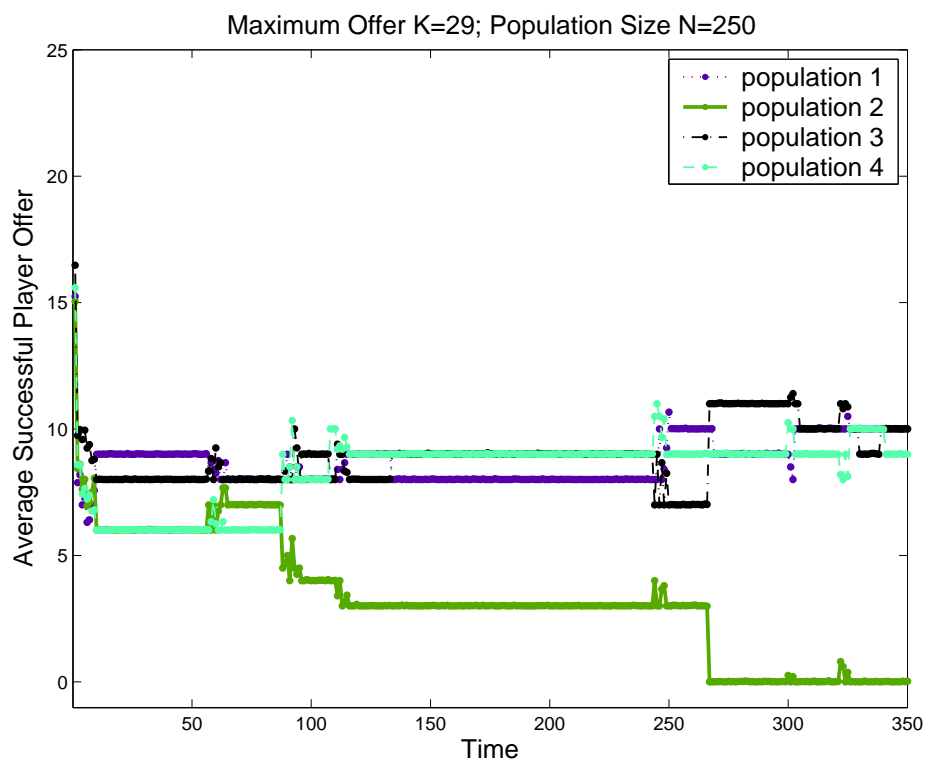
FIG. 1: The defecting behavior of the 4-population game. Here the population size N is 250 and the maximum offer K is 29.
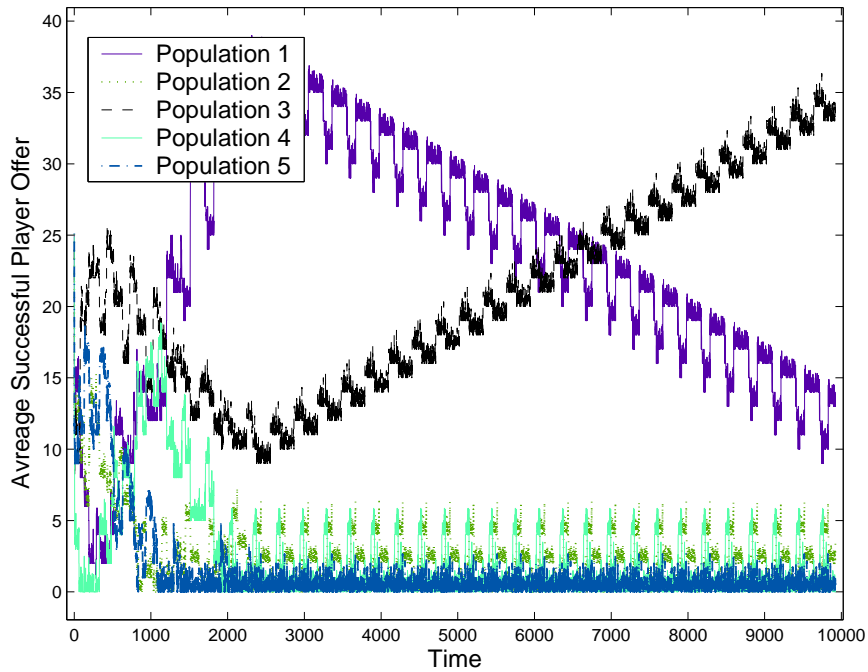
FIG. 2: The defecting behavior of the 5-population game. Here the population size N is 1000 and the maximum offer K is 49.