Stability result for abstract evolution problems

by

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Abstract

Consider an abstract evolution problem in a Hilbert space $H$

\[ \dot{u} = A(t)u + G(t,u) + f(t), \quad u(0) = u_0, \]  

(1)

where $A(t)$ is a linear, closed, densely defined operator in $H$ with domain independent of $t \geq 0$, $G(t,u)$ is a nonlinear operator such that $\|G(t,u)\| \leq a(t)\|u\|^p$, $p = \text{const} > 1$, $\|f(t)\| \leq b(t)$. We allow the spectrum of $A(t)$ to be in the right half-plane $\text{Re}(\lambda) < \lambda_0(t)$, $\lambda_0(t) > 0$, but assume that $\lim_{t \to \infty} \lambda_0(t) = 0$.

Under suitable assumption on $a(t)$ and $b(t)$ we prove boundedness of $\|u(t)\|$ as $t \to \infty$. If $f(t) = 0$, the Lyapunov stability of the zero solution to problem (1) with $u_0 = 0$ is established. For $f \neq 0$, sufficient conditions for Lyapunov stability are given. The novel point in the paper is the possibility for the linear operator $A(t)$ to have spectrum in the half-plane $\text{Re}(\lambda) < \lambda_0(t)$ with $\lambda_0(t) > 0$ and $\lim_{t \to \infty} \lambda_0(t) = 0$ at a suitable rate.

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1 Introduction and the results.

The main results of this paper are formulated in Lemma 1 and Theorem 2. Lemma 1 is proved in Section 2. In Section 3 an example of applications of our result is given.

There is a large literature [1, 2, 3] on the stability of the solutions to differential equation of the form

\[ \dot{u} = A(t)u + G(t,u) + f(t), \quad u(0) = u_0; \quad \dot{u} = \frac{du}{dt}, \quad t \geq 0, \]  

(1)

where $A(t)$ is a linear, closed, densely defined in a Hilbert space $H$ operator, with domain $D(A(t))$ independent of $t$, $G(t,u)$ is a nonlinear operator,

\[ \|G(t,u)\| \leq a(t)\|u\|^p, \quad p > 1, \quad t \geq 0, \]  

(2)

$f(t)$ is a function on $\mathbb{R}_+ = [0, \infty)$ with values in $H$,

\[ \|f(t)\| \leq b(t), \quad t \geq 0, \]  

(3)

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We assume that \( a(t) \) and \( b(t) \) are non-negative continuous functions, that \( G(t, u) \) is locally Lipschitz with respect to \( u \) in the ball \( B(u_0, R) = \{ u : ||u - u_0|| \leq R \} \) and is a continuous in the operator norm function of \( t \). We assume that problem (1) has a unique local solution.

This assumption holds, for example, if \( A(t) \) is a generator of \( C_0 \) semigroup. In this case, problem (1) is equivalent to

\[
\begin{align*}
  u(t) &= U(t, 0)u_0 + \int_0^t U(t, s)G(s, u(s))ds + \int_0^t U(t, s)fds := T(u),
\end{align*}
\]

where

\[
\frac{\partial U(t, s)}{\partial t} = A(t)U(t, s), \quad U(s, s) = I, \quad t \geq s,
\]

\((5)\)

\( I \) is the identity operator, and \( U(t, s) \) is a bounded operator in \( H \):

\[
U(t, s) = I + \int_s^t A(\tau)U(\tau, s)d\tau, \quad t \geq s.
\]

\((6)\)

Indeed, a standard calculation shows that \( u \), defined in (4), solved problem (1):

\[
\begin{align*}
  \frac{d}{dt} \int_0^t U(t, s)f(s)ds &= U(t, t)f + \int_0^t \frac{\partial U(t, s)}{\partial t}f(s)ds \\
  &= f(t) + \int_0^t A(t)U(t, s)fds \\
  &= f(t) + A(t) \int_0^t U(t, s)f(s)ds,
\end{align*}
\]

\[
\frac{d}{dt} U(t, 0)u_0 = A(t)U(t, 0)u_0, \quad U(0, 0)u_0 = u_0,
\]

\[
\frac{d}{dt} \int_0^t U(t, s)G(s, u(s))ds = G(t, u(t)) + A(t) \int_0^t U(t, s)G(s, u(s))ds.
\]

It follows that \( u \), defined in (4), solves problem (1). If problem (1) is equivalent to problem (4) and \( U(t, s) \) are bounded operators, then the operator \( T(u) \) in (4) is locally Lipschitz since \( G(t, u) \) has this property. Consequently, problem (4) (and therefore problem (1)) has a unique local solution. This solution is global if it satisfies a uniform with respect to \( t \) bound

\[
\sup_t ||u(t)|| \leq c,
\]

\((7)\)

where the supremum is taken over all \( t \in [0, T) \) for which \( u(t) \) exists (see, e.g., [5]).

Let us derive bound (7). This derivation is based on an application of a new nonlinear differential inequality, stated in Lemma 1, below. Our basic result, Theorem 2, is also based on this inequality.

Take inner product of (1) with \( u(t) \), then take real part of the resulting equation, and denote \( ||u(t)|| = g(t) \). The result is:

\[
gg \leq \text{Re}(A(t)u, u) + ||G(t, u, u)|| + b(t)g(t).
\]

\((8)\)
Assume that
\[ \text{Re}(A(t)u, u) \leq \gamma(t)g^2, \quad ||G(t, u)|| \leq \alpha(t, g), \] (9)
where \( \gamma(t) \geq 0 \) and \( \alpha(t, g) \geq 0 \) are continuous functions of \( t \), and
\[ \alpha(t, g) \leq a(t)g^p, \quad p > 1. \] (10)
Then inequality (8) implies
\[ \dot{g} \leq \gamma(t)g + a(t)g^p + b(t), \quad g(0) = ||u_0||. \] (11)
We assume that
\[ a(t) > 0, \quad \dot{a}(t) < 0, \] (12)
and \( \gamma(t) \) and \( b(t) \) tend to zero monotonically as \( t \to \infty \).

We need the following lemma, which is proved in Section 2.

**Lemma 1** If there exists a function
\[ \mu(t) > 0, \quad \dot{\mu}(t) < 0, \quad \lim_{t \to \infty} \mu(t) = d > 0, \] (13)
and
\[ \gamma(t) + a(t)\mu^{-p+1}(t) + b(t)\mu(t) + \frac{\dot{\mu}}{\mu} \leq 0, \quad \mu(0)g(0) \leq 1, \] (14)
then any non-negative solution \( g \) to (11) satisfies inequality
\[ 0 \leq g(t) \leq \mu^{-1}(t) \leq \frac{1}{d}. \] (15)

Applying Lemma 1 with \( \mu(t) = d + q(t), \; d = const > 0, \; q(t) > 0, \; \dot{q}(t) < 0, \) one gets from (14) the following inequalities:
\[ [d + q(t)]\gamma(t) + a(t)[d + q(t)]^{2-p} + b(t)[d + q(t)]^2 \leq -\dot{q}(t) = |\dot{q}(t)|, \] (16)
\[ [d + q(0)]g(0) \leq 1. \] (17)
If \( g(0) \neq 0 \) and \( [d + q(0)] \leq g^{-1}(0) \), then (17) holds. If \( g(0) = 0 \), then (17) holds for any \( d \) and \( q(0) \). Fix \( d \) and \( q(0) \) such that (17) holds. Then (16) holds if \( q(t) \) is such that
\[ C[\gamma(t) + a(t) + b(t)] \leq |\dot{q}(t)|. \] (18)
Here \( C = const > 0, \)
\[ C = \max\{d + q(0), [d + q(0)]^2, [d + q(0)]^{2-p}, d^{2-p}\}. \] (19)
We have used monotone decay of \( q(t) \) and the inequalities: \( [d + q(t)]^{2-p} \leq [d + q(0)]^{2-p} \) if \( 2 \geq p \), and \( [d + q(t)]^{2-p} \leq d^{2-p} \) if \( 2 \leq p \).

To satisfy inequality (18), one may choose \( q(t) \) using the relation
\[ -q(t) + q(0) = C \int_0^t [\gamma(s) + a(s) + b(s)]ds := CQ(t), \] (20)
or

\[ q(t) = q(0) - CQ(t). \]  

If the number

\[ Q_\infty := \int_0^\infty [\gamma(t) + a(t) + b(t)] dt \]

is sufficiently small, then one can choose the constant \( q(0) \), so that the function \( q(t) \), defined in (21), is positive for all \( t \geq 0 \).

Applying inequality (15), we obtain the following theorem:

**Theorem 2** Assume that \( u(t) \) solves problem (1) and inequalities (2), (3), (9), (10), hold. If the function \( q(t) \), defined in (21), with \( C \) defined in (19), is positive for all \( t \geq 0 \), then \( u(t) \) exists for all \( t \geq 0 \), and is globally bounded:

\[ ||u(t)|| \leq \frac{1}{d + q(t)} \leq \frac{1}{d}, \quad \forall t \geq 0. \]  

(22)

**Remark.** Let \( \epsilon > 0 \) be a fixed small number. If \( g(0) = ||u(0)|| \leq \delta \), where \( \delta > 0 \) is a sufficiently small number, then one can choose \( d = \delta^{-1} \), define \( \epsilon = \delta \), and obtain:

\[ ||u(t)|| \leq \epsilon, \quad \forall t \geq 0. \]  

(23)

The statement in the above Remark shows that the solution to evolution problem (1) is Lyapunov stable.

2 Proof of Lemma 1.

Consider the problem:

\[ \dot{v}_n = \gamma(t)v_n + a(t)v_n^p + f(t) + \frac{1}{n}, \quad v_n(0) = g(0), \]  

(24)

where \( n > 0 \) is an integer. This problem has a unique local solution. Clearly, \( \dot{v}_n > \dot{g}(0) \).

Therefore, there is an interval \((0, T)\), such that

\[ v_n(t) > g(t), \quad 0 < t < T, \]  

(25)

where \( T > 0 \) is some number. The interval \([0, T]\) is the maximal interval of the existence of the local solution to (24), and if

\[ \sup_{t \geq 0} |v_n(t)| < \infty, \]  

(26)

then \( T = \infty \).

Indeed, if inequality (26) holds but \( T < \infty \), then one can solve the problem

\[ \dot{w} = \gamma(t)w + a(t)w^p + f(t) + \frac{1}{n}, \quad w(T - \frac{l}{2}) = v_n(T - \frac{l}{2}), \]  

(27)

where \( l \) is the length of the interval of the local existence of the solution to problem (27). If (26) holds, then the length \( l \) does not depend on the choice of the Cauchy data. Thus,
Inequality (14) can be written as

\[ \frac{\mu}{\mu^2} = \left( \frac{1}{\mu} \right) \cdot \gamma(t) \frac{a(t)}{\mu(t)} + b(t), \quad \frac{1}{\mu(0)} \geq v(0), \quad t \geq 0. \]  

(30)

By the argument similar to the given above, one obtains

\[ \frac{1}{\mu(t)} \geq v(t) \geq g(t) \geq 0, \quad \forall t \geq 0. \]  

(31)

From (31) the conclusion (15) of Lemma 1 follows, because \( \frac{1}{\mu(t)} \leq \frac{1}{\mu} \). Lemma 1 is proved. \( \square \)

A lemma, similar to Lemma 1, but with \( \gamma(t) < 0 \), was proved in [4] by a different argument. In [2] and [5] one can find proofs of some comparison results for ordinary differential equations. The ideas of these proofs are close to the idea of our proof of Lemma 1.

The principally novel result in our paper is Theorem 2, because it gives sufficient conditions for Lyapunov stability of the solution to evolution problem (1) under the assumptions which allow the linear operator \( A(t) \) to have spectrum in the half-plane \( \text{Re} z > 0 \). Such a result is possible to obtain because this spectrum tends sufficiently fast to the imaginary axis as \( t \to \infty \).

3 Example

Let us illustrate our result by a simple example. As \( H \) we take \( \mathbb{R}^2 \), as \( A(t) \) we take \( m(t)I \), where \( I \) is the unit matrix, \( m(t) = c_1(1 + t)^{-m_1} \), \( G(t,u) \) is a quadratic nonlinearity, \( \| G(t,u) \| \leq n(t)g^2 \), so \( p = 2, \ g^2 = x^2(t) + y^2(t) \), vector \( u \) has two components, \( u(t) := \{ x(t), y(t) \} \), \( n(t) = c_2(1 + t)^{-m_2} \), constants \( c_j, m_j > 0 \) will be chosen later. Inequality (11) with \( b(t) = 0 \) takes the form

\[ \dot{g} \leq c_1(1 + t)^{-m_1}g + c_2(1 + t)^{-m_2}g^2, \quad \forall t \geq 0. \]  

(32)

Choose \( \mu(t) = d + c_3(1 + t)^{-m_3} \), where the constants \( d, c_3, m_3 > 0 \) will be fixed later. Then (14) (with \( b(t) = 0 \)) takes the form:

\[ \frac{c_1}{(1 + t)^{m_1}} + \frac{c_2}{(1 + t)^{m_2}[d + c_3(1 + t)^{-m_3}]} \leq \frac{m_3c_3}{(1 + t)^{m_3+1}[d + c_3(1 + t)^{-m_3}]}, \quad \forall t \geq 0. \]  

(33)
and

\[(d + c_3)g(0) \leq 1. \quad (34)\]

Let us check that inequalities (33) and (34) can be satisfied if the parameters \(d, c_j, m_j\) are properly chosen. Assume that

\[m_3 + 1 \leq \min\{m_1, m_2\}. \quad (35)\]

Then (33) holds if

\[c_1 + c_2 d^{-1} \leq c_3 m_3 (d + c_3)^{-1}. \quad (36)\]

Inequality (34) holds if

\[d + c_3 = \frac{1}{g(0)}. \quad (37)\]

Choose \(c_3 = d\). Then

\[d = \frac{1}{2g(0)}, \quad g(0) = \|u(0)\|. \quad (38)\]

Let \(g(0) \leq \delta\), where \(\delta > 0\) is a small number. Then \(d \geq (2\delta)^{-1}\), and inequality (36) holds if

\[c_1 + 2\delta c_2 \leq c_3 m_3 (2d)^{-1} \leq c_3 m_3 \delta. \quad (39)\]

This inequality holds if \(c_1, c_2\) are sufficiently small. For example, let

\[c_1 \leq 0.5c_3 m_3 \delta, \quad c_2 \leq 0.25c_3 m_3. \quad (40)\]

Then inequalities (33) and (34) hold, and Lemma 1 yields:

\[0 \leq \|u(t)\| \leq [d + c_3 (1 + t)^{-m_3}]^{-1} \leq d^{-1}, \quad \forall t \geq 0. \quad (41)\]

This estimate obviously yields global boundedness of \(\|u(t)\|\), and also Lyapunov stability. Indeed, if \(\|u(0)\| \leq \delta\), and \(\delta > 0\) is sufficiently small, then \(d \geq (2\delta)^{-1}\) can be chosen sufficiently large, and consequently, inequality (41) yields the estimate \(\|u(t)\| \leq \epsilon, \forall t \geq 0\), where \(\epsilon = 2\delta\) is arbitrary small if \(\delta > 0\) is sufficiently small.

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References


