Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Simultaneous Blowup and Mass Separation During Collapse in an Interacting System of Chemotactic Species

by

Elio Eduardo Espejo, Angela Stevens, and Takashi Suzuki

Preprint no.: 64 2011

SIMULTANEOUS BLOWUP AND MASS SEPARATION DURING COLLAPSE IN AN INTERACTING SYSTEM OF CHEMOTACTIC SPECIES

Elio Eduardo Espejo
Department of Systems Innovation, Osaka University
1-3 Machikaneyama-cho, Toyonaka 560-8531, Japan

Angela Stevens
Institut für Numerische und Angewandte Mathematik
Fachbereich für Mathematik und Informatik
der Universität Münster
Einsteinstr. 62, 48149 Münster, Germany

Takashi Suzuki
Department of Systems Innovation, Osaka University
1-3 Machikaneyama-cho, Toyonaka 560-8531, Japan

(Submitted by: ****)

Abstract. We study an interacting system of chemotactic species in two-space dimension. First, we show that there is a parameter region which ensures simultaneous blowup also for non-radially symmetric solutions. If the existence time of the solution is finite, there is a formation of collapse (possibly degenerate) for each component, total mass quantization, and formation of subcollapses. For radially symmetric solutions we can rigorously prove that the collapse concentrates mass on one component if the total masses of the other components are relatively small. Several related results are also shown.

1 Introduction

The Smoluchowski-Poisson equation or drift-diffusion models are systems of elliptic-parabolic equations describing the motion of particle densities in non-equilibrium statistical mechanics. They have been used in semiconductor

Accepted for publication: Xxxxxxxx 20xx.

physics, high-molecular chemistry, and astrophysics (see [33] and the references therein). They arise also in biology, modeling aggregation and self-organization of the cellular slime mold Dictyostelium discoideum (Dd) due to chemotaxis [18, 23, 6]. This model in biology has a crucial negative drift.

The first rigorous proof that chemotaxis can serve as the main mechanism for the onset of self-organization of Dd was given in [16]. Namely, for the critical space dimension two, depending on the L^1-norm of the initial values and on the strength of the chemotactic sensitivity, this model exhibits solutions blowing up in finite time or existing global-in-time. Later studies have clarified the blowup threshold [20, 21, 1, 11, 27], formation of collapses [12, 26], mass quantization [12, 32, 29], and type II blowup rates [12, 25, 22]. On the other hand, [13] gave a family of blowup solutions with precise asymptotics for the full parabolic-parabolic system.

Along this line of thoughts, recently a two-component system for chemotaxis has been studied in [9], to approach the question if cell sorting of Dd in the mound-stage can be mainly due to differential chemotaxis, i.e., the chemotactically weaker cell type sorts out to the bottom of the mound, whereas the chemotactically stronger cell type sorts out of the top, c.f. [34]. Of course this question has finally to be addressed in a spatially three dimensional model. But as a first mathematical test-problem to this question, in [9] it was analyzed if there exists a set of chemotactic sensitivities in the parabolic-elliptic chemotaxis system n the radially symmetric setting, such that the solution for one cell type shows finite time blowup, while the solution for the other cell type still exists at that instant in time, i.e., the first cell type starts to self-organize, while the other cell type does not yet. In the radially symmetric setting this is not possible. It was rigorously proven in [9] that if the solution for one cell type shows blowup in finite time, then the other cell type does so too, and at the same time, no matter how weak its chemotactic sensitivity is. Also sufficient conditions for blowup in finite time for this multi-component system were given. Finally, a formal computation in [9] showed that the blowup mechanism for the two cell types can be different, even up to the situation that one species can exhibit blowup without mass aggregation. Explicit formal asymptotics for the blowup profile for special cases were given to confirm this property using rescaled mass functions, that is, only one component can form a singularity of delta functions at the blowup time.

The considered system is a special form of the multi-component chemotaxis system introduced in [35], where conditions for the existence of global solutions for such systems were derived. Then its stationary state is studied by [15]. Further analysis on the critical mass for blowup for a two-species model for chemotaxis in $\mathbf{R}^2$ have been more recently done [14, 7, 8]). The authors identified a curve in the plane of masses such that outside of this curve there is blow-up of solutions and inside of it the solutions are global in time, if the initial masses satisfy a threshold condition.

This model may also describe a competitive feature of chemotaxis observed in cancer cell biology, especially, in tumor microenvironment at the stage of in-travasation. More precisely, there is an interaction between cancer cells and tumor associated macrophages through chemical substances which causes their localized cell deformations called invadopodia and podosomes, respectively (see [36, 17]). At the tissue level, this phenomenon may be described by a competitive chemotactic feature between two species of cells to chemical substance secreted by themselves. Under this agreement it will be interesting from both the mathematical and biological points of view, if actually only one component of the solution can form δ-functions at the blowup time.

The aim of this paper is to approach such a property of the solution, especially also in the non-radial symmetric situation. We restrict our analysis to the case of two space-dimension which is the critical dimension of the model in such a biological setting. Relatives of this 2-dimensional model also arise as a kinetic mean field equation of point vortices. See [5] and [4] for single and multi-comonent cases, respectively, and also [31] for a higher-dimensional multi-component case.

Given a bounded domain $\Omega \subset \mathbf{R}^2$ with smooth boundary $\partial\Omega$, this model takes the part of Smoluchowski equations:

$$\begin{aligned} \partial_t u_1 &= d_1 \Delta u_1 - \chi_1 \nabla \cdot u_1 \nabla v \\ \partial_t u_2 &= d_2 \Delta u_2 - \chi_2 \nabla \cdot u_2 \nabla v \qquad \text{in } \Omega \times (0,T), \end{aligned} \tag{1}$$

using the chemical gradient term ∇v, with the boundary condition

$$d_1 \frac{\partial u_1}{\partial \nu} - \chi_1 u_1 \frac{\partial v}{\partial \nu} = d_2 \frac{\partial u_2}{\partial \nu} - \chi_2 u_2 \frac{\partial v}{\partial \nu} = 0 \qquad \text{on } \partial\Omega \times (0,T) \tag{2}$$

and the initial condition

$$u_1|_{t=0} = u_{10}(x) \geq 0, \quad u_2|_{t=0} = u_{20}(x) \geq 0 \qquad \text{in } \Omega, \tag{3}$$

where d_1, d_2, χ_1, and χ_2 are positive constants and ν is the unit normal vector. The initial value (u_{10}, u_{20}) satisfies

$$\frac{1}{|\Omega|} \int_\Omega u_{10} + u_{20} \; dx = 1, \tag{4}$$

and then (1), (2), and (3) are coupled with the Poisson equation

$$-\Delta v = u_1 + u_2 - 1, \quad \left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0, \quad \int_\Omega v \; dx = 0. \tag{5}$$

We put the last condition of (5) to normalize an additive constant of v. This normalization is not essential because only the gradient ∇v of v is used in the Smoluchowski equations (1), while condition (4) is necessary for the solvability of (5) at $t = 0$. Since

$$\frac{d}{dt}\int_\Omega u_1 dx = \frac{d}{dt}\int_\Omega u_2 dx = 0 \tag{6}$$

arises for (1) with (2), this compatibility condition for the solvability of (5), that is,

$$\frac{1}{|\Omega|}\int_\Omega u_1 + u_2 \ dx = 1,$$

is kept for all $t > 0$. System (1), (2), (3) and (5) with (4) is thus a generalization of the parabolic-elliptic system of chemotaxis given in [16] for two chemotactic cell types (u_1, u_2). Here we do *not* adopt the normalization $d_1 = 1$ to make the statements below simpler, particularly for more than two-components systems.

We assume that $u_{10}, u_{20} \not\equiv 0$ are in $C^2(\overline{\Omega})$ and satisfy

$$d_1 \frac{\partial u_{10}}{\partial \nu} - \chi_1 u_{10} \frac{\partial v_0}{\partial \nu} = d_2 \frac{\partial u_{20}}{\partial \nu} - \chi_2 u_{20} \frac{\partial v_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega$$

for $v_0 = v_0(x)$ defined by

$$-\Delta v_0 = u_{10} + u_{20} - 1, \quad \left.\frac{\partial v_0}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v_0 \ dx = 0.$$

This assumption guarantees the unique existence of a local-in-time classical solution satisfying

$$u_i(\cdot, t) > 0 \ \text{ on } \overline{\Omega}, \quad 0 < t < T, \ i = 1, 2 \tag{7}$$

with $T = T_{\max} \in (0, +\infty]$ standing for the maximum existence time [9]. Since this T is estimated from below by $\sum_{i=1}^2 \|u_{i0}\|_\infty$, it holds that

$$T < +\infty \quad \Rightarrow \quad \lim_{t \uparrow T} \sum_{i=1}^2 \|u_i(\cdot, t)\|_\infty = +\infty \tag{8}$$

by a standard argument on the continuation of the solution in time (see, for example, [29]).

The solution (u_1, u_2, v) to (1), (2), (3), and (5) with (4) satisfies

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \ dx, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v \ dx = 0, \quad u = u_1 + u_2. \tag{9}$$

Here, the system composed of (1), (2), (3), and (9) may be called interacting system of Smoluchowski-Poisson equations, and henceforth is denoted by (ISP) in short. Similarly, we have a local-in-time well-posedness with the properties (7) and (8) for this system. In particular, $u_{10}, u_{20} > 0$ may be assumed on $\overline{\Omega}$ by the strong maximum principle. Then, there is $A > 0$ such that $Au_{10} \geq u_{20}$.

If $d_1 = d_2 = d$ and $\chi_1 = \chi_2 = \chi$, we obtain

$$\begin{aligned} & w_t = d\Delta w - \chi\nabla\cdot w\nabla v && \text{in } \Omega\times(0,T) \\ & \frac{\partial w}{\partial \nu} = 0 && \text{on } \partial\Omega\times(0,T) \\ & w\vert_{t=0} = Au_{10} - u_{20} \geq 0 && \text{in } \Omega \end{aligned}$$

for $w = Au_1 - u_2$, which implies

$$Au_1 \geq u_2 \qquad \text{on } \overline{\Omega}\times[0,T). \tag{10}$$

If $T < +\infty$ and

$$\liminf_{t\uparrow T} \|u_1(\cdot,t)\|_\infty < +\infty,$$

there is $t_k \uparrow T$ such that

$$\limsup_{k\to\infty} \|u_1(\cdot,t_k)\|_\infty < +\infty,$$

and therefore,

$$\limsup_{k\to\infty} \|u_2(\cdot,t_k)\|_\infty < +\infty$$

by (10). Then it follows that

$$\liminf_{t\uparrow T} \{\|u_1(\cdot,t)\|_\infty + \|u_2(\cdot,t)\|_\infty\} < +\infty,$$

a contradiction to (8). Hence we have

$$\lim_{t\uparrow T} \|u_1(\cdot,t)\|_\infty = +\infty,$$

and similarly,

$$\lim_{t\uparrow T} \|u_2(\cdot,t)\|_\infty = +\infty.$$

The simultaneous blowup

$$T < +\infty \quad \Rightarrow \quad \lim_{t\uparrow T} \|u_1(\cdot,t)\|_\infty = \lim_{t\uparrow T} \|u_2(\cdot,t)\|_\infty = +\infty \tag{11}$$

is thus valid in any dimensions if $d_1 = d_2$ and $\chi_1 = \chi_2$. In two-space dimensions with $u_i = u_i(|x|, t)$, however, a similar property holds for any pairs of (d_i, χ_i), $i = 1, 2$. More precisely, it holds that

$$T < +\infty \quad \Rightarrow \quad \limsup_{t\uparrow T} \|u_1(\cdot, t)\|_\infty = \limsup_{t\uparrow T} \|u_2(\cdot, t)\|_\infty = +\infty, \tag{12}$$

which is not the case of the related drift-diffusion model given in [19] (see [9, 10] for the proof). Our first result shows that there is a parameter region where (11) holds even for non-radially symmetric solutions.

Henceforth, we put

$$\xi_i = d_i/\chi_i, \quad \|u_{i0}\|_1 = \lambda_i, \quad i = 1, 2, \tag{13}$$

indicating inverse motilities and initial masses of the species.

Theorem 1 *If*

$$\lambda_i < 4\pi\xi_i, \quad i = 1, 2 \tag{14}$$

then (11) holds.

Here, condition (14) is consistent with the assumption $T < +\infty$. In fact, if

$$\left(\sum_{i=1}^{2} \lambda_i\right)^2 < 4\pi \sum_{i=1}^{2} \xi_i\lambda_i, \quad \lambda_i < 4\pi\xi_i, \ i = 1, 2$$

then it always hold that $T = +\infty$, while $T < +\infty$ can occur in case

$$\left(\sum_{i=1}^{2} \lambda_i\right)^2 > 4\pi \sum_{i=1}^{2} \xi_i\lambda_i \tag{15}$$

(see Theorems 9 and 10 and the description on the relation between known results, particularly [35], in §2). Then the parameter region defined by (14) and (15) in $\lambda_1\lambda_2$ plane in $\lambda_i > 0$, $i = 1, 2$, is not empty.

Since property (8) means

$$T < +\infty \quad \Rightarrow \quad \lim_{t\uparrow T} \|u(\cdot, t)\|_\infty = +\infty, \tag{16}$$

recalling $u = \sum_{i=1}^{2} u_i$ in (9), the blowup set of (u_1, u_2) defined by

$$\mathcal{S} = \{x_0 \in \overline{\Omega} \mid \exists (x_k, t_k) \to (x_0, T), \ u(x_k, t_k) \to +\infty\} \tag{17}$$

is not empty. In the case of the Smoluchowski-Poisson system for a single unknown species, the formation of collapses occurs with a quantized mass [32, 33, 29]. Our second result shows that this property arises also in (ISP) for each component, with possibly degenerate collapses. Here, we say that the collapse $m_i(x_0)\delta_{x_0}(dx)$, $i = 1, 2$, in (18) below is degenerate if $m_i(x_0) = 0$.

Theorem 2 *If $T < +\infty$, the blowup set $\mathcal{S}$ defined by (17) is finite. It holds that*

$$u_i(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_i(x_0)\delta_{x_0}(dx) + f_i(x)dx, \quad i = 1,2 \tag{18}$$

in $\mathcal{M}(\overline{\Omega}) = C(\overline{\Omega})'$ as $t \uparrow T = T_{\max} < +\infty$, where $m_i(x_0) \geq 0$, $i = 1,2$, are constants satisfying $(m_1(x_0), m_2(x_0)) \neq (0,0)$, and $0 < f_i = f_i(x) \in L^1(\Omega)$, $i = 1,2$, are smooth functions in $\overline{\Omega} \setminus \mathcal{S}$.

Equality (19) in the following theorem may be called a *total mass quantization* because it is an identity involving all the collapse masses $m_i(x_0)$, $i = 1,2$.

Theorem 3 *It holds that*

$$\left(\sum_{i=1}^{2} m_i(x_0)\right)^2 = m_*(x_0)\sum_{i=1}^{2} \xi_i m_i(x_0) \tag{19}$$

for any $x_0 \in \mathcal{S}$, where

$$m_*(x_0) = \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega. \end{cases} \tag{20}$$

We can write (1) as

$$\partial_t u_1 = d_1 \Delta u_1 - \chi_1 \nabla u_1 \cdot \nabla v + \chi_1 u_1 (u - \frac{\lambda}{|\Omega|})$$
$$\partial_t u_2 = d_2 \Delta u_2 - \chi_2 \nabla u_2 \cdot \nabla v + \chi_2 u_2 (u - \frac{\lambda}{|\Omega|}),$$

using $u = u_1 + u_2$ and $\lambda = \lambda_1 + \lambda_2$, where

$$\lambda_i = \frac{1}{|\Omega|}\int_\Omega u_{i0}\ dx, \quad i = 1,2.$$

Thus the ODE part of (ISP) may be defined by

$$\frac{du_1}{dt} = \chi_1 u_1 (u_1 + u_2 - a)$$
$$\frac{du_2}{dt} = \chi_2 u_2 (u_1 + u_2 - a)$$
$$u = \sum_{i=1}^{2} u_i \tag{21}$$

with $a = \frac{\lambda}{|\Omega|}$, where $u_{i0} = u_i(0)$, $i = 1, 2$, are positive constants. Although the solution (u_1, u_2) to (21) is constant: $u_i(t) = u_{i0}$, $i = 1, 2$ if

$$a = \sum_{i=1}^{2} u_{i0}$$

we may have a blowup of the solution to (21). In this connection, we mention that there is actually a blowup of the solution to (ISP) (see Theorems 10 and 11).

In (21) it holds that

$$\frac{d}{dt}(\chi_1^{-1} \log u_1 - \chi_2^{-2} \log u_2) = 0,$$

and hence $u_2 = cu_1^\gamma$ with $\gamma = \chi_2/\chi_1$ and $c = u_{20}/u_{10}^\gamma$. Therefore, the blowup of the solution $(u_1(t), u_2(t))$ to (21) is simultaneous for $u_i(t)$, $i = 1, 2$, if it actually occurs. Assuming $\gamma \leq 1$ without loss of generality, we obtain

$$\frac{du_1}{dt} \sim \chi_1 u_1^2,$$

which implies $u_1(t) \sim \chi_1(T-t)^{-1}$ as $t \uparrow T = T_{\max} < +\infty$. The type (I) blowup rate of (ISP) thus may be defined by

$$\|u(\cdot, t)\|_\infty \sim (T-t)^{-1}, \quad u = u_1 + u_2.$$

The next theorem concerning the formation of subcollapses implies that any blowup rate of (ISP) is not of this type. To state the result, let

$$(u_1, u_2, v) = (u_1(x,t), u_2(x,t), v(x,t))$$

be a solution to (ISP) satisfying $T = T_{\max} < +\infty$, take $x_0 \in \mathcal{S}$, and let

$$\begin{aligned} z_i(y,s) &= (T-t)u_i(x,t), \quad w(y,s) = v(x,t) \\ y &= (x-x_0)/(T-t)^{1/2}, \quad s = -\log(T-t) \end{aligned} \tag{22}$$

$i = 1, 2$, be its backward self-similar transformation. Henceforth, we assume the 0-extensions of $z_i(y,s)$, $i = 1, 2$, where they are not defined. Furthermore, $\mathbf{R}^2 \cup \{\infty\}$ denotes the one-point compactification of $\mathbf{R}^2$, and $C_0(\mathbf{R}^2)$ stands for the set of continuous functions on $\mathbf{R}^2 \cup \{\infty\}$ taking the value 0 at ∞, and $\mathcal{M}_0(\mathbf{R}^2) = C_0(\mathbf{R}^2)'$.

Theorem 4 *We have*

$$z_i(y, s+s')dy \rightharpoonup m_i(x_0)\delta_0(dy), \quad i = 1, 2 \tag{23}$$

in $C_*(-\infty,+\infty;\mathcal{M}_0(\mathbf{R}^2))$ *as* $s'\uparrow+\infty$. *In particular, it holds that*

$$\lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(\Omega\cap B(x_0,b(T-t)^{1/2})}=+\infty \tag{24}$$

for any $b>0$.

Now we approach the property of the collapse mass separation. The first observation is the following theorem. Here we note that the blowup set $\mathcal{S}$ coincides with the origin for radially symmetric solutions satisfying $T<+\infty$.

Theorem 5 *Let* Ω *be a disc with center at the origin,* $u_i=u_i(|x|,t)$, $i=1,2$, *and* $T<+\infty$. *Then* $m_i=m_i(0)$ *must satisfy*

$$m_i\le 8\pi\xi_i,\quad i=1,2 \tag{25}$$

besides (19) with $x_0=0$.

Here we give several remarks. First, inequality (25) arises in the context of a global-in-time continuation of the solution associated with a Trudinger-Moser or logarithmic HLS inequality (see §2). Next, (25) is a consequence of (19) if

$$1/2\le\xi_i/\xi_j\le 2,\quad i,j=1,2. \tag{26}$$

More precisely, if (26) is the case, the curve (a parabola if $\xi_1\neq\xi_2$ and a line in the other case) defined by

$$\left(\sum_{i=1}^2 m_i\right)^2=m_*\sum_{i=1}^2\xi_i m_i,\quad m_*=m_*(x_0), \tag{27}$$

in the m_1m_2-plane in $\{(m_1,m_2)\mid m_i>0,i=1,2\}$ does not cross the lines $m_i=\xi_i m_*$, $i=1,2$. Finally, in the other case of $\xi_i/\xi_j>2$ or $\xi_i/\xi_j<1/2$ for $i\neq j$, one of $(m_1,m_2)=(8\pi\xi_1,0)$ and $(m_1,m_2)=(0,8\pi\xi_2)$ is an isolated point of (27) in the m_1m_2-plane with $\{(m_1,m_2)\mid m_i\ge 0,i=1,2\}$ and (25).

Under such observations we can expect that the above isolated endpoint of (m_1,m_2), which we call *mass separation*, may be actually the case, and is stable under non-radially symmetric perturbations. In this context we recall that *simultaneous blowup* (12) is always the case for radially symmetric solutions, regardless of the parameter region indicated by (14). If both simultaneous blowup and mass separation arise, say, $m_i(x_0)=0$ in (18), then it will hold that $f_i\notin L^\infty(\Omega\cap B(x_0,R))$ for $0<R\ll 1$, where $B(x_0,R)=\{x\mid |x-x_0|<R\}$.

The following theorem shows that the mass separation of radially symmetric solutions, which was formally given in [9], actually occurs if the total mass of one component is relatively small compared with that of the other.

Theorem 6 *Under the assumption of Theorem 5, let*

$$\xi_i/\xi_j > 2$$

for some $i \neq j$. Then $m_i = 0$ and hence $m_j = 8\pi\xi_j$ holds, provided that

$$\|u_{i0}\|_1 < 8\pi(\xi_i - 2\xi_j).$$

We note that a sufficient condition for $T < +\infty$ in the above theorem is given in [9], that is (see Theorem 11 in §5),

$$\|u_{j0}\|_1 > 8\pi\xi_j, \quad \||x|^2 u_{j0}\|_1 \ll 1.$$

Theorem 10 in §3 is also available.

Theorem 1 is proven by the variational structure of (ISP) and the logarithmic HLS inequality derived in [30]. Theorem 2 is obtained by an argument of [26], using an ε-regularity and a monotonicity formula. Then we have the formation of collapses of $\hat{u}(x,t)dx$ as $t \uparrow T$, where $\hat{u} = \sum_{i=1}^{2} \chi_i^{-1} u_i$. A careful analysis then assures this property component-wisely and also

$$m(x_0) \equiv \sum_{i=1}^{2} m_i(x_0) > 0. \tag{28}$$

To prove Theorem 3, first we apply an argument developed for the single component case [32, 33, 29]. We use the backward self-similar transformation, weak scaling limit, scaling back, and translation limit, to obtain a full-orbit defined on the whole (or the half) space domain. Here establishing a *parabolic envelope* is essential. Then, an existence criterion of such orbits follows from the method of local second moments and scaling, which guarantees an estimate of the total collapse mass from above, that is,

$$\left(\sum_{i=1}^{2} m_i(x_0) \right)^2 \leq m_*(x_0) \sum_{i=1}^{2} \xi_i m_i(x_0). \tag{29}$$

We use a new argument to derive the reverse inequality

$$\left(\sum_{i=1}^{2} m_i(x_0) \right)^2 \geq m_*(x_0) \sum_{i=1}^{2} \xi_i m_i(x_0). \tag{30}$$

Namely, we show the boundedness of the total second moment of the rescaled solution and use the scaling limit equation. We have, at the same time, the formation of subcollapses indicated by Theorem 4. We note that a weaker estimate of the total collapse mass from below is obtained similarly to the single component case, that is, either (30) or

$$m_i(x_0) \geq \xi_i m_*(x_0), \quad i = 1, 2 \tag{31}$$

by the logarithmic HLS inequality. Inequality (30), however, is eventually selected for (19) to be established.

The proof of Theorem 5 is based on the fact that the interaction between two-components is neglected in the collapse mass estimate from above for radially symmetric solutions. Then Theorem 6 arises with the total mass conservation of each component of the solution.

This paper is composed of five sections and two appendices. In §2 we take preliminaries and prove Theorem 1. We show Theorem 2 in §3 and then Theorems 3 and 4 in §4. Theorems 5 and 6 are proven in §5. In the first appendix, we show that either (30) or (31) holds by the previous argument used in [26]. The second appendix is devoted to some criteria for mass separation and simultaneous blowup.

Concluding this section, we note that the proof of the above theorems are valid even for the multi-components case, that is, a system of Smoluchowski equations

$$\begin{aligned} &\partial_t u_i = d_i \Delta u_i - \chi_i \nabla \cdot u_i \nabla v && \text{in } \Omega \times (0,T) \\ &d_i \frac{\partial u_i}{\partial \nu} - \chi_i u_i \frac{\partial v}{\partial \nu} = 0 && \text{on } \partial\Omega \times (0,T) \\ &u_i|_{t=0} = u_{i0}(x) \geq 0 && \text{in } \Omega, \end{aligned} \tag{32}$$

$i = 1, 2, \cdots, N$, coupled with the Poisson equation:

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \, dx, \quad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v \, dx = 0, \quad u = \sum_{i=1}^N u_i.$$

Here, $\Omega \subset \mathbf{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$, d_i, χ_i, $i = 1, 2, \cdots, N$, are positive constants, and ν is the unit normal vector.

Thus we have the local-in-time existence of the solution and blowup criterion (16). Next the simultaneous blowup

$$T < +\infty \quad \Rightarrow \quad \lim_{t\uparrow T} \|u_i(\cdot,t)\|_\infty = +\infty, \; i = 1, 2, \cdots, N$$

occurs in the parameter region

$$\left(\sum_{i\in J} \lambda_i\right)^2 < 4\pi \sum_{i\in J} \xi_i \lambda_i, \quad \forall J \subset \{1, 2, \cdots, N\}, \; \sharp J = N - 1.$$

We have finiteness of the blowup set $\mathcal{S}$ defined by (17) and the formation of collapses:

$$u_i(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m_i(x_0)\delta_{x_0}(dx) + f_i(x)dx, \quad i = 1,2,\cdots,N \tag{33}$$

in $\mathcal{M}(\overline{\Omega})$ as $t \uparrow T$, where $m_i(x_0) \geq 0$, $i = 1,2,\cdots,N$, are constants satisfying

$$m(x_0) \equiv \sum_{i=1}^{N} m_i(x_0) > 0$$

and $0 \leq f_i \in L^1(\Omega)$, $i = 1,2,\cdots,N$, are smooth functions in $\overline{\Omega} \setminus \mathcal{S}$.

Letting $\xi_i = d_i/\chi_i$, $i = 1,2,\cdots,N$, it holds that

$$\left(\sum_{i=1}^{N} m_i(x_0)\right)^2 = m_*(x_0)\sum_{i=1}^{N} \xi_i m_i(x_0) \tag{34}$$

and

$$z_i(y,s+s')dy \rightharpoonup m_i(x_0)\delta_0(dy), \quad i = 1,2,\cdots,N \tag{35}$$

in $C_*(-\infty,+\infty;\mathcal{M}_0(\mathbf{R}^2))$ as $s' \uparrow +\infty$ for any $x_0 \in \mathcal{S}$, where $z_i = z_i(y,s)$, $i = 1,2,\cdots,N$, are self-similar transformations of $u_i = u_i(x,t)$ defined by (22), which implies

$$\lim_{t\uparrow T}(T-t)\|u(\cdot,t)\|_{L^\infty(\Omega\cap B(x_0,b(T-t)^{1/2}))} = +\infty, \quad u = \sum_{i=1}^{N} u_i \tag{36}$$

for any $b > 0$. If $u_i = u_i(|x|,t)$, $i = 1,2,\cdots,N$, we have

$$\left(\sum_{i\in K} m_i(0)\right)^2 \leq 8\pi \sum_{i\in K} \xi_i m_i(0)$$

for any $K \subset \{1,2,\cdots,N\}$, $K \neq \emptyset$, besides (34), $x_0 = 0$.

The solution exists global-in-time, provided that

$$\left(\sum_{i\in K} \lambda_i\right)^2 < 4\pi \sum_{i\in K} \xi_i \lambda_i, \quad \forall K \subset \{1,2,\cdots,N\}, \ K \neq \emptyset,$$

where $\lambda_i = \|u_{i0}\|_1$, $i = 1,2,\cdots,N$, while its blowup occurs if

$$\left(\sum_{i=1}^{N} \lambda_i(x_0)\right)^2 > m_*(x_0)\sum_{i=1}^{N} \xi_i \lambda_i(x_0)$$
$$\||x-x_0|^2 u_{i0}\|_{L^1(\Omega\cap B(x_0,2R))} \ll 1, \quad \forall i = 1,2,\cdots,N \tag{37}$$

for some $x_0 \in \overline{\Omega}$, where $\lambda_i(x_0) = \|u_{i0}\|_{L^1(\Omega\cap B(x_0,R))}$.

In the case of $u_i = u_i(|x|,t)$, $i = 1,2,\cdots,N$, we obtain the following. First, the other blowup criterion than (37) arises, that is,

$$\left(\sum_{i\in K}\lambda_i(0)\right)^2 > 8\pi\sum_{i\in K}\xi_i\lambda_i(0)$$
$$\||x-x_0|^2 u_{i0}\|_{L^1(\Omega\cap B(x_0,2R)} \ll 1, \quad \forall i \in K$$

for some $K \subset \{1,2,\cdots,N\}$, $K \neq \emptyset$ implies $T < +\infty$. Next, we obtain always the simultaneous blowup

$$\lim_{t\uparrow T}\|u_i(\cdot,t)\|_\infty = +\infty, \quad \forall i = 1,2,\cdots,N.$$

A collapse mass $m_k = m_k(0)$, $k = 1,2,\cdots,N$, on the other hand, vanishes if $\xi_i > 2\xi_k$ for any $i \neq k$ and

$$\|u_{k0}\|_1 < 8\pi\min_{i\neq k}(\xi_i - 2\xi_k).$$

Consequently, we have mass separation, for example,

$$m_N = 8\pi\xi_N, \quad m_i = 0, \quad \forall i = 1,2,\cdots,N-1,$$

provided that

$$\xi_i > 2\xi_j, \quad \forall j = 1,2,\cdots,N-1, \ \forall i = j+1,\cdots,N,$$

and furthermore,

$$\|u_{j0}\|_1 < 8\pi\min_{i=j+1,\cdots,N}(\xi_i - 2\xi_j), \quad \forall j = 1,2,\cdots,N-1.$$

2 Preliminaries and Proof of Theorem 1

We begin with a description of the variational structure of (ISP). Let $v = (-\Delta)^{-1}u$ stand for (9). First, the total mass conservation follows from (1) and (2) in each component of the solution:

$$\|u_i(\cdot,t)\|_1 = \|u_{i0}\|_1 \equiv \lambda_i, \quad i = 1,2. \tag{38}$$

Next, we obtain

$$\begin{aligned}
&\partial_t u_1 = \nabla\cdot u_1\nabla(d_1\log u_1 - \chi_1 v) && \text{in } \Omega\times(0,T)\\
&u_1\frac{\partial}{\partial\nu}(d_1\log u_1 - \chi_1 v) = 0 && \text{on } \partial\Omega\times(0,T)
\end{aligned} \tag{39}$$

by (7), which implies

$$\frac{d}{dt}\int_\Omega d_1 u_1(\log u_1 - 1)dx - \chi_1\langle v, u_{1t}\rangle = -\int_\Omega u_1|\nabla(d_1\log u_1 - \chi_1 v)|^2 dx,$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2-inner product. Similarly, it holds that

$$\frac{d}{dt}\int_\Omega d_2 u_2(\log u_2 - 1)dx - \chi_2\langle v, u_{2t}\rangle = -\int_\Omega u_2|\nabla(d_2\log u_2 - \chi_2 v)|^2 dx,$$

and hence

$$\begin{aligned}\frac{d}{dt}&\left\{\int_\Omega \sum_{i=1}^2 \xi_i u_i(\log u_i - 1)dx - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle\right\}\\ &= -\int_\Omega \sum_{i=1}^2 \chi_i^{-1} u_i|\nabla(d_i\log u_i - \chi_i v)|^2\ dx\end{aligned}$$

by $\xi_i = d_i/\chi_i$ and

$$\langle v, u_t\rangle = \frac{1}{2}\frac{d}{dt}\langle(-\Delta)^{-1}u, u\rangle.$$

We thus obtain the following lemma using free energy (40) derived in [35].

Lemma 1 *Total mass is conserved for each component of the solution, indicated by (38), and also the decrease of the free energy:*

$$\frac{d}{dt}\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2) \le 0$$

in (ISP), where

$$\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2) = \sum_{i=1}^2\int_\Omega \xi_i u_i(\log u_i - 1)dx - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle \tag{40}$$

for $u = u_1 + u_2$ (recall (13)).

Here we prescribe a parameter region of (ξ_1,ξ_2) which ensures the logarithmic HLS (or dual Trudinger-Moser) inequality

$$\inf\{\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2) \mid u_i \ge 0,\ \|u_i\|_1 = \lambda_i,\ i=1,2\} > -\infty. \tag{41}$$

First, the condition

$$\left(\sum_{i=1}^2 \lambda_i\right)^2 \le 8\pi\sum_{i=1}^2 \xi_i\lambda_i, \quad \lambda_i \le 8\pi\xi_i,\ i=1,2 \tag{42}$$

implies

$$\inf\{\hat{\mathcal{F}}_{\xi_1,\xi_2}(u_1,u_2) \mid u_i \geq 0,\ \|u_i\|_1 = \lambda_i,\ i=1,2\} > -\infty, \tag{43}$$

where

$$\hat{\mathcal{F}}_{\xi_1,\xi_2}(u_1,u_2) = \sum_{i=1}^{2} \int_\Omega \xi_i u_i(\log u_i - 1)dx$$
$$-\frac{1}{2}\iint_{\Omega\times\Omega} \frac{1}{2\pi}\log\frac{1}{|x-x'|}\ u\otimes u\ dxdx'$$

for $u = u_1 + u_2$ and $(u\otimes u)(x,x') = u(x)u(x')$. This property is proven in [30], by using the logarithmic Hardy-Littlewood-Sobolev inequality

$$2\iint_{\Omega\times\Omega} F(x)\ \log\frac{1}{|x-x'|}\ G(x')\ dxdx'$$
$$\leq \int_\Omega (1-\alpha)F\log F + \alpha G\log G\ dx + C_\alpha, \quad 0<\alpha<1,$$

valid to

$$F, G \geq 0, \quad \int_\Omega F\ dx = \int_\Omega G\ dx = 1,$$

and an inequality derived from linear programming.

Second, we deal with $(ISPD)$, which denotes the system of equations combining (1) and (2) with

$$-\Delta v = u, \quad v|_{\partial\Omega} = 0, \quad u = u_1 + u_2. \tag{44}$$

In fact, for this system we also have conservation of total mass (38) and a decreasing free energy

$$\frac{d}{dt}\tilde{\mathcal{F}}_{\xi_1,\xi_2}(u_1,u_2) \leq 0, \tag{45}$$

where

$$\tilde{\mathcal{F}}_{\xi_1,\xi_2}(u_1,u_2) = \sum_{i=1}^{2}\int_\Omega \xi_i u_i(\log u_i - 1)dx - \frac{1}{2}\iint_{\Omega\times\Omega}\tilde{G}(x,x')u\otimes u\ dxdx'$$

with $u = u_1 + u_2$. Here, $\tilde{G} = \tilde{G}(x,x')$ is the Green's function to (44) which satisfies

$$\tilde{G}(x,x') \leq \frac{1}{2\pi}\log\frac{1}{|x-x'|} + C \tag{46}$$

with a constant $C = C(\Omega)$. Hence it follows that

$$\inf\{\tilde{\mathcal{F}}_{\xi_1,\xi_2}(u_1,u_2) \mid u_i \geq 0,\ \|u_i\|_1 = \lambda_i,\ i=1,2\} > -\infty$$

under the assumption of (42). If a slightly stronger condition holds for the total masses of the initial value, that is

$$\left(\sum_{i=1}^{2}\lambda_i\right)^2 < 8\pi\sum_{i=1}^{2}\xi_i\lambda_i, \quad \lambda_i < 8\pi\xi_i, \ i=1,2 \tag{47}$$

for $\lambda_i = \|u_{i0}\|_1$, $i=1,2$, then we have that

$$\sum_{i=1}^{2}\int_\Omega \xi_i u_i(\log u_i - 1)\ dx \le C \tag{48}$$

by (38) and (45), where $(u_1,u_2) = (u_1(\cdot,t),u_2(\cdot,t))$ is the solution to problem $(ISPD)$, and $C>0$ is a constant independent of $t\in[0,T)$. (See the argument for the proof of Theorem 12.)

Now the following theorem is obtained, similarly to the case of a single component [1, 11, 21], using an iterative scheme, Gagliardo-Nirenberg's inequality, and an inequality due to [2], that is, (22) (see also Lemma 4.1 of [32]).

Theorem 7 ([35]) *If (47) is the case, it holds that*

$$T=+\infty, \quad \limsup_{t\uparrow T}\sum_{i=1}^{2}\|u_i(\cdot,t)\|_\infty < +\infty \tag{49}$$

for (ISPD).

Inequalities (45) and (46) imply

$$\hat{\mathcal{F}}_{\xi_1,\xi_2}(u_1,u_2) \le C, \quad t\in[0,T).$$

Here we write

$$\begin{aligned}&\frac{1}{4\pi}\iint_{\Omega\times\Omega} u\otimes u\ \log|x-x'|\ dxdx' \\ &\quad = \frac{1}{4\pi}\sum_{i,j=1}^{2} b_ib_j\iint_{\Omega\times\Omega}\rho_i\otimes\rho_j\ \log|x-x'|\ dxdx'\end{aligned}$$

using $b_i>0$, $i=1,2$, where $\rho_i = u_i/b_i$. Then we examine the criterion due to [30] (the *main theorem*), concerning the boundedness of

$$\Psi(\rho_1,\rho_2) = \sum_{i=1}^{2}\int_\Omega \rho_i\log\rho_i\ dx + \sum_{i,j=1}^{2} a_{ij}\iint_{\Omega\times\Omega}\rho_i\otimes\rho_j\ \log|x-x'|\ dxdx'$$

defined for $\rho_i \geq 0$ with $\|\rho_i\|_1 = M_i$, $i = 1, 2$, where

$$M_i = \frac{\lambda_i}{b_i}, \quad a_{ij} = \frac{1}{4\pi} b_i b_j.$$

In fact, since $a_{ii} = b_i^2/(4\pi) > 0$ this property is controlled by

$$\Lambda_J(\oplus M_i) = 2\sum_{i\in J} M_i - \sum_{i,j\in J} a_{ij} M_i M_j = 2\sum_{i\in J} \frac{\lambda_i}{b_i} - \frac{1}{4\pi}\left(\sum_{i\in J}\lambda_i\right)^2.$$

Thus the above Ψ is bounded below if and only if

$$\Lambda_J(\oplus M_i) \geq 0, \quad \forall J \subset \{1, \cdots, N\},\ J \neq \emptyset,\ N = 2,$$

that is,

$$\left(\sum_{i=1}^2 \lambda_i\right)^2 \leq 8\pi \sum_{i=1}^2 \frac{\lambda_i}{b_i}, \quad \lambda_i \leq \frac{8\pi}{b_i},\ i = 1, 2. \tag{50}$$

Using

$$\int_\Omega \rho_i \log \rho_i \ dx = \frac{1}{b_i}\int_\Omega u_i \log u_i \ dx - \frac{\lambda_i}{b_i}\log b_i,$$

we obtain the following lemma.

Lemma 2 *Under the assumption (50) we have*

$$\sum_{i=1}^2 \int_\Omega (\xi_i - \frac{1}{b_i}) u_i \log u_i \ dx \leq C, \quad t \in [0, T)$$

for (ISPD).

In case $\lambda_1 < 8\pi\xi_1$, we can find $b_i > 0$, $i = 1, 2$, in (50) by taking

$$0 < b_1 - \frac{1}{\xi_1} \ll 1, \quad 0 < b_2 \ll 1.$$

Then it follows that

$$\delta \int_\Omega u_1 \log u_1 \ dx \leq \int_\Omega u_2 \log u_2 \ dx + \delta^{-1}$$

with $0 < \delta \ll 1$. The reverse inequality is similarly obtained if $\lambda_2 < 8\pi\xi_2$, while the inequality in (49) holds if and only if

$$\sup_{t\in[0,T)} \sum_{i=1}^2 \|(u_i \log u_i)(\cdot, t)\|_1 \leq C$$

with $C > 0$ independent of t (see, for example, [32]). We thus end up the following theorem by (16).

Theorem 8 *If*

$$\lambda_i < 8\pi\xi_i, \ i = 1, 2, \tag{51}$$

then we have (11) for $(ISPD)$.

Here we note again that the case $T < +\infty$ can arise in the parameter region (51) with

$$\left(\sum_{i=1}^{2} \lambda_i \right)^2 > 8\pi \sum_{i=1}^{2} \xi_i \lambda_i$$

(see Theorem 10).

To handle the original system (ISP), we use the fact that the Green's function $G = G(x, x')$ of the Poisson equation

$$-\Delta v = u - \frac{1}{|\Omega|} \int_\Omega u \ dx, \quad \left. \frac{\partial v}{\partial \nu} \right|_{\partial\Omega} = 0, \quad \int_\Omega v \ dx = 0$$

satisfies

$$G(x, x') \leq \frac{1}{\pi} \log \frac{1}{|x - x'|} + C' \tag{52}$$

with a constant $C' = C'(\Omega)$ determined by Ω. In fact, each $x_0 \in \partial\Omega$ admits a smooth conformal mapping $X = X(x) : \overline{\Omega \cap B(x_0, 2R)} \to \overline{\mathbf{R}}^2_+$, $0 < R \ll 1$, where $\mathbf{R}^2_+$ stands for the upper half-plane and $0 < R \ll 1$. Then the following relation arises

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')|} + \frac{1}{2\pi} \log \frac{1}{|X(x) - X(x')^*|} + K(x, x') \tag{53}$$

with

$$K = K(x, x') \in C^{1+\theta,\theta}(\overline{\Omega \cap B(x_0, R)} \times \overline{\Omega \cap B(x_0, R)}) \\ \cap C^{\theta,1+\theta}(\overline{\Omega \cap B(x_0, R)} \times \overline{\Omega \cap B(x_0, R)}) \tag{54}$$

for $0 < \theta < 1$, where $X^* = (X_1, -X_2)$ for $X = (X_1, X_2)$ (see [32]). The case $x_0 \in \Omega$ is easier, and it holds that

$$G(x, x') = \frac{1}{2\pi} \log \frac{1}{|x - x'|} + K(x, x') \tag{55}$$

with

$$K = K(x, x') \in C^{1+\theta,\theta}(\overline{B(x_0, R)} \times \overline{B(x_0, R)}) \\ \cap C^{\theta,1+\theta}(\overline{B(x_0, R)} \times \overline{B(x_0, R)}). \tag{56}$$

By a standard covering argument, these relations imply (52). Then we obtain the following lemma.

Lemma 3 *For the free energy $\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2)$ defined by (40), if*

$$\left(\sum_{i=1}^{2}\lambda_i\right)^2 \leq 4\pi\sum_{i=1}^{2}\xi_i\lambda_i, \quad \lambda_i \leq 4\pi\xi_i, \ i=1,2$$

it holds that

$$\inf\{\mathcal{F}_{\xi_1,\xi_2}(u_1,u_2) \mid u_i \geq 0, \ \|u_i\|_1 = \lambda_i, \ i=1,2\} > -\infty. \tag{57}$$

This lemma implies the following theorem and also Theorem 1 similarly to Theorems 7 and 8 derived from (46), respectively.

Theorem 9 *We have (49) in (ISP) if*

$$\left(\sum_{i=1}^{2}\lambda_i\right)^2 < 4\pi\sum_{i=1}^{2}\xi_i\lambda_i, \ \lambda_i < 4\pi\xi_i, \ i=1,2. \tag{58}$$

We proceed to a weak form of (ISP) derived from the symmetry of the Green's function, $G(x,x') = G(x',x)$. Thus we take $\varphi = \varphi(x)$ satisfying

$$\varphi \in C^2(\overline{\Omega}), \quad \left.\frac{\partial\varphi}{\partial\nu}\right|_{\partial\Omega} = 0, \tag{59}$$

to confirm

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_1\varphi \ dx - d_1\int_\Omega u_1\Delta\varphi \ dx &= \chi_1\int_\Omega u_1\nabla v\cdot\nabla\varphi \ dx \\
\frac{d}{dt}\int_\Omega u_2\varphi \ dx - d_2\int_\Omega u_2\Delta\varphi \ dx &= \chi_2\int_\Omega u_2\nabla v\cdot\nabla\varphi \ dx,
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega\left[\sum_{i=1}^{2}\chi_i^{-1}u_i\right]\varphi \ dx - \int_\Omega\left[\sum_{i=1}^{2}\xi_i u_i\right]\Delta\varphi \ dx \\
&\quad = \int_\Omega u\nabla v\cdot\nabla\varphi \ dx = \frac{1}{2}\iint_{\Omega\times\Omega}\rho_\varphi u\otimes u \ dxdx'
\end{aligned} \tag{60}$$

with

$$\rho_\varphi(x,x') = \nabla\varphi(x)\cdot\nabla_x G(x,x') + \nabla\varphi(x')\cdot\nabla_{x'}G(x,x').$$

The weak form (60) can be used to derive a blowup criterion. The argument is similar to that of [27], using (53) (see also [32]). First, we take a nice cut-off function introduced by [26] (see also [32]), which is denoted by $\varphi = \varphi_{x_0,R}(x)$ for $x_0 \in \overline{\Omega}$ and $R > 0$. It is a C^∞-function with support radius $R > 0$, equal to 1 on $\overline{\Omega \cap B(x_0, R/2)}$, and satisfies $0 \leq \varphi \leq 1$, (59), and

$$|\nabla\varphi| \leq O(R^{-1})\varphi^{2/3}, \quad |\nabla^2\varphi| \leq O(R^{-2})\varphi^{5/6} \tag{61}$$

as $R \downarrow 0$. Here, we note the property

$$\lim_{R\downarrow 0} \frac{1}{R^2}\int_\Omega |x-x_0||\nabla\varphi| + |x-x_0|^2|\nabla^2\varphi| \; dx = 0. \tag{62}$$

In fact, if $x_0 \in \Omega$ the above $\varphi = \varphi_{x_0,R}$ is a scaling of a fixed function with respect to R, and therefore, (62) is easy to see. If $x_0 \in \partial\Omega$ is the case, just the above-described conformal mapping $X : \overline{\Omega \cap B(x_0, 2R)} \to \overline{\mathbf{R}}^2_+$ is involved other than the scaling, in constructing $\varphi = \varphi_{x_0,R}$ (see [32]). Then, relation (62) is proven similarly.

Given $x_0 \in \Omega$, we take $\varphi = |x - x_0|^2\varphi_{x_0,R}(x)$ for $R > 0$. Let $B = B(\lambda_1, \lambda_2) \gg 1$ be a constant determined by λ_i, $i = 1, 2$, and let

$$\begin{aligned}
\hat{u}(x,t) &= \sum_{i=1}^{2} \chi_i^{-1} u_i(x,t) \\
I_R(t) &= \int_\Omega |x-x_0|^2 \hat{u}(x,t)\varphi_{x_0,R}(x) \; dx \\
M_R^i(t) &= \int_\Omega u_i(x,t)\varphi_{x_0,R}(x) \; dx, \quad i = 1, 2 \\
J_R(t) &= 4\sum_{i=1}^{2} \xi_i M_R^i(t) - \frac{1}{2\pi}\left(\sum_{i=1}^{2} M_R^i(t)\right)^2 + 8BR^{-1}I_{4R}(t)^{1/2}.
\end{aligned}$$

Then, we can derive

$$\frac{dI_R}{dt}(t) \leq J_R(0) + a(R^{-1}t^{1/2}) + BR^{-1}I_R(t)^{1/2}$$

for $a(s) = B(s^2 + s)$, which implies $T = T_{\max} < +\infty$ in case $J_R(0) < 0$ and $I_R(0) \ll 1$. Given $x_0 \in \partial\Omega$, on the other hand, we take $\varphi = |X(x)|^2\varphi_{x_0,R}(x)$. Then we can argue similarly, and in particular, obtain the following theorem.

Theorem 10 *We have $T < +\infty$ in (ISP) if*

$$\left(\sum_{i=1}^{2} \lambda_i(x_0)\right)^2 > m_*(x_0)\sum_{i=1}^{2} \xi_i\lambda_i(x_0) \tag{63}$$

and

$$\int_{\Omega\cap B(x_0,2R)} |x-x_0|^2 u_{i0}(x)\ dx \ll 1, \quad i=1,2,$$

where $x_0 \in \overline{\Omega}$ *and* $\lambda_i(x_0) = \|u_{i0}\|_{L^1(\Omega\cap B(x_0,R))}$, $i=1,2$.

3 Proof of Theorem 2

In this section, we show the finiteness of the blowup set $\mathcal{S}$ and the formation of collapses indicated by (18).

The first observation is that inequality (58) holds for $0<\lambda_1,\lambda_2 \ll 1$, and therefore, $0<\lambda_1,\lambda_2\ll 1$ implies (49). This property was observed in [16] for the case of a single component. Then, an ε-regularity is proven by the method of localization [26], that is, taking the above $\varphi=\varphi_{x_0,R}$ to localize the global-in-time existence result of [16].

This method is applicable to (ISP), and we obtain the following lemma. Henceforth, $(u_1,u_2)=(u_1(\cdot,t),u_2(\cdot,t))$ denotes the solution to (ISP) with $T<+\infty$.

Lemma 4 *There is* $\varepsilon_0>0$ *such that*

$$\limsup_{t\uparrow T} \sum_{i=1}^{2} \chi_i^{-1}\|u_i(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon_0 \tag{64}$$

implies

$$\limsup_{t\uparrow T} \sum_{i=1}^{2} \|u_i(\cdot,t)\|_{L^\infty(\Omega\cap B(x_0,R/4))} < +\infty, \tag{65}$$

where $x_0\in\overline{\Omega}$ *and* $0<R\le 1$.

For the proof, first, we derive

$$\limsup_{t\uparrow T} \sum_{i=1}^{2} \int_{\Omega\cap B(x_0,R/2)} u_i \log u_i\ dx < +\infty \tag{66}$$

from (64) with $0<\varepsilon_0\ll 1$. Then, (65) is obtained by an iteration scheme. In this process of localization, the cut-off function $\varphi=\varphi_{x_0,R}$, $x_0\in\overline{\Omega}$, $R>0$, satisfying (59) and (61) is used.

The weak form (60) implies an estimate, which may be called the monotonicity formula. More precisely, we have $\rho_\varphi \in L^\infty(\Omega\times\Omega)$ in this formula (see Lemma 5.2 of [32]), and hence

$$\left|\frac{d}{dt}\sum_{i=1}^{2}\int_\Omega \chi_i^{-1}u_i\varphi\ dx\right| \le \left[\sum_{i=1}^{2}\xi_i\lambda_i\right]\|\Delta\varphi\|_\infty + \frac{1}{2}\|\rho_\varphi\|_\infty\left(\sum_{i=1}^{2}\lambda_i\right)^2.$$

The following lemma is thus obtained.

Lemma 5 *There is $C > 0$ such that*

$$\left| \frac{d}{dt} \int_\Omega \hat{u}\varphi \ dx \right| \leq C \|\nabla \varphi\|_{C^1(\overline{\Omega})}$$

for any $\varphi = \varphi(x)$ satisfying (59), where $\hat{u} = \chi_1^{-1} u_1 + \chi_2^{-1} u_2$.

A direct consequence of Lemma 5 is the existence of an extension of $\mu(dx, t) = \hat{u}(x, t)dx$, $0 \leq t < T$, as

$$\mu(dx, t) \in C_*([0, T]; \mathcal{M}(\overline{\Omega})).$$

We have, in particular,

$$\begin{aligned} \mu(\{x_0\}, T) &= \lim_{R\downarrow 0} \limsup_{t\uparrow T} \|\hat{u}(\cdot, t)\|_{L^1(\Omega\cap B(x_0,R))} \\ &= \lim_{R\downarrow 0} \liminf_{t\uparrow T} \|\hat{u}(\cdot, t)\|_{L^1(\Omega\cap B(x_0,R))} \end{aligned}$$

for each $x_0 \in \overline{\Omega}$. Lemma 4, on the other hand, implies

$$x_0 \in \mathcal{S} \quad \Rightarrow \quad \lim_{R\downarrow 0} \limsup_{t\uparrow T} \|\hat{u}(\cdot, t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0,$$

recalling (17), and hence

$$\mu(\{x_0\}, T) \geq \varepsilon_0, \qquad \forall x_0 \in \mathcal{S}.$$

Since $\mu(\overline{\Omega}, T) = \lambda$, we obtain the finiteness of $\mathcal{S}$. Thus the following lemma arises similarly to the case of a single component [26], that is, the formation of collapses of $\hat{u}(x, t)dx$ derived from Lemmas 4 and 5 and the parabolic-elliptic regularity.

Lemma 6 *If $T < +\infty$ in (ISP), it holds that $\sharp\mathcal{S} < +\infty$ and*

$$\hat{u}(x, t)dx \rightharpoonup \sum_{x_0\in\mathcal{S}} \hat{m}(x_0)\delta_{x_0}(dx) + \hat{f}(x)dx \tag{67}$$

in $\mathcal{M}(\overline{\Omega})$ as $t \uparrow T$, where $\hat{m}(x_0) \geq \varepsilon_0$ is a constant and $0 \leq \hat{f} = \hat{f}(x) \in L^1(\Omega)$ is a smooth function in $\overline{\Omega} \setminus \mathcal{S}$.

We are ready to prove Theorem 2. First, we take φ satisfying (59) and $\nabla\varphi = 0$ near $\mathcal{S}$. For such $\varphi = \varphi(x)$, it holds that

$$\frac{d}{dt} \int_\Omega u_i \varphi \ dx = \int_\Omega u_i \Delta\varphi + u_i \nabla v \cdot \nabla\varphi \ dx$$

and hence

$$\left| \frac{d}{dt} \int_\Omega u_i \varphi \ dx \right| \leq C'_\varphi, \quad i = 1, 2$$

with a constant C'_φ, because $(u_1, u_2, v) = (u_1(x,t), u_2(x,t), v(x,t))$ is smooth in $(\overline{\Omega} \times [0,T]) \setminus (\mathcal{S} \times \{T\})$ from the elliptic-parabolic regularity. Since $\sharp \mathcal{S} < +\infty$, the set of such $\varphi = \varphi(x)$ is dense in $C(\overline{\Omega})$. Therefore, like $\hat{u}(x,t)dx$, the measures $u_i(x,t)dx$, $i = 1, 2$, are extended as

$$\mu_i(dx, t) \in C_*([0,T], \mathcal{M}(\overline{\Omega})), \quad i = 1, 2,$$

using (38).

Since $\hat{\mu}(dx,t) = \sum_{i=1}^2 \chi_i^{-1} \mu_i(dx,t)$, the singular parts of $\mu_i(dx,T)$, $i = 1, 2$, are composed of finite sums of delta functions supported on $\mathcal{S}$. We shall write

$$\mu_i(dx, T) = \sum_{x_0 \in \mathcal{S}} m_i(x_0) \delta_{x_0}(dx) + f_i(x) dx, \quad i = 1, 2$$

with $m_i(x_0) \geq 0$ and $0 \leq f_i = f_i(x) \in L^1(\Omega)$, $i = 1, 2$. Then, it holds that

$$\hat{m}(x_0) = \sum_{i=1}^2 \chi_i^{-1} m_i(x_0) \geq \varepsilon_0,$$

and hence $(m_1(x_0), m_2(x_0)) \neq (0,0)$.

Finally, $(u_1, u_2, v) = (u_1(x,t), u_2(x,t), v(x,t))$ satisfies

$$\partial_t u_i = d_i \Delta u_i - \chi_i \nabla \cdot u_i \nabla v \quad \text{in } (\overline{\Omega} \times [0,T]) \setminus (\mathcal{S} \times \{T\}), \quad i = 1, 2.$$

Since (7) holds, therefore, the functions $u_i(x,t) \geq 0$, $i = 1, 2$, cannot attain 0 in $(\overline{\Omega} \setminus \mathcal{S}) \times \{T\}$ from the maximum principle. Hence it holds that $u_i(x,T) = f_i(x) > 0$, $i = 1, 2$, in $\overline{\Omega} \setminus \mathcal{S}$. ∎

4 Proof of Theorems 3 and 4

Theorem 3 is composed of two inequalities, (29) and (30). First, inequality (29) is regarded as a localization of the blowup criterion, Theorem 10. For the proof we use the method of scaling limit developed in the case of a single component [32, 33, 29]. Henceforth, $\varphi = \varphi_{x_0,R}$ denotes the cut-off function mentioned in the previous section satisfying (59) and (61).

Hence we take the backward self-similar transformations (22), to obtain

$$\begin{aligned} &\partial_s z_1 = d_1 \Delta z_1 - \chi_1 \nabla \cdot z_1 \nabla (w + \frac{|y|^2}{4}) \\ &\partial_s z_2 = d_2 \Delta z_2 - \chi_2 \nabla \cdot z_2 \nabla (w + \frac{|y|^2}{4}) \quad \text{in } \textstyle\bigcup_{s > -\log T} \Omega_s \times \{s\} \\ &\frac{\partial z_1}{\partial \nu} = \frac{\partial z_2}{\partial \nu} = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \textstyle\bigcup_{s > -\log T} \partial \Omega_s \times \{s\} \end{aligned} \tag{68}$$

and

$$-\Delta w = z - \frac{1}{|\Omega_s|}\int_{\Omega_s} z\ dy, \ \left.\frac{\partial w}{\partial \nu}\right|_{\partial\Omega_s} = 0, \ \int_{\Omega_s} w\ dy = 0, \tag{69}$$

where $z = z_1 + z_2$ and $\Omega_s = (T-t)^{1/2}(\Omega - \{x_0\})$. As $s \uparrow +\infty$, the domain Ω_s expands to the whole or the half-spaces denoted by L, according to $x_0 \in \Omega$ or $x_0 \in \partial\Omega$, respectively. Then, similarly to the prescaled case, any $s_k \uparrow +\infty$ admits $\{s_k'\} \subset \{s_k\}$ and $\hat{\zeta}(dy, s)$ such that

$$\hat{z}(y, s + s_k')dy \rightharpoonup \hat{\zeta}(dy, s) \quad \text{in } C_*(-\infty, +\infty : \mathcal{M}_0(\mathbf{R}^2)), \tag{70}$$

where $\hat{z} = \chi_1^{-1} z_1 + \chi_2^{-1} z_2$. Here, we recall that the 0-extensions of $z(y, s)$ are taken where `it is` not defined. Hence this $\zeta = \zeta(dy, s)$ has a support contained on $\overline{L}$. We recall also that $\mathcal{M}_0(\mathbf{R}^2) = C_0(\overline{\mathbf{R}^2})'$, where $C_0(\overline{\mathbf{R}^2})$ denotes the set of continuous functions on $\overline{\mathbf{R}^2} = \mathbf{R}^2 \cup \{\infty\}$ vanishing at ∞.

In fact, first, we have

$$\|z_i(\cdot, s)\|_1 = \lambda_i, \quad i = 1, 2. \tag{71}$$

Next, the Green's function to (69) is given by

$$\mathcal{G}_s(y, y') = G(e^{-s/2}y + x_0, e^{-s/2}y' + x_0)$$

where (53) is applicable. In the case of $x_0 \in \Omega$, we thus use

$$\left|\frac{d}{ds}\int_{\mathbf{R}^2} \hat{z}\Phi\ dy\right| \leq C_\Phi \tag{72}$$

valid for $\Phi = \Phi(y) \in C_0^2(L)$. In the other case of $x_0 \in \partial\Omega$, we take the conformal mapping $X : \overline{\Omega \cap B(x_0, R)} \to \mathbf{R}_+^2$ described in §2 and put $Y(y, s) = e^{s/2}X(e^{-s/2}y + x_0)$. Let $C_0^2(\overline{\mathbf{R}}_+^2)$ be the set of C^2-functions on $\overline{L}$ with compact supports. Given

$$\zeta = \zeta(Y) \in C_0^2(\overline{\mathbf{R}}_+^2), \quad \left.\frac{\partial\zeta}{\partial\nu_Y}\right|_{\partial\mathbf{R}_+^2} = 0,$$

we take $\Phi_s = \zeta \circ Y(\cdot, s)$, to obtain

$$\left|\frac{d}{ds}\int_{\Omega_s} \hat{z}\Phi_s\ dy\right| \leq C_\Phi \tag{73}$$

(see [32]). Then inequality (71) with (72) and (73) for $x_0 \in \Omega$ and $x_0 \in \partial\Omega$, respectively, implies the desired convergence (70).

We have also the component-wise convergence of $z_i(y,s)dy$, $i=1,2$. In fact, taking subsequences of $\{s'_k\}$ denoted by the same symbol, we have

$$z_i(dy, s+s'_k) \rightharpoonup \zeta_i(dy,s) \quad \text{in } L^\infty_*(-\infty,+\infty;\mathcal{M}_0(\mathbf{R}^2)), \quad i=1,2 \tag{74}$$

with the limit measures $\zeta_i(dy,s)$, $i=1,2$, of which supports are contained in $\overline{L}$, where

$$L^\infty_*(-\infty,+\infty;\mathcal{M}_0(\mathbf{R}^2)) = L^1(-\infty,+\infty;C_0(\mathbf{R}^2))'.$$

This convergence guarantees

$$\begin{aligned}
\zeta(dy,s) &= \sum_{i=1}^2 \zeta_i(dy,s) \\
\hat{\zeta}(dy,s) &= \sum_{i=1}^2 \chi_i^{-1}\zeta_i(dy,s) \\
\tilde{\zeta}(dy,s) &= \sum_{i=1}^2 \xi_i\zeta_i(dy,s) \\
\zeta_i(\overline{L},s) &\le m_i(x_0), \quad i=1,2
\end{aligned} \tag{75}$$

for a.e. s, recalling (18).

The estimate

$$\left| \frac{d}{dt}\int_\Omega \varphi_{x_0,R}\cdot \hat{u}(x,t)\ dx \right| \le C_7R^{-2}, \quad 0<R\le 1,$$

on the other hand, is derived from Lemma 5 concerning the prescaled solution, which ensures the *parabolic envelope* similarly to the case of a single component [32]. More precisely, it holds that

$$|\langle \varphi_{x_0,R}, \hat{u}(x,t)dx\rangle - \langle \varphi_{x_0,R}, \hat{\mu}(dx,T)\rangle| \le C_7(T-t)R^{-2}$$

for $0\le t<T$. Here, putting $R=b(T-t)^{1/2}$, we make $t\uparrow T$ and then $b\uparrow+\infty$. It follows that

$$\hat{\zeta}(\overline{L},s) = \sum_{i=1}^2 \chi_i^{-1}m_i(x_0), \tag{76}$$

and hence

$$\zeta_i(\overline{L},s) = m_i(x_0), \quad i=1,2 \tag{77}$$

for a.e. s by (75) and (76).

In case $L = \mathbf{R}^2_+$, we use even extensions of $\zeta_i(dy, s)$, $i = 1, 2$. Henceforth, we handle with such $\zeta_i(dy, s)$, $i = 1, 2$, denoted by the same symbols, defined on $L = \mathbf{R}^2$ and put

$$\zeta(dy, s) = \sum_{i=1}^{2} \zeta_i(dy, s)$$
$$\hat{\zeta}(dy, s) = \sum_{i=1}^{2} \chi_i^{-1} \zeta_i(dy, s)$$
$$\tilde{\zeta}(dy, s) = \sum_{i=1}^{2} \xi_i \zeta_i(dy, s).$$

Then it holds that

$$\zeta_i(\mathbf{R}^2, s) = m_i^*(x_0) \equiv \begin{cases} m(x_0), & x_0 \in \Omega \\ 2m(x_0), & x_0 \in \partial\Omega. \end{cases}$$

These measures make up a weak solution to

$$\partial_s \hat{z} = \Delta \tilde{z} - \nabla \cdot z \nabla(\Gamma * z + \frac{|y|^2}{4}) \qquad \text{in } \mathbf{R}^2 \times (-\infty, +\infty)$$
$$\Gamma(y) = \frac{1}{2\pi} \log \frac{1}{|y|}, \tag{78}$$

where $z = \sum_{i=1}^{2} z_i$ and $\tilde{z} = \sum_{i=1}^{2} \xi_i z_i$. This property is formulated as follows, using

$$\mathcal{X} = C_0(\overline{\mathbf{R}^2} \times \overline{\mathbf{R}^2}) \oplus [(C_0(\overline{\mathbf{R}^2}) \oplus \mathbf{R}) \otimes \mathbf{R}] \oplus [\mathbf{R} \otimes (C_0(\overline{\mathbf{R}^2}) \oplus \mathbf{R})].$$

First, let $\mathcal{E}$ be the closed linear hull of

$$\mathcal{E}_0 = \{\rho_\varphi + \psi \mid \varphi \in C_0^2(\mathbf{R}^2),\ \psi \in \mathcal{X}\} \subset L^\infty(\mathbf{R}^2 \times \mathbf{R}^2),$$

where $C_0(\overline{\mathbf{R}^2} \times \overline{\mathbf{R}^2})$ is the set of continuous functions on $\overline{\mathbf{R}^2} \times \overline{\mathbf{R}^2}$ vanishing on $\overline{\mathbf{R}^2} \times \{\infty\} \bigcup \{\infty\} \times \overline{\mathbf{R}^2}$. Then, the mapping $s \in (-\infty, +\infty) \mapsto \langle \varphi, \hat{\zeta}(\cdot, s) \rangle \in \mathbf{R}$ is locally absolutely continuous for any $\varphi \in \mathcal{Y}$. There are, furthermore,

$$0 \le \mathcal{K} = \mathcal{K}(\cdot, \cdot, s) \in \mathcal{E}', \quad \|\mathcal{K}(\cdot, \cdot, s)\|_{\mathcal{E}'} \le \left(\sum_{i=1}^{2} \chi_i^{-1} \lambda_i \right)^2 \quad \text{a.e. } s$$

and

$$0 \le \zeta \in L^\infty_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2)), \quad \zeta(\mathbf{R}^2, s) \le \sum_{i=1}^{2} \lambda_i$$
$$0 \le \tilde{\zeta} \in L^\infty_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2)), \quad \tilde{\zeta}(\mathbf{R}^2, s) \le \sum_{i=1}^{2} \xi_i \lambda_i$$

satisfying

$$\begin{aligned}
&\mathcal{K}(\cdot,\cdot,s)|_X = \zeta(dy,s)\otimes\zeta(dy',s) \\
&\frac{d}{ds}\langle\varphi,\hat{\zeta}(dy,s)\rangle = \langle\Delta\varphi,\tilde{\zeta}(dy,s)\rangle + \langle\frac{1}{2}y\cdot\nabla\varphi,\zeta(dy,s)\rangle \\
&\quad +\frac{1}{2}\langle\rho^0_\varphi,\mathcal{K}(\cdot,\cdot,s)\rangle
\end{aligned} \tag{79}$$

for a.e. s, where

$$\rho^0_\varphi = \rho_\varphi(y,y') = \nabla\Gamma(y-y')\cdot(\nabla\varphi(y)-\nabla\varphi(y')).$$

Now, we take the scaling back transformation used in [33]:

$$\begin{aligned}
&\zeta_i(dy,s) = e^{-s}A_i(dy',s'), \quad i=1,2 \\
&s' = e^{-s/2}y,\ s' = -e^{-s/2}.
\end{aligned}$$

Then, we obtain a weak solution to

$$\partial_s\hat{A} = \Delta\tilde{A} - \nabla\cdot A\nabla\Gamma * A \quad \text{in } \mathbf{R}^2\times(-\infty,0),$$

where

$$A = \sum_{i=1}^2 A_i,\ \hat{A} = \sum_{i=1}^2 \chi_i^{-1}A_i,\ \tilde{A} = \sum_{i=1}^2 \xi_i A_i.$$

Then it holds that

$$A_i(\mathbf{R}^2,s') = \zeta_i(\mathbf{R}^2,s) = m_i^*(x_0).$$

Next, we take the translation limits as in [29]. Thus any $s_k\uparrow+\infty$ admits $\{s'_k\}\subset\{s_k\}$ such that

$$\begin{aligned}
&\hat{A}(dy,s-s'_k) \rightharpoonup \hat{B}(dy,s)\geq 0 && \text{in } C_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2)) \\
&A_i(dy,s-s'_k) \rightharpoonup B_i(dy,s)\geq 0 && \text{in } L^\infty_*(-\infty,+\infty;\mathcal{M}(\mathbf{R}^2)),\ i=1,2
\end{aligned}$$

with the limit measures $B_i(dy,s)$, $i=1,2$, and

$$\hat{B}(dy,s) = \sum_{i=1}^2 \chi_i^{-1}B_i(dy,s),$$

where $\mathcal{M}(\mathbf{R}^2) = [C_0(\overline{\mathbf{R}^2})\oplus\mathbf{R}]'$. Since the space $\mathcal{M}(\mathbf{R}^2)$ envelopes the total masses of $A_i(dy,s)$, $i=1,2$, it holds that

$$B_i(\mathbf{R}^2,s) = m_i^*(x_0), \quad i=1,2.$$

These measures form a weak solution to

$$\partial_s \hat{B} = \Delta \tilde{B} - \nabla \cdot B \nabla \Gamma * B \quad \text{in } \mathbf{R}^2 \times (-\infty, +\infty), \tag{80}$$

where $B = B_1 + B_2$ and $\tilde{B} = \xi_1 B_1 + \xi_2 B_2$. Then, we argue similarly to the case of a single component [19] (see also [32]), using a smooth function $c = c(s)$, $s \geq 0$, satisfying $0 \leq c'(s) \leq 1$, $-1 \leq c(s) \leq 0$, and

$$c(s) = \begin{cases} s - 1, & 0 \leq s \leq 1/4 \\ 0, & s \geq 4. \end{cases}$$

We have $C, \delta > 0$ such that if

$$\sigma_* \equiv \frac{1}{2\pi} \left(\sum_{i=1}^{2} m_i^*(x_0) \right)^2 - 4 \sum_{i=1}^{2} \xi_i m_i^*(x_0) > 0,$$

it holds that

$$\frac{d}{ds} \langle c(|y|^2) + 1, \hat{B}(dy, s) \rangle \leq C \langle c(|y|^2) + 1, \hat{B}(dy, s) \rangle - \delta \sigma_*$$

for a.e. s. Therefore, if

$$\langle c(|y|^2) + 1, \hat{B}(dy, 0) \rangle < \eta \equiv \frac{\delta \sigma_*}{C}$$

is the case, we obtain

$$\langle c(|y|^2) + 1, \hat{B}(dy, s) \rangle < 0, \qquad s \gg 1,$$

a contradiction. Hence it holds that

$$\langle c(|y|^2) + 1, \hat{B}(dy, 0) \rangle \geq \eta. \tag{81}$$

Equation (80), on the other hand, is invariant under the scaling transformation

$$B_i^\mu(y, s) = \mu^2 B_i(\mu y, \mu^2 s), \quad i = 1, 2,$$

where $\mu > 0$ is a constant. Since $C, \delta > 0$ are absolute constants and σ_* is invariant under this transformation, inequality (81) applied for $B_i^\mu(dy, s)$, $i = 1, 2$, takes the form

$$\langle c(|y|^2) + 1, \hat{B}^\mu(dy, 0) \rangle = \langle c(\mu^{-2}|y|^2) + 1, \hat{B}(dy, 0) \geq \eta, \tag{82}$$

where

$$\hat{B}^\mu(dy, s) = \sum_{i=1}^{2} \chi_i^{-1} B_i^\mu(dy, s).$$

However, since $0 \leq c(\mu^{-2}|y|^2) + 1 \leq 1$ and

$$\lim_{\mu \uparrow +\infty} c(\mu^{-2}|y|^2) + 1 = 0, \quad \forall y \in \mathbf{R}^2$$

we obtain $\eta \leq 0$ by the dominated convergence theorem, a contradiction. Hence it holds that $\sigma_* \leq 0$, that is, (29).

For the proof of (30), we use the total second moment of the scaled solution. First, similarly to the scalar case [25, 22] (see also [33]), Lemma 5 applied to $\varphi = |x - x_0|^2 \varphi_{x_0,R}$ implies the convergence and the uniform boundedness of the global second moment of $\hat{\zeta}(dy, s)$:

$$0 \leq \langle |y|^2, \hat{\zeta}(dy, s) \rangle = I(s) \leq C \tag{83}$$

with a constant $C > 0$. If $x_0 \in \partial\Omega$, we modify the above $\varphi = |x - x_0|^2 \varphi_{x_0,R}$, using a conformal mapping as in the proof of Theorem 10. Then, we obtain

$$\frac{dI}{ds} \geq \min_{i=1,2} \chi_i \cdot I - \sigma(x_0), \qquad \text{a.e. } s \in (-\infty, +\infty)$$

by (79), where

$$\sigma(x_0) \equiv \frac{4}{m_*(x_0)} \left(\sum_{i=1}^{2} m_i(x_0) \right)^2 - 4 \sum_{i=1}^{2} \xi_i m_i(x_0).$$

Then it follows that

$$I(s) \leq \max_{i=1,2} \chi_i^{-1} \cdot \sigma(x_0)$$

from (83). Then, since $I(s) \geq 0$, we obtain (30). ∎

Now we turn to the proof of Theorem 4. In fact, having proven Theorem 3, we obtain $\sigma(x_0) = 0$, and hence $I \equiv 0$ which implies

$$\hat{\zeta}(dy, s) = \hat{\zeta}(\mathbf{R}^2, s) \delta_0(dy), \quad \forall s \in \mathbf{R}.$$

Then it holds that supp $\zeta_i(dy, s) \subset \{0\}$, and hence

$$\zeta_i(dy, s) = m_i(x_0) \delta_0(dy), \quad \text{a.e. } s, \ i = 1, 2. \tag{84}$$

The convergence (74) is thus refined as

$$z_i(dy, s + s'_k) \rightharpoonup \zeta_i(dy, s) \quad \text{in } C_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2))$$

similarly to the prescaled case. Furthermore, the limit measures are prescribed as in (84), and hence we obtain (23) in $C_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2))$ as $s' \uparrow +\infty$. The latter part of Theorem 4, relation (24), is obvious from this convergence and (28). ∎

5 Proof of Theorems 5 and 6

Let $\Omega = B(0,1)$ and $u_i = u_i(|x|,t)$, $i = 1,2$, in (ISP). In this case, the blowup criterion (63) of Theorem 10 is reduced to a component-wise condition [9]. First, we confirm this property, using a method of symmetrization introduced by [24].

In fact, the Poisson equation (9) is reduced to

$$-rv_r(r,t) = \int_0^r s\left(u(s,t) - \frac{\lambda}{|\Omega|}\right) ds \geq \int_0^r s\left(u_1(s,t) - \frac{\lambda}{|\Omega|}\right) ds,$$

recalling $\lambda = \lambda_1 + \lambda_2$ and $\lambda_i = \|u_{i0}\|_1$, $i = 1,2$. Then, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{|x|<\ell} |x|^2 u_1(x,t)\ dx &= \int_{|x|<\ell} |x|^2 \nabla\cdot(d_1\nabla u_1 - \chi_1 u_1 \nabla v)\ dx \\ &= \int_{|x|=\ell} |x|^2 (d_1 u_{1r} - \chi_1 u_1 v_r)\ ds - \int_{|x|<\ell} 2x\cdot(d_1\nabla u_1 - \chi_1 u_1\nabla v)\ dx \\ &= \int_{|x|=\ell} \ell^2(d_1 u_{1r} - \chi_1 u_1 v_r) - 2d_1(x\cdot\nu)u_1\ ds + 4d_1\int_{|x|<\ell} u\ dx \\ &\quad +4\pi\chi_1\int_0^\ell r^2 u_1 v_r\ dr\end{aligned}$$

for $0 < \ell \leq 1$ and $r = |x|$ with

$$\begin{aligned}\int_{|x|=\ell}(d_1 u_{1r} - \chi_1 u_{1r} v_r)\ ds &= \int_{|x|<\ell} \nabla\cdot(d_1\nabla u_1 - \chi_1\nabla v)\ dx \\ &= \frac{d}{dt}\int_{|x|<\ell} u_1\ dx\end{aligned}$$

and

$$\begin{aligned}\int_0^\ell r^2 u_1 v_r\ dr &\leq -\int_0^\ell r u_1(r,t)dr\cdot\int_0^r s\left(u_1(s,t) - \frac{\lambda}{|\Omega|}\right)\ ds \\ &= -\frac{1}{2}\left\{\int_0^\ell r u_1(r,t)\ dr\right\}^2 + \frac{\lambda}{2|\Omega|}\int_0^\ell r^3 u_1(r,t)\ dr.\end{aligned}$$

Thus, it follows that

$$\begin{aligned}\frac{d}{dt}\int_{|x|<\ell}|x|^2 u_i(x,t)\ dx &\leq \ell^2\frac{d}{dt}\int_{|x|<\ell} u_i(x,t)\ dx + 4d_i\int_{|x|<\ell} u_i(x,t)\ dx \\ &\quad -\frac{\chi_i}{2\pi}\left\{\int_{|x|<\ell} u_i(x,t)\ dx\right\}^2 + \frac{\chi_i\lambda}{|\Omega|}\int_{|x|<\ell}|x|^2 u_i(x,t)\ dx \qquad (85)\end{aligned}$$

for $i = 1$.

This inequality is also valid to $i = 2$, and thus we obtain the following theorem. Here, differently from Theorem 10, the blowup criterion is taken only for one component.

Theorem 11 ([9]) *Given a radially symmetric solution, let* $\lambda_i = \|u_{i0}\|_1$ *and* $\xi_i = d_i/\chi_i$. *If* $\lambda_i > 8\pi\xi_i$ *and* $\||x|^2 u_{i0}\|_1 \ll 1$, *then* $u_i = u_i(|x|, t)$ *cannot exist global-in-time, where* $i = 1, 2$.

To prove Theorem 5, we derive

$$\left[\int_{|x|<\ell} |x|^2 u_i(x,t)\ dx\right]_{t=t_1}^{t=t_2} \leq \left[\ell^2 \int_{|x|<\ell} u_i(x,t)\ dx\right]_{t=t_1}^{t=t_2}$$
$$+\int_{t_1}^{t_2} \left[4d_i \int_{|x|<\ell} u_i(x,t)\ dx - \frac{\chi_i}{2\pi}(\int_{|x|<\ell} u_i(x,t)\ dx)^2\right.$$
$$\left.+\frac{\chi_i \lambda}{|\Omega|} \int_{|x|<\ell} |x|^2 u_i(x,t)\ dx\right] dt$$

from (85), where $0 \leq t_1 < t_2 < T$. Using the backward self-similar transformations (22) with $x_0 = 0$, this inequality means

$$\left[e^{-s} \int_{|y|<\ell e^{-s/2}} |y|^2 z_i(y,s)\ dy\right]_{s=s_1}^{s=s_2} \leq \left[\ell^2 \int_{|y|<\ell e^{-s/2}} z_i(y,s)\ dy\right]_{s=s_1}^{s=s_2}$$
$$+\int_{s_1}^{s_2} e^{-s} \left[4d_i \int_{|y|<\ell e^{-s/2}} z_i(y,s)\ dy - \frac{\chi_i}{2\pi}(\int_{|y|<\ell e^{-s/2}} z_i(y,s)\ dy)^2\right.$$
$$\left.+\frac{\chi_i \lambda}{|\Omega|} e^{-s} \int_{|y|<\ell e^{-s/2}} |y|^2 z_i(y,s)\ dy\right] ds$$

for $-\log T \leq s_1 < s_2 < +\infty$. Here, we put $\ell = be^{-s'}$, $s_1 = s'$, and $s_2 = s' + 1$, to make $s' \uparrow +\infty$. Then, convergence (23) implies

$$0 \leq 4d_i m_i - \frac{\chi_i}{2\pi} m_i^2, \quad i = 1, 2,$$

where $m_i = m_i(x_0)$, $x_0 = 0$. Since $\mathcal{S}$ is finite, it holds that $\mathcal{S} = \{0\}$, and hence we obtain (25). ∎

We may assume $i = 1$ to prove Theorem 6, that is, $\xi_1/\xi_2 > 2$ and $\|u_{10}\|_1 < 8\pi(\xi_1 - 2\xi_2)$. By $\xi_1/\xi_2 > 2$, as we have noticed, $(m_1, m_2) = (0, 8\pi\xi_2)$ is the only point on the curve

$$\left(\sum_{i=1}^{2} m_i\right)^2 = 8\pi \sum_{i=1}^{2} \xi_i m_i$$

satisfying

$$0 \leq m_i \leq 8\pi\xi_i,\ i = 1, 2, \quad m_1 < 8\pi(\xi_1 - 2\xi_2).$$

Using (38), on the other hand, we have

$$m_1 \leq \lambda_1 < 8\pi(\xi_1 - 2\xi_2),$$

and hence it holds that

$$(m_1, m_2) = (0, 8\pi\xi_2).$$

The proof of Theorem 6 is complete. ∎

A A total collapse mass estimate from below

Here we confirm that the localization of Theorem 9 concerning the existence of a global-in-time solution implies either (30) or (31). This argument is used for the single component case [32, 33, 29], which, however, does not provide a sharp inequality in the multi-component case. Inequality (30) is thus always selected. We note, however, that (31) is also true for radially symmetric solutions, which has not been known for the other case. If it is valid, then Theorem 6 holds even for non-radially symmetric solutions.

First observation is the following lemma.

Lemma 7 *Given $x_0 \in \mathcal{S}$, let $\varphi = \varphi_{x_0,R}$, $0 < R \ll 1$, and let $\tilde{u}_i = u_i\varphi$, $i = 1, 2$. Then, it holds that*

$$\limsup_{t \uparrow T} \mathcal{F}_{\xi_1,\xi_2}(\tilde{u}_1(\cdot,t), \tilde{u}_2(\cdot,t)) < +\infty. \tag{86}$$

Proof: From (39) it follows that

$$\begin{aligned}
&\int_\Omega u_{1t}(d_1 \log u_1 - \chi_1 v)\varphi \, dx \\
&= -\int_\Omega u_1 \nabla(d_1 \log u_1 - \chi_1 v) \cdot \nabla[(d_1 \log u_1 - \chi_1 v)\varphi] dx \\
&= -\int_\Omega u_1 |\nabla(d_1 \log u_1 - \chi_1 v)|^2 \varphi \\
&\quad - u_1(d_1 \log u_1 - \chi_1 v)\nabla(d_1 \log u_1 - \chi_1 v) \cdot \nabla\varphi \, dx \\
&\leq -\int_\Omega u_1(d_1 \log u_1 - \chi_1 v)\nabla(d_1 \log u_1 - \chi_1 v) \cdot \nabla\varphi \, dx.
\end{aligned} \tag{87}$$

We have $\mathcal{S} \cap B(x_0, 2R) = \{x_0\}$ for $0 < R \ll 1$, and then the right-hand side is estimated from above by a constant independent of t, using parabolic-elliptic regularity of (u_1, u_2, v).

Henceforth, C_i, $i = 1, 2, \cdots, 6$, denote positive constants independent of t and R. First, the left-hand side of (87) is equal to

$$\int_\Omega u_{1t}(d_1 \log u_1 - \chi_1 v)\varphi \ dx = \int_\Omega \tilde{u}_{1t}(d_1 \log \tilde{u}_1 - d_1 \log \varphi - \chi_1 v) \ dx$$
$$= \int_\Omega \tilde{u}_{1t}(d_1 \log \tilde{u}_1 - \chi_1 v) \ dx - \frac{d}{dt}\int_\Omega d_1 u_1(\varphi \log \varphi) \ dx,$$

which implies

$$\frac{d}{dt}\int_\Omega d_1 \tilde{u}_1(\log \tilde{u}_1 - 1) \ dx - \chi_1 \langle v, \tilde{u}_{1t}\rangle \leq C_1 + \frac{d}{dt}\int_\Omega d_1 u_1(\varphi \log \varphi) \ dx.$$

Similarly, it holds that

$$\frac{d}{dt}\int_\Omega d_2 \tilde{u}_2(\log \tilde{u}_2 - 1) \ dx - \chi_2 \langle v, \tilde{u}_{2t}\rangle \leq C_2 + \frac{d}{dt}\int_\Omega d_2 u_2(\varphi \log \varphi) \ dx$$

and hence

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^2 \xi_i \tilde{u}_i(\log \tilde{u}_i - 1) \ dx - \langle v, \tilde{u}_t\rangle \leq C_3 + \frac{d}{dt}\int_\Omega \left[\sum_{i=1}^2 \xi_i u_i\right] \varphi \log \varphi \ dx.$$

Here, the parabolic-elliptic regularity ensures

$$\langle v, \tilde{u}_t\rangle = \langle \varphi v, \varphi u_t\rangle + \langle (1-\varphi)v, \varphi u_t\rangle = \langle \varphi v, \varphi u_t\rangle + O(1)$$

and also

$$-\Delta(\varphi v) = \varphi u + h, \quad \left.\frac{\partial(\varphi v)}{\partial \nu}\right|_{\partial\Omega} = 0$$

for $h = h(x,t)$ satisfying

$$\|h\|_{L^\infty(\Omega\times(0,T))} + \|h_t\|_{L^\infty(\Omega\times(0,T))} \leq C_4.$$

Therefore, from the elliptic regularity it follows that

$$\varphi v = (-\Delta)^{-1}(\varphi u) + g$$

with $g = g(x,t)$ such that

$$\|g\|_{L^\infty(\Omega\times(0,T))} + \|g_t\|_{L^\infty(\Omega\times(0,T))} \leq C_5. \tag{88}$$

Hence we obtain

$$\begin{aligned}\langle \tilde{u}_t, v\rangle &= \langle \tilde{u}_t, \varphi v\rangle + O(1) = \langle \tilde{u}_t, (-\Delta)^{-1}\tilde{u} + g(\cdot,t)\rangle + O(1) \\ &= \frac{1}{2}\frac{d}{dt}\langle \tilde{u}, (-\Delta)^{-1}\tilde{u}\rangle + \frac{d}{dt}\langle \tilde{u}, g\rangle - \langle \tilde{u}, g_t\rangle + O(1) \\ &= \frac{1}{2}\frac{d}{dt}\langle \tilde{u}, (-\Delta)^{-1}\tilde{u}\rangle + \frac{d}{dt}\langle \tilde{u}, g\rangle + O(1)\end{aligned}$$

by (38) and (88). We thus end up with

$$\frac{d}{dt}\mathcal{F}_{\xi_1,\xi_2}(\tilde{u}_1(\cdot,t), \tilde{u}_2(\cdot,t)) \le C_6 + \frac{d}{dt}\int_\Omega \left[\sum_{i=1}^2 \xi_i u_i(\cdot,t)\right]\varphi\log\varphi + ug\varphi\ dx,$$

and then (86) follows from (38) and (88). ∎

Theorem 12 *We have either (30) or (31) for any $x_0 \in \mathcal{S}$.*

Proof: If these inequalities are not fulfilled, there is $\sigma_i > 1$, $i = 1, 2$, such that

$$\left(\sum_{i=1}^2 m_i\right)^2 \le m_*(x_0)\sum_{i=1}^2 \xi_i m_i$$

for any $0 < m_i < \sigma_i m_i(x_0)$, $i = 1, 2$, where $\xi_i = d_i/\chi_i$. Then (18) guarantees the existence of $\theta_i > 1$, $i = 1, 2$, such that $\|\theta_i\tilde{u}_i(\cdot,t)\|_1 < \sigma_i m_i(x_0)$ for $0 < T - t \ll 1$ and $0 < R \ll 1$.

If $x_0 \in \partial\Omega$, we have $m_*(x_0) = 4\pi$. Then Lemma 3 is applicable. We have that

$$\liminf_{t\uparrow T}\mathcal{F}_{\xi_1,\xi_2}(\theta_1\tilde{u}_1(\cdot,t), \theta_2\tilde{u}_2(\cdot,t)) > -\infty. \tag{89}$$

Combining (89) with (86), we obtain

$$\limsup_{t\uparrow T}\sum_{i=1}^2\int_\Omega \tilde{u}_i(\log\tilde{u}_i - 1)\ dx < +\infty$$

by $\theta_i > 1$, $i = 1, 2$. This inequality implies (66), and hence $x_0 \notin \mathcal{S}$, a contradiction.

In the case of $x_0 \in \Omega$, we use Lemma 8 below in stead of Lemma 3. Actually this lemma is a direct consequence of equality (55) with (54) and inequality (43) valid to (λ_1, λ_2) in the parameter region (42). Then, either (30) or (31) is obtained with $m_*(x_0) = 8\pi$. ∎

Lemma 8 *If λ_i, $i = 1, 2$, are in the parameter region (42), then it holds that*

$$\inf\{\mathcal{F}_{\xi_1,\xi_2}(u_1, u_2) \mid u_i \ge 0,\ \|u_i\|_1 = \lambda_i,\ \textit{supp}\ u_i \subset \Omega,\ i = 1, 2\} > -\infty.$$

B Criteria for mass separation and simultaneous blowup

First, mass separation is the case of radially symmetric solution if and only if the aggregation to the component forming a collapse is slower than that of the other. It is proven by the same identity used for the proof of Theorem 5.

Theorem 13 *Let Ω be a disc with center at the origin, $u_i = u_i(|x|, t)$, $i = 1, 2$, and $T < +\infty$. Then, there is $k \in \{1, 2\}$ such that $m_i(0) = 0$ for $i \neq k$ (and hence $m_k(0) = 8\pi\xi_k$ by (19)) if and only if*

$$\lim_{t \uparrow T} \iint_{|x|<|x'|<b(T-t)^{1/2}} u_k(|x|, t) u_i(|x'|, t) dx dx' = 0 \tag{90}$$

for any $b > 0$.

Proof: We have only to use a refined form of (85),

$$\begin{aligned} \frac{d}{dt} \int_{|x|<\ell} |x|^2 u_i(x,t)\ dx &= \ell^2 \frac{d}{dt} \int_{|x|<\ell} u_i(x,t)\ dx + 4d_i \int_{|x|<\ell} u_i(x,t)\ dx \\ & -\frac{\chi_i}{2\pi} \left\{ \int_{|x|<\ell} u_i(x,t)\ dx \right\}^2 - 4\pi\xi \int_0^\ell r u_i(r,t) dr \cdot \int_0^r s u_j(s,t) ds \\ & + \frac{\chi_i \lambda}{|\Omega|} \int_{|x|<\ell} |x|^2 u_i(x,t)\ dx \end{aligned}$$

and apply the same argument to the proof of Theorem 11. ∎

The final theorem shows that the simultaneous blowup holds even for non-radially symmetric solutions if one can prescribe the rate of convergence in (18).

To state the result, let

$$\begin{aligned} \mu_j(dx, T) &= \sum_{x_0 \in \mathcal{S}} m_j(x_0) \delta_{x_0}(dx) + f_j(x) dx, \quad j = 1, 2 \\ \mu(dx, T) &= \sum_{i=1}^{2} \mu_i(dx, T). \end{aligned}$$

It is proven by the total mass quantization, (19), combined with a linear analysis to each component $u_i(x,t)$, $i = 1, 2$.

Theorem 14 *If (25) and*

$$\liminf_{r \downarrow 0} \inf_{B(x_0, r)} f_i > 0 \tag{91}$$

in (18), and if

$$\|u(\cdot,t)dx - \mu(dx,T)\|_{\mathcal{M}(\overline{\Omega})} = o((T-t)^{\beta}) \tag{92}$$

as $t \uparrow T$ with $\beta > 1$, it holds that

$$\limsup_{t\uparrow T} \|u_i(\cdot,t)\|_{L^\infty(\Omega\cap B(x_0,R))} = +\infty \tag{93}$$

where $x_0 \in \mathcal{S}$, $R > 0$, and $i = 1, 2$.

Proof: The L^1-estimate to the Poisson part (9) implies

$$\limsup_{t\uparrow T} \|v(\cdot,t)\|_{W^{1,q}(\Omega)} < +\infty, \quad 1 \le q < 2 \tag{94}$$

by $\|u(\cdot,t)\|_1 = \lambda$ (see [3]). We recall that this elliptic estimate is derived from the duality argument, using the boundedness of $(-\Delta)^{-1} : W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)' \to L^\infty(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$, that is,

$$\|(-\Delta)^{-1} f\|_\infty \le C_{q'} \|f\|_{W^{-1,q'}},$$

which is derived from Stampacchia's truncation method. Since we use the triple $W_0^{1,q}(\Omega) \hookrightarrow L^2(\Omega) \cong L^2(\Omega)' \hookrightarrow W^{-1,q'}(\Omega)$, the space $W_0^{1,q}(\Omega)$ is dense in $W^{-1,q'}(\Omega)$. We use the Sobolev imbedding $W_0^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ and the L^p-elliptic regularity with $p > 2$, which implies the boundedness of $(-\Delta)^{-1} : L^p(\Omega) \to C(\overline{\Omega})$. Since $L^p(\Omega)$ is dense in $W^{-1,q'}(\Omega)$ by the above reason, we obtain $(-\Delta)^{-1} : W^{-1,q'} \to C(\overline{\Omega})$, which implies the boundedness of $(-\Delta)^{-1} : W_0^{1,q}(\Omega) \to \mathcal{M}(\overline{\Omega})$, $1 < q < 2$. Hence (92) implies

$$\|v(\cdot,t) - v_T\|_{W^{1,q}(\Omega)} = o((T-t)^{\beta}) \tag{95}$$

for any $1 \le q < 2$, where

$$v_T(x) = \sum_{x_0\in\mathcal{S}} m(x_0) G(x,x_0) + \int_\Omega G(x,x') f(x') dx'$$

with $m(x_0) = \sum_{i=1}^2 m_i(x_0)$ and $f(x) = \sum_{i=1}^2 f_i(x)$.

Given $\varphi = \varphi_{x_0,R}$ with $x_0 \in \mathcal{S}$, $0 < R \ll 1$, it holds that

$$\nabla v(\cdot,t)\varphi = \nabla \left[\gamma \log \frac{1}{|x-x_0|} + g\right] \varphi + o((T-t)^{\beta})$$

in $L^q(\Omega)$, $1 < q < 2$, as $t \uparrow T$, where $g = g(x)$ is a smooth function independent of $0 < R \ll 1$ and

$$\gamma = \begin{cases} m(x_0)/2\pi, & x_0 \in \Omega \\ m(x_0)/\pi, & x_0 \in \partial\Omega. \end{cases}$$

We assume the existence of $R_0 > 0$ satisfying

$$\limsup_{t\uparrow T} \|u_1(\cdot,t)\|_{L^\infty(\Omega\cap B(x_0,R_0))} < +\infty. \tag{96}$$

Here we use

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |x-x_0|^2\varphi\cdot u_1(x,t)dx \\
&= \int_\Omega [d_1\Delta(|x-x_0|^2\varphi) + \chi_1\nabla v\cdot\nabla(|x-x_0|^2\varphi)]u_1(x,t)\ dx \\
&= \int_\Omega [(4d_1 + 2\chi_1(x-x_0)\cdot\nabla v)\varphi + 4d_1(x-x_0)\cdot\nabla\varphi \\
&\quad + d_1|x-x_0|^2\Delta\varphi + \chi_1|x-x_0|^2\nabla v\cdot\nabla\varphi]u_1(x,t)\ dx.
\end{aligned}$$

First, we have $\|\nabla\varphi\|_\infty = O(R^{-1})$, and therefore,

$$\sup_{t\in[0,T)}\int_\Omega |x-x_0|^2[\nabla v\cdot\nabla\varphi]u_1(x,t)\ dx = O(R^3)$$

as $R\downarrow 0$. Next, we obtain

$$\sup_{t\in[0,T)}\int_\Omega [4(x-x_0)\cdot\nabla\varphi + |x-x_0|^2\Delta\varphi]u_1(x,t)\ dx = o(R^2)$$

by (62). It also holds that

$$\begin{aligned}
&\|[4d_1 + 2\chi_1(x-x_0)\cdot\nabla v]\varphi \\
&\quad -(4d_1 - 2\chi_1\gamma)\varphi + [2(x-x_0)\cdot\nabla g]\varphi\|_q \le \tilde{c}(t)
\end{aligned}$$

for $1<q<2$ with $\tilde{c}(t) = o((T-t)^\beta)$.

Here, we have

$$\int_\Omega |(x-x_0)\cdot\nabla g|\varphi\ dx = O(R^3),$$

while (19) with $(m_1(x_0), m_2(x_0)) \neq (0,0)$ implies

$$m(x_0) \equiv \sum_{i=1}^2 m_i(x_0) \ge \min\{\xi_1 m_*(x_0), \xi_2 m_*(x_0)\}$$

by (25) and in particular, $\gamma > 2\xi_1$. From the assumption (96), therefore, we have $0 < A(R) = o(R^2)$, $\delta_1 > 0$, and $0 < c(t) = o((T-t)^\beta)$ such that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |x-x_0|^2\varphi_{x_0,R}\cdot u_1(x,t)dx \\
&\le -\delta_1\int_\Omega \varphi\cdot u_1(x,t)dx + A(R) + c(t)R^{2/q'} \\
&\le -\delta_1\int_\Omega \varphi\cdot f_1(x)dx + o((T-t)^\beta) + A(R) + c(t)R^{2/q'}
\end{aligned}$$

by (92) with $j = 1$.

From (91) with $i = 1$, we have $\delta_2 > 0$ such that

$$\int_\Omega \varphi \cdot f_1(x)dx \geq \delta_2 R^2$$

for $0 < R \ll 1$. We thus end up with

$$\frac{d}{dt}\int_\Omega |x - x_0|^2 \varphi_{x_0,R} \cdot u_1(x,t)dx \leq -\delta R^2 + o((T-t)^\beta) + c(t)R^{2/q'},$$

where $\delta > 0$ is a constant. So we obtain

$$\int_\Omega |x - x_0|^2 \varphi \cdot f_1(x)dx \leq -\delta(T-t)R^2 + C_9 R^4 + o((T-t)^{1+\beta}) + R^{2/q'}\int_t^T c(t')dt'. \quad (97)$$

We set $R = b(T-t)^{1/2}$ with $0 < b < C_9/\delta$. Then

$$R^{2/q'}\int_t^T c(t')dt' = o((T-t)^{1/q'+1+\beta}) = o((T-t)^2),$$

holds for $0 < q' - 2 \ll 1$ by $\beta > 0$. Since $\beta > 1$, however, the right-hand side of (97) is negative for $0 < T - t \ll 1$, which is impossible. Hence (91) with $j = 1$ implies

$$\limsup_{t \uparrow T} \|u_1(\cdot,t)\|_{L^\infty(\Omega \cap B(x_0,R))} = +\infty.$$

The proof for the other case, $j = 2$, is similar. ∎

ACKNOWLEDGEMENTS

This research is sponsored by JST project, CREST (Alliance for Breakthrough Between Mathematics and Sciences).

REFERENCES

[1] P. Biler, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl., **8** (1998), 715–743.

[2] P. Biler, W. Hebisch, and T. Nadzieja, *The Debye system: existence and large time behavior of solutions*, Nonlinear Analysis, **23** (1994), 1189–1209.

[3] H. Brezis and W. Strauss, *Semi-linear second-order elliptic equations in* L^1, J. Math. Soc. Japan, **25** (1973), 565–590.

[4] P.-H. Chavanis, *Two-dimensional Brownian vortices*, Physica A, **387** (2008), 6917–6942.

[5] P.-H. Chavanis, P. Laurençot, and M. Lemou, *Chapman-Enskog derivation of the generalized Smoluchowski equation*, Physica A, **341** (2004), 145–164.

[6] S. Childress and J.K. Percus, *Nonlinear aspects of chemotaxis*, Math. Biosci., **56** (1981), 217–237.

[7] C. Conca, E.E. Espejo, and K. Vilches, *Remarks on the blow-up and global existence for a two-species chemotactic Keller-Segel system in* $\mathbf{R}^2$, to appear in; Euro. J. Appl. Math..

[8] C. Conca, E.E. Espejo, and K. Vilches, *Sharp condition for blow-up and global existence in a two species chemotactic Keller-Segel system in* $\mathbf{R}^2$, preprint.

[9] E.E. Espejo, A. Stevens, and J.J.L. Velázquez, *Simultaneous finite time blow-up in a two-species model for chemotaxis*, Analysis, **29** (2009), 317–338.

[10] E.E. Espejo, A. Stevens, and J.J.L. Velázquez, *A note on non-simultaneous blow-up for a drift-diffusion model*, Differential and Integral Equations, **23** (2010), 451–462.

[11] H. Gajewski and K. Zacharias, *Global behaviour of a reaction-diffusion system modelling chemotaxis*, Math. Nachr., **195** (1998), 77–114.

[12] M.A. Herrero and J.J.L. Velázquez, *Singularity patterns in a chemotaxis model*, Math. Ann., **306** (1996), 583–623.

[13] M.A. Herrero and J.J.L. Velázquez, *A blowup mechanism for a chemotaxis model*, Ann. Scoul. Norm. Sup. Pisa, **IV 24** (1997), 633–683.

[14] D. Horstmann, *Generalizing Keller-Segel: Lyapunov functionals, steady state analysis and blow-up results for multi-species chemotaxis in the presence of attraction and repulsion between competitive interacting species*, J. Nonlinear Science, **21** (2011), 231–270.

[15] D. Horstmann and M. Lucia, *Uniqueness and symmetry of equilibria in a chemotaxis model*, J. Reine Angew. Math., **654** (2011), 82–124.

[16] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc., **329** (1992), 819–824.

[17] J.A. Joyce and J.W. Pollard, *Microenvironmental regulation of metastasis*, Nature Reviews Cancer, **9** (2009), 239-252.

[18] E.F. Keller and L.A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol., **26** (1970), 399–415.

[19] M. Kurokiba and T. Ogawa, *Finite time blow-up of the solution for a nonlinear parabolic equation of drift-diffusion type*, Differential and Integral Equations, **16** (2003), 427–452.

[20] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl., **5** (1995), 581–601.

[21] T. Nagai, T. Senba, and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis*, Funkcial. Ekvac., **40** (1997), 411–433.

[22] Y. Naito and T. Suzuki, *Self-similarity in chemotaxis systems*, Colloquium Mathematics, **111** (2008), 11–34.

[23] V. Nanjundiah, *Chemotaxis, signal relaying, and aggregation morphology*, J. Theor. Biol., **42** (1973), 63–105.

[24] H. Ohtsuka, T. Senba, and T. Suzuki, *Blowup in infinite time in the simplified system of chemotaxis*, Adv. Math. Sci. Appl., **17** (2007), 445–472.

[25] T. Senba, *Type II blowup of solutions to a simplified Keller-Segel system in two dimensions*, Nonlinear Analysis, **66** (2007), 1817–1839.

[26] T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of mathematical biology*, Adv. Differential Equations, **6** (2001), 21–50.

[27] T. Senba and T. Suzuki, *Parabolic system of chemotaxis; blowup in a finite and in the infinite time*, Meth. Appl. Anal., **8** (2001), 349–368.

[28] T. Senba and T. Suzuki, *Blowup behavior of solutions to re-scaled Jäger-Luckhaus system*, Adv. Differential Equations, **8** (2003), 787–820.

[29] T. Senba and T. Suzuki, "Applied Analysis, Mathematical Methods in Natural Science," second edition, Imperial College Press, London, 2011.

[30] I. Shafrir and G. Wolansky, *Moser-Trudinger and logarithmic HLS inequalities for systems*, J. Euro. Math. Soc., 7 (2005), 413–448.

[31] J. Sopik, C. Sire, and P.-H. Chavanis, *Self-gravitating Brownian systems and bacterial populations with two or more types particles*, Phys. Rev. E, **72** (2005), 026105.

[32] T. Suzuki, "Free Energy and Self-Interacting Particles," Birkhäuser, Boston, 2005.

[33] T. Suzuki, "Mean Field Theories and Dual Variation," Atlantis Press, Paris-Amsterdam, 2008.

[34] C. Weijer, *Dictyostelium morphogenesis*, Curr. Opin. Genet. Dev., **14** (2004), 392–398.

[35] G. Wolansky, *Multi-components chemotactic system in the absence of conflicts*, Euro. J. Appl. Math., **13** (2002), 641–661.

[36] H. Yamaguchi, F. Pixley, and J. Condeelis, *Invadopodia and podosomes in tumor invasion*, Euro. J. Cell Biology, **85** (2006), 213–218.