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by

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Abstract. We investigate the possibility and conditions to factorize the entanglement evolution of a multiqubit system passing through multi-sided noisy channels. By means of a lower bound of concurrence (LBC) as entanglement measure, we derive an explicit formula of LBC evolution of the N -qubit generalized Greenberger-Horne-Zeilinger (GGHZ) state under some typical noisy channels, based on which two kinds of factorizing conditions for the LBC evolution are presented. In this case, the time-dependent LBC can be determined by a product of initial LBC of the system and the LBC evolution of a maximally entangled GHZ state under the same multi-sided noisy channels. We analyze the realistic situations where these two kinds of factorizing conditions can be satisfied. In addition, we also discuss the dependence of entanglement robustness on the number of the qubits and that of the noisy channels.

1. Introduction

Quantum entanglement, as a type of distributed or nonlocal coherence among several quantum subsystems, have been more and more recognized as an indispensable resource in realizing many intriguing quantum information processing and quantum computation [1]. Realistically, however, the unavoidable coupling of the entangled system to environment will lead to a destruction of the necessary entanglement. Therefore, a deeper understanding of entanglement dynamics is of great importance not only in the foundation of quantum mechanics [2] but also in the rapidly developing quantum technologies [1].

The usual way in studying entanglement evolution is first to deduce the evolved state of the system and then to calculate its entanglement [3, 4, 5, 6, 7, 8, 9, 10]. However, as far as multipartite systems (or higher dimensional systems) be concerned, it is generally very hard to solve the state equation and therefore the evolution of entanglement can be determined only in very special cases. Different from the aforementioned state-evolution technique, it was found [11] for any two-qubit system with only one qubit being subjected to a noisy environment, i.e., under one-sided noisy channel, the evolution of system's entanglement in terms of concurrence [12] can be completely determined by the product of the system's initial concurrence and the concurrence evolution of a maximally entangled state under also the one-sided channel. Through this channel-dependent technique one can characterize the entanglement dynamics under unknown channels by probing the entanglement evolution of a maximally entangled state alone without exploring the concrete action of the channel on all initial states [11]. The entanglement factorization law [11] has been experimentally verified in an amplitude decay channel [13], and in the combined channel of phase damping and amplitude decay [14]. The factorization law has been generalized to finite-dimensional bipartite system for both initially pure state [15] and mixed state [16, 17], and to determine the evolution of the G concurrence [18] of two qudits [19]. In Ref. [20], the author studied the multipartite system with only one subsystem undergoing an arbitrary physical process and found that the entanglement evolution of the multipartite system can be determined uniquely by a single function of the quantum channel alone, irrespective to the number of qudits in the system. Therefore, in this case, the evolution of multipartite entanglement is in exactly the same way as that of bipartite (two qudits) entanglement [20].

In many practical situations, each particle of an entangled system is coupled with a local noisy environment, therefore the factorization law of entanglement evolution under one-sided noisy channel should be generalized to the case of multi-sided channels. It would be of great interest if the entanglement evolution under multi-sided noisy channels could be factorized as that in the one-sided channel. Unfortunately, however, even for the simplest case of two-qubit entanglement under two-sided noisy channels, the factorization law [11] does not hold any more except for some special forms of two-qubit state under special environments [21, 22]. It was shown that [21] for the

two-sided amplitude damping channels, only for the NOE (the number of excitations in the system is not more than one) state the factorization law is valid. For the two-sided pure dephasing channel case, the factorization law is valid not only for the NOE state but also for all the three basis states [22]. Therefore, a general factorization law of entanglement evolution does not hold for multi-sided channels. However, it is still useful to explore the factorization law for special forms of entangled states, especially the multipartite entanglement, under special noisy channels. This study may shed some light on the understanding of entanglement dynamics of multipartite system. In this work, focusing on the N -qubit generalized Greenberger-Horne-Zeilinger (GGHZ) state [23], we shall investigate the possibility and conditions to factorize the entanglement evolution of multiqubit system under multi-sided noisy channels.

In studying the dynamics of multipartite entanglement, one of the biggest obstacles is the lack of a computable entanglement measure. Most studies [24, 25, 26, 27, 28] on the entanglement dynamics of multiqubit system are based on the strategy of bipartition and the measure of bipartite entanglement. Another strategy one usually adopts is to generalize the bipartite entanglement measure directly to the multipartite case [29, 30]. However, the calculation for the mixed state of a multipartite system requires an optimization process, which is very difficult to solve exactly and can only be determined purely algebraically in the regime where the mixing is moderate [29]. To be able to assess the global entanglement of a multiqubit system, we adopt the concept of lower bound of concurrence (LBC)[31, 32]. Based on the derived formula of LBC evolution of a N -qubit system under some typical noisy channels, we find two kinds of sufficient conditions under which the factorization law holds, namely, the time-dependent LBC can be determined by the product of initial LBC of the system and the LBC evolution of a maximally entangled N -qubit state under the same multi-sided noisy channels. Then we consider the realistic situations under which the two kinds of factorization conditions can be satisfied. The dependence of entanglement robustness on the number of qubits and that of the noisy channels is also discussed.

2. LBC evolution and the conditions for its factorization

2.1. Time-dependent LBC of a N -qubit system under multi-sided noisy channels

In this work, we adopt the LBC proposed in [32] to quantify the global entanglement of a multiqubit system, defined for an arbitrary N -qubit mixed state ρ as

$$\underline{C}_N(\rho) = \sqrt{\frac{1}{2^{N-1} - 1} \sum_{\mathcal{P}} \sum_{l_1=1}^{k_1} \sum_{l_2=1}^{k_2} (\mathcal{C}_{l_1 l_2}^{\mathcal{P}[k|N-k]}(\rho))^2}, \quad (1)$$

with

$$\mathcal{C}_{l_1 l_2}^{\mathcal{P}[k|N-k]}(\rho) = \max\{0, \sqrt{\lambda_{l_1 l_2, 1}^{\mathcal{P}[k|N-k]}} - \sum_{i>1} \sqrt{\lambda_{l_1 l_2, i}^{\mathcal{P}[k|N-k]}}\}. \quad (2)$$

In Eqs. (1) and (2), $k|N - k$ stands for the bipartitions with k qubits in one block and $N - k$ ones in another. In view of the distinctions of the N qubits, we use

$\mathcal{P}[k|N-k]$ to specify a concrete combination of k and $N-k$ qubits in constituting the bipartitions $k|N-k$. Thus, $\sum_{\mathcal{P}}$ stands for the summation over all possible concrete bipartitions of $\mathcal{P}[k|N-k]$. In Eq.(2), $\lambda_{l_1 l_2, i}^{\mathcal{P}[k|N-k]}$ are the eigenvalues, in decreasing order, of the non-Hermitian matrix $\rho(L_{l_1}^k \otimes L_{l_2}^{N-k})\rho^*(L_{l_1}^k \otimes L_{l_2}^{N-k})$ with $\{L_{l_1}^k; l_1 = 1, 2 \dots k_1\}$ and $\{L_{l_2}^{N-k}; l_2 = 1, 2 \dots k_2\}$ the generators of group $\text{SO}(2^k)$ and $\text{SO}(2^{N-k})$ acting on the k and $N-k$ qubits of a concrete bipartition $\mathcal{P}[k|N-k]$. Surely, $\underline{\mathcal{C}}_N(\rho) > 0$ signifies that ρ is entangled and a separable state ρ always has $\underline{\mathcal{C}}_N(\rho) = 0$. Yet, $\underline{\mathcal{C}}_N(\rho) = 0$ does not necessarily imply separability of ρ . The LBC of three-qubit X states was analyzed in Ref. [33] to demonstrate when it goes to zero.

It is known the forms of multiqubit entanglement are diverse. In this work, the consideration is restricted to the GHZ state of N qubits in the form

$$|\Psi_0\rangle_{1,2\dots N} = \alpha |i_1, i_2, \dots, i_N\rangle_{1,2\dots N} + \beta |\bar{i}_1, \bar{i}_2, \dots, \bar{i}_N\rangle_{1,2\dots N}, \quad (3)$$

where $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$, $i_1, i_2 \dots i_N \in \{0, 1\}$ and $\bar{i}_1 = 1 \oplus i_1, \bar{i}_2 = 1 \oplus i_2 \dots \bar{i}_N = 1 \oplus i_N$ with \oplus being an addition mod 2. Consider the N qubits are locally subjected to $M \leq N$ independent noisy channels without mutual interactions between the qubits. The dynamics of the j -th qubit is governed by a master equation that gives rise to a completely positive trace-preserving map (or channel) \mathcal{E}_j describing the evolution as $\rho_j(t) = \mathcal{E}_j \rho_j(0)$, where $\rho_j(0)$ and $\rho_j(t)$ are, respectively, the initial and evolved states of the j -th qubit. Under the actions of M independent noisy channels, the density matrix of the initial state (3) of the N -qubit system

$$\begin{aligned} \rho_N(0) &\equiv |\Psi_0\rangle_{1,2\dots N} \langle\Psi_0| \\ &= |\alpha|^2 |i_1, i_2, \dots, i_N\rangle_{1,2\dots N} \langle i_1, i_2, \dots, i_N| + |\beta|^2 |\bar{i}_1, \bar{i}_2, \dots, \bar{i}_N\rangle_{1,2\dots N} \langle \bar{i}_1, \bar{i}_2, \dots, \bar{i}_N| \\ &\quad + \alpha\beta^* |i_1, i_2, \dots, i_N\rangle_{1,2\dots N} \langle \bar{i}_1, \bar{i}_2, \dots, \bar{i}_N| + \alpha^*\beta |\bar{i}_1, \bar{i}_2, \dots, \bar{i}_N\rangle_{1,2\dots N} \langle i_1, i_2, \dots, i_N| \end{aligned} \quad (4)$$

will evolve into a mixed state $\rho_N(t)$ given simply by the composition of the M individual maps:

$$\rho_N(t) = \mathcal{E}_1 \mathcal{E}_2 \dots \mathcal{E}_M \rho_N(0). \quad (5)$$

In this work, we shall consider several paradigmatic types of noisy channels, such as the amplitude-damping (AD), depolarization (D) and phase-damping (PD) (dephasing) channels. Under the action of any one of these channels, it never appears new off-diagonal terms in $\rho_N(t)$ but multiply the initial off-diagonal element of $\rho_N(0)$ (4) by a time-dependent factor, whereas the diagonal elements of $\rho_N(0)$, may give rise to new diagonal terms [27]. For convenience, in the representation spanned by the N -qubit product states $\{|00\dots 00\rangle, |00\dots 01\rangle, \dots, |11\dots 11\rangle\}$, we label the 2^N diagonal elements of $\rho_N(t)$ as $a_1(t), \dots, a_m(t), \dots, a_{2^{N-1}}(t), b_{2^{N-1}}(t), \dots, b_m(t), \dots, b_1(t)$ (from the top left one to the lower right one) and the two off-diagonal elements as $d_m(t)$ and $d_m^*(t)$. Here, we specify the elements $a_m(t)$ and $b_m(t)$ ($d_m(t)$ and $d_m^*(t)$) as the time-evolution of the initial nonzero diagonal (off-diagonal) elements of $\rho_N(0)$. The other ones $a_{m'}(t)$ and $b_{m'}(t)$ (with $m' \neq m \in \{1, 2, \dots, 2^{N-1}\}$) are the new diagonal elements derived in the time evolution. Also, note that the two diagonal terms of $\rho_N(t)$ corresponding to the

elements $a_i(t)$ and $b_i(t)$ ($i = 1, 2, \dots, 2^{N-1}$) have completely opposite marginals for all the N individual qubits. As an example, suppose the initial density operator of a three-qubit system is $\rho_3(0) = |\alpha|^2|000\rangle\langle 000| + |\beta|^2|111\rangle\langle 111| + \alpha\beta^*|000\rangle\langle 111| + \alpha^*\beta|111\rangle\langle 000|$, the evolved density operator of which is $\rho_3(t) = a_1(t)|000\rangle\langle 000| + a_2(t)|001\rangle\langle 001| + a_3(t)|010\rangle\langle 010| + a_4(t)|011\rangle\langle 011| + b_4(t)|100\rangle\langle 100| + b_3(t)|101\rangle\langle 101| + b_2(t)|110\rangle\langle 110| + b_1(t)|111\rangle\langle 111| + d_1(t)|000\rangle\langle 111| + d_1^*(t)|111\rangle\langle 000|$. In this case, the time-evolution of the initial nonzero diagonal (off-diagonal) elements are obviously $a_1(t)$ and $b_1(t)$ ($d_1(t)$ and $d_1^*(t)$). The other diagonal elements $a_{m'}(t)$ and $b_{m'}(t)$ with $m' = 2, 3, 4$ are the new diagonal elements derived in the time evolution. The two diagonal terms, such as $|001\rangle\langle 001|$ and $|110\rangle\langle 110|$ with the coefficients $a_2(t)$ and $b_2(t)$ (the same for the other three pairs of diagonal terms with the coefficients $a_i(t)$ and $b_i(t)$, $i = 1, 3, 4$), involve completely opposite reduced states for all the three individual qubits.

By virtue of Eq. (2), we obtain the concurrence of a concrete bipartition $\mathcal{P}[k|N-k]$ of the evolved state $\rho_N(t)$ as

$$\mathcal{C}^{\mathcal{P}[k|N-k]}(\rho_N(t)) = 2 \max\{0, |d_m(t)| - \sqrt{a_{m'}(t)b_{m'}(t)}\}, \quad (6)$$

where $d_m(t)$ is the time-evolution of the initial off-diagonal element of the GHZ state (4), while $a_{m'}(t)$ and $b_{m'}(t)$ (with $m' \neq m$) are the diagonal elements derived in the evolution. The value of $m' \in \{1, 2, \dots, 2^{N-1}\}$ is determined by the definite bipartition $\mathcal{P}[k|N-k]$. By virtue of Eqs. (1) and (6), the LBC of the N -qubit system can be expressed as

$$\underline{\mathcal{C}}_N(\rho_N(t)) = \sqrt{\frac{1}{2^{N-1} - 1} \sum_{\mathcal{P}} [\mathcal{C}^{\mathcal{P}[k|N-k]}(\rho_N(t))]^2}. \quad (7)$$

So far, we have derived a general formula (7) of the time-dependent LBC of a N -qubit system under multi-sided noisy channels. It should be pointed out that we have not solved the state evolution equation of the qubits' system in the sense that we do not know the explicit forms of the matrix elements of the evolved density operator $\rho_N(t)$. Actually, we are aware of the structure of the evolved density operator $\rho_N(t)$ since we are restricted to the GHZ initial state of the N qubits and to the special noisy channels, which ensures the derivation of the formula (7). Inspections of the formula (7) show that to make the factorization law hold for the LBC evolution of the N -qubit system under multi-sided noisy channels, the associated bipartite concurrence $\mathcal{C}^{\mathcal{P}[k|N-k]}(\rho_N(t))$ (6) should satisfy either one of the following two kinds of conditions.

2.2. The first kind of condition

An obvious situation under which the LBC (7) can be factorized is that the product of $a_{m'}(t)$ and $b_{m'}(t)$ in the formula (6) is equal to zero, i.e., $a_{m'}(t)b_{m'}(t) = 0 \forall m' \neq m \in \{1, 2, \dots, 2^{N-1}\}$. In this case, we have $\mathcal{C}^{\mathcal{P}[k|N-k]}(\rho_N(t)) = 2|d_m(t)|$ which is independent of the concrete bipartition $\mathcal{P}[k|N-k]$. Accordingly, the LBC (7) of the N -qubit system is reduced to a simple form as

$$\underline{\mathcal{C}}_N(\rho_N(t)) = 2|d_m(t)|, \quad (8)$$

in which $|d_m(t)|$ is obviously the time-evolution of the off-diagonal element $|d_m(0)| = |\alpha\beta|$ of the N -qubit system. That is, the LBC of the N -qubit system can be determined completely by the time-evolution of the off-diagonal element $|d_m(t)|$. As mentioned above, application of any one of these noisy channels to the N -qubit system will multiply the off-diagonal element of $\rho_N(0)$ (4) by a time-dependent factor, namely, $d_m(t) = d_m(0)\mathcal{D}(t)$. Therefore, the LBC (8) can be further reexpressed as

$$\underline{\mathcal{C}}_N(\rho_N(t)) = \underline{\mathcal{C}}_N(\rho_N(0))|\mathcal{D}(t)| \quad (9)$$

where $\underline{\mathcal{C}}_N(\rho_N(0)) = 2|d_m(0)| = 2|\alpha\beta|$ is the initial LBC of the GHZ state (3). By virtue of Eq. (9), for the initially maximal entangled state of the N -qubit system with $\underline{\mathcal{C}}_N(\rho_N(0)) = 1$, we have $\underline{\mathcal{C}}_N(\rho_N(t)) = |\mathcal{D}(t)|$, which implies $|\mathcal{D}(t)|$ is the LBC evolution of the maximally entangled state of the N -qubit system under the same multi-sided noisy channels. Therefore, the LBC evolution of the N -qubit system under multi-sided noisy channels can be determined by the product of the initial LBC and the LBC evolution of a maximally entangled state. Actually, as we shall show in next section, the factor $\mathcal{D}(t)$ can be determined by the parameters of the independent noisy channels that act on the qubits. Of course, the factorization expression Eq. (9) is conditioned on the condition $a_{m'}(t)b_{m'}(t) = 0$ in (6). In the next section, we shall show that this condition can be satisfied in AD and PD channels.

2.3. The second kind of condition

In some situations, the first kind of condition $a_{m'}(t)b_{m'}(t) = 0 \forall m' \neq m \in \{1, 2, \dots, 2^{N-1}\}$ may do not hold and the LBC evolution cannot be factorized as the expression in Eq. (9). In this case, if $\sqrt{a_{m'}(t)b_{m'}(t)}$ can be decomposed to $|d_m(0)||\mathcal{F}_{m'}(t)|$, the concurrence (6) of a definite bipartition $\mathcal{P}[k|N-k]$ will take the form

$$\mathcal{C}^{\mathcal{P}[k|N-k]}(\rho_N(t)) = 2|d_m(0)|\mathcal{Q}^{\mathcal{P}[k|N-k]}(t), \quad (10)$$

with $\mathcal{Q}^{\mathcal{P}[k|N-k]}(t) = \max\{0, |\mathcal{D}(t)| - |\mathcal{F}_{m'}(t)|\}$. By virtue of Eq. (7), LBC of the N -qubit system is thus reduced to

$$\underline{\mathcal{C}}_N(\rho_N(t)) = \underline{\mathcal{C}}_N(\rho_N(0))\sqrt{\frac{1}{2^{N-1}-1} \sum_{\mathcal{P}} (\mathcal{Q}^{\mathcal{P}[k|N-k]}(t))^2}, \quad (11)$$

with $\underline{\mathcal{C}}_N(\rho_N(0)) = 2|d_m(0)| = 2|\alpha\beta|$ being still the initial LBC of the GHZ state (3). From Eq. (11), for $\underline{\mathcal{C}}_N(\rho_N(0)) = 1$ corresponding to an initially maximal entangled state of the N -qubit system, we obtain its the LBC evolution as $\underline{\mathcal{C}}_N(\rho_N(t)) = \sqrt{\frac{1}{2^{N-1}-1} \sum_{\mathcal{P}} (\mathcal{Q}^{\mathcal{P}[k|N-k]}(t))^2}$. Therefore, the LBC evolution of an arbitrary N -qubit system under multi-sided noisy channels can still be determined by the product of the initial LBC and the LBC evolution of a maximally entangled state. In the next section, we shall show that the LBC evolution of the N -qubit system under D channels can satisfy the above condition and exhibit the factorized form in (11).

3. The factorization LBC evolution in various noisy channels

We have shown theoretically that under certain conditions the LBC evolution of a N -qubit system under multi-sided noisy channels may be factorized in such a way that the LBC is determined by the product of the initial LBC and the LBC evolution of a maximally entangled state under the same multi-sided channels. Now, an immediate question arises that in what realistic situations these conditions can be satisfied. In the following, we provide an answer to this question by analyzing the evolution of the N -qubit system in the AD, D and PD channels, respectively.

3.1. Amplitude-damping channel

At first, we consider the AD channel which correspond to the zero-temperature dissipative reservoir. The factorization law of two-qubit under one-sided noisy channel [11] has been verified in the AD channel [13, 14]. The action of an AD channel j for a qubit is described by a map $\mathcal{E}_j^{\text{AD}} : \rho(t) = \mathcal{E}_j^{\text{AD}}\rho(0)$ with $\rho(0)$ and $\rho(t)$ the initial and evolved density matrixes of the qubit. During the time evolution a qubit decays from its excited state $|1\rangle$ to ground state $|0\rangle$ by emitting an excitation, with a probability $p_j(t) = 1 - \exp(-\Gamma_j t)$, where Γ_j is the decay rate of the noisy channel. The action of $\mathcal{E}_j^{\text{AD}}$ on elements of the reduced density matrix of a qubit reads:

$$\mathcal{E}_j^{\text{AD}} : \begin{cases} |0\rangle\langle 0| \rightarrow |0\rangle\langle 0|, \\ |0\rangle\langle 1| \rightarrow \sqrt{1-p_j(t)}|0\rangle\langle 1|, \\ |1\rangle\langle 0| \rightarrow \sqrt{1-p_j(t)}|1\rangle\langle 0|, \\ |1\rangle\langle 1| \rightarrow p_j(t)|0\rangle\langle 0| + (1-p_j(t))|1\rangle\langle 1|. \end{cases} \quad (12)$$

Different forms of initial GHZ state (3), such as $|\psi_s\rangle = \alpha|000\rangle + \beta|111\rangle$ and $|\psi_a\rangle = \alpha|001\rangle + \beta|110\rangle$ for a three-qubit system, will exhibit different dynamical evolution in the AD channels [28]. Therefore, we classify the initial GHZ state of the N -qubit system into symmetrical and asymmetrical cases: in the symmetrical GHZ state (13) all the N qubits are equivalent with the same reduced states, while in the asymmetrical state (16) the N qubits can be divided to two asymmetrical blocks with opposite reduced states.

The symmetrical GHZ state can be expressed as

$$|\Psi_I\rangle_{1,2\dots N} = \alpha|0,0,\dots,0\rangle_{1,2\dots N} + \beta|1,1,\dots,1\rangle_{1,2\dots N} \quad (13)$$

Without loss of generality, suppose the qubits $1, 2, \dots, M$ ($M \leq N$) are locally coupled to M independent and different AD channels, respectively. Under the actions of the M AD channels the initial density matrix $\rho_N^I(0) = |\Psi_I\rangle_{1,2\dots N}\langle\Psi_I|$ will map onto $\rho_N^I(t)$ which can be obtained as

$$\begin{aligned} \rho_N^I(t) = & \prod_{j=1}^M [1-p_j(t)]^{\frac{1}{2}} (\alpha\beta^* |00\dots 0\rangle_{1,2\dots N} \langle 11\dots 1| + \alpha^*\beta |11\dots 1\rangle_{1,2\dots N} \langle 00\dots 0|) \\ & + |\beta|^2 \sum_{i_1\dots i_M=0}^1 q_1(t)\dots q_M(t) |i_1\dots i_M, 1\dots 1\rangle_{1\dots M, M+1\dots N} \langle i_1\dots i_M, 1\dots 1| \end{aligned}$$

$$+ |\alpha|^2 |0\dots 0\rangle_{1\dots N} \langle 0\dots 0|, \quad (14)$$

where the coefficients $q_j(t) \equiv p_j(t) \forall i_j = 0$ and $q_j(t) \equiv 1 - p_j(t) \forall i_j = 1$ for $j = 1, 2, \dots, M$. Recall that we specify $a_m(t)$ and $b_m(t)$ ($d_m(t)$ and $d_m^*(t)$) as the time evolution of the nonzero diagonal (off-diagonal) elements of initial density operator $\rho_N(0)$, while $a_{m'}(t)$ and $b_{m'}(t)$ (with $m' \neq m$) are the derived diagonal elements in the evolution. From (14) we can see that the initial off-diagonal element $|d_m(0)| = |\alpha\beta|$ has been evolved to $|d_m(t)| = |\alpha\beta| \prod_{j=1}^M [1 - p_j(t)]^{\frac{1}{2}}$, while the initial diagonal elements $a_m(0) = |\alpha|^2$ keep invariant and $b_m(0) = |\beta|^2$ has been evolved to $b_m(t) = |\beta|^2 (1 - p_1(t)) \dots (1 - p_M(t))$. To decide if the LBC of $\rho_N^I(t)$ can be factorized, we should check if the aforementioned condition $a_{m'}(t)b_{m'}(t) = 0 \forall m' \neq m \in \{1, 2, \dots, 2^{N-1}\}$ can be satisfied. From the second term of the right-hand side (RHS) of Eq. (14), we observe that if not all the N qubits are subjected to the AD channels, i.e., $M < N$, all these diagonal terms have the same reduced density operators $|11\dots 1\rangle_{M+1, M+2\dots N} \langle 11\dots 1|$ for the qubits $M+1, M+2\dots N$ that are not subjected to the noisy channels. In other words, the matrix $\rho_N^I(t)$ (14) does not involve the derived diagonal terms whose reduced density operators for the qubits $M+1, M+2, \dots, N$ are $|00\dots 0\rangle_{M+1, M+2\dots N} \langle 00\dots 0|$. Recall that the elements $a_{m'}(t)$ and $b_{m'}(t)$ are the derived diagonal terms that should possess opposite reduced density operators for all the N individual qubits, therefore in the case of $M < N$, we always have $a_{m'}(t)b_{m'}(t) = 0$. The LBC of the N -qubit system can thus be obtained by virtue of Eq. (9) as

$$\underline{\mathcal{C}}_N(\rho_N^I(t)) = 2|\alpha\beta| \prod_{j=1}^M [1 - p_j(t)]^{\frac{1}{2}}. \quad (15)$$

Obviously, the $2|\alpha\beta|$ in (15) is the initial LBC of the GHZ state (13) while the term $\prod_{j=1}^M (1 - p_j(t))^{\frac{1}{2}}$ comprises the parameters of the M AD channels.

In contrast to the case of $M < N$, if all the N qubits are coupled to the AD channels, i.e., $M = N$, then all the 2^N diagonal elements of the states (14) are nonzero, which implies that the condition $a_{m'}(t)b_{m'}(t) = 0$ cannot be satisfied. Therefore, the LBC of the system cannot be factorized to the form (15). Actually, it has been shown [27] under the action of N independent AD channels, the N -qubit system in the GHZ state (13) will suffer from the entanglement sudden death [7] when $|\beta| > |\alpha|$. At the same time, the robustness of the N -qubit system, i.e., the time at which the entanglement become arbitrarily small, decrease with the increase of the number N of qubits [27]. Here, for the case of $M < N$, we can see from the factorization expression (15) that the LBC is independent of the system size N and only related to the number M of the AD channels. In Fig. 1, we plot the evolution of LBC under $M = 1, \dots, 5$ identical AD channels for the systems with $N > M$ qubits. For convenience, we have parameterized the time dependence in terms of $p \equiv p(t)$ instead of t noticing that $p = 0$ when $t = 0$ and $p \rightarrow 1$ when $t \rightarrow \infty$, i.e., $p \in [0, 1]$ for $t \in [0, \infty]$. From Fig.1, we can see that the decay rate of the system increase with the number M of the AD channels. It deserves noting that for $M = 1$, i.e., under one-sided channel, the LBC evolution of any system with $N > 2$ qubits are equivalent to the concurrence evolution of a two-qubit

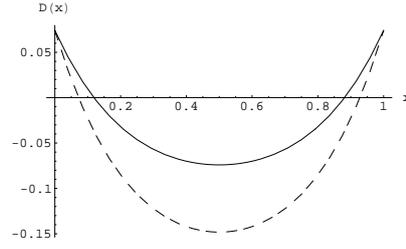


Figure 1. The LBC evolution of a system with $N > M$ qubits that are initially in the symmetrical GHZ state (13) under actions of M independent AD channels. The parameter of the initial GHZ state is $|\alpha| = |\beta| = 1/\sqrt{2}$ and $p \in [0, 1]$ for $t \in [0, \infty]$.

system under also the one-sided channel, which is consistent with the result of Gour in [20].

Next, we investigate the asymmetrical GHZ state in which the reduced states of the N qubits are not equivalent. Without loss of generality, we assume the reduced states of the qubits $1, 2, \dots, n$ with $1 \leq n < N$ are all $|0\rangle$, while the qubits $n+1, n+2, \dots, N$ are all $|1\rangle$, therefore the second class of GHZ state can be expressed as

$$|\Psi_{\text{II}}\rangle_{1,2\dots N} = \alpha |0\dots 0, 1\dots 1\rangle_{1\dots n, n+1, \dots, N} + \beta |1\dots 1, 0\dots 0\rangle_{1\dots n, n+1, \dots, N}. \quad (16)$$

Suppose all the N qubits are coupled with their own AD channels. Then the initial density matrix $\rho_N^{\text{II}}(0) = |\Psi_{\text{II}}\rangle_{1,2\dots N} \langle\Psi_{\text{II}}|$ will map onto $\rho_N^{\text{II}}(t)$ which can be obtained as

$$\begin{aligned} \rho_N^{\text{II}}(t) = & \prod_{j=1}^N (1 - p_j(t))^{\frac{1}{2}} (\alpha\beta^* |0\dots 0, 1\dots 1\rangle_{1\dots n, n+1, \dots, N} \langle 1\dots 1, 0\dots 0| + h.c.) \\ & + |\alpha|^2 \sum_{i_{n+1}\dots i_N=0}^1 q_{n+1}(t)\dots q_N(t) |0\dots 0, i_{n+1}\dots i_N\rangle_{1\dots n, n+1, \dots, N} \langle 0\dots 0, i_{n+1}\dots i_N| \\ & + |\beta|^2 \sum_{i_1\dots i_n=0}^1 q_1(t)\dots q_n(t) |i_1\dots i_n, 0\dots 0\rangle_{1\dots n, n+1, \dots, N} \langle i_1\dots i_n, 0\dots 0|, \end{aligned} \quad (17)$$

where the coefficients $q_j(t) \equiv p_j(t) \forall i_j = 0$ and $q_j(t) \equiv 1 - p_j(t) \forall i_j = 1$ with $j \in \{1, 2, \dots, N\}$. Here, we directly check if the first kind of condition $a_{m'}(t)b_{m'}(t) = 0 \forall m' \neq m \in \{1, 2, \dots, 2^{N-1}\}$ can be satisfied by $\rho_N^{\text{II}}(t)$. From Eq. (17), we observe that there exist only two diagonal terms, i.e., $|0\dots 0, 1\dots 1\rangle_{1\dots n, n+1, \dots, N} \langle 0\dots 0, 1\dots 1|$ and $|1\dots 1, 0\dots 0\rangle_{1\dots n, n+1, \dots, N} \langle 1\dots 1, 0\dots 0|$, which involve completely opposite reduced states for all the N individual qubits. However, these two terms are merely the nonzero diagonal terms of the initial density operator $\rho_N^{\text{II}}(0)$ rather than the derived terms in the evolution. That is, the derived diagonal terms with elements $a_{m'}(t)$ and $b_{m'}(t)$, such as $|0\dots 0, 1\dots 0\rangle_{1\dots n, n+1, \dots, N} \langle 0\dots 0, 1\dots 0|$ and $|1\dots 1, 0\dots 1\rangle_{1\dots n, n+1, \dots, N} \langle 1\dots 1, 0\dots 1|$, which involve completely opposite reduced states for all the N individual qubits will not appear simultaneously (for the present example only the former term can appear while its counterpart does not). Therefore, the condition $a_{m'}(t)b_{m'}(t) = 0$ can be satisfied for the N -qubit system in the GHZ states (16) under the action of N independent AD

channels. By virtue of (9), we can get the factorized form of LBC for the state $\rho_N^{\text{II}}(t)$ as

$$\underline{\mathcal{C}}_N(\rho_N^{\text{II}}(t)) = 2|\alpha\beta| \prod_{j=1}^N (1 - p_j(t))^{\frac{1}{2}}. \quad (18)$$

Obviously, $2|\alpha\beta|$ in (18) is the initial LBC of the state (16) while the term $\prod_{j=1}^N (1 - p_j(t))^{\frac{1}{2}}$ reflects the actions of the N AD channels. Here we have considered all the N qubits are subjected to the independent AD channels, i.e., $M = N$. For the case of $M < N$, the factorized form in (18) still hold but the RHS of (18) should be replaced by $2|\alpha\beta| \prod_{j=1}^M (1 - p_j(t))^{\frac{1}{2}}$ implying the action of M AD channels.

3.2. Depolarization channel

The depolarizing channel j describes the situation in which a qubit remains untouched with probability $1 - p_j(t)$, or is depolarized (white noise), i.e., its state is taken to the maximally mixed state, with probability $p_j(t)$. The action of the map \mathcal{E}_j^D of the D channel j on elements of the reduced density matrix of a qubit reads:

$$\mathcal{E}_j^D : \begin{cases} |0\rangle\langle 0| \rightarrow (1 - \frac{p_j(t)}{2}) |0\rangle\langle 0| + \frac{p_j(t)}{2} |1\rangle\langle 1|, \\ |1\rangle\langle 1| \rightarrow (1 - \frac{p_j(t)}{2}) |1\rangle\langle 1| + \frac{p_j(t)}{2} |0\rangle\langle 0|, \\ |0\rangle\langle 1| \rightarrow (1 - p_j(t)) |0\rangle\langle 1|, \\ |1\rangle\langle 0| \rightarrow (1 - p_j(t)) |1\rangle\langle 0|. \end{cases} \quad (19)$$

Without loss of generality, we suppose the qubits $1, 2, \dots, M$ are coupled to M ($M \leq N$) independent D channels, respectively. Then under the actions of the M different D channels the initial density matrix $\rho_N(0)$ (4) of the N -qubit system will map onto $\rho_N(t)$ which can be obtained as

$$\begin{aligned} \rho_N^D(t) &= \prod_{j=1}^M (1 - p_j(t)) (\alpha\beta^* |i_1 \dots i_N\rangle_{1\dots N} \langle \bar{i}_1 \dots \bar{i}_N| + \alpha^* \beta |\bar{i}_1 \dots \bar{i}_N\rangle_{1\dots N} \langle i_1 \dots i_N|) \\ &+ |\alpha|^2 \sum_{i'_1 \dots i'_M=0}^1 q_1(t) \dots q_M(t) |i'_1 \dots i'_M, i_{M+1} \dots i_N\rangle_{1\dots M, M+1 \dots N} \langle i'_1 \dots i'_M, i_{M+1} \dots i_N| \\ &+ |\beta|^2 \sum_{\bar{i}'_1 \dots \bar{i}'_M=0}^1 g_1(t) \dots g_M(t) |\bar{i}'_1 \dots \bar{i}'_M, \bar{i}_{M+1} \dots \bar{i}_N\rangle_{1\dots M, M+1 \dots N} \langle \bar{i}'_1 \dots \bar{i}'_M, \bar{i}_{M+1} \dots \bar{i}_N| \end{aligned}$$

where the coefficients $q_j(t) \equiv 1 - \frac{p_j(t)}{2}$ ($\frac{p_j(t)}{2}$) when $i'_j = i_j$ ($i'_j = i_j \oplus 1$), $g_j(t) \equiv 1 - \frac{p_j(t)}{2}$ ($\frac{p_j(t)}{2}$) when $\bar{i}'_j = \bar{i}_j$ ($\bar{i}'_j = \bar{i}_j \oplus 1$) with $j = 1, 2, \dots, M$. Before discussing the general case, we first take a three-qubit system in the initial state $\rho_3(0) = |\alpha|^2 |000\rangle_{123} \langle 000| + |\beta|^2 |111\rangle_{123} \langle 111| + \alpha\beta^* |000\rangle_{123} \langle 111| + \alpha^*\beta |111\rangle_{123} \langle 000|$ as an example and suppose only the former two qubits are subjected to the D channels. By virtue of Eq.(20), the evolved density operator read

$$\rho_3(t) = \prod_{j=1}^2 (1 - p_j(t)) (\alpha\beta^* |000\rangle_{123} \langle 111| + \alpha^*\beta |111\rangle_{123} \langle 000|)$$

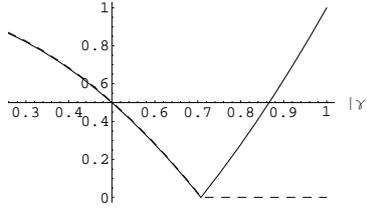


Figure 2. (a) The time-evolution of LBC for a four-qubit system in the GHZ state (13) under $M = 1$ (solid line), $M = 2$ (dashed line) and $M = 3$ (dotted line) D channels. (b) The time-evolution of LBC for the system with $N = 4$ (solid line), $N = 3$ (dashed line) and $N = 2$ (dotted line) qubits under a single D channel. (c) The time-evolution of LBC for the system with $N = 4$ (solid line), $N = 3$ (dotted line) qubits under two D channels. The parameters used are $|\alpha| = |\beta| = 1/\sqrt{2}$ and $p \in [0, 1]$ for $t \in [0, \infty]$.

$$\begin{aligned}
 & + |\alpha|^2 \left[\left(1 - \frac{p_1(t)}{2}\right) \left(1 - \frac{p_2(t)}{2}\right) |000\rangle\langle 000| + \frac{p_1(t)}{2} \frac{p_2(t)}{2} |110\rangle\langle 110| \right. \\
 & + \left. \left(1 - \frac{p_1(t)}{2}\right) \frac{p_2(t)}{2} |010\rangle\langle 010| + \frac{p_1(t)}{2} \left(1 - \frac{p_2(t)}{2}\right) |100\rangle\langle 100| \right] \\
 & + |\beta|^2 \left[\left(1 - \frac{p_1(t)}{2}\right) \left(1 - \frac{p_2(t)}{2}\right) |111\rangle\langle 111| + \frac{p_1(t)}{2} \frac{p_2(t)}{2} |001\rangle\langle 001| \right. \\
 & + \left. \left(1 - \frac{p_1(t)}{2}\right) \frac{p_2(t)}{2} |101\rangle\langle 101| + \frac{p_1(t)}{2} \left(1 - \frac{p_2(t)}{2}\right) |011\rangle\langle 011| \right]. \quad (21)
 \end{aligned}$$

From Eq. (21), we can see that any one of the former four diagonal terms with a common coefficient $|\alpha|^2$ has a counterpart in group of the latter four terms with a common coefficient $|\beta|^2$, such as $|110\rangle\langle 110|$ and $|001\rangle\langle 001|$, so as to these two terms have completely opposite reduced states for all the three qubits. Therefore, the second factorization condition, i.e., $\sqrt{a_{m'}(t)b_{m'}(t)}$ can be decomposed to $|d_m(0)\mathcal{F}_{m'}(t)|$, is satisfied for the present instance and we have $\sqrt{a_{m'}(t)b_{m'}(t)} = |\alpha\beta|\mathcal{F}_{m'}(t)$ with $\mathcal{F}_{m'}(t) = \frac{p_1(t)p_2(t)}{2}$, $\left(1 - \frac{p_1(t)}{2}\right)\frac{p_2(t)}{2}$ or $\frac{p_1(t)}{2}\left(1 - \frac{p_2(t)}{2}\right)$ being determined by the concrete value of m' . This specific example can be generalized to the general case. Actually, for the evolved density operator (20), we observe that if $M < N$ the second factorization condition hold. In this case, we have $a_{m'}(t) = |\alpha|^2 q_1(t) \dots q_M(t)$, $b_{m'}(t) = |\beta|^2 g_1(t) \dots g_M(t)$, and $\sqrt{a_{m'}(t)b_{m'}(t)} = |\alpha\beta|\mathcal{F}_{m'}(t)$ with $|\mathcal{F}_{m'}(t)| = \sqrt{q_1(t) \dots q_M(t)g_1(t) \dots g_M(t)}$. In addition, we note that the two diagonal elements $a_{m'}(t)$ and $b_{m'}(t)$ are equivalent so that $|\mathcal{F}_{m'}(t)| = |q_1(t) \dots q_M(t)| = |g_1(t) \dots g_M(t)|$. By virtue of Eq. (11), the LBC of the state $\rho_N^D(t)$ can be expressed in the form

$$\underline{\mathcal{C}}_N(\rho_N^D(t)) = 2|\alpha\beta| \sqrt{\frac{1}{2^{N-1} - 1} \sum_{\mathcal{P}} (\mathcal{Q}^{\mathcal{P}[k|N-k]}(t))^2}, \quad (22)$$

where $2|\alpha\beta|$ is the initial LBC of the GHZ state (3) and $\mathcal{Q}^{\mathcal{P}[k|N-k]}(t) = \max\{0, |\prod_{j=1}^M (1 - p_j(t))| - |\prod_{j=1}^M q_j(t)|\}$. As for the case of $M = N$, the second kind of condition does not hold any more, therefore the LBC evolution of the N -qubit system cannot be factorized as the form (22).

In the following, we make a brief discussion on the relations between qubits' LBC and the system's size N as well as the number of the noisy channels M . From (22), we conclude that in the time-evolution of the N -qubit system, the LBC increases with the qubit's number N for the fixed number $M (< N)$ of the D channels, while decreases with the number M for the fixed qubit's number N . In Fig.2 (a), we have shown the evolution of LBC of a four-qubit system in the GHZ state (13) under $M = 1, 2, 3$ identical D channels, where we can see the LBC decreases with an increase of M . Fig.2 (b) is for the evolution of LBC for the systems with $N = 4$, $N = 3$ and $N = 2$ qubits under one-sided D channel, from which we can see the robustness of LBC increases with N . In Fig.2(c) we plot the evolution of LBC for the system with $N = 4$ and $N = 3$ qubits under two identical D channels, which still shows that the robustness of LBC increases with N . Here, we have parameterized the time dependence in terms of $p \equiv p(t)$ instead of t noticing that $p = 0$ when $t = 0$ and $p \rightarrow 1$ when $t \rightarrow \infty$, i.e., $p \in [0, 1]$ for $t \in [0, \infty]$.

3.3. Dephasing channel

The PD (or dephasing) channel represents the situation in which there is loss of quantum coherence with probability $p(t)$, but without any energy exchange. The action of the map \mathcal{E}_j^{PD} of the PD channel j on the density matrix of a qubit multiplies the off-diagonal elements by the factor $(1 - p(t))$ while remaining the diagonal elements invariant. Since the PD channels cannot lead to the population evolution of the N -qubit system, both the elements $a_{m'}(t)$ and $b_{m'}(t)$ representing the derived diagonal terms in the evolution are zero. This implies that the first kind of condition for the factorization of the LBC of the N -qubit system can always be satisfied when the N -qubit system are coupled with independent PD channels. Therefore, the LBC of the N -qubit system after evolution has the factorization form in Eq. (9).

4. Conclusion

In conclusion, by using LBC as a measure of multiqubit entanglement, we have explored the possibility to generalize the factorization law of two-qubit under one-sided channel to multiqubit under multi-sided channels. Instead of a general answer to this issue, we have considered a special form of multiqubit entanglement, i.e., the GHZ state (3), and three typical types of noisy channels, i.e., the amplitude-damping (AD), depolarizing (D) and phasing damping (PD) (dephasing) channels. The explicit formulae (7) for the evolution of LBC as well as the associated bipartite concurrence (6) for an arbitrary bipartition are derived, based on which we observe that under two kinds of conditions the LBC can be factorized. That is, the LBC evolution of the N -qubit system can be determined by the product of its initial LBC and the LBC evolution of a maximal entangled state under the same multi-sided noisy channels. We then study the realistic situations in which these two kinds of conditions can be satisfied. We have shown that (i) the N -qubit system in symmetrical GHZ state (13) can satisfy the first kind of factorization condition when

the number M of the independent AD channels is lesser than the number N of the qubits, while the N -qubit system in asymmetrical GHZ state (16) can always satisfy the first kind of condition irrespective of the number M of the AD channels; (ii) in the D channels, the N -qubit system in the general GHZ state (3) can satisfy the second factorization condition when the number M of the D channels is lesser than N ; (iii) the PD channels can always lead to a factorization of LBC of the N -qubit system in GHZ state (3) with a satisfaction of the first kind of condition. By virtue of the concise expressions of the LBC under different noisy channels, we have discussed the dependence of the entanglement robustness on the system size in terms of N and the number M of the noisy channels. The GHZ state, though only be a special class of various types of multipartite entanglement, have attracted extensive researches from different aspects, such as the preparation [34], application [35], dynamics[24, 25, 26, 27, 29, 28] and so on. Therefore, the study on the factorization law of special multiqubit entanglement under some typical multi-sided noisy channels can not only deepen the understanding of multipartite entanglement dynamics but also facilitate the corresponding calculation.

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