Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Local Unitary Equivalence of Arbitrary Dimensional Bipartite Quantum States

by

Chunqin Zhou, Ting-Gui Zhang, Shao-Ming Fei, Naihuan Jing, and Xianqing Li-Jost

Local Unitary Equivalence of Arbitrary Dimensional Bipartite Quantum States

Chunqin Zhou[1], Ting-Gui Zhang[2], Shao-Ming Fei[2,3], Naihuan Jing[4,5], and Xianqing Li-Jost[2]

[1]*Department of Mathematics, Shanghai Jiaotong University, Shanghai 200240, China*
[2]*Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*
[3]*School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[4]*School of Sciences, South China University of Technology, Guangzhou 510640, China*
[5]*Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA*

The nonlocal properties of arbitrary dimensional bipartite quantum systems are investigated. A complete set of invariants under local unitary transformations is presented. These invariants give rise to both sufficient and necessary conditions for the equivalence of quantum states under local unitary transformations: two density matrices are locally equivalent if and only if all these invariants have equal values.

Introduction. — As a fundamental phenomenon in quantum mechanics, the quantum nonlocality has been recently extensively investigated. Nonlocally quantum correlated states, like quantum entangled states [1] or states with nonzero quantum discord [2], play very important roles in many quantum information processing such as quantum computation [3], teleportation [4], dense coding [5], cryptography [6] and assisted optimal state discrimination [7].

Due to the fact that the nonlocal properties, e.g. the quantum entanglement of two parts of a quantum system, remain invariant under local unitary transformations, they can be characterized in principle by the complete set of invariants under local unitary transformations. For instance, the trace norms of realigned or partial transposed density matrices in entanglement measure [8] and the separability criteria [9] are some of these invariants. A complete set of invariants gives rise to full classification of the quantum states under local unitary transformations.

Many approaches to construct invariants of local unitary transformations have been presented in recent years. The method developed in [10, 11], in principle, allows one to compute all the invariants of local unitary transformations, though it is not easy to perform it operationally. In [12], a complete set of 18 polynomial invariants is presented for the locally unitary equivalence of two qubit mixed states. Partial results have been obtained for three qubits states [13], some generic mixed states [14–16], tripartite pure and mixed states [17]. The local unitary equivalence problem for multipartite pure qubit states has been solved in [18]. Then the problem for arbitrary dimensional multipartite pure states are also solved recently [19]. However, for mixed sates, generally we still have no operational criteria to judge the equivalence of two arbitrary dimensional bipartite states under local unitary transformations.

In this letter, we study the nonlocal properties of arbitrary dimensional bipartite quantum systems and solve the local equivalence problem by presenting a complete set of invariants such that two density matrices are locally equivalent if and only if all these invariants have the equal values in these density matrices. These invariants can be explicitly calculated and give rise to an operational way to judge the local unitary equivalence for nondegenerate density matrices. For degenerate case, due to the eigenvector decompositions of a given state are not unique in the degenerate eigenvectors' subspace, the approach is no longer operational since the expressions of the set of invariants are not unique. Nevertheless the set of invariants is still complete in the sense that if, in a suitable eigenvector decompositions, two states have the same values of the invariants, they must be equivalent under local unitary transformations. In particular, we also present a set of invariants that are independent on the detailed eigenvector decompositions of a quantum state. These invariants present a necessary criterion on local unitary equivalence.

The linear space formed by the invariants. — We first establish a linear space spanned by matrices whose traces are the invariants under local unitary transformations.

Let H be an N-dimensional complex Hilbert space, with $|i\rangle$, $i = 1, ..., N$, an orthonormal basis. Let ρ be a density matrix defined on $H \otimes H$ with $rank(\rho) = n \leq N^2$. ρ can be generally written as

$$\rho = \sum_{i=1}^{n} \lambda_i |v_i\rangle\langle v_i|, \tag{1}$$

where $|v_i\rangle$ is the eigenvector with respect to the nonzero eigenvalue λ_i. $|v_i\rangle$ is a normalized bipartite pure state of the form:

$$|v_i\rangle = \sum_{k,l=1}^{N} a_{kl}^i |kl\rangle, \quad a_{kl}^i \in \mathbb{C}, \quad \sum_{k,l=1}^{N} a_{kl}^i a_{kl}^{i*} = 1,$$

where $*$ stands for complex conjugation.

Two density matrices ρ and ρ' are said to be equivalent under local unitary transformations if there exist unitary operators U_1 (resp. U_2) on the first (resp. second) space

of $H \otimes H$ such that

$$\rho' = (U_1 \otimes U_2)\rho(U_1 \otimes U_2)^{\dagger}, \qquad (2)$$

where $\dagger$ denotes transpose and complex conjugation.

To solve the local equivalence problem of the density matrices ρ and ρ', it is sufficient to find the complete set of invariants under local unitary transformation $U_1 \otimes U_2$.

Lemma 1. The following quantities are invariants under local unitary transformations:

$$\begin{aligned} J^s(\rho) &= Tr_1(Tr_2\rho^s), \\ &Tr[(A_iA_j^{\dagger})(A_kA_l^{\dagger})\cdots(A_hA_p^{\dagger})], \end{aligned} \qquad (3)$$

where $s = 1, ..., N^2$, $i, j, k, l, ..., h, p = 1, ..., n$, Tr_1 and Tr_2 stand for the traces over the first and second Hilbert spaces respectively, A_i denotes the matrix with entries given by $(A_i)_{kl} = a_{kl}^i$, $i = 1, 2, \cdots, n$.

[Proof]. Let U_1 and U_2 be unitary transformations satisfying $U_1U_1^{\dagger} = U_1^{\dagger}U_1 = U_2U_2^{\dagger} = U_2^{\dagger}U_2 = 1$. Under the local unitary transformation $U_1 \otimes U_2$, we have $\rho \to \rho' = U_1 \otimes U_2 \, \rho \, U_1^{\dagger} \otimes U_2^{\dagger}$. Correspondingly, we have $|\nu_i\rangle \to |\nu_i'\rangle = U_1 \otimes U_2 |\nu_i\rangle$, or equivalently A_i is mapped to $A_i' = U_1 A_i U_2^t$, where U_2^t is the transpose of U_2. Therefore

$$A_i'A_j'^{\dagger} = U_1 A_i A_j^{\dagger} U_1^{\dagger}, \quad A_i'^{\dagger}A_j' = U_2^* A_i^{\dagger}A_j U_2^t \qquad (4)$$

for any $i, j = 1, ..., n$. By using relations (4) and taking into account that $Tr_2|v_i\rangle\langle v_i| = A_iA_i^{\dagger}$, it is straightforward to verify that $J^s(\rho') = Tr_1[\sum_{i=1}^n \lambda_i^s Tr_2(|\nu_i'\rangle\langle\nu_i'|)] = Tr_1[\sum_{i=1}^n \lambda_i^s A_i'A_i'^{\dagger}] = J^s(\rho)$, $Tr[(A_i'A_j'^{\dagger})(A_k'A_l'^{\dagger})\cdots(A_h'A_p'^{\dagger})] = Tr[U_1(A_iA_j^{\dagger})(A_kA_l^{\dagger})\cdots(A_hA_p^{\dagger})U_1^{\dagger}] = Tr[(A_iA_j^{\dagger})(A_kA_l^{\dagger})\cdots(A_hA_p^{\dagger})]$. Hence the quantities in (3) are invariants of local unitary transformations. $\square$

If two density matrices are equivalent under local unitary transformations, then their corresponding invariants in (3) must have the same values. Before proving that if two density matrices have the same values of all the invariants in (3), they must be equivalent under local unitary transformations, we first claim that the set of independent invariants in (3) is finite. In fact, each factor $A_iA_j^{\dagger}$ belongs to the finite dimensional algebra $Mat(N)$, which has a linear basis E_{ij}, $i, j = 1, \cdots, N$. Therefore, denoting τ the number of factors like $(A_iA_j^{\dagger})$ in $Tr[(A_iA_j^{\dagger})(A_kA_l^{\dagger})\cdots(A_hA_p^{\dagger})]$, we have that τ is at most N^2. Subjecting to the variations of the subindices $i, j, k, l, ..., h, p$, there could be many invariants of this form. However, τ may be much less than N^2 for given $i, j, k, l, ..., h, p$. For instance for the case of $i = j = k = l... = h = p$, τ takes values from 2 to N (the case $\tau = 1$ is trivial since $Tr(A_iA_i^{\dagger}) = 1$ due to the normalization of the state $|v_i\rangle$). The same is true also for the invariants $Tr[(A_i^{\dagger}A_j)(A_k^{\dagger}A_l)\cdots(A_h^{\dagger}A_p)]$.

Denote the subalgebra of $Mat(N)$ spanned by products of $A_iA_j^{\dagger}$ by

$$I\!R(\rho) = span\{(A_iA_j^{\dagger})(A_kA_l^{\dagger})\cdots(A_hA_p^{\dagger})\},$$

$i, j, k, l, ..., h, p = 1, ..., n$. Obviously $I\!R(\rho)$ is a finite dimensional associative algebra. Set $m = \dim I\!R(\rho)$. Let $\{\rho_1, \rho_2, \cdots, \rho_m\}$ be a basis of the linear space $I\!R(\rho)$.

Lemma 2. The metric tensor matrix Ω, with entries given by $\Omega(\rho)_{ij} = Tr(\rho_i\rho_j)$, $i, j = 1, 2, \cdots, m$, is nonsingular.

Proof. We prove by contradiction. If $\det[\Omega(\rho)] = 0$, then the m row vectors of the matrix $\Omega(\rho)$ are linear dependent. There exist $c_1, c_2, \cdots, c_m \in \mathbb{C}$ which are not all zero, such that

$$\sum_{i=1}^m c_i Tr(\rho_i\rho_j) = 0, \quad j = 1, ..., m.$$

It follows that

$$Tr((c_1\rho_1 + c_2\rho_2 + \cdots + c_m\rho_m)\rho_j) = 0, \quad j = 1, ..., m. \quad (5)$$

Since $(A_{i_1}A_{j_1}^{\dagger}\cdots A_{i_r}A_{j_r}^{\dagger})^{\dagger} \in I\!R(\rho)$ and ρ_i are linear combinations of the terms like $A_{i_1}A_{j_1}^{\dagger}\cdots A_{i_r}A_{j_r}^{\dagger}$, we have $(c_1\rho_1 + c_2\rho_2 + \cdots + c_m\rho_m)^{\dagger} \in I\!R(\rho)$ and

$$(c_1\rho_1 + c_2\rho_2 + \cdots + c_m\rho_m)^{\dagger} = h_1\rho_1 + h_2\rho_2 + \cdots + h_m\rho_m$$

for some $h_i \in \mathbb{C}$, $i = 1, 2, \cdots, m$. It follows form (5) that

$$Tr[(c_1\rho_1 + c_2\rho_2 + \cdots + c_m\rho_m)(c_1\rho_1 + c_2\rho_2 + \cdots + c_m\rho_m)^{\dagger}] = 0,$$

which implies $c_1\rho_1 + c_2\rho_2 + \cdots + c_m\rho_m = 0$. Hence one concludes that $\{\rho_i, i = 1, 2, \cdots, m\}$ are linear dependent, which contradicts that $\{\rho_1, \rho_2, \cdots, \rho_m\}$ is a basis of the linear space $I\!R(\rho)$. $\square$

Actually, one can also prove that if $\{\rho_i, i = 1, 2, \cdots, m\}$ is a sequence of matrices such that $\det[\Omega(\rho)] \neq 0$, then $\{\rho_i, i = 1, 2, \cdots, m\}$ is linear independent.

The invariants $Tr[(A_iA_j^{\dagger})(A_kA_l^{\dagger})\cdots(A_hA_p^{\dagger})]$ can be equivalently written as $Tr[(A_i^{\dagger}A_j)(A_k^{\dagger}A_l)\cdots(A_h^{\dagger}A_p)]$. Correspondingly one has the linear space defined by

$$I\!N(\rho) = span\{(A_i^{\dagger}A_j)(A_k^{\dagger}A_l)\cdots(A_h^{\dagger}A_p)\},$$

$i, j, k, l, ..., h, p = 1, ..., n$, with finite dimension $d = \dim I\!N(\rho)$. If $\{\theta_1, \theta_2, \cdots, \theta_d\}$ is a basis of $I\!N(\rho)$ like $I\!R(\rho)$, we similarly have that the matrix Θ with entries given by $\Theta(\rho)_{ij} = Tr(\theta_i\theta_j)$, $i, j = 1, 2, \cdots, d$, satisfies $\det[\Theta(\rho)] \neq 0$.

Lemma 3. If two density matrices $\rho = \sum_{i=1}^n \lambda_i|\nu_i\rangle\langle\nu_i|$ and $\rho' = \sum_{i=1}^n \lambda_i|\nu_i'\rangle\langle\nu_i'|$ have the same values for the following invariants:

$$Tr[(A_iA_j^{\dagger})(A_kA_l^{\dagger})\cdots(A_hA_p^{\dagger})], \qquad (6)$$

$i, j, k, l, ..., h, p = 1, ..., n$, then the corresponding linear spaces $\mathbb{R}(\rho)$ and $\mathbb{R}(\rho')$ have the same dimension. Moreover, the bases of $\mathbb{R}(\rho)$ and $\mathbb{R}(\rho')$ have a one to one correspondence: if $(A_i A_j^\dagger)(A_k A_l^\dagger)\cdots(A_h A_p^\dagger)$ is a basis element of $\mathbb{R}(\rho)$, then $(A_i' A_j'^\dagger)(A_k' A_l'^\dagger)\cdots(A_h' A_p'^\dagger)$ is a basis element of $\mathbb{R}(\rho')$. Similar results hold between the linear spaces $\mathbb{N}(\rho)$ and $\mathbb{N}(\rho')$.

[Proof]. Assume that $\{\rho_\alpha, \alpha = 1, 2, \cdots, m\}$ is a basis of $\mathbb{R}(\rho)$. Each ρ_α has a form of $(A_i A_j^\dagger)(A_k A_l^\dagger)\cdots(A_h A_p^\dagger)$. Denote $\rho_\alpha' = (A_i' A_j'^\dagger)(A_k' A_l'^\dagger)\cdots(A_h' A_p'^\dagger)$, $\alpha = 1, 2, \cdots, m$. Since $\det[\Omega(\rho)] = \det[\Omega(\rho')] \neq 0$ from Lemma 1 and the condition (6), we have $\{\rho_\alpha', \alpha = 1, 2, \cdots, m\}$ is linearly independent. Hence $\dim \mathbb{R}(\rho') \geq \dim \mathbb{R}(\rho)$. In a similar way one can prove that $\dim \mathbb{R}(\rho) \geq \dim \mathbb{R}(\rho')$. Therefore $\dim \mathbb{R}(\rho') = \dim \mathbb{R}(\rho)$ and $\{\rho_\alpha', \alpha = 1, 2, \cdots, m\}$ is a basis of $\mathbb{R}(\rho')$.

Similar results between the linear spaces $\mathbb{N}(\rho)$ and $\mathbb{N}(\rho')$ can be obtained from the expression of invariants $Tr[(A_i^\dagger A_j)(A_k^\dagger A_l)\cdots(A_h^\dagger A_p)]$. □

Local equivalence of bipartite states. — We now give the necessary and sufficient condition for the equivalence of bipartite states under local unitary transformations.

Theorem Two arbitrary dimensional bipartite density matrices are equivalent under local unitary transformations if and only if there exit eigenstate decompositions (1) such that the following invariants have the same values for both density matrices:

$$J^s(\rho) = Tr_2(Tr_1 \rho^s), \quad s = 1, ..., N^2;$$
$$Tr[(A_i A_j^\dagger)(A_k A_l^\dagger)\cdots(A_h A_p^\dagger)], \tag{7}$$

where $i, j, k, l, ..., h, p = 1, ..., n$.

[Proof]. We have proved that the quantities in (7) are invariants under local unitary transformations. We now prove that these invariants are complete. Suppose that the states $\rho = \sum_{i=1}^n \lambda_i |\nu_i\rangle\langle\nu_i|$ and $\rho' = \sum_{i=1}^n \lambda_i' |\nu_i'\rangle\langle\nu_i'|$ have the same values to the invariants in (7). From $J^s(\rho) = J^s(\rho')$, $s = 1, ..., N^2$, we have that the two density matrices ρ and ρ' have the same set of eigenvalues, i.e. $\lambda_i = \lambda_i'$ for $i = 1, 2, \cdots, n$.

Next we introduce the dual basis ρ_i^* in $\mathbb{R}(\rho)$ such that $Tr(\rho_i \rho_j^*) = \delta_{ij}$. In fact, let $\Omega(\rho)^{-1} = [\Omega^{ij}(\rho)]$, then

$$\rho_i^* = \sum_{j=1}^m \Omega^{ij}(\rho)\rho_j. \tag{8}$$

By Cramer's rule $\Omega(\rho)^{-1} = |\Omega(\rho)|^{-1} Adj(\Omega)(\rho)$, so $\Omega^{ij}(\rho)$ are given by polynomials in $\Omega_{ij}(\rho)$, consequently ρ_i^* are expressed as polynomials in the basis element ρ_i.

Now we study the algebra structure of $\mathbb{R}(\rho)$. First we define the bilinear form $\langle \ , \ \rangle$ on $\mathbb{R}(\rho)$ by

$$\langle \sigma, \tau \rangle = Tr(\sigma\tau^\dagger), \qquad \sigma, \tau \in \mathbb{R}(\rho). \tag{9}$$

Then $\mathbb{R}(\rho)$ becomes a Hilbert space and each operator ρ_i is bounded under the norm $||\rho|| = \langle\rho, \rho\rangle^{1/2}$. Next we

have

$$\rho_i \rho_j = \sum_{k=1}^m Tr(\rho_i \rho_j \rho_k^*)\rho_k. \tag{10}$$

The invariance of the traces (7) implies that in the algebra $\mathbb{R}(\rho')$ the element $\rho_i'^*$ is given by the same formula as ρ_i^*:

$$\rho_i'^* = \sum_{j=1}^m \Omega^{ij}(\rho')\rho_j' = \sum_{j=1}^m \Omega^{ij}(\rho)\rho_j'. \tag{11}$$

Moreover $\rho_i' \rho_j'$ are also given by the same structure constants:

$$\rho_i' \rho_j' = \sum_{k=1}^m Tr(\rho_i' \rho_j' \rho_k'^*)\rho_k' = \sum_{k=1}^m Tr(\rho_i \rho_j \rho_k^*)\rho_k'. \tag{12}$$

Eqs (10) and (12) imply that the map $\rho_i \to \rho_i'$ gives an isomorphism from the algebra $\mathbb{R}(\rho)$ onto the algebra $\mathbb{R}(\rho')$. Therefore $\mathbb{R}(\rho)$ and $\mathbb{R}(\rho')$ are two equivalent representations of the same underlying associative algebra. Thus there exists a non-singular matrix T such that $\rho_i = T\rho_i' T^{-1}$ for all $i = 1, ..., m$. In particular, we have $A_i A_i^\dagger = T A_i' A_i'^\dagger T^{-1}$, $i = 1, ..., m$. As $A_i A_i^\dagger$ are hermitian, due to the algebraic property (9) and using Theorem 12.36 in [20] we have

$$A_i A_i^\dagger = u A_i' A_i'^\dagger u^\dagger, \tag{13}$$

where u is the unitary part of T in the polar decomposition.

Similarly from isomorphism of the algebras $\mathbb{N}(\rho) \simeq \mathbb{N}(\rho')$, one has

$$A_i^\dagger A_i = w A_i'^\dagger A_i' w^\dagger, \tag{14}$$

for some unitary matrix w and all i.

By using relations (13) and (14) we can show

$$A_i = u A_i'(w^*)^t, \ i = 1, ..., n.$$

In fact, let u_i and u_i' be the unitary matrices that diagonalize the hermitian matrices $A_i A_i^\dagger$ and $A_i^\dagger A_i$ respectively, $u_i A_i A_i^\dagger u_i^\dagger = diag\{\eta_{i_1}^2, ..., \eta_{i_N}^2\}$, $u_i' A_i^\dagger A_i u_i'^\dagger = diag\{\eta_{i_1}^2, ..., \eta_{i_N}^2\}$. From the procedure of the singular value decomposition of matrices [21], we have $u_i A_i u_i'^\dagger = diag\{\eta_{i_1}, ..., \eta_{i_N}\}$ with $\eta_k \geq 0$. From (13) and (14), we have $u_i A_i A_i^\dagger u_i^\dagger = u_i u A_i' A_i'^\dagger u^\dagger u_i^\dagger = diag\{\eta_{i_1}^2, ..., \eta_{i_N}^2\}$ and $u_i' A_i^\dagger A_i u_i'^\dagger = u_i' w A_i'^\dagger A_i' w^\dagger u_i'^\dagger = diag\{\eta_{i_1}^2, ..., \eta_{i_N}^2\}$. Therefore from singular value decomposition we have $u_i u A_i' w^\dagger u_i'^\dagger = diag\{\eta_{i_1}, ..., \eta_{i_N}\}$. Hence we obtain $u_i A_i u_i'^\dagger = u_i u A_i' w^\dagger u_i'^\dagger$, i.e. $A_i = u A_i' w^\dagger = u A_i'(w^*)^t$.

Since $A_i = u A_i'(w^*)^t$, we have $|\nu_i'\rangle = u^\dagger \otimes (w^*)^\dagger |\nu_i\rangle$, $i = 1, ..., n$, and $\rho' = u^\dagger \otimes (w^*)^\dagger \rho \, u \otimes w^*$. Therefore ρ' and ρ are equivalent under local unitary transformations. □

From the proof one sees that the invariants (7) are complete, finite and can be easily calculated. For non-degenerate states ρ and ρ', the eigenvector $|\nu_i\rangle$ of ρ corresponds uniquely to the eigenvector $|\nu_i'\rangle$ of ρ'. Hence the correspondence between A_i and A_i' is also unique. It is straightforward to judge the local unitary equivalence of ρ and ρ' by simply comparing the values of invariants from A_i and A_i' one by one.

For degenerate states, it becomes less operational since the eigenvector decompositions (1) is no longer unique. If $|v_i\rangle$, $i = 1, ..., r$, are the orthogonal eigenvectors with respect to a same eigenvalue. then $\widetilde{|v_i\rangle} = \sum_j U_{ij}|v_j\rangle$ is also a set of linearly independent orthogonal eigenvectors to the same eigenvalue for any unitary matrix $(U)_{ij} = U_{ij}$. Therefore the expressions of the invariants (7) are not unique in general. However, by linear combinations of the invariants in (7) we can get a set of invariants which does not depend on the detailed eigenvector decompositions,

$$\sum_{i,i',j,j',...,k,k'=1}^{r} Tr(A_i A_{i'}^\dagger A_j A_{j'}^\dagger \cdots A_k A_{k'}^\dagger),$$
$$\sum_{i,i',j,j',...,k,k'=1}^{r} Tr(A_i^\dagger A_{i'} A_j^\dagger A_{j'} \cdots A_k^\dagger A_{k'}), \quad (15)$$

where the indices $\{i', j', ..., k'\}$ is any given permutation of the indices $\{i, j, ..., k\}$. Different given permutation gives different invariants. Since the invariants (15) are independent on the detailed eigenvector decompositions, they gives an operational necessary condition for the local equivalence, with respect to the subspaces spanned by the degenerate eigenvectors. If two density matrices with the same degenerate eigenvalue are equivalent under local unitary transformations, they must have the same values of all the invariants in (15). As an simple example, consider $\rho = diag\{1/2, 1/2, 0, 0\}$, $\rho' = diag\{1/2, 0, 1/2, 0\}$. We have $A_1 = A_1' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = (A_2')^t = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, From (15) we have that ρ and ρ' are not local unitary equivalent, since $Tr(A_1 A_1^\dagger A_1 A_1^\dagger + A_1 A_2^\dagger A_2 A_1^\dagger + A_2 A_1^\dagger A_1 A_2^\dagger + A_2 A_2^\dagger A_2 A_2^\dagger) = 2$, while $Tr(A_1' A_1'^\dagger A_1' A_1'^\dagger + A_1' A_2'^\dagger A_2' A_1'^\dagger + A_2' A_1'^\dagger A_1' A_2'^\dagger + A_2' A_2'^\dagger A_2' A_2'^\dagger) = 4$.

Conclusion and remarks. — We have investigated the nonlocal properties of arbitrary dimensional bipartite quantum systems and solved the local equivalence problem by presenting a complete set of invariants such that two density matrices are locally equivalent if and only if all these invariants have the equal values. Although the independent invariants may vary with the detailed bipartite states, the number of the invariants one needs to check is finite. Here we have dealt with the case that the dimensions of both Hilbert spaces are the same for simplicity. Nevertheless the case that the Hilbert spaces have different dimensions can be similarly discussed. Our approach of constructing local invariants and algebraic proof of the sufficiency may also shed light on multipartite case, for which only the multipartite pure state case has been extensively studied [18, 19].

[1] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, Rev. Mod. Phys. 81(2009).
[2] See, for example, H. Ollivier and W.H. Zurek, Phys. Rev. Lett. 88, 017901(2001).
B. Dakíc, V. Vedral, and Č. Brukner, Phys. Rev. Lett. 105, 190502(2010).
S. Luo, Phys. Rev. A **77**, 042303 (2008).
M.D. Lang, and C.M. Caves, Phys. Rev. Lett. 105, 150501(2010).
B. Li, Z.X. Wang and S.M. Fei, Phys. Rev. A 83, 022321(2011).
[3] See, for example, D.P. DiVincenzo, Science 270, 255(1995).
[4] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters, Phys. Rev. Lett. 70, 1895(1993).
S. Albeverio and S.M. Fei, Phys. Lett. A 276, 8(2000).
S. Albeverio and S.M. Fei and W.L. Yang, Phys. Rev. A 66, 012301(2002).
[5] C.H. Bennett and S.J. Wiesner, Phys. Rev. Lett. 69, 2881(1992).
[6] A. Ekert, Phys. Rev. Lett. **67**, 661(1991).
D. Deutsch, A. Ekert, P. Rozsa, C. Macchiavello, S. Popescu and A. Sanpera, Phys. Rev. Lett. 77, 2818(1996).
[7] L. Roa1, J. C. Retamal, and M. Alid-Vaccarezza, Phys. Rev. Lett. 107, 080401(2011).
B. Li, S.M. Fei, Z.X. Wang and H. Fan, Phys. Rev. A 85, 022328(2012).
[8] K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. Lett. 95, 040504 (2005);
K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. Lett. 95, 210501 (2005).
[9] K.Chen and L.A.Wu, Quant. Inf. Comput.3,193(2003).
K.Chen and L.A.Wu, Phys. Lett. A 306, 14(2002).
O. Rudolph, Phys. Rev. A 67,032312(2003).
K. Chen, S. Albeverio and S.M. Fei, Phys. Rev. A 68, 062313(2003).
[10] E.M. Rains, IEEE Transactions on Information Theory, 46, 54-59 (2000).
[11] M. Grassl, M. Rötteler and T. Beth, Phys. Rev. A 58, 1833 (1998).
[12] Y. Makhlin, Quant. Info. Proc. 1, 243-252 (2002).
[13] N. Linden, S. Popescu and A. Sudbery, Phys. Rev. Lett. 83, 243 (1999).
[14] S. Albeverio, S.M. Fei, P. Parashar,W.L. Yang, Phys. Rev. A 68, 010303(2003).
[15] S. Albeverio, S.M. Fei, and D. Goswami, Phys. Lett. A 340 37-42(2005).
[16] B.Z. Sun, S.M. Fei, X.Q. Li-Jost and Z.X. Wang, J. Phys. A 39, L43-L47(2006).

[17] S. Albeverio, L. Cattaneo, S.M. Fei and X.H. Wang, Int. J. Quant. Inform. 3, 603-609(2005).

[18] B. Kraus, Phys. Rev. Lett. 104, 020504(2010); Phys. Rev. A 82, 032121(2010).

[19] B. Liu, J.L. Li, X.K. Li and C.F. Qiao, Phys. Rev. Lett. 108, 050501 (2012).

[20] W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.

[21] R.A. Horn, C.R. Johnson, *Matrix Analysis*, Combridge University Press, 1990.