Continuation of infinitesimal bendings on developable surfaces and equilibrium equations for nonlinear bending theory of plates

by

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Preprint no.: 46 2012
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Abstract
We introduce a natural concept of stationarity for the nonlinear bending theory of elastic plates, and we derive the equilibrium equations satisfied by stationary points. A key ingredient is a geometric result about the continuation of infinitesimal bendings on developable surfaces.

1 Introduction
In 1850 Kirchhoff proposed the following mathematical model to describe the behaviour of thin elastic plates: The reference configuration of the plate is modelled by a bounded domain \( S \subset \mathbb{R}^2 \) and its deformation is modelled by an isometric immersion \( u : S \rightarrow \mathbb{R}^3 \), i.e. by a solution of the PDE system
\[
\partial_i u \cdot \partial_j u = g_{ij},
\]
where \( g_{ij} = \delta_{ij} \) is the standard flat metric in \( \mathbb{R}^2 \). The elastic energy stored in the deformed configuration \( u(S) \) is given by
\[
E(u) = \int_S Q(A),
\]
where \( A \) is the second fundamental form of the surface \( u \) and \( Q \) is the quadratic form of linearized elasticity. This model was justified rigorously in [1]. According to this model, in order to understand the shape of thin elastic plates under external forces or boundary conditions, one has to understand the minimizers of the functional \( E \) within the admissible class of deformations \( u \). A common way to extract information about the shape of minimizers is to deduce the equilibrium (or Euler-Lagrange) equations satisfied by them.

In this paper, we derive such equilibrium equations. In fact, certain equilibrium equations for minimizers of \( E \) were derived earlier in [5] (cf. also [16]). The derivation in [5] works under minimal hypotheses and leads to an equilibrium equation that is a system of ODEs for lines of curvature on the minimizing deformation. This system carries enough information to derive an (essentially optimal) regularity result, cf. [5]. The derivations of the equilibrium equations for \( E \) provided in [5] and in [16]

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seem to be the only ones currently available in the literature. They represent the stationary deformation $u$ in terms of a framed curve, and then (roughly speaking) they derive the Euler-Lagrange equations satisfied by framed curves which are stationary for the transformed energy functional obtained by expressing $E$ in terms of the framed curve representing $u$. From a surface theoretic or calculus of variations viewpoint, such a derivation is not very natural. In particular, the relation between the equilibrium equations for the functional $E$ derived in [5] and the equilibrium equations of the functional $\int_S |A|^2$ without the isometry constraint (the Willmore functional from differential geometry) remained unclear.

Here we adopt a new viewpoint. Conceptually, it is more natural than the one in [5, 16]. It is ‘surface-theoretic’ and it has the advantage that the relation to the Willmore equation becomes clear. Moreover, the new approach yields a very simple formal derivation of the equilibrium equations for general functionals with the isometry constraint (1). It directly yields equations involving expressions that have a natural interpretation in terms of the surface variables and other data.

We also provide the key geometric result that gives a rigorous justification of this formal derivation. This result asserts that infinitesimal bendings on developable surfaces are continuable. (We refer to the body of the paper for the precise meaning of this.) This question of ‘continuability’ (or ‘extension’) of infinitesimal bendings has been studied for many years, cf. e.g. [9, 11] and the references therein. Recently, the term ‘matching’ has been proposed in [12, 8] to denote the same process, but here we prefer to stick to the established terminology. The result about continuation of infinitesimal bendings on developable surfaces obtained in this paper is of independent geometric interest. In addition, it is also fundamental for the analysis of developable shells in the (linearized) Föppl-von Kármán setting, cf. [8].

In the case of convex surfaces (this corresponds to a metric $g$ in (1) that has positive Gauss curvature), it has been known for several decades that all infinitesimal bendings are ‘continuable’, cf. [11, 12]. The relevant PDEs arising in the context of such metrics with positive Gauss curvature are elliptic, and they were already well understood in [13]. In the case of metrics with zero Gauss curvature studied here, the relevant PDEs are degenerate. The most advanced results towards a resolution of the continuation problem in this situation seem to be those in [8]. There it is shown that every regular infinitesimal bending $\tau$ can be corrected to become an infinitesimal bending of arbitrary (but finite) higher order. Such ‘approximate’ bendings are enough in order to understand linearized (Föppl-von Kármán-) versions of the ‘correct’ nonlinear bending energy. However, for a rigorous analysis of the fully nonlinear bending theory studied in the present paper, approximate bendings are not enough, as the hard isometry constraint allows no error tolerance.

A drawback of the approach in the present paper as compared to the approach in [5] is that it requires a degree of regularity which does not follow from finiteness of the energy alone. However, in view of the regularity results in [5], the regularity assumptions made here are justified.

Notation. For a vector $w \in \mathbb{R}^2$ we set $\partial_w f = w_i \partial_i f$. Here and elsewhere repeated indices are summed from 1 to 2. We will frequently write $f(\Psi)$ instead of $f \circ \Psi$. The letter $T$ has two meanings, which should not create confusion: it
occurs as the endpoint of intervals \((0, T)\) and it denotes a vector field \((T = \Gamma'\) in the notation introduced below).

## 2 Geometric structure of nonlinear plates

A modern formulation of the nonlinear bending energy functional for isotropic materials is this:

\[
\mathcal{E}(u) = \begin{cases} 
\int_S |\nabla^2 u(x)|^2 \, dx & \text{if } u \in W^{2,2}_\delta(S) \\
+\infty & \text{otherwise.}
\end{cases} \tag{2}
\]

This functional was derived in [1] by \(\Gamma\)-convergence from 3d nonlinear elasticity. In [5] the variational properties of this functional were studied. A main result of that paper asserts that minimizers of \(\mathcal{E}\) under given boundary conditions are \(C^3\) on developable regions, and that (locally) on such developable regions the mean curvature of a minimizer vanishes on finitely many line segments. And away from these segments, minimizers are \(C^1\). These results in fact apply to all solutions of the Euler-Lagrange equations derived in [5].

In view of these results it is natural to restrict ourselves in much of what follows to developable regions (i.e., domains of the form \([\Gamma]\) for some curve \(\Gamma\) in \(\mathbb{R}^2\)), sometimes we will also assume non-vanishing mean curvature. We use a setting that is compatible with the representation of intrinsically flat surfaces used in [5, 6, 4], as this has proven useful for analysis questions.

### 2.1 Structure of nonlinear plates

Let \(S \subset \mathbb{R}^2\) be a bounded Lipschitz domain, and denote by \(W^{2,2}_\delta(S)\) the subset of maps \(u \in W^{2,2}(S, \mathbb{R}^3)\) satisfying (1) pointwise almost everywhere on \(S\), where \((g_{ij}) = (\delta_{ij})\) is the identity matrix. This is the set of deformations with zero stretching energy and with finite bending energy. A fundamental property of surfaces \(u \in W^{2,2}_\delta(S)\) is that they are developable away from locally affine regions. This result is classical for \(C^2\) solutions of (1), cf. [3]. It carries over to \(W^{2,2}_\delta\), cf. [15, 10, 14]. A detailed analysis of such surfaces can be found in [6].

From now on \(u\) will always denote a surface in \(W^{2,2}_\delta(S)\). We denote by \(n := \partial_1 u \wedge \partial_2 u\) the normal to this surface and by

\[
h_{ij} = n \cdot \partial_i \partial_j u
\]

the entries of its second fundamental form. The mean curvature of \(u\) is given by

\[
H := \frac{\text{Tr} h}{2}.
\]

A basic result about intrinsically flat surfaces is the following one (cf. e.g. [2, 10]):

**Lemma 2.1** If \(S \subset \mathbb{R}^2\) is a bounded domain and \(u \in W^{2,2}_\delta(S)\), then

\[
\partial_i \partial_j u = h_{ij} n \text{ almost everywhere on } S, \tag{3}
\]

and

\[
\text{curl } h = 0 \text{ in the sense of distributions on } S.
\]
Moreover, there exists a \( a : S \to \mathbb{R}^2 \) such that 
\[
h(x) = a(x) \otimes a(x) \text{ almost everywhere on } S.
\]
In what follows we will also need some facts about fine properties of maps in \( W^{2,2}_\delta \); we refer to [6] for many more details. We define 
\[
C_{\nabla u} = \{ x \in S : \nabla u \text{ is constant in a neighbourhood of } x \};
\]
this is the set of points on which the surface \( u \) is locally a plane. Away from this set, the surface \( u \) is developable and admits local line of curvature parameters: for all \( x \in S \setminus \overline{C_{\nabla u}} \) there exists \( T > 0 \) and an arclength parametrized curve \( \Gamma \in W^{2,\infty}([0,T],S) \) with \( x = \Gamma(T/2) \) and such that \( \nabla u \) is constant on the open segments 
\[
[\Gamma(t)] = \{ \Gamma(t) + sN(t) : s \in (s^-(t),s^+(t)) \}.
\]
Here \( N = (\Gamma')^\perp \) denotes the normal to \( \Gamma \) and \( s^\pm(t) \) denote the directed distances of \( \Gamma(t) \) from \( \partial S \) in the direction of \( N(t) \). More precisely, following [6] we introduce the directed distance function \( S \) by setting 
\[
S(x;\alpha) = \inf \{ \theta > 0 : x + \theta \alpha \notin S \}
\]
for all \( (x,\alpha) \in S \times (\mathbb{R}^2 \setminus \{0\}) \). We will often omit the index \( S \). Then we define 
\[
s^+(t) = \nu(\Gamma(t),N(t)) \text{ and } s^-(t) = -\nu(\Gamma(t),-N(t)).
\]
Observe that the open line segment with endpoints \( x \pm \nu(x,\pm \alpha)\alpha \) is just the maximal subinterval of the line \( x + \mathbb{R}\alpha \) contained in \( S \) and which itself contains the point \( x \). As in [4] we define 
\[
M_{s^\pm} = \{ (s,t) : t \in (0,T) \text{ and } s \in (s^-(t),s^+(t)) \}.
\]
As \( M_{s^\pm} \) is the intersection of the subgraph of \( s^+ \) with the epigraph of \( s^- \), it is a Lipschitz domain if \( s^\pm \) are both Lipschitz. If \( S \) is a Lipschitz domain and if \( [\Gamma(t)] \) intersects \( \partial S \) transversally at all \( t \in [0,T] \), then \( s^\pm \) are Lipschitz functions, cf. [6, Proposition 14].

As \( \nabla u \) is constant on the segment (4), we have 
\[
u \Gamma(t) + sN(t)) = u(\Gamma(t)) + s(\partial_{N(t)}u)(\Gamma(t)) \text{ for all } (s,t) \in M_{s^\pm}.
\]
Now define 
\[
\gamma(t) = u(\Gamma(t)) \quad v(t) = (\partial_{N(t)}u)(\Gamma(t))
\]
and introduce the change of coordinates 
\[
\Phi : M_{s^\pm} \to \mathbb{R}^2 \\
(s,t) \mapsto \Gamma(t) + sN(t).
\]
Then (6) can be written as 
\[
u(\Phi(s,t)) = \gamma(t) + sv(t) \text{ for all } (s,t) \in M_{s^\pm}.
\]
This suggests that \( u \) is fully determined on the bounded set 
\[
[\Gamma(0,T)] := [\Gamma] := \Phi(M_{s^\pm}) \subset S
\]
by the curve \( \gamma : [0,T] \to \mathbb{R}^3 \) and the vector field \( v : [0,T] \to \mathbb{S}^2 \). This process can be reversed, as in the following consequence of [4, Proposition 2].
Proposition 2.2 Let $s^\pm : [0, T] \to \mathbb{R}$ be Lipschitz functions satisfying $s^- < 0 < s^+$ on $[0, T]$, let $\Gamma \in W^{2,\infty}((0, T); \mathbb{R}^2)$ be parametrized by arclength and such that
\[
[\Gamma(t_1)] \cap [\Gamma(t_2)] = \emptyset \quad \text{for all unequal } t_1, t_2 \in [0, T],
\]
and let $\mu \in L^2(0, T)$ and denote by $\kappa \in L^\infty(0, T)$ the curvature of $\Gamma$. Denote by $r \in W^{1,2}((0, T); SO(3))$ the solution of
\[
r' = \begin{pmatrix} 0 & \kappa & \mu \\ -\kappa & 0 & 0 \\ -\mu & 0 & 0 \end{pmatrix} r
\]
with initial value $r(0) = Id$. Define
\[(\gamma', v, n)^T := r,
\]
define $\gamma(t) := \int_0^t \gamma'$, and define the mapping $u : [\Gamma] \to \mathbb{R}^3$ by setting
\[u(\Phi(s, t)) := \gamma(t) + sv(t)\]
for all $(s, t) \in M_{s^\pm}$. Then the following are true:
We have
\[
\kappa(t) \in \left[ \frac{1}{s^-(t)}, \frac{1}{s^+(t)} \right] \quad \text{for almost every } t \in (0, T),
\]
and the map $u$ is well defined on $[\Gamma]$, with $u \in W^{2,2}_{\delta, loc}(\Gamma)$. Moreover,
\[
\nabla u(\Phi(s, t)) = \gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t)
\]
for all $(s, t) \in \bar{M}_{s^\pm}$, and
\[
(\partial_i \partial_j u)(\Phi(s, t)) = \left( \frac{\mu(t)}{1 - sk(t)} \Gamma_i'(t) \Gamma_j'(t) \right) n(t) \quad \text{for a.e. } (s, t) \in M_{s^\pm},
\]
and
\[
\int_{[\Gamma]} |\nabla^2 u(x)|^2 \, dx = \int_0^T \left( \int_{s^-(t)}^{s^+(t)} \frac{\mu^2(t)}{1 - sk(t)} \, ds \right) \, dt.
\]
In particular, if $1 - sk \geq c > 0$ for almost every $(s, t) \in M_{s^\pm}$, then $u \in W^{2,2}_{\delta}(\Gamma)$.

For $s^\pm$, $\Gamma$ and $\mu$ as in the hypothesis of Proposition 2.2, the corresponding map $u \in W^{2,2}_{\delta}(\Gamma)$ from Proposition 2.2 is said to be induced by $s^\pm$, $\Gamma$ and $\mu$. We say such a $u$ is induced by $\Gamma$ and $s^\pm$ if there exists some $\mu \in L^2(0, T)$ such that $u$ is induced by $\Gamma$, $s^\pm$ and $\mu$. We denote the lateral boundary of $[\Gamma]$ by
\[
\partial_L[\Gamma] = [\Gamma(0)] \cup [\Gamma(T)].
\]
Requiring the existence of a curve $\Gamma$ such that $u \in W^{2,2}_{\delta}(\Gamma)$ is induced by $\Gamma$ (and some $s^\pm$ and some $\mu$) amounts to the requirement that there exist a global line of curvature chart for $u$. And the change of variables between the standard coordinates and line of curvature coordinates is just $\Phi$. Assuming the existence of such a global chart is natural because in view of the decomposition results in [6], every flat $W^{2,2}$-immersion of some Lipschitz domain $S \subset \mathbb{R}^2$ is locally of that form. Those decomposition results show that maps $u \in W^{2,2}_{\delta}(\Gamma)$ induced by $\Gamma$, $s^\pm$, $\mu$ as in Proposition 2.2 are the natural building blocks of an (intrinsically flat) isometric immersion of a Lipschitz domain $S \subset \mathbb{R}^2$. 

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2.2 Properties of the change of variables $\Phi$

From now on, $T > 0$ and $\Gamma \in W^{2,\infty}((0,T),\mathbb{R}^2)$ denotes the arclength parametrized curve with $\Gamma(0) = 0$ and $\Gamma'(0) = (1,0)^T$, and with curvature $\kappa$. We denote its normal by $N$. We fix Lipschitz functions $s^\pm : [0,T] \to \mathbb{R}$ with $s^- < 0 < s^+$ on $[0,T]$. The curve $\Gamma$ will always be assumed to satisfy (8), so it induces the parametrization $\Phi : M_{s^\pm} \to [\Gamma]$ defined in (7). For given $f : [\Gamma] \to \mathbb{R}^n$ we will use the notation

$$f := f \circ \Phi.$$ Moreover, $\Gamma$ determines a natural orthonormal frame field $(T,N)$ on $[\Gamma]$: the vector fields $T,N : [\Gamma] \to S^1$ are defined by setting $T((s,t)) = 0(t)$ and $N((s,t)) = N(t)$ for all $(s,t) \in M_{s^\pm}$.

We are abusing notation in the definition of $N$, and we will sometimes abuse notation similarly for $T$ and write $T(t)$ instead of $T((s,t))$. Similarly, we write $n(t)$ instead of $n((s,t))$.

In what follows we write $s_x$ to denote the first component of $\Phi^{-1}(x)$ and $t_x$ to denote the second one. In other words, for $x \in [\Gamma]$ we define $s_x$, $t_x$ by $\Phi(s_x, t_x) = x$. Observe that (13) and (3) imply that for almost every $x \in [\Gamma]$ we have

$$h(x) = 2H(x)(T(x) \otimes T(x)) \quad \text{and} \quad 2H(x) = \frac{\mu(t_x)}{1 - s_x \kappa(t_x)}. \quad (15)$$

We will frequently use the change of variables

$$\int_{[\Gamma]} F(x) \, dx = \int_{M_{s^\pm}} (1 - s(t)) F(s,t) \, ds \, dt. \quad (16)$$

**Lemma 2.3** Let $U, V \subset \mathbb{R}^n$ be bounded domains. If $\Psi : U \to V$ is a homeomorphism such that $\Psi$ is locally Lipschitz on $U$ and $\Psi^{-1}$ is locally Lipschitz on $V$, then we have:

(i) If $p \geq 1$ then

$$f \in W^{1,p}_{loc}(V) \iff f(\Psi) \in W^{1,p}_{loc}(U),$$

and

$$\nabla (f(\Psi)) = (\nabla f)(\Psi) \nabla \Psi \text{ almost everywhere on } U. \quad (17)$$

(ii) Assume, moreover, that $\Psi \in W^{2,\infty}(U,V)$ and that

$$\det \nabla \Psi \geq c \text{ almost everywhere on } U \quad (18)$$

for some constant $c > 0$. Then $\Psi^{-1} \in W^{2,\infty}(V,U)$ and

$$f \in W^{2,2}(V) \iff f(\Psi) \in W^{2,2}(U).$$

In both cases, the usual chain rules apply.

**Proof.** Part (i) is well-known, cf. e.g. [17, Theorem 2.2.2]. In order to prove part (ii) note that part (i) implies $f(\Psi) \in W^{1,2}_{loc}(U)$ and (17) holds. Since $\nabla f \in W^{1,2}_{loc}(V,\mathbb{R}^n)$, part (i) implies that $(\nabla f)(\Psi) \in W^{1,2}_{loc}(U,\mathbb{R}^n)$ with

$$\nabla ((\nabla f)(\Psi)) = (\nabla^2 f)(\Psi)(\nabla \Psi, \cdot) \text{ a.e. on } U.$$

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Hence by the Leibniz rule for Sobolev functions, we have
\[ \nabla^2 (f(\Psi)) = (\nabla^2 f) (\Psi) (\nabla \Psi, \nabla \Psi) + (\partial_{\nabla^2 \Psi} f) (\Psi). \] (19)

Since \( \nabla^2 f \in L^2(V, \mathbb{R}^{n \times n}) \), we have \( (\nabla^2 f)(\Psi) \in L^2(U, \mathbb{R}^{n \times n}) \) by (18) and by the change of variables formula. Similarly, \( (\nabla f)(\Psi) \in L^2(U, \mathbb{R}^n) \). Since \( \Psi \in W^{2,\infty}(U, \mathbb{R}^n) \), the right-hand side of (19) is square integrable.

The version of Lemma 2.3 for \( W^{2,\infty} \) instead of \( W^{2,2} \) is the following trivial remark:

**Remark 2.4** If \( k \geq 1 \) is an integer and \( \Psi \in W^{k,\infty}(U, V) \) and \( f \in W^{k,\infty}(V) \) then \( f \circ \Psi \in W^{k,\infty}(U) \).

In the following lemma we set \( W^{0,\infty} := L^\infty \), and we write inf instead of ess inf.

**Lemma 2.5** Let \( k \geq 1 \) be an integer and let \( \kappa \in W^{k-1,\infty}(0, T) \). Then \( \Phi \in W^{k,\infty}(M_{s+}, [\Gamma]) \) is bijective, and
\[ \Phi^{-1} \in W^{k,\infty}(\Phi(M_{s+}, \mathbb{R}^2), \mathbb{R}^2) \] (20)
for all \( \varepsilon > 0 \) smaller than \( \inf_{(0,T)} s^+ \) and \( \inf_{(0,T)} |s^-| \). If, moreover,
\[ \inf_{(s,t) \in M_{s+}} (1 - s \kappa(t)) > 0, \] (21)
then
\[ \Phi^{-1} \in W^{k,\infty}((\Gamma], \mathbb{R}^2). \] (22)

**Proof.** For \( k = 1 \) the claim is a consequence of [6, Proposition 10]. If \( k > 1 \) then it is clear that \( \Phi \in W^{k,\infty} \), because \( N \in W^{k,\infty} \) and \( \Gamma \in W^{k+1,\infty} \). Let \( \varepsilon > 0 \) (or \( \varepsilon = 0 \) if (21) holds). Since \( |\det \nabla \Phi| = 1 - s \kappa \) is bounded from below by a positive constant on \( \Phi(M_{s+}, \mathbb{R}^2) \), we conclude that (20) holds. In case \( \varepsilon = 0 \) this is just (22).

For future reference, we state and prove the following lemmas without regularity assumptions on \( \kappa \).

**Lemma 2.6** We have
\[ f \in W^{1,2}_{\text{loc}}([\Gamma]) \iff \bar{f} \in W^{1,2}_{\text{loc}}(M_{s\pm}). \] (23)

If these are satisfied, then
\[ (\partial_t \bar{f})(s, t) = (1 - s \kappa(t))(\partial_T f)(\Phi(s, t)) \] (24)
\[ (\partial_s \bar{f})(s, t) = (\partial_N f)(\Phi(s, t)) \] (25)
for almost every \( (s, t) \in M_{s\pm} \). In particular, for any Lipschitz subdomain \( \widetilde{M} \subset M_{s\pm} \) we have
\[ \int_{\widetilde{M}} (\partial_s \bar{f})^2 + (\partial_t \bar{f})^2 \, dsdt = \int_{\Phi(\widetilde{M})} ((1 - s \kappa(t_x))(\partial_T f)(x))^2 + \frac{|(\partial_N f)(x)|^2}{1 - s \kappa(t_x)} \, dx. \] (26)
Proof. The equivalence (23) follows from Lemma 2.3 and Lemma 2.5. Equations (24), (25) follow from \( \nabla f = (\nabla f)(\Phi)\nabla \Phi \), because \( \partial_s \Phi = N \) and \( \partial_t \Phi = (1 - sk)T \). Equation (26) follows from (16).

\[ \Box \]

**Lemma 2.7** Let \( I \subset (0, T) \) be a set of positive measure and let \( f \in W^{2,2}_{\text{loc}}([\Gamma]) \) be such that

\[
(N \otimes N) : (\nabla^2 f)(\Phi(s, t)) = 0 \text{ for almost every } (s, t) \in M_{s\pm} \text{ with } t \in I.
\]

Then there exist \( p, q : I \to \mathbb{R} \) such that

\[
\tilde{f}(s, t) = p(t) + sq(t) \text{ for almost every } (s, t) \in M_{s\pm}
\]

with \( t \in I \).

**Proof.** Since \( f \in W^{2,2}_{\text{loc}} \) we have \( \partial_N f \in W^{1,2}_{\text{loc}} \) because \( N \) is locally Lipschitz on \( S \). Since \( \Phi \) is Bilipschitz we conclude that \( (\partial_N f)(\Phi) \in W^{1,2}_{\text{loc}} \), and the chain rule applies:

\[
\partial_s ((\partial_N f)(\Phi)) = (\partial_N \partial_N f)(\Phi) \text{ a.e. on } M_{s\pm}.
\]

But the right-hand side is zero for almost every \((s, t)\) with \( t \in I \). Hence for such \( t \) there exists \( q(t) \in \mathbb{R} \) such that

\[
(\partial_N f)(\Phi(s, t)) = q(t) \text{ for almost every } s \in (s^-(t), s^+(t)).
\]

On the left we can apply the chain rule again to see that it agrees with \( (\partial_s \tilde{f})(s, t) \).

Hence there is \( p(t) \) such that (28) holds. \( \Box \)

**Lemma 2.8** For all \( f \in W^{2,2}_{\text{loc}}([\Gamma]) \) we have almost everywhere on \([\Gamma]\):

\[
\nabla^2 f = (\partial_N \partial_N f)(N \otimes N) + (\partial_N \partial_T f)(T \otimes N + N \otimes T) + \left( \partial_{tt} \partial_T f - \frac{\kappa(t_x)}{1 - sk} \partial_N f \right) (T \otimes T).
\]

Moreover:

(i) We have \( \tilde{f}, \partial_s \tilde{f}, \frac{\partial_s \tilde{f}}{1 - sk} \in W^{1,2}_{\text{loc}}(M_{s\pm}) \), and almost everywhere on \( M_{s\pm} \):

\[
(\nabla^2 f)(\Phi) = (\partial_s \partial_s \tilde{f})(N \otimes N) + \partial_s \left( \frac{\partial_s \tilde{f}}{1 - sk} \right) (T \otimes N + N \otimes T) + \frac{1}{1 - sk} \left[ \partial_t \left( \frac{\partial_t \tilde{f}}{1 - sk} \right) - \kappa \partial_s \tilde{f} \right] (T \otimes T).
\]

(ii) If \( f \in W^{2,2}([\Gamma]) \) and if there exist \( p, q : [0, T] \to \mathbb{R} \) such that

\[
\tilde{f}(s, t) = p(t) + sq(t) \text{ for almost every } (s, t) \in M_{s\pm},
\]

then \( p, q \in W^{1,2}(0, T) \) and

\[
(N \otimes N) : \nabla^2 f = 0
\]

\[
[(T \otimes N) : \nabla^2 f](\Phi) = [(N \otimes T) : \nabla^2 f](\Phi) = \left( \frac{1}{1 - sk} \right)^2 (q' + \kappa p').
\]
(iii) If, in addition to the hypotheses of part (ii), we have \( \kappa \in W^{1.\infty}(0, T) \), then

\[ [(T \otimes T) : \nabla^2 f](\Phi) = \frac{1}{1 - s\kappa} \left\{ \partial_t \left( \frac{1}{1 - s\kappa} \right) (p' + sq') + \frac{p'' + sq''}{1 - s\kappa} - \kappa q \right\}. \tag{34} \]

Proof. Since \( f \in W^{2,2}_{\text{loc}}([\Gamma]) \), we have \( \partial_T f, \partial_N f \in W^{1,2}_{\text{loc}}([\Gamma]) \) by the Leibniz rule and because \( N, T \) are locally Lipschitz on \( \Gamma \). Applying Lemma 2.6 to these functions we see that \( (\partial_N f)(\Phi), (\partial_T f)(\Phi) \in W^{1,2}_{\text{loc}}(M_{s^2}) \) with

\[ \begin{align*}
&\partial_s ((\partial_N f)(\Phi)) = (\partial_N \partial_N f)(\Phi) \\
&\partial_s ((\partial_T f)(\Phi)) = (\partial_N \partial_T f)(\Phi) \\
&\partial_t ((\partial_T f)(\Phi)) = (1 - s\kappa) (\partial_T \partial_T f)(\Phi)
\end{align*} \]

almost everywhere on \( M_{s^2} \). Applying Lemma 2.6 to \( f \) itself shows that \( (\partial_N f)(\Phi) = \dot{\partial}_s \tilde{f} \) and \( (\partial_T f)(\Phi) = \frac{\partial_T f}{1 - s\kappa} \) almost everywhere. In particular, \( \dot{\partial}_s \tilde{f}, \frac{\partial_T f}{1 - s\kappa} \in W^{1,2}_{\text{loc}}(M_{s^2}) \). (Note that this does not imply \( \partial_T f \in W^{1,2}_{\text{loc}} \) and in particular, in general we cannot conclude that \( f \in W^{2,2}_{\text{loc}} \).) From the above equalities we see:

\[ \begin{align*}
(\partial_N \partial_N f)(\Phi) &= \partial_s \partial_s \tilde{f} \\
(\partial_N \partial_T f)(\Phi) &= \partial_s \left( \frac{\partial_T \tilde{f}}{1 - s\kappa} \right) \\
(\partial_T \partial_T f)(\Phi) &= \frac{1}{1 - s\kappa} \partial_t \left( \frac{\partial_T \tilde{f}}{1 - s\kappa} \right)
\end{align*} \]

On the other hand, the following formulae are easily seen to be satisfied almost everywhere on \([\Gamma] \):

\[ (T \otimes T) : \nabla^2 f = \partial_T \partial_T f - \frac{\kappa(t_x)}{1 - s\kappa(t_x)} \partial_N f. \tag{35} \]

\[ (T \otimes N) : \nabla^2 f = (N \otimes T) : \nabla^2 f = \partial_N \partial_T f \tag{36} \]

\[ (N \otimes N) : \nabla^2 f = \partial_N \partial_N f. \tag{37} \]

To prove e.g. (35) note that

\[ (T \otimes T) : \nabla^2 f = T_i T_j \partial_i \partial_j f = T_i \partial_i (T_j \partial_j f) - (\partial_T T_j) \partial_j f. \]

Since

\[ (\partial_T T)(\Phi) = \frac{\partial_T \tilde{T}}{1 - s\kappa} = \frac{\Gamma'}{1 - s\kappa} = \frac{\kappa}{1 - s\kappa} N, \]

we obtain (35). The other equations follow from similar computations. Since \( N, T \) form an orthonormal basis of \( \mathbb{R}^2 \), we have

\[ \nabla^2 f = \sum_{a, b \in \{T, N\}} [(a \otimes b) : \nabla^2 f] (a \otimes b). \]

Hence (29) follows from (35) through (37). And inserting the earlier computations into (29), we find (30).
Now assume that $f \in W^{2,2}([\Gamma])$ and suppose there are $p, q$ satisfying (31) and let $\varepsilon > 0$ be smaller than $\inf |s^-| (0, T)$ and smaller than $\inf s^+ (0, T)$, and set $\tilde{M} = (-\varepsilon, \varepsilon) \times (0, T)$. Applying (26) with this $\tilde{M}$ and recalling (11), we see that $\tilde{f} \in W^{1,2}(\tilde{M})$. Hence also $(s, t) \mapsto \tilde{f}(-s, t) \in W^{1,2}(\tilde{M})$. Since

$$p(t) = \frac{1}{2} \left( \tilde{f}(s, t) + \tilde{f}(-s, t) \right),$$

we conclude that $p \in W^{1,2}(0, T)$. Since $\tilde{f}(s, \cdot) \in W^{1,2}(0, T)$ for almost every $s \in (-\varepsilon, \varepsilon)$, we deduce that $q \in W^{1,2}(0, T)$.

Equations (32), (33) follow at once from (30). For instance, (30) implies that the left-hand sides of (33) equal

$$\partial_s \left( \frac{\partial_1 \tilde{f}}{1 - sk} \right) = \partial_s \left( \frac{p' + sq'}{1 - sk} \right) = \frac{q'}{1 - sk} + \frac{\kappa}{(1 - sk)^2} (p' + sq'),$$

which is the right-hand side of (33).

Now assume, in addition, that $\kappa \in W^{1,\infty}$ and suppose that $f \in W^{2,2}([\Gamma])$ and let $\tilde{M}$ as above. Lemma 2.5 implies that

$$\Phi^{-1} \in W^{2,\infty} \left( \Phi(\tilde{M}) \right).$$

Lemma 2.3 thus implies that $\tilde{f} \in W^{2,2}(\tilde{M})$. If $p$ and $q$ are as in the hypothesis, then $p, q \in W^{1,2}(0, T)$ by part (ii). Since $\tilde{f} \in W^{2,2}(\tilde{M})$ we also have

$$p' + sq' = \partial_1 \tilde{f} \in W^{1,2}(\tilde{M}).$$

Arguing as in the proof of part (ii) we deduce from this that $p' \in W^{1,2}(0, T)$ and then that $q' \in W^{1,2}(0, T)$. Finally, note that equation (34) follows at once from (30).

Lemma 2.9 **Suppose that**

$$\kappa \in W^{1,\infty}(0, T) \text{ and } \inf_{(s, t) \in M_s \pm} (1 - sk(t)) > 0. \quad (38)$$

**Then:**

(i) We have

$$f \in W^{2,\infty}([\Gamma]) \iff \tilde{f} \in W^{2,\infty}(M_s \pm). \quad (39)$$

(ii) If $f : [\Gamma] \to \mathbb{R}$ and $p, q : (0, T) \to \mathbb{R}$ are such that

$$\tilde{f}(s, t) = p(t) + sq(t) \text{ for all } (s, t) \in M_s \pm,$$

then

$$f \in W^{2,\infty}([\Gamma]) \iff p, q \in W^{2,\infty}(0, T).$$

If the latter are satisfied and if we define

$$A(s, t) = \tilde{H}(s, t) \partial_1 \left( \frac{1}{1 - sk} \right) \quad (40)$$

$$B(s, t) = \frac{\tilde{H}(s, t)}{1 - sk} = \frac{\mu}{2(1 - sk)^2}, \quad (41)$$

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then
\[
\frac{1}{2} \int_{\Gamma} h(x) : (\nabla^2 f)(x) \, dx = \\
= \int_0^T \left( \int_{s^-}^{s^+} A(s, t) \, ds \right) p' + \left( \int_{s^-}^{s^+} sA(s, t) \, ds \right) q' \, dt \\
+ \int_0^T \left( \int_{s^-}^{s^+} B(s, t) \, ds \right) p'' + \left( \int_{s^-}^{s^+} sB(s, t) \, ds \right) q'' \, dt \\
- \int_0^T \left( \int_{s^-}^{s^+} H \, ds \right) \kappa q \, dt. \tag{42}
\]

**Proof.** By (38), Lemma 2.5 implies that \( \Phi \in W^{2,\infty}(M_{s\pm}, \mathbb{R}^2) \) and \( \Phi^{-1} \in W^{2,\infty}([\Gamma], \mathbb{R}^2) \). Hence (39) follows from Remark 2.4.

To prove (ii), suppose that \( f \in W^{2,\infty}([\Gamma]) \). Then (39) implies \( \tilde{f} \in W^{2,\infty}([\Gamma]) \).

Hence \( p' + sq' = \partial_t \tilde{f} \) is Lipschitz on \( M_{s\pm} \).

Hence \( p' \) is Lipschitz, hence also \( q' \) is Lipschitz. Conversely, if \( p, q \in W^{2,\infty} \) then clearly \( \tilde{f} = p + sq \) belongs to \( W^{2,\infty}(M_{s\pm}) \), so \( f \in W^{2,\infty}([\Gamma]) \) by (39).

To prove (42), note that by (34), by (16) and by (15):
\[
\frac{1}{2} \int_{\Gamma} h(x) : (\nabla^2 f)(x) \, dx = \int H(T \otimes T) : \nabla^2 f \, dx \\
= \int_{M_{s\pm}} \tilde{H} \left\{ \partial_t \left( \frac{1}{1 - sk} \right) (p' + sq') + \frac{pq'' + q''}{1 - sk} - \kappa q \right\} \, dsdt \\
= \int_{M_{s\pm}} A(s, t)(p'(t) + sq'(t)) + B(s, t)(p''(t) + sq''(t)) - \kappa(t)\tilde{H}(s, t)q(t) \, dsdt
\]

Hence (42) follows from Fubini’s Theorem. \( \square \)

## 3 Structure of infinitesimal bendings

The following general definition of infinitesimal bendings was adopted in [7]:

**Definition 3.1** For a given immersion \( u \in W^{2,2}(S, \mathbb{R}^3) \) a map \( \tau \in L^2(S, \mathbb{R}^3) \) is an infinitesimal bending of \( u \) provided that
\[
\partial_i(\tau \cdot \partial_j u) + \partial_j(\tau \cdot \partial_i u) = 2\tau \cdot \partial_i \partial_j u \text{ in } D'(S) \text{ for } i, j = 1, 2.
\]

It is immediate from this definition that a map \( \tau \in L^2(S, \mathbb{R}^3) \) is an infinitesimal bending of \( u \in W^{2,2}_\delta(S) \) if and only if \( V := \tau \cdot \nabla u \) and \( \Psi := \tau \cdot n \) satisfy
\[
\text{sym } \nabla V = \Psi h \tag{43}
\]
in the distributional sense. Following [7], we adapt the notion of bendings from geometry to our setting:
\textbf{Definition 3.2} A bending of a deformation \( u \in W^{2,2}_s(S) \) is a strongly \( W^{2,2}_s \)-continuous one-parameter family \( \{u_t\}_{t \in (-1,1)} \subset W^{2,2}_s(S) \) satisfying \( u_0 = u \) and which is such that the weak \( W^{2,2} \)-limit
\[
\tau = \lim_{t \to 0} \frac{1}{t} (u_t - u_0) \tag{44}
\]
eexists. The vector field \( \tau \) is called the \textit{infinitesimal bending field} induced by the bending \( \{u_t\}_{t \in (-1,1)} \).

An infinitesimal bending \( \tau \) is said to be \textit{continuable} if it is induced by some bending.

### 3.1 Structure of infinitesimal bendings in general domains

The following lemma shows that infinitesimal bendings are essentially fully determined by their normal component. It will be used in Section 4.

\textbf{Lemma 3.3} Let \( u \in W^{2,2}_s(S) \) and let \( \tau^{(1)}, \tau^{(2)} \in L^2(S, \mathbb{R}^3) \) be infinitesimal bendings of \( u \) with \( n \cdot \tau^{(1)} = n \cdot \tau^{(2)} \) almost everywhere on \( S \). Then there exist \( k \in \mathbb{R}^2 \) and \( \lambda \in \mathbb{R} \) such that
\[
\tau^{(1)}(x) - \tau^{(2)}(x) = \langle k + \lambda x^\perp, \nabla u(x) \rangle \quad \text{for almost every } x \in S.
\]

\textbf{Proof.} Set \( \tau = \tau^{(1)} - \tau^{(2)} \), which by linearity is again an infinitesimal bending of \( u \), and we have \( n \cdot \tau = 0 \). From (43) and Korn’s inequality we deduce that \( V = \tau \cdot \nabla u \in L^2 \) is in fact a skew affine map from \( \mathbb{R}^2 \) into \( \mathbb{R}^2 \), i.e., there exist \( k \in \mathbb{R}^2 \) and \( \lambda \in \mathbb{R} \) such that \( V(x) = k + \lambda x^\perp \). \( \square \)

\textbf{Lemma 3.4} Let \( \psi \in W^{2,2}_s(S) \) and let \( B \in L^2(S, \mathbb{R}^{2 \times 2}_{\text{sym}}) \) be such that \( \text{curl} \, B = 0 \) in the sense of distributions on \( S \). Then
\[
\text{curl}(\text{curl}(\psi B)) = \nabla^2 \psi : \text{cof} \, B \quad \text{in the sense of distributions on } S. \tag{45}
\]

\textbf{Proof.} If \( D \subset S \) is a simply connected subdomain, then there exists \( f \in W^{2,2}(D) \) such that \( B = \nabla^2 f \) almost everywhere on \( D \). We claim that
\[
\text{curl}(\text{curl}(\psi \nabla^2 f)) = \nabla^2 \psi : \text{cof} \, \nabla^2 f \quad \text{in the sense of distributions on } D. \tag{46}
\]

In fact, (46) is easily verified if \( \psi, f \) are smooth, and the general case follows by approximating \( f \) and \( \psi \) by smooth functions.

So we have shown that (45) holds on every simply connected subdomain \( D \) of \( S \). Covering the support of an arbitrary test function on \( S \) by finitely many balls contained in \( S \) and using a partition of unity subordinate to this covering, the claim follows. \( \square \)

\textbf{Proposition 3.5} Let \( u \in W^{2,2}_s(S) \) and let \( a \) be as in the conclusion of Lemma 2.1. Let \( \tau \in L^2(S, \mathbb{R}^3) \) be an infinitesimal bending of \( u \) with \( \Psi := n \cdot \tau \in W^{2,2}(S) \). Then \( \tau \in (W^{1,2} : W^{1,2})(S, \mathbb{R}^3) \) and
\[
(a^\perp \otimes a^\perp) : \nabla^2 \Psi = 0 \quad \text{almost everywhere in } S \tag{47}
\]

Conversely, if \( S \) is simply connected and if \( \Psi \in W^{2,2}(S) \) satisfies (47) then there exists \( V \in W^{1,2}(S, \mathbb{R}^2) \) (namely the solution \( V \in L^2 \) of (43), which is unique up to addition of a skew affine map) such that \( \tau := \Psi n + \partial_V u \) is an infinitesimal bending of \( u \).
Proof. Under slightly stronger hypotheses, this result is implicit in [8]. As noted above, $\tau$ is an infinitesimal bending of $u$ precisely if (43) is satisfied. But the left-hand side of (43) is a symmetrized gradient. Hence
\begin{equation}
\text{curl(curl}(\Psi h)) = 0 \text{ in } \mathcal{D}'(S).
\end{equation}
By Lemma 2.1 we know that $h$ is weakly curl-free. Hence we can apply (45) to deduce
\begin{equation}
(\text{cof } h) : \nabla^2 \Psi = 0 \text{ almost everywhere on } S.
\end{equation}
Since $\text{cof } h = a^\perp \otimes a^\perp$, this implies (47).

Conversely, if $\Psi \in W^{2,2}$ is given and (47) is satisfied, then (49) holds, hence (48) holds, hence (using that $S$ is simply connected) there exists $V \in W^{1,2}$ such that (43) is satisfied, and $V$ is unique (up to a skew affine map) among all $L^2$ distributional solutions of (43). This follows from Korn’s inequality as stated in Proposition 1 of [2]. Clearly $\tau = \partial_V u + \Psi n$ is an infinitesimal bending of $u$.

\[ \square \]

3.2 Structure of infinitesimal bendings in line of curvature charts

The objects $\Gamma$, $N$, $T$ and $s^\pm$ continue to have the same properties as in the previous section. In addition, from now on (unless stated otherwise) $u$ denotes the immersion in $W^{2,2}_{\delta}([\Gamma])$ induced by some function $\mu \in L^2(0,T)$ and by $s^\pm$ and $\Gamma$ via Proposition 2.2. That is, $\Phi$ is a global line of curvature chart for $u$.

Lemma 3.6 The following are true:

(i) If $u \in W^{3,\infty}_\delta([\Gamma])$ then $\mu \in W^{1,\infty}(0,T)$.

(ii) If $k \geq 3$ is an integer and
\begin{equation}
\begin{align*}
\mu &\in W^{k,\infty}_\delta([\Gamma]) \text{ and } H \neq 0 \text{ on } \overline{\Gamma}, \\
\kappa &\in W^{k-2,\infty}(0,T) \text{ with } \mu \neq 0 \text{ on } [0,T], \text{ and} \\
\inf_{(s,t) \in M_{s^\pm}} (1 - s\kappa(t)) &> 0.
\end{align*}
\end{equation}

In particular, $\Phi \in W^{2,\infty}(M_{s^\pm}, \mathbb{R}^2)$ and $\Phi^{-1} \in W^{2,\infty}([\Gamma], \mathbb{R}^2)$.

Proof. We refer to [8, Lemma 2.4] for a similar result. The hypotheses imply that $h = n \cdot \nabla^2 u$ is Lipschitz on $[\Gamma]$. Hence $H = \frac{\partial_n \Delta}{\kappa}$ is Lipschitz, too. Since $\Phi$ is Lipschitz on $M_{s^\pm}$, we conclude that
\begin{equation}
\frac{\mu}{1 - s\kappa} = 2\bar{H}
\end{equation}
is Lipschitz on $M_{s^\pm}$. Thus $\mu = 2\bar{H}(0,\cdot)$ is Lipschitz on $(0,T)$. This proves the first part.

For the second part, assume first that $k = 3$. We can write (52) in the form
\begin{equation}
s_0\kappa = 1 - \frac{\mu}{2\bar{H}}
\end{equation}
where $s_0 > 0$ is some constant that is smaller that $\inf s^+$ and $\inf |s^-|$. Thus $\kappa$ is Lipschitz because by continuity and the hypotheses $\dot{H}$ is bounded away from zero. We also have $1 - s_0 \kappa \geq c > 0$ by the choice of $s_0$, so (52) and $\dot{H} \neq 0$ imply that $\mu \neq 0$ on $[0, T]$. Hence (51) follows from (52).

Now let $k \geq 4$ and let us argue inductively. By the induction hypothesis we have $\kappa \in W^{k-3, \infty}$, hence Lemma 2.5 implies that $\Phi \in W^{k-2, \infty}$. Since $H \in W^{k-2, \infty}$, this implies that $\dot{H} \in W^{k-2, \infty}$. Thus (52) implies that $\mu \in W^{k-2, \infty}$. Hence $\kappa \in W^{k-2, \infty}$ by (53). By this and by (51), Lemma 2.5 implies that $\Phi$ and $\Phi^{-1}$ belong to $W^{2, \infty}$.

**Proposition 3.7** Assume (50) is satisfied. If $\tau \in W^{2, \infty}(\Gamma, \mathbb{R}^3)$ is an infinitesimal bending of $u$ and we set $\Psi = \tau \cdot n$ and $V = \tau \cdot \partial_t u$, then $\Psi$, $V \in W^{2, \infty}(\Gamma)$ and there exist $p, q \in W^{2, \infty}(0, T)$ and constants $M_0 \in \mathbb{R}^2$, $L_0 \in \mathbb{R}$ such that

$$
\begin{align*}
\bar{\Psi}(s, t) &= p(t) + sq(t) \text{ for all } (s, t) \in M_{s \pm} \\
\bar{V}(s, t) &= M(t) + sL(t)T(t) \text{ for all } (s, t) \in M_{s \pm},
\end{align*}
$$

where we have introduced

$$
\begin{align*}
L(t) &= L_0 + \int_0^t q(\sigma) \mu(\sigma) \, d\sigma \\
M(t) &= M_0 + \int_0^t (\mu(\sigma)p(\sigma)T(\sigma) - L(\sigma)N(\sigma)) \, d\sigma.
\end{align*}
$$

**Proof.** By the Leibniz rule we see that $V, \Psi \in W^{2, \infty}$. Hence Proposition 3.5 and the facts that $H \neq 0$ and that $h = 2HT \otimes T$ imply that

$$(N \otimes N) : \nabla^2 \Psi = 0 \text{ almost everywhere on } [\Gamma].$$

Hence Lemma 2.7 implies that there exist $p, q \in (0, T) \to \mathbb{R}$ such that (54) is satisfied. (Formula (54) was in fact obtained under weaker hypotheses in [8, Theorem 4.1].) The regularity of $p$ and $q$ follows from Lemma 2.9.

By Definition 3.1 we know that $V$ satisfies (43) on $[\Gamma]$. But it is easy to check that $V$ as defined in (55) (with $L_0 = 0$ and $M_0 = 0$, say) is a solution of (43), see below. On the other hand, if $Q(x) = k - \lambda x^\perp$ is a skew affine map then

$$
\begin{align*}
\bar{Q}(s, t) &= k - \lambda (\Phi(s, t))^\perp = k - \lambda H^\perp(t) + s\lambda T(t).
\end{align*}
$$

Hence by uniqueness up to skew affine maps (which follows from Korn’s inequality) we see that (55) must be satisfied for suitable $M_0$ and $L_0$.

To check that $V$ defined by (55) indeed solves (43), note first that

$$
2 \text{sym} \nabla V = T \otimes \nabla T + \partial_T V \otimes T + N \otimes \partial_N V + \partial_N V \otimes N. \tag{58}
$$

In view of Lemma 2.6 and (55) we have

$$
\begin{align*}
\partial_T V &= \frac{\mu \cdot (p + sq)}{1 - s^2 \kappa} T - LN \\
\partial_N V &= LT.
\end{align*}
$$
Inserting these into (58) and dividing by 2 gives
\[
\text{(sym } \nabla V)\Phi = \frac{\mu}{1 - s\kappa} (T \otimes T)(p + sq) = h(\Phi)\widetilde{\Psi}.
\]

Now follows the converse of Proposition 3.7. It will be used to construct infinitesimal bendings and to derive the first variation under infinitesimal bendings, cf. Lemma 5.3.

**Proposition 3.8** Assume that \( u \in W^{3,\infty}_\delta([\Gamma]) \) and let \( p, q \in W^{2,\infty}(0, T) \). If \( \Psi \) is given via (54) and \( V \) is given via (55), then \( \Psi \in W^{2,\infty}([\Gamma]) \) and \( V \in W^{2,\infty}([\Gamma], \mathbb{R}^2) \), and \( \tau = \Psi n + \partial_V u \) is an infinitesimal bending of \( u \), and \( \tau \in W^{2,\infty}([\Gamma], \mathbb{R}^3) \).

Assume, moreover, that (38) is satisfied. Then the following are satisfied:

(i) Let \( A, B \) be given by (40), (41) and define
\[
E(t) = -\int_{t}^{T} \left[ \left( \int_{s^{-}(\sigma)}^{s^{+}(\sigma)} \partial_\ell(\widetilde{H}^2) \right) T(\sigma) \right. \\
+ \left. \left( \int_{s^{-}(\sigma)}^{s^{+}(\sigma)} (1 - s\kappa)\partial_\ell(\widetilde{H}^2) \right) N(\sigma) \right] d\sigma
\]
\[
D(t) = -\int_{t}^{T} \left[ \left( \int_{s^{-}(\sigma)}^{s^{+}(\sigma)} s\partial_\ell(\widetilde{H}^2) \right) + E(\sigma) \cdot N(\sigma) \right] d\sigma.
\]

Then we have
\[
\int_{[\Gamma]} H(\Delta \Psi + 4H^3 \Psi + \partial_V (H^2)) \, dx
\]
\[
= \int_{0}^{T} \left( \int_{s^{-}}^{s^{+}} A(s, t) \, ds \right) p' + \left( \int_{s^{-}}^{s^{+}} sA(s, t) \, ds \right) q' \, dt
\]
\[
+ \int_{0}^{T} \left( \int_{s^{-}}^{s^{+}} B(s, t) \, ds \right) p'' + \left( \int_{s^{-}}^{s^{+}} sB(s, t) \, ds \right) q'' \, dt
\]
\[
+ \int_{0}^{T} \left\{ 4 \left( \int_{s^{-}}^{s^{+}} s\widetilde{H}^3(1 - s\kappa) \, ds \right) - \mu D - \kappa \left( \int_{s^{-}}^{s^{+}} \widetilde{H} \, ds \right) \right\} q \, dt
\]
\[
- \left( \int_{0}^{T} \left( \int_{s^{-}}^{s^{+}} \widetilde{H}^3(1 - s\kappa) \, ds \right) - \mu(T \cdot E) \right) p \, dt
\]
\[
- E(0) \cdot M(0) - D(0)L(0).
\]

(ii) If
\[
p = q = p' = q' = 0 \text{ on } \{0, T\}
\]
and \( M(0) = 0, L(0) = 0 \) as well as
\[
\int_{0}^{T} q(t) \mu(t) \, dt = 0
\]
\[
\int_{0}^{T} \mu(t) \left( p(t)\Gamma'(t) + q(t)\Gamma^L(t) \right) \, dt = 0,
\]
then  

\[ (\tau, \nabla \tau) = 0 \text{ on } \partial_L[\Gamma]. \]  

(65)

(iii) Conversely, if (65) is satisfied, then (62), (63), (64) are satisfied, and  

\[ M(0) = 0 \text{ and } L(0) = 0. \]

Proof. If \( p, q \in W^{2,\infty}(0, T) \) are given and \( \Psi \) satisfies (54), then \( \Psi \in W^{2,\infty}(\Gamma) \) by Lemma 2.9, which we can apply because we are assuming (38). And if \( V \) is given by (55) then we can apply Lemma 2.9 to each of its components to see that \( V \in W^{2,\infty}(\Gamma, \mathbb{R}^2) \). Since by (54), (55) we know that \( V, \Psi \) satisfy the system (43), the map \( \tau = \Psi n + \partial_V u \) is an infinitesimal bending of \( u \). If \( u \in W^{3,\infty} \) then clearly \( \tau \in W^{2,\infty} \) because then it is a linear combination of products of maps belonging to \( W^{2,\infty} \).

From the definition of \( \Psi \) we see from (32) that \( (N \otimes N) : \nabla^2 \Psi = 0 \) everywhere on \( [\Gamma] \). Hence

\[ \Delta \Psi = (N \otimes N + T \otimes T) : \nabla^2 \Psi = (T \otimes T) : \nabla^2 \Psi, \]

so that \( h : \nabla^2 \Psi = 2 H \Delta \Psi \). As (38) is satisfied, we can apply Lemma 2.9 with \( f = \Psi \) to conclude that

\[ \int_{[\Gamma]} H \Delta \Psi \, dx = \frac{1}{2} \int_{[\Gamma]} h : \nabla^2 \Psi \, dx \]

equals the right-hand side of (42). Next, from (16) we see

\[ \int_{[\Gamma]} H^3 \Psi \, dx = \int_{[\Gamma]} \bar{H}^3(p + sq)(1 - sk) \, ds \, dt \]

\[ = \int_0^T \left( \int_{s^{-}}^{s^{+}} \bar{H}^3(1 - sk) \, ds \right) p(t) \, dt \]

\[ + \int_0^T \left( \int_{s^{-}}^{s^{+}} s \bar{H}^3(1 - sk) \, ds \right) q(t) \, dt. \]

Finally, we compute \( \partial_V(H^2) \). Generally, for Lipschitz continuous \( f : [\Gamma] \to \mathbb{R} \) we have \( \partial_V f = (V \cdot T) \partial_T f + (V \cdot N) \partial_N f \). Hence

\[ (\partial_V f)(\Phi) = (T \cdot M + sL) \frac{\partial \bar{f}}{1 - sk} + (N \cdot M) \partial_s \bar{f}. \]

Thus by (16) we see

\[ \int_{[\Gamma]} \partial_V f \, dx = \int_{M_{+}} (T \cdot M + sL) \partial_t \bar{f} + (1 - sk)(N \cdot M) \partial_s \bar{f} \, ds \, dt \]

\[ = \int_0^T \left\{ \left( \int_{s^{-}(t)}^{s^{+}(t)} \partial_t \bar{f} \, ds \right) T(t) + \left( \int_{s^{-}(t)}^{s^{+}(t)} (1 - sk) \partial_s \bar{f} \, ds \right) N(t) \right\} \cdot M(t) \, dt \]

\[ + \int_0^T \left\{ \int_{s^{-}(t)}^{s^{+}(t)} s \partial_s \bar{f} \, ds \right\} L(t) \, dt. \]
By partial integration, for all regular enough $G : [0, T] \to \mathbb{R}$ and $E : [0, T] \to \mathbb{R}^2$ we have
\[
\int_0^T L(t)G'(t) \, dt = L(T)G(T) - L(0)G(0) - \int_0^T G(t)\mu(t) \, q(t) \, dt
\]
\[
\int_0^T M(t) \cdot E'(t) \, dt = M(T) \cdot E(T) - M(0) \cdot E(0)
\]
\[
+ \int_0^T (E(t) \cdot N(t)) L(t) - (E(t) \cdot T(t))\mu(t)p(t) \, dt.
\]
Hence, for any primitive $D$ of $G' + E \cdot N$ we have
\[
\int_0^T M \cdot E' + LG' = [M \cdot E + LD]^T_0 - \int_0^T (E \cdot T)\mu p - \int_0^T D\mu q. \tag{66}
\]
Hence with $G' = \int_s^T \partial_s (\overline{H}^2) \, ds$ and with $E, D$ as in the hypotheses, we find from (66) (and since by definition $E(T) = 0$ and $D(T) = 0$):
\[
\int_{[T(0), T]} \partial_V (\overline{H}^2) \, dx = -\int_0^T (E \cdot T)\mu p - \int_0^T D\mu q - M(0) \cdot E(0) - L(0)D(0).
\]
Combining the above equalities, (61) follows.

Now suppose that the hypotheses of part (ii) are satisfied. Observe that (63), (64) ensure that $M(T) = 0$ and $L(T) = 0$ as well. Hence $(V, \Psi) = 0$ on $\partial_L[\Gamma]$, so that $\tau = 0$ on $\partial_L[\Gamma]$. By (62),
\[
\langle \nabla \Psi \rangle (\Phi(s, t)) = \frac{p'(t) + sq'(t)}{1 - sq(t)} T(t) + q(t) N(t) \tag{67}
\]
is zero whenever $t \in \{0, T\}$, so $\nabla \Psi = 0$ on $\partial_L[\Gamma]$. We have
\[
\langle \nabla V \rangle (\Phi(s, t)) = \frac{M'(t) + s(LT)'(t)}{1 - sq(t)} \otimes T(t) + L(t)T(t) \otimes N(t) \tag{68}
\]
As the hypotheses imply that $L = L' = 0$ and $M = M' = 0$ on $\{0, T\}$, we conclude that $\nabla V = 0$ on $\partial_L[\Gamma]$. Since
\[
\partial_j \tau = (\partial_j \Psi + h_{ji} V_i) u + (-h_{ij} \Psi + \partial_j V_i) \partial_i u, \tag{69}
\]
it follows that $\nabla \tau = 0$ on $\partial_L[\Gamma]$.

In order to prove the converse, assume that $\tau$ satisfies (65). Since $\tau = 0$ on $[\Gamma(0)]$, also $\Psi$ vanishes on this set, hence $p(0) + sq(0) = 0$ for all $s \in (s^-(0), s^+(0))$. Thus $p = q = 0$ at 0. The same argument applies at $T$. It also applies to $V$ instead of $\Psi$, giving $M(0) = M(T) = 0$ and $L(T) = L(0) = 0$. Since $\nabla \tau$ as well as $V$ vanish on $[\Gamma(0)]$, we have by (69) that $\nabla \Psi$ vanishes on this set, too. Hence (67), (68) together with (38) imply that indeed $p' = q' = 0$ at 0. A similar reasoning applies at $T$. \qed
4 Continuation of infinitesimal bendings

In this section we continue to assume $\Gamma \in W^{2,\infty}((0,T),\mathbb{R}^2)$ and $s^\pm \in W^{1,\infty}(0,T)$ with $s^- < 0 < s^+$ on $[0,T]$. We let $\mu \in L^\infty(0,T)$ and as before we let $u \in W^{2,\infty}_0(\Gamma]$ be induced from $s^\pm$, $\Gamma$ and $\mu$ via Proposition 2.2. Hence

$$\bar{u} = \gamma + sv.$$ 

Figures and detailed explanations about the techniques applied in this section can be found in [5], [4], [6]. The purpose of this section is to prove the following result:

**Theorem 4.1** Let $\Gamma \in W^{3,\infty}((0,T)$ be parametrized by arclength, let $s^\pm \in W^{1,\infty}(0,T)$ be such that $s^- < 0 < s^+$ on $[0,T]$, assume that

$$\inf_{(s,t) \in M_s^\pm} (1 - sk(t)) > 0,$$ 

(70)

and assume that $u \in W^{3,\infty}_0([\Gamma])$ is induced by $\Gamma$ and $s^\pm$ via Proposition 2.2. Let $P, Q \in W^{2,\infty}(0,T)$, set

$$p(t) = \int_0^t \mu P$$

$$q(t) = \int_0^t \mu Q,$$ 

and assume that

$$p = P = q = Q = 0 \text{ on } \{0,T\},$$ 

(71)

Introduce

$$L(t) = \int_0^t q(r)\mu(r) \, dr$$

(72)

$$M(t) = \int_0^t (\mu(r)p(r)T(r) - L(r)N(r)) \, dr,$$ 

(73)

define $V : [\Gamma] \to \mathbb{R}^2$ and $\Psi : [\Gamma] \to \mathbb{R}$ by setting

$$\bar{V}(s,t) = p(t) + sq(t) \text{ for all } (s,t) \in M_s^\pm$$

(74)

$$\bar{V}(s,t) = L(t)\Gamma'(t) + sM(t) \text{ for all } (s,t) \in M_s^\pm,$$ 

(75)

and set

$$\tau = \Psi n + \partial_{\nu} u.$$ 

(76)

Assume that $P, Q$ are such that $M(T) = L(T) = 0$. Then $\tau \in W^{2,\infty}([\Gamma],\mathbb{R}^3)$ is a $W^{2,\infty}$-continuable infinitesimal bending of $u$.

More precisely: there exists $\varepsilon_0 > 0$ and a $W^{2,2}$-continuous 1-parameter family

$$\{\overline{a}_\varepsilon\} \in [-\varepsilon_0,\varepsilon_0] \subset W^{2,\infty}_0([\Gamma])$$

such that

$$\frac{1}{\varepsilon} (\overline{a}_\varepsilon - u) \rightharpoonup \tau \text{ weakly-* in } W^{2,\infty}([\Gamma],\mathbb{R}^3),$$ 

(77)

and

$$(\overline{a}_\varepsilon, \nabla \overline{a}_\varepsilon) = (u, \nabla u) \text{ on } \partial_L[\Gamma]$$

(78)

for all $\varepsilon \in [-\varepsilon_0,\varepsilon_0]$. 

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Remarks.

(i) For $\varepsilon \neq 0$ the map $\pi_\varepsilon$ is, in general, not induced by $\Gamma$ (i.e. $\Gamma$ is not a line of curvature for $\pi_\varepsilon$). It is, however, a well-defined isometric immersion into $\mathbb{R}^3$ of the domain $[\Gamma]$ parametrized by $\Gamma$ and $s^\pm$ (endowed with the standard flat metric).

(ii) The hypotheses on $p$ and $q$ made in Theorem 4.1 are formulated in a way that allows for poor information about $\mu$. If we know that $\mu \in W^{2,\infty}(0, T)$ satisfies $\mu \neq 0$ on $[0, T]$, then all $p, q \in W^{3,\infty}$ with

$$p(t) = q(t) = p'(t) = q'(t) = 0 \text{ for } t \in \{0, T\}$$

satisfy the hypotheses of Theorem 4.1; simply set $Q = \frac{q'}{p'}$ and $P = \frac{p'}{p}$.

(iii) The assumption that $u$ is induced by $\Gamma$ and by Lipschitz continuous functions $s^\pm$ is natural, see for instance [5, Lemma 2.2].

(iv) By Lemma 3.6, the hypothesis (70) is satisfied, e.g., if $H \neq 0$ on $\overline{[\Gamma]}$. This latter assumption on the mean curvature is indeed quite natural, for the following reasons:

- One of the main results in [5] is that for minimizers the mean curvature vanishes only on finitely many line segments (within one patch $\overline{[\Gamma]}$).
- If $H = 0$ on an open set, then $u$ is affine on this set. But infinitesimal bendings of a plane are in general not continuable.

(v) Theorem 4.1 should be seen together with Proposition 3.7, as the latter shows that every infinitesimal bending of $u$ is of the form (74), (75), (76) for suitable $p, q$.

(vi) It is crucial that the first order boundary conditions on the infinitesimal bending $\tau$ (expressed by (71)) can actually be preserved by the inducing family of bendings $\{\pi_\varepsilon\}$, i.e., that (78) holds.

(vii) In order to get a stronger convergence than (77) one would need higher regularity of $u$ and $\tau$. In contrast, obtaining a bending $\{\pi_\varepsilon\}$ with smooth maps $\pi_\varepsilon$ could be achieved by applying the arguments from [4], see for instance [4, Lemma 6]. Similar results as in Theorem 4.1 can be derived by the same arguments also under stronger regularity assumptions on the data, leading to higher regularity of the bendings and a stronger topology for the convergence.

We can formulate Theorem 4.1 without any explicit reference to the developable structure of flat immersions (except assuming the existence of a global line of curvature chart):

**Corollary 4.2** If $u \in W^{4,\infty}_k([\Gamma])$ satisfies $H \neq 0$ everywhere on $\overline{[\Gamma]}$, then every infinitesimal bending $\tau \in W^{3,\infty}([\Gamma], \mathbb{R}^3)$ of $u$ with $(\tau, \nabla \tau) = 0$ on $\partial_L[\Gamma]$ is $W^{2,\infty}$-continuable by bendings preserving first order boundary conditions on $\partial_L[\Gamma]$. 

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Proof. Since \( u \in W^{4,\infty} \) and \( H \neq 0 \) on \( \Gamma \), Lemma 3.6 implies that \( \kappa, \mu \in W^{2,\infty}(0,T) \) with \( \mu \neq 0 \) on \( [0,T] \) and such that (70) holds.

Defining \( \Psi = \tau \cdot n \) and \( V = \tau \cdot \nabla u \), and applying Proposition 3.7, we see that there exist \( p, q \in W^{2,\infty}(0,T) \) and constants \( M_0, L_0 \) such that (54), (55) hold with \( L, M \) given by (56), (57). Since \( \tau \in W^{2,\infty} \) and \( (\tau, \nabla \tau) = 0 \) on \( \partial L[\Gamma] \), Proposition 3.8 implies that \( M_0 = 0, L_0 = 0 \) and that (71) is satisfied (we set \( P = \frac{\nu}{\mu} \) and \( Q = \frac{\eta}{\mu} \)), and that \( M(T) = 0 \) and \( L(T) = 0 \).

Since in fact \( u \in W^{3,\infty} \) and \( n \in W^{3,\infty} \), we see that \( \Phi \in W^{3,\infty} \). Arguing as usual, we deduce from this that \( p, q \in W^{3,\infty} \).

Hence all hypotheses of Theorem 4.1 are satisfied. The claim follows from this theorem. \( \square \)

4.1 Bendings via the framed curve approach

We use the technology developed in [5], [4] to construct bendings. For simplicity, we assume throughout that \( 1 - s \kappa \) is uniformly bounded from below on \( M_\varepsilon \) by a positive constant. Let \( \varepsilon_0 > 0 \) and let

\[
\{ \psi_\varepsilon \}_{\varepsilon \in (-\varepsilon_0,\varepsilon_0)} \subset L^\infty((0,T),\mathbb{R}^3)
\]

be a continuous 1-parameter family. With it we associate the 1-parameter families

\[
Y_\varepsilon, \hat{R}_\varepsilon, \hat{\Gamma}_\varepsilon, \Phi_\varepsilon, \Gamma_\varepsilon, ...
\]

defined as follows:

Define \( \hat{\kappa}_\varepsilon = \kappa + (\psi_\varepsilon)_1 \) and \( \hat{\kappa}_\varepsilon(t) = \int_0^t \hat{\kappa}_\varepsilon \), and set \( \hat{\mu}_\varepsilon = \mu + (\psi_\varepsilon)_2 \), and define

\[
\hat{R}_\varepsilon(t) = \begin{pmatrix} e^{i\hat{\kappa}_\varepsilon(t)} & 0 \\ 0 & e^{i\hat{\mu}_\varepsilon(t)} \end{pmatrix}.
\]

Define \( \hat{\Gamma}_\varepsilon \) by integration of the first row of \( \hat{R}_\varepsilon \). Define the vector field \( Y_\varepsilon : (0,T) \rightarrow \mathbb{R}^2 \) by

\[
Y_\varepsilon(t) = \int_0^t (\psi_\varepsilon)_3(\hat{\Gamma}_\varepsilon)'.
\]

The Darboux frame \( r = (\gamma' \mid v \mid n)^T \) solves the ODE system

\[
r' = \begin{pmatrix} 0 & \kappa & \mu \\ -\kappa & 0 & 0 \\ -\mu & 0 & 0 \end{pmatrix} r.
\]

By definition, the frame \( \hat{\gamma}_\varepsilon \) solves the analogous system with \( \kappa \) and \( \mu \) replaced by \( \hat{\kappa}_\varepsilon \) and \( \hat{\mu}_\varepsilon \), respectively, and with initial value \( \hat{\gamma}_\varepsilon(0) = r(0) \). Then define

\[
(\hat{\gamma}_\varepsilon' \mid \hat{\varepsilon}_\varepsilon \mid \hat{\kappa}_\varepsilon)^T := \hat{\gamma}_\varepsilon,
\]

and define \( \hat{\gamma}_\varepsilon \) by integration.

Define

\[
\sigma_\varepsilon(t) = \int_0^t (1 + (\psi_\varepsilon)_3)
\]
For uniformly small $\psi_\varepsilon$ this defines a homeomorphism of $(0, T)$ onto $(0, \sigma_\varepsilon(T))$. Hence we can define 

$$\tilde{R}_\varepsilon(\sigma_\varepsilon(t)) := \tilde{R}_\varepsilon(t) \text{ and } \sigma_\varepsilon(t) := \tilde{\sigma}_\varepsilon(t)$$

and set $(\sigma_\varepsilon, \pi_\varepsilon) = \sigma_\varepsilon$. We introduce the Lipschitz continuous functions

$$\tilde{\pi}_\varepsilon^\pm(t) = \pm \nu_{[\Gamma]} (\tilde{\Gamma}_\varepsilon(t), \pm \tilde{N}_\varepsilon(t)),$$

where $\nu_{[\Gamma]}$ is as defined in (5) for $S = [\Gamma]$. We introduce

$$M_{\pi_\varepsilon} := \{(s, t) \in \mathbb{R} \times (0, \sigma_\varepsilon(T)) : s \in (\sigma_\varepsilon^-(t), \sigma_\varepsilon^+(t))\}$$

and the maps

$$\Phi_\varepsilon(s, t) = \Gamma_\varepsilon(t) + s \tilde{N}_\varepsilon(t) \text{ for all } (s, t) \in M_{\pi_\varepsilon}$$

and the domains

$$[\tilde{\Gamma}_\varepsilon] = [\Gamma_\varepsilon(0, \sigma_\varepsilon(T))] := \Phi_\varepsilon \left( M_{\pi_\varepsilon} \right).$$

For $i = 1, ..., 4$ define the functionals $G_i$ on $L^\infty((0, T), \mathbb{R}^3)$ by setting

$$G_1(\psi_\varepsilon) = \tilde{\gamma}_\varepsilon(T) + y_\varepsilon(T) - \left(v(T) \otimes N(T)\right)(\tilde{\Gamma}_\varepsilon(T) + Y_\varepsilon(T)).$$

$$G_2(\psi_\varepsilon) = \left(\tilde{\pi}_\varepsilon(T) \cdot v(T)\right) \gamma'(T) - \left(\tilde{\gamma}_\varepsilon(T) \cdot n(T)\right) v(T) + \left(\tilde{\gamma}_\varepsilon(T) \cdot v(T)\right) n(T)$$

$$G_3(\psi_\varepsilon) = \tilde{\Gamma}_\varepsilon(T) + Y_\varepsilon(T) \cdot \Gamma'(T)$$

$$G_4(\psi_\varepsilon) = \int_0^T (\psi_\varepsilon)_1(t) \ dt$$

By [5, Lemma 3.11], if $\psi_\varepsilon$ is small enough in $L^\infty(0, T)$ and if $G_4(\psi_\varepsilon) = 0$ and $G_3(\psi_\varepsilon) = G_3(0)$, then the framed curve $(\tilde{\Gamma}_\varepsilon, \tilde{R}_\varepsilon)$ satisfies the endpoint conditions

$$\tilde{\Gamma}_\varepsilon(\sigma_\varepsilon(T)) - \Gamma(T) \parallel N(T)$$

$$\tilde{R}_\varepsilon(\sigma_\varepsilon(T)) = R(T).$$

And as shown in [5] this ensures that the domain is kept invariant, i.e.,

$$[\tilde{\Gamma}_\varepsilon(0, \sigma_\varepsilon(T))] = [\Gamma(0, T)].$$

This will be crucial in Proposition 4.6, because the modified surfaces $\pi_\varepsilon$ introduced there are defined on $[\Phi_\varepsilon(0, \sigma_\varepsilon(T))]$, but we need isometric immersions defined on $[\Gamma(0, T)]$.

Note that since the family $\{\psi_\varepsilon\}$ was arbitrary, the above definitions in fact determine functionals on $L^\infty((0, T), \mathbb{R}^3)$. We have chosen the above notation in order to avoid cumbersome expressions and because we will always consider these functionals along such a given family $\{\psi_\varepsilon\}$.
For given \( \psi \in L^\infty((0, T), \mathbb{R}^3) \) define

\[
\xi = \dot{\psi}_1 n - \dot{\psi}_2 v \quad \text{and} \quad \Xi(t) = \int_0^t \xi,
\]  
(79)

\[
\dot{R}(t) = \left( \int_0^t \dot{\psi}_1 \right) N
\]  
(80)

\[
\dot{r} = (\Xi \wedge \gamma' \mid \Xi \wedge v) \Xi \wedge n^T
\]  
(81)

\[
\dot{Y}(t) = \int_0^t \dot{\psi}_3 \Gamma'
\]  
(82)

\[
\dot{y}(t) = \int_0^t \dot{\psi}_3 \gamma'.
\]  
(83)

We define \((\hat{\Gamma}', \hat{N})^T := \hat{R} \quad \text{and} \quad (\hat{\gamma}', \hat{v}, \hat{n})^T := \hat{r} \). It is easy to see (cf. [5]) that the functionals \( \mathcal{G}_i \) are continuously Fréchet differentiable in a neighbourhood of the origin. Their linearizations are given by (cf. [5, Section 3.4] and also Lemma 4.3 below):

\[
\mathcal{G}_1(\psi) = \dot{\gamma}(T) + \dot{y}(T) - \left( v(T) \otimes N(T) \right) \left( \hat{\Gamma}(T) + \dot{Y}(T) \right),
\]  
(84)

\[
\mathcal{G}_2(\psi) = \int_0^T (\dot{\psi}_1 n - \dot{\psi}_2 v),
\]  
(85)

\[
\mathcal{G}_3(\psi) = \left( \hat{\Gamma}(T) + \dot{Y}(T) \right) \cdot \Gamma'(T),
\]  
(86)

and obviously

\[
\mathcal{G}_4(\psi) = \int_0^T \dot{\psi}_1.
\]  
(87)

Define

\[
\beta_\varepsilon : \mathbb{R} \times (0, T) \to \mathbb{R} \times (0, \sigma_\varepsilon(T)) \quad (s, t) \mapsto (s, \sigma_\varepsilon(t))
\]  
(88)

and define

\[
Q_\varepsilon = \beta_\varepsilon^{-1} \circ \mathcal{F}_\varepsilon^{-1}.
\]  
(89)

Then

\[
Q_\varepsilon : [\Gamma_\varepsilon] \to \mathbb{R}^2 \quad \hat{\Gamma}(t) + Y(t) + s\hat{N}(t) \mapsto (s, t).
\]

**Lemma 4.3** Suppose \( \kappa, \mu \in W^{1,\infty}(0, T) \) and let \( \{\psi_\varepsilon\}_{\varepsilon \in (0, 1)} \subset W^{1,\infty}((0, T), \mathbb{R}^3) \) be a strongly \( C^1 \) family with \( \psi_0 = 0 \). Define \( \dot{\psi} \) by the condition

\[
\frac{\dot{\psi}_\varepsilon}{\varepsilon} \to \dot{\psi} \quad \text{pointwise on} \quad (0, T)
\]

and obviously

\[
\dot{\psi}_1 = \dot{\psi}_1.
\]  
(87)
and make the definitions (79) through (83). Then, as $\varepsilon \to 0$, we have

\begin{align}
\frac{1}{\varepsilon} (\vec{R}_\varepsilon - \vec{R}) & \to \vec{R} \text{ weakly-* in } W^{2,\infty}(]0, T[; \mathbb{R}^{2\times 2}) \tag{90} \\
\frac{1}{\varepsilon} (\hat{r}_\varepsilon - r) & \to \hat{r} \text{ weakly-* in } W^{2,\infty}(]0, T[; \mathbb{R}^{3\times 3}) \tag{91} \\
\frac{\hat{Y}_\varepsilon}{\varepsilon} & \to \hat{Y} \text{ weakly-* in } W^{2,\infty}(]0, T[; \mathbb{R}^2) \tag{92} \\
\frac{\hat{y}_\varepsilon}{\varepsilon} & \to \hat{y} \text{ weakly-* in } W^{2,\infty}(]0, T[; \mathbb{R}) \tag{93}.
\end{align}

Assume, in addition, that $G_i(\psi_\varepsilon) = G_i(0)$ for $i = 3, 4$. Then

\begin{equation}
Q_\varepsilon \to \Phi^{-1} \text{ weakly-* in } W^{2,\infty}(\mathbb{R}; \mathbb{R}^2). \tag{94}
\end{equation}

**Proof.** We omit the index $\varepsilon$. To prove (90) we define $K(t) = \int_0^t \kappa$ and $\hat{K}(t) = \int_0^t \hat{\kappa}$. Recall that

\[ \hat{R}(t) = \begin{pmatrix} e^{i\hat{K}(t)} \\ ie^{i\hat{K}(t)} \end{pmatrix}, \]

and a similar equation for $R$. By definition we have

\[ \hat{R}(t) = \begin{pmatrix} e^{iK(t)} \\ ie^{iK(t)} \end{pmatrix}, \]

where $\hat{K}(t) = \int_0^t \hat{\psi}_1$. Hence pointwise convergence in (90) is clear. Hence by Lemma 4.8 it remains to prove

\[ \|e^{i\hat{K}} - e^{iK}\|_{W^{2,\infty}} \leq C\varepsilon. \tag{95} \]

Since

\[ (e^{iK})'' = (i\kappa' - \kappa^2)e^{iK}, \]

we see that

\[ (e^{i\hat{K}} - e^{iK})'' = (\kappa^2 - \hat{\kappa}^2 + i(\hat{\kappa}' - \kappa')) e^{iK} + (i\kappa' - \hat{\kappa}) (e^{i\hat{K}} - e^{iK}) \cdot \]

As $\sin$ and $\cos$ are 1-Lipschitz, we have

\[ \|e^{i\hat{K}} - e^{iK}\|_{L^\infty} \leq C\varepsilon. \]

The bounds on the other terms are as obvious. Similar estimates on the lower derivatives are clear, so we have proven (95). Hence (90) follows. The proof of (91) is easy as well (although less explicit) and is left to the reader.

Set $\rho = (\psi_\varepsilon)_3$. To prove (92) it is again enough to obtain a bound on $Y''$. But

\[ |Y''| = |\kappa \hat{N} + \rho \hat{\Gamma}| \leq C\varepsilon \]

because $\|\rho\|_{W^{1,\infty}} \leq C\varepsilon$. The proof of (93) is essentially the same.

To prove (94) we introduce $P = Q^{-1} = \Phi \circ \beta$, so that

\[ P(s, t) = \hat{\Gamma}(\sigma(t)) + s\hat{N}(\sigma(t)) = \hat{\Gamma}(t) + Y(t) + s\hat{N}(t). \tag{96} \]
Hence
\[ P \to \Gamma + s N = \Phi \] uniformly on \( M_{s,\pm} \).

The hypotheses \( G_i(\psi) = G_i(0) \) for \( i = 3, 4 \) ensure that \( \bar{\Gamma} = [\Gamma] \). Hence we conclude:
\[ Q = P^{-1} \to \Phi^{-1} \] pointwise on \( [\Gamma] \).

By Lemma 4.8 it therefore remains to show that there exists a constant \( C \) independent of \( \varepsilon \) such that
\[ \|Q_\varepsilon\|_{W^{2,\infty}([\Gamma])} \leq C \] (97)
for all \( \varepsilon \) small enough in modulus. Since \( Y' = \rho \hat{\Gamma}' \), we see
\[ \nabla P = \left( \hat{N} \mid (\rho + 1 - s \kappa) \hat{\Gamma}' \right). \] (98)

As \( \hat{\Gamma}' \) and \( \hat{N} \) are orthonormal, this shows that
\[ \det ((\nabla P)(s, t)) = \rho(t) + 1 - s \kappa(t). \]

By hypothesis \( 1 - s \kappa \) is uniformly bounded from below by a positive constant. Hence
\[ \left\| \frac{1}{\det \nabla P} \right\|_{L^\infty(Q([\Gamma]))} \leq C \] (99)
for all \( \varepsilon \) small enough. From (96) we compute that \( \partial_s \partial_t P = 0 \) and
\[ \partial_t \partial_s P = -s \kappa \hat{\Gamma}' \]
\[ \partial_t \partial_t P = (\rho' - s \kappa') \hat{\Gamma}' + (\rho + 1 - s \kappa) \kappa \hat{N}. \]

Hence the bounds on \( \psi' \) show that there exists an \( \varepsilon \)-independent constant \( C \) such that
\[ \|\nabla^2 P\|_{L^\infty} \leq C. \] (100)

Using that
\[ (\nabla Q)(P) = (\nabla P)^{-1} = \frac{(\text{cof} \nabla P)^T}{\det (\nabla P)}, \]
one sees that \( \|\nabla^2 Q\|_{L^\infty([\Gamma])} \) is bounded by a polynomial in
\[ \left\| P \right\|_{W^{2,\infty}(Q([\Gamma]))}, \left\| \frac{1}{\det \nabla P} \right\|_{L^\infty(Q([\Gamma]))}. \]

Together with the obvious uniform Lipschitzness of \( P \), with (100) and with (99) this implies that \( \|\nabla^2 Q\|_{L^\infty} \) is bounded by an \( \varepsilon \)-independent constant. The first derivatives and \( Q \) itself are obviously uniformly bounded in the same way. Hence (97) is proven, and (94) follows. \( \square \)

We collect all functionals \( G_1, \ldots, G_4 \) into a single \( \mathbb{R}^8 \)-valued functional \( G \) in the obvious way. The following Lemma is [5, Proposition 3.15]; recall that we are assuming \( 1 - s \kappa \geq c > 0 \).
Lemma 4.4 If $\mu$ differs from zero on a set of positive measure in $(0,T)$, then following non-degeneracy condition is satisfied:

$$\left\{ \hat{\mathcal{G}}(\psi) : \psi \in C_0^\infty((0,T), \mathbb{R}^3) \right\} = \mathbb{R}^8.$$ 

The next result is essentially [5, Lemma 3.17]; as we need the precise result, we include its proof.

Lemma 4.5 There exists $\varepsilon_0 > 0$ and for $j = 1, \ldots, 8$ there exist $\hat{\psi}_j \in C_0^\infty((0,T), \mathbb{R}^3)$ such that the following is true:

If $\psi \in L^\infty((0,T), \mathbb{R}^3)$ is such that $\hat{\mathcal{G}}(\psi) = 0$, then for $j = 1, \ldots, 8$ there exist $\eta_j \in C^1(-\varepsilon_0, \varepsilon_0)$ with $\eta_j(0) = 0$ and such that the family $\{\psi_\varepsilon\}_{\varepsilon \in (-\varepsilon_0, \varepsilon_0)}$ defined by

$$\psi_\varepsilon = \varepsilon \hat{\psi} + \sum_{j=1}^8 \eta_j(\varepsilon) \hat{\psi}_j$$

satisfies

$$\mathcal{G}_i(\psi_\varepsilon) = 0 \text{ for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0), \ i = 1, 2, 3, 4.$$ 

Clearly,

$$\psi_\varepsilon \to \hat{\psi} \text{ in } L^\infty((0,T), \mathbb{R}^3).$$

Proof. By Lemma 4.4, for $i = 1, \ldots, 8$ there exist $\hat{\psi}^{(i)} \in C_0^\infty((0,T), \mathbb{R}^3)$ such that the matrix with entries

$$\hat{\mathcal{G}}_j(\hat{\psi}^{(i)})$$

agrees with the unit matrix. For $\eta \in \mathbb{R}^8$ define

$$F_i(\varepsilon, \eta) = \mathcal{G}_i \left( \varepsilon \hat{\psi} + \sum_{j=1}^8 \eta_j \hat{\psi}_j \right).$$

By the invertibility of (101), the Jacobian with respect to the variable $\eta$ does not vanish at $(\varepsilon, \eta) = 0$. Hence the implicit function theorem yields $\varepsilon_0 > 0$ and a $C^1$-function $\eta$ such that

$$F_i(\varepsilon, \eta(\varepsilon)) = \mathcal{G}_i(0,0) \text{ for all } \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

As usual, we have $\eta'_0(0) = 0$ because $\hat{\mathcal{G}}_i(\hat{\psi}) = 0$. Hence $|\eta(\varepsilon)| \ll |\varepsilon|$ as $\varepsilon \to 0$.

Proposition 4.6 Let $u \in W_+^{2,\infty}(\Gamma)$ and assume that $1 - sk \geq c > 0$ and let $\{\psi_\varepsilon\}_{\varepsilon \in (-1,1)} \subset L^\infty((0,T), \mathbb{R}^3)$ be a strongly continuous 1-parameter family satisfying $\psi_0 = 0$ and

$$\mathcal{G}(\psi_\varepsilon) = \mathcal{G}(0) \text{ for all } \varepsilon \in (-1,1).$$

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the map $\pi_\varepsilon$ defined via Proposition 2.2 by the formula

$$\pi_\varepsilon \left( \Gamma_\varepsilon(t) + s \mathcal{N}_\varepsilon(t) \right) = \Gamma_\varepsilon(t) + s \mathcal{N}_\varepsilon(t) \text{ for all } (s,t) \in M_{\pi_\varepsilon}.$$ 

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is well-defined on $[\Gamma]$, and  
\[
\pi_\varepsilon \in W^{2,\infty}_\delta([\Gamma])
\]
with
\[
(\pi_\varepsilon, \nabla \pi_\varepsilon) = (u, \nabla u) \text{ on } \partial_L[\Gamma].
\]  
(102)
Moreover, the map $\varepsilon \mapsto \pi_\varepsilon$ is strongly continuous into $W^{2,2}([\Gamma], \mathbb{R}^3)$.

**Proof.** We associate with $\{\psi_\varepsilon\}$ the family of surfaces 
\[
\{\pi_\varepsilon\} \subset W^{2,2}_\delta([\Gamma_\varepsilon])
\]
defined via Proposition 2.2 by setting 
\[
\pi_\varepsilon \left(\Gamma_\varepsilon(t) + s\mathcal{N}_\varepsilon(t) \right) := \tau_\varepsilon(t) + s\nu_\varepsilon(t)
\]
for all $(s, t) \in \mathbb{R} \times [0, \sigma_\varepsilon(T)]$ with $s \in (\tilde{s}_-^\varepsilon(t), \tilde{s}_+^\varepsilon(t))$. If $|\varepsilon|$ is small, then by strong continuity $\psi_\varepsilon$ is small in $L^\infty$. Hence from the uniform lower bound on $1 - s\kappa$ one obtains the uniform lower bound 
\[
1 - s\kappa_\varepsilon(t) \geq c > 0 \text{ for all } (s, t) \in M_{s^\varepsilon}\).
\]
This also implies that the parametrization $\overline{\Phi}_\varepsilon$ is injective on $M_{s^\varepsilon}$, see for instance [6, Proposition 13] or the proof of [5, Lemma 2.2].

The constraints $G_3(\psi_\varepsilon) = G_4(\psi_\varepsilon) = 0$ ensure that 
\[
\overline{\Phi}_\varepsilon \left( M_{s^\varepsilon} \right) = [\Gamma],
\]
hence Proposition 2.2 implies that $\pi_\varepsilon$ is a well-defined surface in $W^{2,2}_\delta([\Gamma])$. But in fact $\pi_\varepsilon \in W^{2,\infty}$ because by construction we have (cf. [5, formula (3.8)])
\[
\pi_\varepsilon(\sigma_\varepsilon(t)) = \frac{\mu + (\psi_\varepsilon)_2}{1 + (\psi_\varepsilon)_3}
\]
and $\psi_\varepsilon$ is small in $L^\infty$, and $\mu \in L^\infty$ because $u \in W^{2,\infty}$. A similar formula holds for $\kappa_\varepsilon$. Since the mean curvature of $\pi_\varepsilon$ satisfies 
\[
2\overline{H}_\varepsilon(\overline{\Phi}_\varepsilon) = \frac{\overline{\pi}_\varepsilon}{1 - s\kappa_\varepsilon},
\]
we see that $\pi_\varepsilon \in W^{2,\infty}$.

The constraint $G(\psi_\varepsilon) = 0$ ensures that (102) is satisfied, cf. [5, Lemma 3.11]. Finally, the strong $W^{2,2}$ continuity is proven in [5, Remark 3.4 (iii)].  

\[\square\]

### 4.2 Continuation of infinitesimal bendings

**Lemma 4.7** Assume that $\kappa, \mu \in W^{1,\infty}(0, T)$, let $P, Q \in W^{2,\infty}(0, T)$, set 
\[
p(t) = \int_0^t \mu P
\]
\[
q(t) = \int_0^t \mu Q
\]

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and define $M$ and $L$ by (57), (56) with $M_0 = 0$ and $L_0 = 0$.

Assume that
\[ p(t) = q(t) = P(t) = Q(t) = 0 \text{ for } t \in \{0, T\}, \]  
and that
\[
L(T) = 0 \quad (104) \\
M(T) = 0. \quad (105)
\]

Then there exists $\psi \in W^{1,\infty}(0, T)$ such that
\[ n \cdot (\dot{\gamma} + \dot{y}) = p \quad (106) \]
\[ n \cdot \dot{v} = q \quad (107) \]
pointwise on $[0, T]$ and
\[ G_i(\psi) = 0 \text{ for } i = 1, 2, 3, 4. \quad (108) \]

**Proof.** Since $\mu, P, Q \in W^{1,\infty}$, we see that $p, q \in W^{2,\infty}$. We define $f : [0, T] \to \mathbb{R}$ by setting
\[ f(t) = \int_0^t (Q + \kappa P). \]

The hypotheses imply that $f \in W^{3,\infty}$. Define
\[ \varphi = \mu q + Q, \quad \alpha = -\kappa q, \quad \rho = -\mu p - \kappa f - P. \quad (109) \]

The hypotheses imply that $\alpha, \rho$ and $\varphi$ are Lipschitz.

In order to emphasize the arbitrariness of $\psi$ in the following computations, we let $g \in W^{2,\infty}$ be any function with $g(0) = g(T) = 0$ and define
\[ \varphi = \mu q + f'' + (\kappa g)' \]
\[ \alpha = -\kappa q + (p' + \mu g)' \]
\[ \rho = -\mu p - \kappa f + g'. \]

This does not make sense unless $p$ has two derivatives, and it is only used in order to keep track of the role of $g$; the earnest proof works with $\varphi, \alpha$ and $\rho$ as defined in (109), which is obtained by choosing $g = -P$.

We define $\psi = (\varphi, \alpha, \rho)$ and we define $L$ by (56) and $M$ by (57). We claim that (106), (107) are satisfied for this choice of $\varphi, \alpha, \rho$ (whichever the choice of $g \in W^{2,\infty}$).

To prove this, let us first verify that
\[ \Xi = q\gamma' + (f' + \kappa g)n - (p' + \mu g)v. \quad (110) \]

In fact, using $(f + \kappa g)' = \varphi - \mu q$ and $(p' + \mu g)' = \psi + \kappa q$, we see that the derivative of the right-hand side of (110) equals
\[ q'\gamma' + q\gamma'' + (f' + \kappa g)n' + (\varphi - \mu q)n - (\alpha + \kappa q)v - (p' + \mu g)v' \]
\[ = (q' + \kappa p' - \mu f')\gamma' + \varphi n - \psi v = \xi \]
with $\xi$ as in (79). As both sides of (110) vanish at 0, this concludes the proof of (110).
Since \( \psi = \Xi \wedge v \), we see that (107) is equivalent to \( \Xi \cdot \gamma' = q \), which by (110) is obviously satisfied.

From (110) and the definition of \( \rho \) we compute
\[
\dot{\gamma}' + \dot{y}' = \Xi \wedge \gamma' + \rho \gamma' = (-\mu p - \kappa f + g')\gamma' + (f' + \kappa g)v + (p' + \mu g)n = (pn + f v + g\gamma')'.
\]
Since \( p(0) = f(0) = g(0) = 0 \) and \( \dot{\gamma}(0) = \dot{y}(0) = 0 \) as well, this is equivalent to
\[
\dot{\gamma} + \dot{y} = pn + f v + g\gamma', \tag{111}
\]
from which (106) follows at once.

It remains to verify (108). We compute from the definitions of \( \rho, \varphi, \alpha \):
\[
\int_0^t \varphi = \int_0^t \mu q + f'' + (\kappa g)' \, d\sigma = L(t) + f'(t) + \kappa(t)g(t), \tag{112}
\]
because \( f'(0) = 0 \) by (103) and \( g(0) = 0 \). Hence (104) implies that
\[
\hat{G}_3(\varphi, \rho) = \int_0^T \varphi = f'(T) + \kappa(T)g(T) = 0,
\]
because \( f'(T) = 0 \) by (103) and \( g(T) = 0 \).

We compute
\[
(\Gamma + \dot{Y})' = \left( \int_0^t \varphi \right) N + \rho \Gamma' = \left( \int_0^t \mu q + f'' + (\kappa g)' \right) N + (g' - \kappa f - \mu p)\Gamma' = LN + (f' + \kappa g)N + (g' - \kappa f)\Gamma' - \mu p \Gamma' = -M' + (f N + g \Gamma')'.
\]
Since \( M(0) = 0 \) and \( f'(0) = g(0) = 0 \) this implies
\[
\Gamma(t) + \dot{Y}(t) = f(t)N(t) + g(t)\Gamma'(t) - M(t) \quad \text{for all } t \in [0, T]. \tag{113}
\]
Hence,
\[
\hat{G}_4(\varphi, \rho) \equiv \left( \Gamma(t) + \dot{Y}(T) \right) \cdot \Gamma'(T) = g(T) - M(T) \cdot \Gamma'(T).
\]
This is zero because of (105) and because \( g(T) = 0 \).

Since by (103) we have \( f'(T) = 0 \) and \( q(T) = p'(T) = 0 \), we see from (110) that
\[
\hat{G}_2(\varphi, \rho, \alpha) \equiv \Xi(T) = q(T)\gamma'(T) + (f' + \kappa g)(T)n(T) - (p' + \mu g)(T)v(T) = 0.
\]
Finally, from (111), (113) and using again (103) and \( g(T) = 0 \):
\[
\hat{G}_1(\varphi, \rho, \alpha) \equiv (\dot{\gamma} + \dot{y})(T) - (v \otimes N)(T)(\Gamma + \dot{Y})(T) = p(T)n(T) + f(T)v(T) + g(T)\gamma'(T) - (f(T) - M(T) \cdot N(T)) \cdot v(T) = 0.
\]
We will keep using the following fact, so we give it a name. Its proof is straightforward.

**Lemma 4.8** Assume that \( f_n \to f \) almost everywhere and that \( \|f_n\|_{W^{2, \infty}} \leq C \). Then \( f_n \rightharpoonup f \) in \( W^{2, \infty} \).

**Lemma 4.9** Let \( \Omega, \Omega' \subset \mathbb{R}^m \) be bounded domains and for all \( \varepsilon \in [0, 1) \) let \( \overline{G}_\varepsilon, G \in W^{2, \infty}(\Omega) \) and \( Q_\varepsilon \in W^{2, \infty}(\Omega', \Omega) \), and assume that
\[
\|\overline{G}_\varepsilon - G\|_{W^{2, \infty}(\Omega)} \leq C\varepsilon
\]
and that
\[
\|Q_\varepsilon\|_{W^{2, \infty}(\Omega')} \leq C.
\]
Then
\[
\|\overline{G}_\varepsilon \circ Q_\varepsilon - G \circ Q_\varepsilon\|_{W^{2, \infty}(\Omega')} \leq C\varepsilon.
\]
If, moreover,
\[
\frac{\overline{G}_\varepsilon - G_\varepsilon}{\varepsilon} \rightharpoonup \hat{G} \text{ weakly-* in } W^{2, \infty}(\Omega)
\]
and
\[
Q_\varepsilon \rightharpoonup \hat{Q} \text{ weakly-* in } W^{2, \infty}(\Omega', \mathbb{R}^m),
\]
then
\[
\frac{1}{\varepsilon} \left( \overline{G}_\varepsilon \circ Q_\varepsilon - G \circ Q \right) \rightharpoonup \hat{G} \circ \hat{Q} \text{ weakly-* in } W^{2, \infty}(\Omega'). \quad (114)
\]

**Proof.** Without loss of generality we may assume that \( G = 0 \). We omit the index \( \varepsilon \). We have \( \nabla (\overline{G} \circ Q) = (\nabla G)(Q)\nabla Q \), so
\[
\|\nabla (\overline{G} \circ Q)\|_{L^\infty} \leq \|\nabla G\|_{L^\infty} \|\nabla Q\|_{L^\infty}.
\]
Similarly,
\[
\|\nabla^2 (\overline{G} \circ Q)\|_{L^\infty} \leq \|\nabla^2 G\|_{L^\infty} \|\nabla Q\|_{L^\infty}^2 + \|\nabla G\|_{L^\infty} \|\nabla^2 Q\|_{L^\infty}.
\]
The last part of the claim follows from Lemma 4.8 because the convergence \((114)\) clearly holds pointwise. \(\square\)

**Lemma 4.10** Let \( \Omega \subset \mathbb{R}^m \) be a bounded domain, let \( U \in W^{3, \infty}(\Omega) \) and for all \( \varepsilon \in (0, 1) \) let \( F_\varepsilon \in W^{2, \infty}(\Omega, \Omega) \) be invertible with \( F_\varepsilon^{-1} \in W^{2, \infty}(\Omega, \Omega) \). Assume that
\[
\|F_\varepsilon - id\|_{W^{2, \infty}(\Omega)} \leq C\varepsilon, \quad (115)
\]
and let \( \overline{U}_\varepsilon \in W^{2, \infty}(\Omega) \) be such that
\[
\|\overline{U}_\varepsilon - U(F_\varepsilon)\|_{W^{2, \infty}(\Omega)} \leq C\varepsilon.
\]
Then
\[
\|\overline{U}_\varepsilon - U\|_{W^{2, \infty}(\Omega)} \leq C\varepsilon.
\]
Moreover, if
\[
\frac{1}{\varepsilon} (F_\varepsilon - id) \rightharpoonup \hat{F} \text{ in } W^{2, \infty}(\Omega)
\]
and
\[ \frac{1}{\varepsilon} (U - U(F_\varepsilon)) \xrightarrow{\varepsilon \to 0} f \text{ in } W^{2,\infty}(\Omega), \]

then
\[ \frac{1}{\varepsilon} (U - U) \xrightarrow{\varepsilon \to 0} f + \partial_p U \text{ in } W^{2,\infty}(\Omega) \]
as \( \varepsilon \to 0 \).

**Proof.** We omit the index \( \varepsilon \). We write
\[ U = U(U(F)) + U(U(F)), \]

By hypothesis,
\[ \frac{U - U(F)}{\varepsilon} \xrightarrow{\varepsilon \to 0} f \text{ in } W^{2,\infty}(\Omega). \]

As \( U \in C^1(\Omega) \), we have for all \( x \in \Omega \):
\[ \frac{1}{\varepsilon} (U(F(x)) - U(x)) = \left( \int_0^1 (\nabla U)(x + t(F(x) - x)) \frac{F(x) - x}{\varepsilon} \right). \]

Since \( F \to id \) uniformly and since \( \nabla u \) is uniformly continuous, we have
\[ \int_0^1 (\nabla U)(x + t(F(x) - x)) \to (\nabla U)(x) \]
uniformly in \( x \in \Omega \). Since
\[ F(x) - x \xrightarrow{\varepsilon \to 0} \dot{F}(x) \]
uniformly in \( x \in \Omega \), we conclude that
\[ \frac{1}{\varepsilon} (U(F) - U) \to \partial_p U \text{ uniformly on } \Omega. \]

In view of Lemma 4.8 it remains to show that
\[ \|U(U(F) - U)\|_{W^{2,\infty}(\Omega)} \leq C\varepsilon. \] (116)

By the Leibniz rule we have
\[ \nabla^2 (U(F) - U) = (\nabla^2 U)(F)(\nabla F, \nabla F) - (\nabla^2 U)(F) \]
\[ - \nabla^2 U + (\nabla U)(F)\nabla^2 F. \]

Hence
\[ \|\nabla^2 (U(F) - U)\|_{L^\infty} \leq \|\nabla^2 U\|_{L^\infty} \|\nabla F - I\|_{L^\infty}^2 + \| (\nabla^2 U)(F) - (\nabla^2 U)(x) \|_{L^\infty} \]
\[ + \|\nabla U\|_{L^\infty} \|\nabla^2 F\|_{L^\infty}. \]

But
\[ \| (\nabla^2 U)(F) - (\nabla^2 U)(x) \|_{L^\infty} \leq \|\nabla^3 U\|_{L^\infty} \|F - id\|_{L^\infty} \]
because \( \nabla^2 U \) is Lipschitz with Lipschitz constant not exceeding \( \|\nabla^3 U\|_{L^\infty} \).

From (115) we therefore deduce (116). \( \square \)
Proposition 4.11 Suppose that $\kappa \in W^{1,\infty}(0, T)$ is such that (70) is satisfied, and let $u \in W^{3,\infty}_h([\Gamma])$. Let $\{\psi_\varepsilon\}_{\varepsilon \in (-1, 1)} \subset W^{1,\infty}((0, T), \mathbb{R}^3)$ be such that

$$G(\psi_\varepsilon) = G(0) \text{ for all } \varepsilon \in (-1, 1).$$

(117)

Then the immersions $\pi_\varepsilon$ induced by $\psi_\varepsilon$ from $u$ via Proposition 4.6 constitute a strongly $W^{2,2}$-continuous 1-parameter family

$$\{\pi_\varepsilon\}_{\varepsilon \in (-1, 1)} \subset W^{2,\infty}_h([\Gamma]),$$

(118)

and we have

$$\frac{1}{\varepsilon} (\pi_\varepsilon - u) \rightharpoonup \tau \text{ in } W^{2,\infty}([\Gamma], \mathbb{R}^3).$$

(119)

Here, $\tau : [\Gamma] \to \mathbb{R}^3$ is defined by setting

$$\tau(s, t) = \dot{\gamma}(t) + \dot{y}(t) + s\dot{v}(t) - (\gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t)) \left(\dot{\Gamma}(t) + \dot{Y}(t) + s\dot{N}(t)\right)$$

(120)

for all $(s, t) \in M_{s \pm}$.

**Proof.** Lemma 3.6 implies that $\mu \in W^{1,\infty}(0, T)$. The first part of the statement (until (118)) follows from Proposition 4.6.

Recall the definition of $\beta_\varepsilon$ in (88) and define $F_\varepsilon : [\Gamma] \to [\Gamma]$ by

$$F_\varepsilon = \Phi \circ \beta_\varepsilon^{-1} \circ \Phi_\varepsilon^{-1}.$$  

Define $m : M_{s \pm} \to \mathbb{R}^3$ by setting

$$m = \dot{\gamma}(t) + \dot{y}(t) + s\dot{v}(t) \text{ for all } (s, t) \in M_{s \pm}.$$  

Since

$$\pi_\varepsilon(\Phi_\varepsilon(\beta_\varepsilon)) - u(\Phi) = \tilde{\gamma}_\varepsilon + \tilde{y}_\varepsilon + s\tilde{v}_\varepsilon - \gamma - sv,$$

(121)

from (91), (93) we see that

$$\frac{1}{\varepsilon} (\pi_\varepsilon(\Phi_\varepsilon \circ \beta_\varepsilon) - u(\Phi)) \rightharpoonup m \text{ weakly-* in } W^{2,\infty}(M_{s \pm}, \mathbb{R}^3).$$

(122)

Define $\hat{F} : [\Gamma] \to \mathbb{R}^2$ by setting

$$\hat{F}(\Phi) = -(\dot{\Gamma} + \dot{Y} + s\dot{N}).$$

Defining $Q_\varepsilon$ as in (89) and writing

$$\pi_\varepsilon - u(F_\varepsilon) = (\pi_\varepsilon(\Phi_\varepsilon \circ \beta_\varepsilon) - u(\Phi)) \circ Q_\varepsilon$$

and using (94), we deduce from (122) with the aid of Lemma 4.9 that

$$\frac{1}{\varepsilon} (\pi_\varepsilon - u(F_\varepsilon)) \rightharpoonup m(\Phi^{-1}) \text{ in } W^{2,\infty}([\Gamma], \mathbb{R}^3).$$

(123)

Similarly, we have

$$(F_\varepsilon - id) \circ Q_\varepsilon^{-1} = \Phi - \Phi_\varepsilon(\beta_\varepsilon) = \Gamma + sN - \tilde{\Gamma}_\varepsilon - Y_\varepsilon - s\tilde{N}_\varepsilon.$$
Hence (90), (92) imply
\[
\frac{1}{\varepsilon}(F_\varepsilon - \text{id}) \circ Q^{-1}_\varepsilon \rightharpoonup \hat{F}(\Phi) \text{ weakly-* in } W^{2,\infty}(M_{\varepsilon}, \mathbb{R}^2).
\]
This together with (94) and with Lemma 4.9 implies
\[
\frac{F_\varepsilon - \text{id}}{\varepsilon} \rightharpoonup \hat{F} \text{ weakly-* in } W^{2,\infty}([\Gamma], \mathbb{R}^2). \tag{124}
\]
Lemma 4.10 together with (123), (124) implies
\[
\frac{1}{\varepsilon} (\pi_\varepsilon - u) \rightharpoonup m(\Phi^{-1}) + \partial_{\varepsilon} u \text{ in } W^{2,\infty}([\Gamma], \mathbb{R}^3).
\]
But the right-hand side is just \(\tau\) as defined by (120). Indeed, by definition the right-hand side evaluated at \(\Phi(s, t)\) equals
\[
m(s, t) + (\partial_{\varepsilon} u)(\Phi(s, t)) = m(s, t) + (\gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t)) \hat{F}(\Phi(s, t))
\]
\[= \dot{\gamma}(t) + \dot{y}(t) + s\dot{\gamma}(t) - (\gamma'(t) \otimes \Gamma'(t) + v(t) \otimes N(t)) \left(\hat{\Gamma}(t) + \hat{Y}(t) + s\hat{N}(t)\right).\]

**Proof of Theorem 4.1.** Combining the hypotheses with Lemma 3.6 we have that \(\kappa\) and \(\mu\) are Lipschitz and that \(1 - s\kappa(t)\) is uniformly bounded from below by a positive constant. The hypotheses of Lemma 4.7 are satisfied. It yields \(\hat{\psi} \in W^{1,\infty}((0, T), \mathbb{R}^3)\) such that \(\hat{G}(\hat{\psi}) = 0\) and
\[
n \cdot (\dot{\gamma} + \dot{y}) = p
\]
\[
n \cdot \dot{\psi} = q
\]
pointwise on \([0, T]\). Lemma 4.5 yields a corresponding family \(\{\psi_\varepsilon\} \subset W^{1,\infty}\) with \(\hat{G}(\psi_\varepsilon) = 0\). By its definition (cf. Lemma 4.5) it is clearly strongly \(C^1\) into \(W^{1,\infty}\). Proposition 4.11 shows that the family
\[
\{\pi_\varepsilon\} \subset W^{2,\infty}_{\varepsilon}([\Gamma])
\]
generated as in Proposition 4.6 from \(u\) via \(\{\psi_\varepsilon\}\) is strongly \(W_{\varepsilon}^{2,2}\)-continuous and satisfies
\[
\frac{1}{\varepsilon} (\pi_\varepsilon - u) \rightharpoonup \tau^{(2)} \text{ in } W^{2,\infty}([\Gamma], \mathbb{R}^3),
\]
where \(\tau^{(2)}\) is given by the right-hand side of (120). By Proposition 4.6 we know that (78) is satisfied. By (125) we see that \(n \cdot \tau = n \cdot \tau^{(2)}\). Applying Lemma 3.3 to the infinitesimal bendings \(\tau\) and \(\tau^{(2)}\), we conclude that there exist \(k \in \mathbb{R}^2\) and \(\lambda \in \mathbb{R}\) such that
\[
\tau - \tau^{(2)} = \langle k + \lambda x^1, (\nabla u)(x) \rangle
\]
for all \(x \in [\Gamma]\). But the left-hand side vanishes at \(\Gamma(0)\), which we choose to be the origin. So \(k = 0\) because \((\nabla u)(0)\) is an isometry. Moreover, \(\tau(s\hat{N}(0)) = \tau^{(2)}(s\hat{N}(0))\) for some \(s \neq 0\). Thus
\[
0 = s\lambda \langle \Gamma'(0), (\nabla u)(0) \rangle = s\lambda \gamma'(0),
\]
which implies that \(\lambda = 0\). Hence \(\tau = \tau^{(2)}\).

\[\square\]
5 Equilibrium equations

In this section we provide a simple derivation of the equilibrium equations for the nonlinear bending functional augmented by a term modelling external forces $F$:

\[
J(u; S) = \begin{cases}
\frac{1}{2} \int_{S} |\nabla^2 u|^2 \, dx + \int_{S} F \cdot u \, dx & \text{if } u \in W^{2,2}_{\delta}(S) \\
+\infty & \text{otherwise}.
\end{cases}
\] (126)

The derivation provided here is very simple: it reduces to testing a general first variation formula with test functions enjoying a particular structure. The first variation formula is related to the classical Willmore equation.

A priori, this derivation is only formal, which is expressed by the hypothesis of 'stationarity under infinitesimal bendings', as opposed to the natural condition of stationarity (cf. below). However, thanks to Theorem 4.1, these two notions agree in the presence of sufficient regularity; therefore, the formal derivation is in fact rigorous for regular enough equilibria. For vanishing external forces, it was shown in [5] that minimizers enjoy the required regularity away from the set where they are planar.

5.1 Bendings, infinitesimal bendings and equilibria

Proposition 5.1 The functional $J$ attains a global minimum on the space

\[
\mathcal{A}_0 = \left\{ u \in W^{2,2}_{\delta}(S) : \int_{S} u \, dx = 0 \right\}.
\]

Proof. By the Poincaré inequality and since $\nabla u$ is uniformly bounded due to the isometry constraint, we see that on $\mathcal{A}_0$ the full $W^{2,2}$-norm is equivalent to the $L^2$-norm of the Hessian. As $\mathcal{A}_0$ is closed under weak $W^{2,2}$-convergence, the claim follows from the direct method. \qed

Following [7], a deformation $u \in W^{2,2}_{\delta}(S)$ will be called a stationary (equilibrium) point of $J$ if

\[
\frac{d}{dt} \bigg|_{t=0} J(u_t) = 0 \text{ for all bendings } \{u_t\}_{t \in (-1,1)} \text{ of } u.
\] (127)

A deformation $u \in W^{2,2}_{\delta}(S)$ is said to be stationary for $J$ under infinitesimal bendings of a certain kind (of a certain regularity, satisfying some given boundary conditions...) if

\[
\frac{1}{2} \int_{S} \nabla^2 u : \nabla^2 \tau \, dx + \int_{S} F \cdot \tau = 0
\]

for all infinitesimal bendings $\tau : S \to \mathbb{R}^3$ of this kind. The following first variation formula is easily derived, cf. [7]:

Proposition 5.2 If $\{u_t\}_{t \in (-1,1)}$ is a bending of $u \in W^{2,2}_{\delta}(S)$ inducing the continu-able infinitesimal bending $\tau \in W^{2,2}(S, \mathbb{R}^3)$, then

\[
\frac{d}{dt} \bigg|_{t=0} J(u_t; S) = \frac{1}{2} \int_{S} \nabla^2 u : \nabla^2 \tau \, dx + \int_{S} F \cdot \tau.
\] (128)

In particular, we have the following:
(i) An immersion \( u \in W^{2,2}_S(S) \) is a stationary point of \( J \) precisely if
\[
\frac{1}{2} \int_S \nabla^2 u : \nabla^2 \tau + \int_S F \cdot \tau = 0
\]  
(129)
for all infinitesimal bendings \( \tau \) which are induced by some bending.

(ii) Every \( u \in W^{2,2}_S(S) \) that is stationary for \( J \) under \( W^{2,2} \)-infinitesimal bendings is a stationary point of \( J \).

Corollary 4.2 provides a (partial) converse to Proposition 5.2 (ii).

5.2 First variation formula

We compute explicitly the first variation of the functional (2) under infinitesimal bendings.

Lemma 5.3 Let \( u \in W^{3,\infty}_S(S) \) and let \( V_1, V_2, \Psi \in (W^{2,2} \cap W^{1,\infty})(S) \), be such that (43) is satisfied almost everywhere on \( S \), and define \( \tau = \Psi u + \partial V u \). Then \( \tau \in W^{2,2}(S, \mathbb{R}^3) \) is an infinitesimal bending of \( u \), and we have
\[
\frac{1}{2} \int_S \langle \nabla^2 u, \nabla^2 \tau \rangle \, dx = \int_S H \Delta \Psi + 4H^3 \Psi + \partial V \left( H^2 \right) \, dx.
\]  
(130)

Proof. This result is, in fact, a particular case of \([7, \text{Proposition 6.4}]\). As its derivation is quite simple in the intrinsically flat case considered here, we provide a self-contained derivation.

First note that by the Leibniz rule it is clear that \( \partial_i \partial_j u \), so by Definition 3.1 we know that \( \tau \) is an infinitesimal bending of \( u \). Since \( \partial_i \partial_j u = h_{ij} n \), we have
\[
\langle \nabla^2 u, \nabla^2 \tau \rangle = h_{ij} n \cdot \partial_i \partial_j \tau.
\]  
(131)

We compute from \( \tau = \partial V u + \Psi u \):
\[
\partial_i \tau = (\partial_i \Psi + V_j h_{ij}) n + (\partial_i V_j - h_{ij} \Psi) \partial_j u.
\]

By the regularity of \( h \) and \( V \), we can apply the Leibniz rule to conclude:
\[
n \cdot \partial_k \partial_i \tau = \partial_k \partial_i \Psi + (\partial_k V_j) h_{ij} + (\partial_i V_j) h_{kj} + V_j \partial_k h_{ij} - \Psi h_{ij} h_{jk}.
\]  
(132)

By Lemma 2.1 there exists \( a : S \to \mathbb{S}^1 \) such that \( h = 2H a \otimes a \). Proposition 3.5 implies that \( Ha^+ \otimes a^+ : \nabla^2 \Psi = 0 \). Hence
\[
H \Delta \Psi = H \text{Tr} \nabla^2 \Psi = H (a \otimes a + a^+ \otimes a^+) : \nabla^2 \Psi = H(a \otimes a) : \nabla^2 \Psi.
\]

That is,
\[
h : \nabla^2 \Psi = 2H \Delta \Psi.
\]  
(133)

Next,
\[
h_{ki}(\partial_k V_j) h_{ij} + h_{ki}(\partial_i V_j) h_{kj} = h^2 : (\nabla V)
\]
\[
= 2h^2 : (\text{sym} \nabla V) = 2h^2 : h \Psi
\]
by (43). Since \( h = 2H a \otimes a \) we see that \( h^2 : h = 8H^3 \). Thus
\[
h_{ki}(\partial_k V_j) h_{ij} + h_{ki}(\partial_i V_j) h_{kj} = 16H^3 \Psi.
\]  
(134)
Next since $h$ is symmetric and curl-free we have
\[ h_{ik} \partial_k h_{ij} = h_{ik} \partial_j h_{ik} = 2 \partial_j (H^2). \] (135)
Using (133), (134), (135) and $h_{ik} h_{ij} h_{jk} = 8 H^3$, we deduce from (132) that
\[ h_{ik} n \cdot \partial_k \partial_j \tau = 2 H \Delta \Psi + 16 H^3 + 2 \partial_{\nu'} (H^2) - 8 H^3 \Psi. \]
Inserting this into (131) and integrating yields (130).

\[ \square \]

**Remarks.**

(i) For $g = \delta$, the following expression was derived in [7] for the left-hand side of (130) (for all pairs $(V, \Psi)$ satisfying (43)):
\[
\int_S \left( \Delta H + 2 H^3 \right) \Psi \, dx + \int_{\partial S} H^2 \langle \nu, V \rangle_{\mathbb{R}^2} + H \langle \nu, \nabla \Psi \rangle_{\mathbb{R}^2} - \langle \nu, \nabla H \rangle_{\mathbb{R}^2} \Psi \, d\mathcal{H}^1. \] (136)
After a partial integration and using (43) one recovers the right-hand side of (130).

(ii) The expression (136) shows the relation to the classical Willmore equation. Indeed, term $\Delta H + 2 H^3$ is just the Willmore Lagrangian because the Gauss curvature is zero. So if $\Psi$ could range through all compactly supported smooth functions, (136) would lead to the Willmore equation. However, we know from Proposition 3.7 that the admissible test functions must be of the form (54), (55).

### 5.3 Equilibrium equations in line of curvature charts

As before, $\Gamma \in W^{2,\infty}((0, T), \mathbb{R}^2)$ and $s^+$ are Lipschitz. If $u \in W^{2,\infty}_g((\Gamma])$ is induced by $\Gamma$ and $s^+$, and if it is stationary for $J(\cdot; [\Gamma])$ under infinitesimal bendings, then
\[
\int_{[\Gamma]} \nabla^2 u : \nabla^2 \tau + F \cdot \tau = 0 \] (137)
for all infinitesimal bendings $\tau$ of $u$. Introduce the average $F$ and the first moment $\bar{F}$ of $F$ along rulings:
\[
\bar{F}(t) = \int_{s^-(t)}^{s^+(t)} (1 - s \kappa) \bar{F}(s, t) \, ds
\]
\[
\bar{F}(t) = \int_{s^-(t)}^{s^+(t)} (1 - s \kappa) s \bar{F}(s, t) \, ds
\]
We introduce the pulled-back average tangential force
\[
\mathcal{E}(t) = - \int_t^T (T \otimes \gamma' + N \otimes v) F \, d\sigma \] (138)
and the related force
\[
\mathcal{D}(t) = - \int_t^T \bar{F} \cdot \gamma' + \mathcal{E} \cdot N \, d\sigma. \] (139)
We recall from (40), (41) the definitions of $A$ and $B$:

$$A(s,t) = \tilde{H}(s,t) \partial_t \left( \frac{1}{1 - sk} \right)$$

(140)

$$B(s,t) = \frac{\tilde{H}(s,t)}{1 - sk} = \frac{\mu}{2(1 - sk)^2},$$

(141)

and from (59) and (60) the definitions

$$E(t) = \int_{s(t)}^{s(t)} \left( \int_{s(t)}^{s(\sigma)} \partial_k(\tilde{H}^2) \, ds \right) T(\sigma)$$

$$+ \left( \int_{s(t)}^{s(\sigma)} (1 - sk)\partial_k(\tilde{H}^2) \, ds \right) N(\sigma) \right) d\sigma$$

(142)

$$D(t) = - \int_{s(t)}^{s(t)} \left[ \left( \int_{s(t)}^{s(\sigma)} s\partial_k(\tilde{H}^2) \, ds \right) + E(\sigma) \cdot N(\sigma) \right] d\sigma.$$  

(143)

We define the averages and first moments of $A$:

$$\overline{A}(t) = \int_{s(t)}^{s(t)} A(s,t) \, ds$$

and

$$\widehat{A}(t) = \int_{s(t)}^{s(t)} sA(s,t) \, ds$$

The quantities $\overline{B}$ and $\widehat{B}$ are defined similarly.

**Theorem 5.4** Assume that $u \in W_{\delta}^{3,\infty}(\Gamma)$ is not affine, that (38) is satisfied and that $u$ is stationary for $J(\cdot; [\Gamma])$ under infinitesimal bendings $\tau \in W^{2,\infty}(\Gamma, \mathbb{R}^3)$ which vanish identically in a neighbourhood of $\partial_L[\Gamma]$. Define $A$, $B$, $E$, $D$, $E$, $D$ as in (140), (141), (141), (143), (138), (139).

Then there exist constants $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}^2$ such that

$$(\lambda_1 + \lambda_2 \cdot \Gamma^\perp + D + D) \mu = 4 \int_{s(t)}^{s(t)} s\tilde{H}^3(1 - sk) \, ds - \kappa \int_{s(t)}^{s(t)} \tilde{H} \, ds$$

$$- \overline{A}' + \overline{B}' + \overline{F} \cdot n$$

(144)

and

$$T \cdot (\lambda_2 + E + \mathcal{E})\mu = 4 \int_{s(t)}^{s(t)} \tilde{H}^3(1 - sk) \, ds - \overline{A}' + \overline{B}' + \overline{F} \cdot n$$

(145)

pointwise on $(0,T)$.

**Proof.** Let $p$, $q \in C_0^\infty(0,T)$ satisfy (63), (64). Define $\Psi$ and $V$ by (54), (55) with $M_0 = 0$ and $L_0 = 0$. Then Proposition 3.8 implies that $\tau := \Psi n + \partial_V u \in W^{2,\infty}$ is an infinitesimal bending of $u$. By (63), (64) and by Proposition 3.8 (ii) we know that $(\tau, \nabla \tau) = 0$ on $\partial_L[\Gamma]$. Since $p$ and $q$ have compact support, this readily implies that $\tau$ vanishes in a neighbourhood of $\partial_L[\Gamma]$. The stationarity hypothesis on $u$ therefore implies that (137) is satisfied for this $\tau$.

By the first variation formula (130) this means that

$$\int_{[\Gamma]} H \Delta \Psi + 4H^3 \Psi + \partial_V(H^2) + F \cdot (\Psi n + \partial_V u) = 0.$$  

(146)
Now we apply equation (61), which in view of $M_0 = 0$ and $L_0 = 0$ and by trivial integration by parts reduces to

\[
\int_{[T]} H\Delta\Psi + 4H^3\Psi + \partial_V(H^2) \, dx
\]
\[
= \int_0^T \left\{ \tilde{B}'' - \tilde{A}' + 4\left( \int_{s^-}^{s^+} s\tilde{H}^3(1 - s\kappa) \, ds \right) - \mu D - \kappa \left( \int_{s^-}^{s^+} \tilde{H} \, ds \right) \right\} q \, dt
\]
\[
+ \int_0^T \left\{ \tilde{B}' - \tilde{A} + 4\left( \int_{s^-}^{s^+} \tilde{H}^3(1 - s\kappa) \, ds \right) - \mu(T \cdot E) \right\} p \, dt.
\]

(147)

By (16) the second term on the left-hand side of (146) is

\[
\int_{[T']} F \cdot (\Psi n + \partial_V u) \, dx = \int_{M_{s^+}} (\tilde{F} \cdot n) (1 - s\kappa) \, ds dt
\]
\[
+ \int_{M_{s^-}} \left( (\tilde{F} \cdot \gamma')(T \cdot \tilde{V}) + (\tilde{F} \cdot v) (N \cdot \tilde{V}) \right) (1 - s\kappa) \, ds dt
\]
\[
= \int_0^T \left( \int_{s^-}^{s^+} (1 - s\kappa)\tilde{F} \, ds \right) \cdot n \, p \, dt + \int_0^T \left( \int_{s^-}^{s^+} (1 - s\kappa)s\tilde{F} \, ds \right) \cdot n \, q \, dt
\]
\[
+ \int_0^T \left( \int_{s^-}^{s^+} (1 - s\kappa)\tilde{F} \, ds \right) \cdot \gamma'(M \cdot T) \, dt + \int_0^T \left( \int_{s^-}^{s^+} (1 - s\kappa)\tilde{F} \, ds \right) \cdot v(M \cdot N) \, dt
\]
\[
+ \int_0^T \left( \int_{s^-}^{s^+} (1 - s\kappa)s\tilde{F} \, ds \right) \cdot \gamma' L \, dt.
\]

With $E$ as in the hypotheses, we find

\[
\int_{[T']} F \cdot (\Psi n + \partial_V u) \, dx
\]
\[
= \int_0^T \tilde{F} \cdot n \, p \, dt + \int_0^T \tilde{F} \cdot n \, q \, dt + \int_0^T \tilde{F}' \cdot M \, dt + \int_0^T \tilde{F} \cdot \gamma' L \, dt \quad (148)
\]
\[
= \int_0^T \left\{ \tilde{F} \cdot n - \mu E \cdot T \right\} p \, dt + \int_0^T \left\{ \tilde{F} \cdot n - \mu D \right\} q \, dt.
\]

In the last step we used (the analogue of) (66) together with the fact that $E(T) = D(T) = 0$. By (146) the sum of (147) and (148) vanishes for all $p, q \in C^\infty_0(0, T)$ satisfying the isoperimetric constraints (63), (64). Hence, by the Lagrange multiplier rule (we exclude the degenerate case at the end of this proof), there exist constants $\lambda_1 \in \mathbb{R}$ and $\lambda_2 \in \mathbb{R}^2$ such that the sum of (147) and (148) equals

\[
\int_0^T \left\{ \lambda_1 \mu + \lambda_2 \cdot \Gamma^\perp \mu \right\} q \, dt + \int_0^T \mu(\lambda_2 \cdot T) \, p \, dt \quad (149)
\]

for all $p, q \in C^\infty_0(0, T)$. This implies the pointwise equalities in the claim.

In order to justify the use of the Lagrange multiplier rule invoked above, assume that (149) is zero for all $p, q \in C^\infty_0(0, T)$. We must show that then $\lambda_1 = 0$ and
\( \lambda_2 = 0 \). But in fact, since \( u \) is not affine, we know that \( \mu \), which by Lemma 3.6 is continuous, differs from zero on an open interval \( I \). Hence

\[
\begin{align*}
\lambda_1 + \lambda_2 \cdot \Gamma^\perp &= 0 \quad (150) \\
\lambda_2 \cdot T &= 0 \quad (151)
\end{align*}
\]
on \( I \). Differentiating (150) we get \( \lambda_2 \cdot N = 0 \) on \( I \), which together with (151) implies that \( \lambda_2 = 0 \). And then also \( \lambda_1 = 0 \) by (150). \( \square \)

**Remarks.**

(i) If \( u \in W^{4,\infty}(\Gamma) \) is locally minimizing for \( J(\cdot; \Gamma) \) under its own first order boundary conditions on \( \partial_L[\Gamma] \) and if \( H \neq 0 \) on \( \overline{\Gamma} \), then Corollary 4.2 shows that \( u \) satisfies the hypotheses of Theorem 5.4.

(ii) Let \( S \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary and let \( \tilde{u} \in W^{2,2}_\delta(S) \) be a local minimizer of \( J(\cdot; S) \). Let \( x \in S \setminus \mathcal{C}_u \) be such that the level segment of \( \nabla u \) containing \( x \) does not intersect \( \partial S \) tangentially. Then there exists a line of curvature \( \Gamma \in W^{2,\infty}([0, T], S) \) and \( s^{\pm} \in W^{1,\infty}(0, T) \) with \( s^< < 0 < s^> \) such that \( x = \Gamma(T/2) \) (and \( [\Gamma] \) it is the connected component of \( S \setminus \partial_L[\Gamma] \) containing \( x \), and such that \( u := \tilde{u}|[\Gamma] \) is induced by \( \Gamma \), \( s^{\pm} \). Moreover, \( u \) is locally minimizing for \( J(\cdot; \Gamma) \) under its own first order boundary conditions on \( \partial_L[\Gamma] \).

**References**


