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by

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Entanglement of formation is an important measure of quantum entanglement. We present an experimental way to measure the entanglement of formation for arbitrary dimensional pure states. The measurement only evolves local quantum mechanical observables.

Quantum entangled states have become the most important physical resources in quantum communication, information processing and quantum computing. One of the most difficult and fundamental problems in entanglement theory is to quantify the quantum entanglement. A number of entanglement measures such as the entanglement of formation and distillation [1, 2], negativity [3] and concurrence [4] have been proposed. Among these entanglement measures, the entanglement of formation, which quantifies the required minimally physical resources to prepare a quantum state, plays important roles in quantum phase transition for various interacting quantum many-body systems [5] and may significantly affect macroscopic properties of solids [6]. Thus the quantitative evaluation of entanglement of formation is of great significance.

Comparing with the concurrence, entanglement of formation is more difficult to deal with, and less results have been derived. The experimental measurement of concurrence has been proposed in [7] for pure two-qubit systems by using two copies of the unknown quantum state. In [8] the authors presented an approach of measuring concurrence for arbitrary dimensional pure multipartite systems in terms of only one copy of the unknown quantum state. However, due to the complicated expression of entanglement of formation, there is no experimental way yet to determine the entanglement of formation for an unknown quantum pure state, except for the case of two-qubit systems for which the concurrence and entanglement of formation have a simple monotonic relations [2].

In this brief report we present an experimental determination of the entanglement of formation for arbitrary dimensional pure quantum states. The measurement only evolves local quantum mechanical observables and the entanglement of formation can be obtained according to the mean values of these observables.

The entanglement of formation is defined for bipartite systems. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be m and n $(m \leq n)$ dimensional complex Hilbert spaces with orthonormal basis $|i\rangle$, $i = 1, ..., m$, and $|j\rangle$, $i = 1, ..., n$ respectively. A pure quantum state on $\mathcal{H}_A \otimes \mathcal{H}_B$ is generally of the form,

$$|\psi\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} |ij\rangle, \qquad a_{ij} \in \mathbb{C} \qquad (1)$$

with normalization

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} a_{ij}^* = 1. \qquad (2)$$

The entanglement of formation of $|\psi\rangle$ is defined as the partial entropy with respect to the subsystems [1],

$$E(|\psi\rangle) = -\text{Tr}\left(\rho^A \log_2 \rho^A\right) = -\text{Tr}\left(\rho^B \log_2 \rho^B\right), \qquad (3)$$

where ρ^A (resp. ρ^B) is the reduced density matrix obtained by tracing $|\psi\rangle\langle\psi|$ over the space $\mathcal{H}_B$ (resp. $\mathcal{H}_A$). This definition can be extended to mixed states ρ by the convex roof,

$$E(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \qquad (4)$$

where the minimization goes over all possible ensemble realizations of ρ,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \ \ p_i \geq 0, \ \ \sum_i p_i = 1. \qquad (5)$$

A bipartite quantum state $|\psi\rangle$ can be written in the Schmidt form $|\psi\rangle = \sum_{i=1}^{m} \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, under suitable basis $|i_A\rangle \in \mathcal{H}_A$ and $|i_B\rangle \in \mathcal{H}_B$. λ_i, $i = 1, ..., m$, are also the eigenvalues of ρ^A. $E(|\psi\rangle)$ can be further expressed as

$$E(|\psi\rangle) = S(\rho^A) = -\sum_{i=1}^{m} \lambda_i \log \lambda_i. \qquad (6)$$

For two-qubit case, $m = n = 2$, $|\psi\rangle = a_{11}|00\rangle + a_{12}|01\rangle + a_{21}|10\rangle + a_{22}|11\rangle$, $|a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2 = 1$. (3) can be written as [2],

$$E(|\psi\rangle) = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right), \qquad (7)$$

where $h(x) = -x \log_2 x - (1-x) \log_2(1-x)$, $C = 2|a_{11}a_{22} - a_{12}a_{21}|$ is the concurrence. In this special case $E(|\psi\rangle)$ is just a monotonically increasing function of the concurrence C. However for $m \geq 3$, there is no such relations like (7) between the entanglement of formation and concurrence in general. Since for the case $m = 2$, due to the normalization condition, $\lambda_1 + \lambda_2 = 1$, only one free parameter is left in the formula (6). For general high dimensional case, $E(|\psi\rangle)$ depends on more free parameters. Nevertheless, if ρ^A has only two non-zero eigenvalues (each of which may be degenerate), the maximal non-zero diagonal determinant D of ρ^A is a generalized concurrence, namely, the corresponding entanglement of formation is again a monotonically increasing function of D [9]. The construction of such kind of states is presented in [10]. In [11], the results are generalized to more general case: relations like (7) holds for states with ρ^A having more non-zero eigenvalues such that all these eigenvalues are functions of two independent parameters.

To measure the quantity (6) experimentally, we first rewrite the expression (6) according to the entanglement of formation of some "two-qubit" states. Let L_α^A and L_β^B be the generators of special unitary groups $SO(m)$ and $SO(n)$, with the $m(m-1)/2$ generators L_α^A given by $\{|i\rangle\langle j| - |j\rangle\langle i|\}$, $1 \leq i < j \leq m$, and the $n(n-1)/2$ generators L_β^B given by $\{|k\rangle\langle l| - |l\rangle\langle k|\}$, $1 \leq k < l \leq n$, respectively. The matrix operators L_α^A (resp. L_β^B) have $m-2$ (resp. $n-2$) rows and $m-2$ (resp. $n-2$) columns that are identically zero.

Let $\rho = |\psi\rangle\langle\psi|$ be the density matrix with respect to the pure state $|\psi\rangle$. We define

$$\rho_{\alpha\beta} = \frac{L_\alpha^A \otimes L_\beta^B \, \rho \, (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger}{||L_\alpha^A \otimes L_\beta^B \, \rho \, (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger||}, \qquad (8)$$

where $\alpha = 1, 2, \cdots, \frac{m(m-1)}{2}$; $\beta = 1, 2, \cdots, \frac{n(n-1)}{2}$, and $||X|| = \sqrt{Tr(XX^\dagger)}$ is the trace norm of matrix X. As the matrix $L_\alpha^A \otimes L_\beta^B$ has $mn-4$ rows and $mn-4$ columns that are identically zero, $\rho_{\alpha\beta}$ has at most $4 \times 4 = 16$ nonzero elements and is called "two-qubit" state. $\rho_{\alpha\beta}$ is still a normalized pure state.

Theorem 1 *For any $m \otimes n$ ($m \leq n$) pure quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,*

$$E(|\psi\rangle) = \frac{1}{(m-1)^2} \sum_{\alpha\beta} \frac{E(\rho_{\alpha\beta}) + \log(C_{\alpha\beta})}{C_{\alpha\beta}}, \quad (9)$$

where $C_{\alpha\beta} = 1/Tr\{L_\alpha^A \otimes L_\beta^B |\psi\rangle\langle\psi| (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger\}$.

Proof. To calculate $E(\rho_{\alpha\beta})$ we denote $L_\alpha^A = |a\rangle\langle b| - |b\rangle\langle a|$ and $L_\beta^B = |c\rangle\langle d| - |d\rangle\langle c|$ for convenience, where $1 \leq a < b \leq m$ and $1 \leq c < d \leq m$. Set

$$\rho'_{\alpha\beta} = L_\alpha^A \otimes L_\beta^B |\psi\rangle\langle\psi| (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger. \qquad (10)$$

It is direct to verify that

$$\rho'_{\alpha\beta} = |\psi\rangle_{\alpha\beta}\langle\psi|, \qquad (11)$$

where $|\psi\rangle_{\alpha\beta} = \lambda_b \delta_{bd}|ac\rangle - \lambda_b \delta_{bc}|ad\rangle - \lambda_a \delta_{ad}|bc\rangle + \lambda_a \delta_{ac}|bd\rangle$.

We now compute the eigenvalues of $\rho'^A_{\alpha\beta} = Tr_B(\rho'_{\alpha\beta})$ according to the values of a, b, c and d:

i). $a \neq b \neq c \neq d$. We have $|\psi\rangle_{\alpha\beta} = 0$.

ii). $b > a = c < d$ and $b \neq d$. We get $|\psi\rangle_{\alpha\beta} = \sqrt{\lambda_a}|bd\rangle$. The eigenvalue of $\rho'_{\alpha\beta}$ corresponding to this case is λ_a. As $a = c$ can be chosen to be $1, 2, \cdots, m-2$, b and d have only $m-k$ and $m-k-1$ (corresponding to $a = c = k$, $k = 1, 2, \cdots, m-2$) kinds of choices. Altogether we have $(m-k)(m-k-1)$ eigenvalues of $\rho'^A_{\alpha\beta}$ to be λ_k in this case, with $k = 1, 2, \cdots, m-2$.

iii). $a < b = d > c$ and $a \neq c$. We have $|\psi\rangle_{\alpha\beta} = \sqrt{\lambda_b}|ac\rangle$. The eigenvalue of $\rho'^A_{\alpha\beta}$ is λ_b. In this case $b = d$ can be $3, 4, \cdots, m$. Then a and c have only $k-1$ and $k-2$ (corresponding to $b = d = k$, $k = 3, 4, \cdots, m$) kinds of choices. Hence we have $(k-1)(k-2)$ eigenvalues of $\rho'^A_{\alpha\beta}$ to be λ_k in this case, $k = 3, 4, \cdots, m$.

iv). $b > a = c < d$ and $b = d$. We obtain $|\psi\rangle_{\alpha\beta} = \sqrt{\lambda_b}|ac\rangle + \sqrt{\lambda_a}|bd\rangle$. The eigenvalues of $\rho'^A_{\alpha\beta}$ are λ_a and λ_b, and $a = c$ can be $1, 2, \cdots, m-1$. Then $b = d$ can be $k+1, k+2, \cdots, m$ (corresponding to $a = c = k, k = 1, 2, \cdots, m-1$). We have $(m-1)$ eigenvalues of $\rho'^A_{\alpha\beta}$ to be λ_k, $k = 1, 2, \cdots, m$.

v). $a < b = c < d$. We have $|\psi\rangle_{\alpha\beta} = -\sqrt{\lambda_b}|ad\rangle$. The eigenvalue of $\rho'^A_{\alpha\beta}$ is λ_b. $b = c$ can be $2, 3, \cdots, m-1$. Then a and d have only $k-1$ and $m-k$ (corresponding to $b = c = k$, $k = 2, 3, \cdots, m-1$) kinds of choices. We have $(k-1)(m-k)$ eigenvalues of $\rho'^A_{\alpha\beta}$ to be λ_k, $k = 2, 3, \cdots, m-1$.

vi). $c < d = a < b$. We have $|\psi\rangle_{\alpha\beta} = -\sqrt{\lambda_a}|bc\rangle$. The eigenvalue of $\rho'^A_{\alpha\beta}$ is λ_a. In this case $a = d$ can be $2, 3, \cdots, m-1$. c and b have only $k-1$ and $m-k$ (corresponding to $b = c = k$, $k = 2, 3, \cdots, m-1$) kinds of choices. Therefore we have $(k-1)(m-k)$ eigenvalues of $\rho'^A_{\alpha\beta}$ to be λ_k, with $k = 2, 3, \cdots, m-1$.

Let $\lambda^i_{\alpha\beta}$ stand for the eigenvalues of $\rho^A_{\alpha\beta}$. From the analysis of cases i)-vi) and formula (6), we get

$$E(|\psi\rangle) = -\frac{1}{(m-1)^2} \sum_{\alpha\beta} \sum_i \lambda^i_{\alpha\beta} \log(\lambda^i_{\alpha\beta}). \qquad (12)$$

Since

$$\rho_{\alpha\beta} = \frac{\rho'_{\alpha\beta}}{Tr\{\rho'_{\alpha\beta}\}} = C_{\alpha\beta}\rho'_{\alpha\beta}, \qquad (13)$$

we have $\sum_i \lambda^i_{\alpha\beta} C_{\alpha\beta} = 1$ for any α and β. Therefore

$$\begin{aligned}
E(\rho_{\alpha\beta}) &= -\sum_{i=1} C_{\alpha\beta}\lambda^i_{\alpha\beta} \log(C_{\alpha\beta}\lambda^i_{\alpha\beta}) \\
&= -\sum_{i=1} C_{\alpha\beta}\lambda^i_{\alpha\beta} \log(C_{\alpha\beta}) - \sum_{i=1} C_{\alpha\beta}\lambda^i_{\alpha\beta} \log(\lambda^i_{\alpha\beta}) \\
&= -\log(C_{\alpha\beta}) - C_{\alpha\beta} \sum_{i=1} \lambda^i_{\alpha\beta} \log(\lambda^i_{\alpha\beta}).
\end{aligned}$$

That is

$$-\sum_{i=1}^{m} \lambda_{\alpha\beta}^i \log(\lambda_{\alpha\beta}^i) = \frac{E(\rho_{\alpha\beta}) + \log(C_{\alpha\beta})}{C_{\alpha\beta}}. \qquad (14)$$

Substituting (14) into (12), we obtain that

$$E(|\psi\rangle) = \frac{1}{(m-1)^2} \sum_{\alpha\beta} \frac{E(\rho_{\alpha\beta}) + \log(C_{\alpha\beta})}{C_{\alpha\beta}}, \qquad (15)$$

which proves the theorem. $\qquad\square$

The theorem shows that one can derive the entanglement of formation of a pure quantum state by measuring the values of the entanglement of formation of all the states $\rho_{\alpha\beta}$ and values of $C_{\alpha\beta}$. Here if $|\psi\rangle_{\alpha\beta} = 0$, then $C_{\alpha\beta}$ goes to infinity and this term does not contribute to the summation in (9). Hence the summation $\sum_{\alpha\beta}$ in (9) simply goes over all the terms such that $Tr\{L_\alpha^A \otimes L_\beta^B |\psi\rangle\langle\psi|(L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger\} \neq 0$.

With formula (9), we now show how to get the value of $E(|\psi\rangle)$ experimentally by measuring the quantities on the right hand side of (9).

The quantity $C_{\alpha\beta} = 1/Tr\{\rho'_{\alpha\beta}\}$ can be determined by $Tr\{\rho'_{\alpha\beta}\}$. Since $Tr\{\rho'_{\alpha\beta}\} = \langle\psi|(L_\alpha^A)^\dagger L_\alpha^A \otimes (L_\beta^B)^\dagger L_\beta^B|\psi\rangle$, one can obtain $C_{\alpha\beta}$ by measuring the local Hermitian operator $(L_\alpha^A)^\dagger L_\alpha^A \otimes (L_\beta^B)^\dagger L_\beta^B$ associated with the state $|\psi\rangle$.

To measure $E(\rho_{\alpha\beta})$, we first note that although $\rho_{\alpha\beta}$ are $m \otimes n$ bipartite quantum states, they are basically "two-qubit" ones. For given $L_\alpha = |i\rangle\langle j| - |j\rangle\langle i|$ and $L_\beta = |k\rangle\langle l| - |l\rangle\langle k|$, $i \neq j$, $k \neq l$, the non-zero elements of $\rho_{\alpha\beta}$ are located at the $i*(m-1)+k$th, $i*(m-1)+l$th, $j*(m-1)+k$th, and $j*(m-1)+l$th rows and the $i*(m-1)+k$th, $i*(m-1)+l$th, $j*(m-1)+k$th, and $j*(m-1)+l$th columns. They constitute a 4×4 matrix,

$$\sigma'_{\alpha\beta} = \begin{pmatrix} \rho_{ik,ik} & \rho_{ik,il} & \rho_{ik,jk} & \rho_{ik,jl} \\ \rho_{il,ik} & \rho_{il,il} & \rho_{il,jk} & \rho_{il,jl} \\ \rho_{jk,ik} & \rho_{jk,il} & \rho_{jk,jk} & \rho_{jk,jl} \\ \rho_{jl,ik} & \rho_{jl,il} & \rho_{jl,jk} & \rho_{jl,jl} \end{pmatrix}.$$

Set $\sigma_{\alpha\beta} = \sigma'_{\alpha\beta}/Tr\{\sigma'_{\alpha\beta}\}$. Obviously $E(\rho_{\alpha\beta}) = E(\sigma_{\alpha\beta})$. But $\sigma_{\alpha\beta}$ are actually two-qubit pure states. According to the formula (7), $E(\sigma_{\alpha\beta})$ is determined by the concurrence $C(\sigma_{\alpha\beta}) = C(\rho_{\alpha\beta})$. Therefore if we can measure the quantity $C(\rho_{\alpha\beta})$, we can obtain $E(\rho_{\alpha\beta})$.

The quantity $C(\rho_{\alpha\beta})$ can be measured experimentally in terms of the method introduced in [8], with a few modifications of the measurement operators. Corresponding to the case of $L_\alpha = |i\rangle\langle j| - |j\rangle\langle i|$ and $L_\beta = |k\rangle\langle l| - |l\rangle\langle k|$, we define $m \times m$ matrix operators Σ_s, $s = 0, 1, 2, 3$, such that $(\Sigma_0)_{pq} = \delta_{pi}\delta_{qi} + \delta_{pj}\delta_{qj}$, $(\Sigma_1)_{pq} = \delta_{pi}\delta_{qj} + \delta_{pj}\delta_{qi}$, $(\Sigma_2)_{pq} = I\delta_{pi}\delta_{qj} - I\delta_{pj}\delta_{qi}$, $(\Sigma_3)_{pq} = \delta_{pi}\delta_{qi} - \delta_{pj}\delta_{qj}$, $p, q = 1, ..., m$. Similarly we define $n \times n$ matrix operators $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3 by replacing the indices i, j in $\Sigma_0, \Sigma_1, \Sigma_2$ and Σ_3 with k, l respectively, and setting $p, q = 1, ..., n$. It is straightforward to derive that $C(\rho_{\alpha\beta})$ can be expressed as the mean values of the above local observables,

$$\begin{aligned} C^2(\rho_{\alpha\beta}) = {} & \frac{1}{2} + \frac{C_{\alpha\beta}^2}{2} \left(\langle\Sigma_3 \otimes \Gamma_3\rangle^2 - \langle\Sigma_3 \otimes \Gamma_0\rangle^2 \right. \\ & - \langle\Sigma_0 \otimes \Gamma_3\rangle^2 - \langle\Sigma_0 \otimes \Gamma_1\rangle^2 + \langle\Sigma_3 \otimes \Gamma_1\rangle^2 \\ & \left. - \langle\Sigma_0 \otimes \Gamma_2\rangle^2 + \langle\Sigma_3 \otimes \Gamma_2\rangle^2 \right). \end{aligned} \qquad (16)$$

In summary we have presented an experimental way to measure the entanglement of formation for arbitrary dimensional pure states, by measuring some local quantum mechanical observables. We reduced the difficult problem to find the concurrence of "two-qubit" states for which many results have been already derived. Recently high dimensional bipartite systems like in NMR and nitrogen-vacancy defect center have been successfully used for quantum computation and simulation experiments [18]. Our results present a plausible way to measure the entanglement of formation in these systems and to investigate the roles played by the entanglement of formation in these quantum information processing.

So far experimental measurement on entanglement of formation and concurrence concerns only pure states. For mixed states, less is known except for experimental determination of separability, both sufficiently and necessary, for two-qubit [19] and qubit-qutrit systems [20]. Generally (4) has only analytical results for some special states [21] and analytical lower bounds [22] which are not experimentally measurable. Recently in [23] we have presented a measurable lower bound of entanglement of formation. The formula (9) for pure state may also help to study measurable lower bounds of entanglement of formation for mixed states.

[1] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, Phys. Rev. A 54, 3824 (1996).
C.H. Bennett, H.J. Bernstein, S. Popescu, and B. Schumacher, Phys. Rev. A 53, 2046 (1996).
V. Vedral, M.B. Plenio, M.A. Rippin, and P.L. Knight, Phys. Rev. Lett. 78, 2275 (1997).

[2] W. K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[3] A. Peres, Phys. Rev. Lett. 77 1413 (1996).
K. życzkowski and P. Horodecki, Phys. Rev. A 58, 883 (1998).
[4] A. Uhlmann, Phys. Rev. A 62, 032307 (2000).
P. Rungta, V. Bužek, C.M. Caves, M. Hillery, and G.J.

Milburn, Phys. Rev. A **64**, 042315 (2001).

S. Albeverio and S. M. Fei, J. Opt. B: Quantum Semiclass. Opt. 3, 223(2001).

[5] A. Osterloh et al., Nature 416 , 608 (2002).

L. A. Wu, M. S. Sarandy, and D. A. Lidar, Phys. Rev. Lett. 93, 250404 (2004).

[6] S. Ghosh, T. F. Rosenbaum, G. Aeppli, S. N. Coppersmith, Nature 425, 48 (2003).

V. Vedral, Nature 425, 28 (2003).

[7] F. Mintert, M. Kuś and A. Buchleitner, Phys. Rev. Lett. 95, 260502 (2005).

S. P. Walborn, P. H. Souto Ribeiro, L. Davidovich, F. Mintert and A. Buchleitner, Nature. 440, 20(2006).

[8] S. M. Fei, M. J. Zhao, K. Chen and Z. X. Wang, Phys. Rev. A 80, 032320(2009).

[9] S. M. Fei, J. Jost, X. Q. Li-Jost and G. F. Wang, Phys. Lett. A 310, 333(2003).

[10] S. M. Fei and X. Q. Li-Jost, Rep. Math. Phys. 53, 195(2004).

[11] S. M. Fei, Z. X. Wang and H. Zhao, Phys. Lett. A 329, 414(2004).

[12] E. Gerjuoy Phys. Rev. A 67, 052308 (2003).

[13] K. Chen, S. Albeverio, and S. M. Fei, Phys. Rev. Lett. 95, 210501(2005).

[14] B. M. Terhal and K. H. Vollbrecht, Phys. Rev. Lett. 85, 2625(2000).

[15] F. Mintert and A. Buchleitner, Phys. Rev. Lett. 98, 140505 (2007).

[16] C. J. Zhang, Y. X. Gong, Y. S. Zhang, and G. C. Guo, Phys. Rev. A 78, 042308(2008).

[17] L. Aolita, A. Buchleitner, and F. Mintert, Phys. Rev. A 78, 022308(2008).

[18] F. Shi et al, Phys. Rev. Lett. 105, 040504 (2010).

J. F. Du et al, Phys. Rev. Lett. 104, 030502 (2010).

[19] S. Yu, J. W. Pan, Z. B. Chen and Y. D. Zhang, Phys. Rev. Lett. 91, 21(2003).

[20] S. M. Fei, M. J. Zhao, K. Chen and Z. X. Wang, Phys. Rev. A 80, 032320(2009).

[21] B. M. Terhal and Karl G. H. Vollbrecht, Phys. Rev. Lett. 85, 2625 (2000).

K. G. H. Vollbrecht and R. F. Werner, Phys. Rev. A 64, 062307(2001).

[22] K. Chen, S. Albeverio and S. M. Fei, Phys. Rev. Lett. 95, 210501(2005).

[23] M. Li and S. M. Fei, Phys. Rev. A 82, 044303(2010).