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by

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CLOSURE MEASURES AND THE TENT MAP

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ABSTRACT. We quantify the relationship between the dynamics of a particular time-discrete dynamical system, the tent map, and the induced dynamics at a symbolical level in information theoretical terms. The symbol dynamics is obtained by choosing a partition point $\alpha \in [0, 1]$ and lumping together the points in the intervals $[0, \alpha]$ or $(\alpha, 1]$, resp. Interpreting the original dynamics and the symbolic one as different levels, this allows us to quantitatively evaluate and compare various closure measures that have been proposed for identifying emergent macro-levels of a dynamical system. In particular, we can see how these measures depend on the choice of the partition point α, with $\alpha = \frac{2}{3}$ yielding a minimum. Also, when we study the iterated tent map, i.e., probe more time steps, we get more refined insights into this relationship between the two levels, and even a whole hierarchy of mesoscopic levels.

1. INTRODUCTION

Consider a dynamical system $T : X \to X$ on a probability space X with measure μ where T can be a measurable map, a Markov kernel, etc. Suppose we have an operator $\phi : X \to \hat{X}$ – for instance a coarse-graining, aggregation, averaging, etc. – of the lower, microscopic level X onto an upper level $\hat{X}$. As the dynamics evolves on the lower level, an induced dynamics can be observed on the upper state space $\hat{X}$. We say that the upper level is *closed* if it can be also represented by a dynamical system: there is a measurable map, a Markov kernel, etc. $\hat{T} : \hat{X} \to \hat{X}$ such that $\phi \circ T = \hat{T} \circ \phi$. The maps ϕ_α may correspond to

$$
\begin{array}{ccc}
\hat{X} & \dashrightarrow{\hat{T}} & \hat{X} \\
\phi \uparrow & & \uparrow \phi \\
X & \xrightarrow{T} & X
\end{array}
$$

FIGURE 1. Basic setup of multilevel dynamical system.

operations like coarse-graining, aggregation, averaging etc., and X and $\hat{X}$ represent, respectively, microscopic and macroscopic state spaces. Furthermore, the maps ϕ_α, with their scalar parameter $\alpha \geq 0$, refer to different *scales* where the coarse-graining, etc. is carried out. We characterize a relevant scale as one where special structural or dynamical regularities can be detected.

Closure measures provide a link between the two concepts of "levels" and "scales" because they should allow us to identify emergent levels, i.e scales for which a(n approximately) closed description exists, by means of quantifying to which extent the induced system deviates from being closed. The following closure measures have been proposed so far:

Informational closure: In [11] we called the higher process to be informationally closed, if there is no information flow from the lower to the higher level. Knowledge of the microstate will not improve predictions of the macrostate, i.e for $s_t = \phi_\alpha(x_t)$ we have
(1.1)
$$
I(s_{t+1} : x_t | s_t) = H(s_{t+1}|s_t) - H(s_{t+1}|s_t, x_t) = 0 \,,
$$

where I denotes the conditional mutual information, and H the entropy.

The entropy of a random variable $Y : X \to \mathbb{R}$ on a probability space X with measure μ is defined by

$$
H(Y) = - \sum_y p(y) \log(p(y))
$$

where $p(y) = \mu(Y = y)$ denotes the distribution on $\mathbb{R}$ induced by Y – the probability mass function of Y. We use logarithms to base 2. The entropy will then be measured in bits. The entropy is a measure of the average uncertainty in the random variable.

Conditional entropy $H(Z|Y)$ for two random variables Z and Y with conditional distribution $p(z|y)$ is defined as

$$
H(Z|Y) = - \sum_y p(y) \sum_z p(z|y) \log(p(z|y)) \,,
$$

which is the average uncertainty of a random variable Z conditional on the knowledge of another random variable Y.

The reduction in uncertainty due to another random variable is called the mutual information

$$
I(Z : Y) = H(Z) - H(Z|Y) \,.
$$

The mutual information $I(Z : Y)$ is a measure of the dependence between the two random variables. It is symmetric in Z and Y and always non negative and is

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equal to zero if and only if Z and Y are independent, see [4].

Markovianity: In [13] Shalizi and Moore, and in [7] Görnerup and Jacobi proposed Markovianity of the upper process $s_t \to s_{t+1}$ as a property of an emergent level. In this case s_{t+2} is independent of s_t given s_{t+1}, which can be expressed again in terms of the conditional mutual information as

$$(1.2) \qquad I(s_t : s_{t+2}|s_{t+1}) = 0\,.$$

Hence, it is reasonable to measure the deviation of the process from being Markovian by means of the conditional mutual information Eq. (1.2).

Predictive Efficiency: In his PhD thesis Shalizi [12] proposed to measure the "predictive efficiency" to identify emergent levels. A relevant scale would correspond to a description with a high predictive efficiency. The basic idea behind predictive efficiency is to quantify the trade-off between state complexity and predictive information. Or in other words — how much predictability one gains by increasing the state space. We will discuss two possibilities to define the predictive efficiency formally: On the one hand the ratio between the excess entropy [5], and statistical complexity [6] — this is also the measure proposed by Shalizi [12]. [1] On the other hand we will consider a variational functional that can be related to the information bottleneck method introduced in [15].

Whereas our previous paper [11] was dedicated to uncover different notions of closure [2] and their interdependencies in general, in this paper we focus on the explicit computation of the three closure measures listed above for the dynamical system induced by the tent map. More precisely, we set $X = [0,1]$ to be the unit interval endowed with the Lebesgue measure, the unique ergodic measure on X with respect to the tent map

$$T(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2 - 2x & \text{else} \end{cases}\,,$$

---

[1] Note that the excess entropy is also known under the names effective measure complexity [8] or predictive information [3] while the statistical complexity was also introduced as true measure complexity [8] or forecast complexity [16].

[2] Beside the already mentioned notions of closure we defined the one we called "observational commutativity". But due to theorem 4.1 in [11] is equivalent to informational closure Eq. (1.1) if the coarse-graining $\phi_\alpha : X \to \hat{X}$ is deterministic. This is the case if one aggregates all points contained in the interval $[0,\alpha]$ or $(\alpha,1]$, respectively. Therefore, we skipped mentioning a corresponding closure measure in the main text.

with full support, and for an $\alpha \in [0,1]$ we define $\phi_\alpha : X \to \hat{X} = \{0,1\}$ as

$$\phi_\alpha(x) = \begin{cases} 0 & \text{if } 0 \le x \le \alpha \\ 1 & \text{else} \end{cases}\,.$$

So, from the sequence $x_n = T^n(x)$, for an initial value $x \in X$, one obtains a derived symbol dynamics $s_n = \phi_\alpha(x_n) \in \{0,1\}$ and the probability of finding s_n in the state 0 is given by the probability that x_n lies in the interval $[0,\alpha]$ which is equal α.
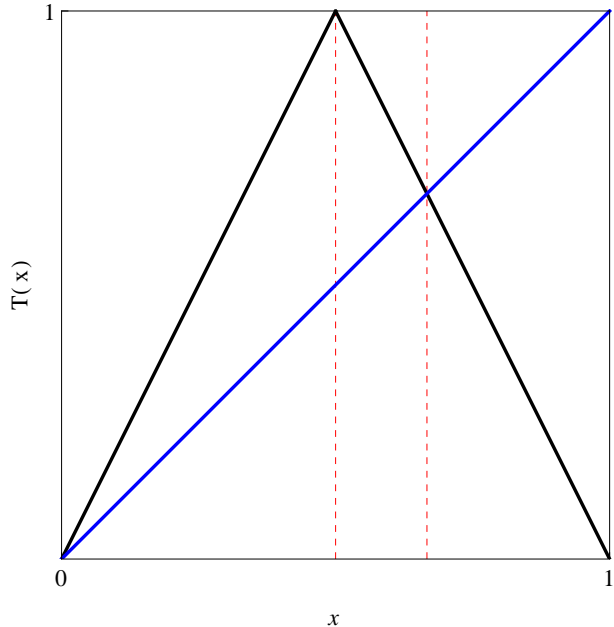

FIGURE 2. The graph of the tent map has its peak at $1/2$ and intersects the diagonal at $2/3$

In the present paper we proceed as follows. Firstly, we compute the joint probability of one consecutive time step in the macrostate, i.e we compute the probability distribution $p(s_{n+1}, s_n)$ for all $\alpha \in [0,1]$. This allows us to compute Eq. (1.1) the information flow from the lower to the upper level for all α. It turns out that Eq. (1.1) has a local minimum in $\alpha = 2/3$, i.e the point where the tent map intersects the diagonal. This might be related to the fact that for $s_n = 1$ we have $2/3 \le x_n$ hence $x_{n+1} = T(x_n) \le 2/3$ and therefore always $s_{n+1} = 0$, i.e the successor state of $s_n = 1$ is always 0. This fact was used in [2] to distinguish the symbol dynamics generated by the tent map from a purely random one. Furthermore, we compute a first order approximation of the predictive efficiency given as the ratio of the one step mutual information and the state entropy, and in addition the variational functional related to the information bottleneck method.

Secondly, we determine for all $\alpha \in [0,1]$ the joint probability $p(s_{n+2}, s_{n+1}, s_n)$ of two consecutive time steps in the macrostate which leads us to the computation of Eq. (1.2) as function in $\alpha \in [0,1]$, with the result that beside the well known Markovianity of the symbol dynamics obtained from the choice $\alpha = 1/2$ as partitioning point, also $\alpha = 2/3$ yields a Markovian symbol dynamics.

Next, we move to the general case of arbitrary many time steps and investigate for which choices of the partitioning point α one obtains forbidden sequences of a certain length. We have already mentioned that for $0 < \alpha < 2/3$ every sequence of length two may occur. Beside this property, 0 and $2/3$ are the two fix points of the tent map. This observation leads to a generalization: Let us denote with p_0 the greatest m-periodic point of the tent map, i.e m is the smallest integer such that $T^m(p_0) = p_0$, smaller $1/2$ and p_1 the smallest m-periodic point of the tent map bigger than $1/2$. Then for all partitioning points $\alpha \in (p_0, p_1)$ all sequences of length $\leq m+1$ may occur. Furthermore, we prove that this result is closely related to the solution of the problem to find partitioning points α such that the dynamics of the extended states $(s_{n+m}, \ldots, s_n)$ turns out to be Markovian. We prove that this is the case for all $m+1$-periodic points of the tent map contained in the interval (p_0, p_1). This result provides a countable amount of partitioning points whose aggregations lead to a Markovian dynamics, at least on an extended state space. We conclude with a brief study of possible aggregations of states within these extended state spaces which lead again to a Markovian dynamical system. "Possible" in the sense of two different points of view: Firstly, the Markovian dynamics on the aggregated states leads to a closed description mentioned previously, i.e such that the diagram Fig. (1) commutes. Secondly, instead of obtaining a closed description we look for aggregations such that the partition of the unit interval is a Markovian partition. That the first aspect differs from the second one is shown by a simple example coming from the Markovian dynamics induced by the partitioning point $\alpha = 2/5$.

2. ONE TIME STEP

We adopt the previous notation: $x_n \in X = [0,1]$ denotes a microstate of the system, $s_n = \phi_\alpha(x_n) \in \hat{X} = \{0,1\}$ the corresponding macrostate which depends on the choice of the partition point $\alpha \in [0,1]$. With p we denote the distribution induced on $\hat{X}$ by the aggregation map ϕ_α and the uniform Lebesgue measure λ on X, which yields $p(0) = \lambda([0,\alpha]) = \alpha$

	$\alpha \leq 2/3$	$\alpha > 2/3$
$p(0,0)$	$\alpha/2$	$2\alpha - 1$
$p(1,0)$	$\alpha/2$	$1 - \alpha$
$p(0,1)$	$\alpha/2$	$1 - \alpha$
$p(1,1)$	$1 - 3/2\alpha$	0

TABLE 1. The joint probabilites $p(s_{n+1}, s_n)$ for one time step.

and $p(1) = 1 - \alpha$, respectively. In the present section we compute the joint distribution of one time step, i.e $p(s_{n+1}, s_n)$.

The joint distribution induces a partition $\mathcal{A}$, consisting of at most four sets, of the unit interval $X = [0,1]$ via

$$A_{\epsilon_1 \epsilon_0} = \{x \in X : (\phi_\alpha(x), \phi_\alpha \circ T(x)) = (\epsilon_1, \epsilon_0)\} ,$$

with $\epsilon_0, \epsilon_1 \in \{0,1\}$. Let be $\gamma_0 = \alpha/2$ and $\gamma_1 = 1 - \alpha/2$ the two preimages of α with respect to the tent map, i.e $T(\gamma_{0,1}) = \alpha$. One checks easily that the partition sets $A_{\epsilon_1 \epsilon_0} \in \mathcal{A}$ are unions of intervals whose endpoints are contained in the set $\{0, 1, \alpha, \gamma_0, \gamma_1\}$. This, and some combinatorial considerations, yields the following values for the joint distribution $p(\epsilon_1, \epsilon_0) = \lambda(A_{\epsilon_1, \epsilon_0})$, see table 1. This enables us to compute the complexity measures mentioned in the introduction. We start with the entropy. For $\alpha \leq 2/3$ we have

$$H(s_n) = -\alpha \log(\alpha) - (1-\alpha)\log(1-\alpha)$$
$$H(s_{n+1}, s_n) = -3/2\alpha \log(\alpha/2)$$
$$- (1 - 3/2\alpha)\log(1 - 3/2\alpha)$$

The conditional entropy becomes

$$H(s_{n+1}|s_n) = H(s_{n+1}, s_n) - H(s_n)$$
$$= 3/2\alpha - \alpha/2 \log(\alpha) + (1-\alpha)\log(1-\alpha)$$
$$- (1 - 3/2\alpha)\log(1 - 3/2\alpha)$$

and the mutual information turns out to be

$$I(s_{n+1}, s_n) = H(s_{n+1}) - H(s_{n+1}|s_n)$$
$$= -3/2\alpha - \alpha/2 \log \alpha$$
$$- 2(1-\alpha)\log(1-\alpha)$$
$$+ (1 - 3/2\alpha)\log(1 - 3/2\alpha)$$

For $\alpha > 2/3$ we have

$$H(s_n) = -\alpha \log(\alpha) - (1-\alpha)\log(1-\alpha)$$
$$H(s_{n+1}, s_n) = -(2\alpha - 1)\log(2\alpha - 1)$$
$$- 2(1-\alpha)\log(1-\alpha)$$

The conditional entropy becomes

$$H(s_{n+1}|s_n) = -(2\alpha - 1)\log(2\alpha - 1) + \alpha\log(\alpha)$$
$$- (1 - \alpha)\log(1 - \alpha)$$

and the mutual information

$$I(s_{n+1}, s_n) = -2\alpha\log\alpha + (2\alpha - 1)\log(2\alpha - 1)$$

2.1. Informational flow.

The information flow between the micro-level corresponding to state x_n and the coarse-grained level s_n is defined by the conditional mutual information

$$I(s_{n+1} : x_n|s_n) = H(s_{n+1}|s_n) - H(s_{n+1}|s_n, x_n).$$

Since s_{n+1} is fully determined by x_n, the second term vanishes and we have

$$I(s_{n+1} : x_n|s_n) = H(s_{n+1}|s_n),$$

i.e. the information flow is equal to the conditional entropy on the coarse grained level, which has a local minimum in $\alpha = 2/3$.
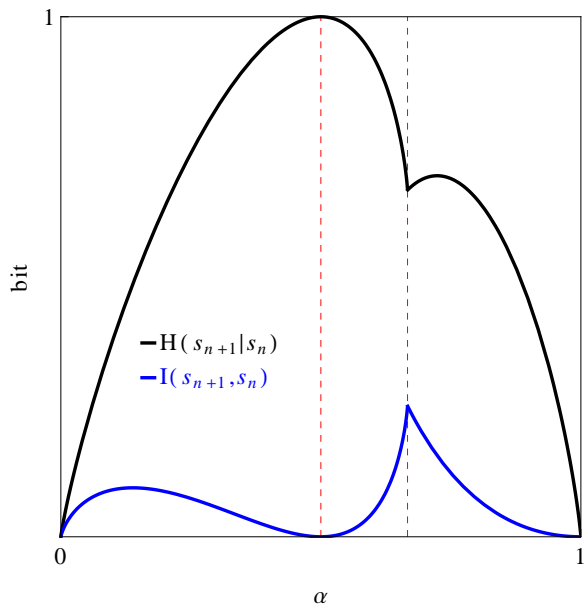

FIGURE 3. Conditional entropy and mutual information as functions of the partition parameter α. maximum and minimum turn out to be in $1/2$ and $2/3$.

2.2. Prediction efficiency.

Prediction efficiency tries to quantify the trade-off between the size of the state space - measured by its entropy - and the corresponding mutual information between the state and the future. Here we restrict ourselves to a first order approximation using only the one-step mutual information instead of the full predictive information. Nevertheless, both are equal if the system is Markovian.

Moreover, we use the entropy of the given state instead of the entropy of the causal states [6]. Subject to these assumptions we quantify predictive efficiency in two different ways. On the one hand as the ratio

$$(2.1) \qquad PE1(\alpha) = \frac{I(s_{n+1}, s_n)}{H(s_n)},$$

a proxy of prediction efficiency [12] based on the one step mutual information, or on the other hand as a variational functional

$$(2.2) \qquad PE2(\alpha, \beta) = I(s_{n+1}, s_n) - \beta H(s_n)$$

corresponding to the information bottleneck method [15]. Since $s_n = \phi_\alpha(x_n)$, we have $H(s_n) = I(s_n, x_n)$ and can interpret s_n as the bottleneck variable which strives to extract the information x_n contains about s_{n+1}. Here, the additional parameter β acts as a Lagrangian multiplier which adjusts the tradeoff between preserving information about s_{n+1} and reducing information about x_n. One should observe that for $\beta = 0$ the term (2.2) concurs with the mutual information and for $\beta = 1$ with the negative conditional entropy.

In general, depending on the tradeoff parameter β a different coarse graining α could be an optimal information bottleneck. But in the case of the tent map, $\alpha = 2/3$ is a local maximum in any case (see Fig. 5).
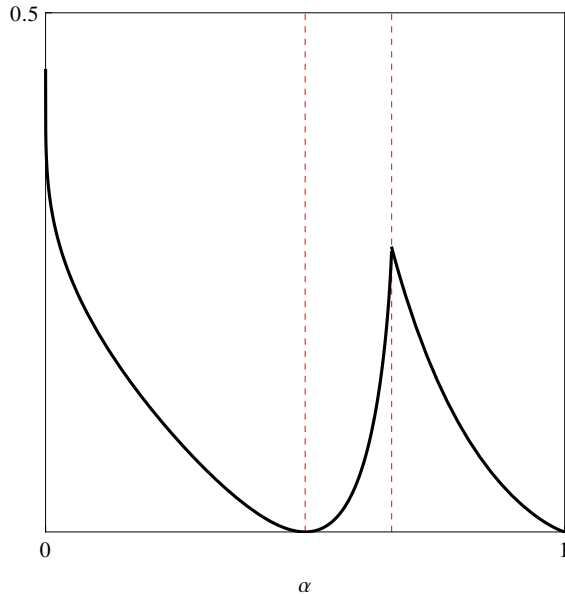

FIGURE 4. Predictive efficiency (2.1) as function of the partition parameter α. The values $\alpha = 1/2, 2/3$ are labeled.

From the analytical expressions for the mutual information $I(s_{n+1}, s_n)$ and the entropy $H(s_n)$ of the macrostate space one verifies the values $PE1(0) =$

$1/2$, $PE1(1/2) = 0$ and that the function (2.1) has a local maximum in $\alpha = 2/3$.
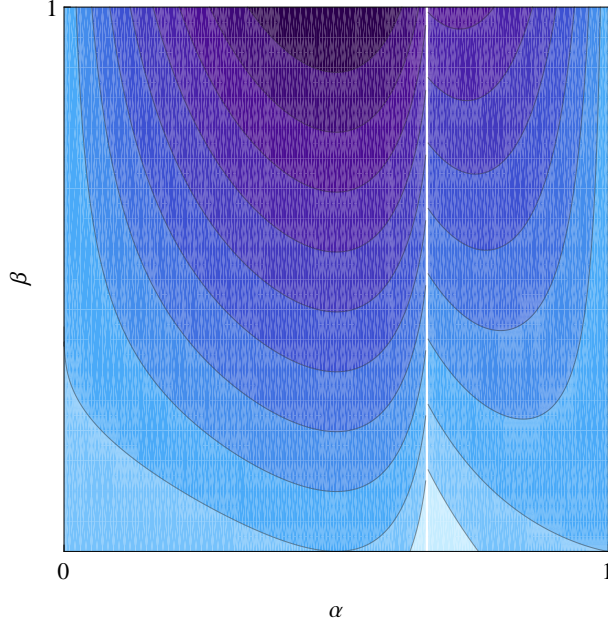

FIGURE 5. Predictive efficiency (2.2) as function of the partition threshold α and the parameter β as a contour plot where brighter colors indicate increasing values. The isoclines vary from 0.2, 0.1, …, −0.8

3. Two Time Steps

We tackle the problem to check when the macro-level process $s_n \to s_{n+1}$ is Markovian. Markovianity holds if the term $I(s_n, s_{n+2}|s_{n+1})$ in Eq. (1.2) vanishes, and the deviation from 0 measures to which extent the symbol dynamics drifts away from being Markovian. To compute the mutual information term a further refinement of the partition is needed, induced by the joint distribution $p(s_{n+2}, s_{n+1}, s_n)$ of two consecutive time steps. Assembling all sets
(3.1)
$$A_{\epsilon_2\epsilon_1\epsilon_0} = \left\{ x \in X : \phi_\alpha \circ T^i(x) = \epsilon_i, \, i = 0, 1, 2 \right\}$$

with $\epsilon_0, \epsilon_1, \epsilon_2 \in \{0, 1\}$ and $T^0 = \mathrm{id}_{[0,1]}$, yields this partition. Analogously to the one time step , we need to compute the preimage $\{\gamma_{00}, \gamma_{01}\}$ of γ_0 and the preimage $\{\gamma_{10}, \gamma_{11}\}$ of γ_1 and obtain

$$\begin{array}{cc} \gamma_{00} = \frac{\alpha}{4} & \gamma_{01} = 1 - \frac{\alpha}{4} \\ \gamma_{10} = \frac{1}{2} - \frac{\alpha}{4} & \gamma_{11} = \frac{1}{2} + \frac{\alpha}{4} \end{array}$$

Again, the sets $A_{\epsilon_2\epsilon_1\epsilon_0}$ of the partition are unions of intervals whose endpoints are contained in the set $\{0, 1, \alpha, \gamma_0, \gamma_1, \gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}\}$. Some combinatorics yields the values of the joint distribution $p(\epsilon_2, \epsilon_1, \epsilon_0) =$

	$0 < \alpha < 2/5$	$2/5 < \alpha < 2/3$
$p(0,0,0)$	$\alpha/4$	$\alpha/4$
$p(1,0,0)$	$\alpha/4$	$\alpha/4$
$p(0,1,0)$	0	$5/4\alpha - 1/2$
$p(1,1,0)$	$\alpha/2$	$1/2 - 3/4\alpha$
$p(0,0,1)$	$\alpha/4$	$\alpha/4$
$p(1,0,1)$	$\alpha/4$	$\alpha/4$
$p(0,1,1)$	$\alpha/2$	$1/2 - 3/4\alpha$
$p(1,1,1)$	$1 - 2\alpha$	$1/2 - 3/4\alpha$

	$2/3 < \alpha < 4/5$	$4/5 < \alpha < 1$
$p(0,0,0)$	$7/4\alpha - 1$	$3\alpha - 2$
$p(1,0,0)$	$\alpha/4$	$1 - \alpha$
$p(0,1,0)$	$1 - \alpha$	$1 - \alpha$
$p(1,1,0)$	0	0
$p(0,0,1)$	$\alpha/4$	$1 - \alpha$
$p(1,0,1)$	$1 - 5/4\alpha$	0
$p(0,1,1)$	0	0
$p(1,1,1)$	0	0

TABLE 2. The joint probabilities $p(s_{n+2}, s_{n+1}, s_n)$ for two consecutive time steps

$\lambda(A_{\epsilon_2\epsilon_1\epsilon_0})$ which are listed in tabel 2.

We can decompose the mutual information Eq. (1.2)

$$\begin{aligned} I(s_n, & s_{n+2}|s_{n+1}) \\ &= H(s_{n+2}|s_{n+1}) - H(s_{n+2}|s_n, s_{n+1}) \\ &= H(s_{n+2}, s_{n+1}) - H(s_{n+1}) - H(s_{n+2}, s_{n+1}, s_n) \\ &\quad + H(s_{n+1}, s_n) \\ &= 2H(s_{n+1}, s_n) - H(s_{n+2}, s_{n+1}, s_n) - H(s_n), \end{aligned}$$

where the last step follows from the stationarity of the process which ensures $H(s_{n+2}, s_{n+1}) = H(s_{n+1}, s_n)$ and $H(s_{n+1}) = H(s_n)$. The previously computed joint distributions for the one- and two-step leads us to the following analytical expressions for the mutual information.

$0 < \alpha < 2/5$.

$$\begin{aligned} I(s_n, s_{n+2}|s_{n+1}) = & -(2 - 3\alpha)\log(1 - 3/2\alpha) \\ & + (1 - 2\alpha)\log(1 - 2\alpha) + (1 - \alpha)\log(1 - \alpha) \end{aligned}$$

$2/5 < \alpha < 2/3$.

$$\begin{aligned} I(s_n, s_{n+2}|s_{n+1}) = & -3\alpha\log(\alpha/2) \\ & - (2 - 3\alpha)\log(1 - 3/2\alpha) + \alpha\log(\alpha/4) \\ & + (5/4\alpha - 1/2)\log(5/4\alpha - 1/2)) \\ & + (3/2 - 9/4\alpha)\log(1/2 - 3/4\alpha) \\ & + \alpha\log(\alpha) + (1 - \alpha)\log(1 - \alpha) \end{aligned}$$

$2/3 < \alpha < 4/5$.

$$\begin{aligned}
I(s_n, s_{n+2}|s_{n+1}) = &-2(2\alpha - 1)\log(2\alpha - 1) \\
&- 2(1 - \alpha)\log(1 - \alpha) + \alpha/2\log(\alpha/4) \\
&+ (7/4\alpha - 1)\log(7/4\alpha - 1) + \alpha\log(\alpha) \\
&+ (1 - 5/4\alpha)\log(1 - 5/4\alpha)
\end{aligned}$$

$4/5 < \alpha < 1$.

$$\begin{aligned}
I(s_n, s_{n+2}|s_{n+1}) = &-2(2\alpha - 1)\log(2\alpha - 1) \\
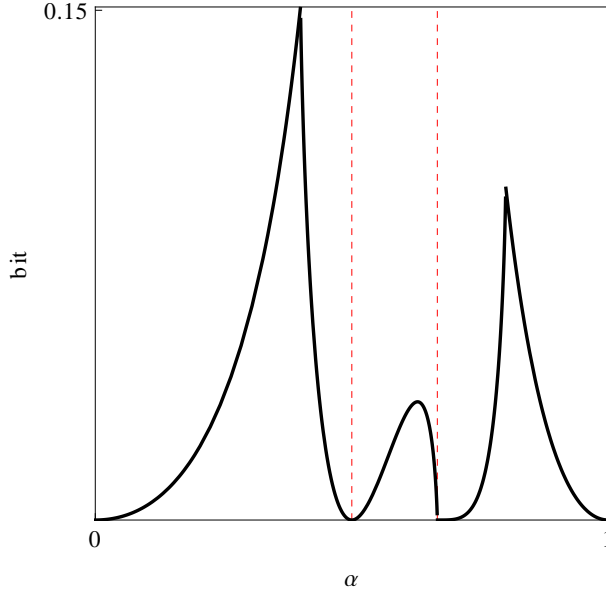&+ (3\alpha - 2)\log(3\alpha - 2) + \alpha\log(\alpha)
\end{aligned}$$

FIGURE 6. The mutual information Eq. (1.2) as a function of the partition parameter α. The values $\alpha = 1/2$, $2/3$ are labeled.

As one can read of the graphic, and also check analytically, the mutual information Eq. (1.2) is zero when $\alpha = 1/2$ and $\alpha = 2/3$, hence the symbol dynamics coming from these choices of the partition point are Markovian. For $\alpha = 2/3$, the Markov kernel of the upper process $\hat{T} : \hat{X} \to \hat{X}$ is described by the 2×2-matrix $(p(\epsilon_0|\epsilon_1))_{\epsilon_0, \epsilon_1 \in \{0,1\}}$

(3.2)
$$\begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}.$$

If one considers the microstate dynamics $T : X \to X$ induced by the deterministic tent map also as Markovian process, one could ask whether the micro and macro processes commute. Is the distribution p on $\hat{X}$ induced by $\phi_{2/3} \circ T$, i.e the push forward $(\phi_{2/3} \circ T)_*(\lambda)$ of the Lebesgue measure λ of $X = [0, 1]$ onto $\hat{X} = \{0, 1\}$, the same induced by $\hat{T} \circ \phi_{2/3}$? Since λ is T-invariant we obtain on the one hand

$$(\phi_{2/3} \circ T)_*(\lambda) = \phi_{2/3_*} \circ T_*(\lambda) = \phi_{2/3_*}(\lambda) = (2/3, 1/3)$$

where the notation $(2/3, 1/3)$ means $p(0) = 2/3$ and $p(1) = 1/3$. On the other hand, we get

$$\begin{aligned}
(\hat{T} \circ \phi_{2/3_*})(\lambda) &= \hat{T}_* \circ \phi_{2/3_*}(\lambda) \\
&= \hat{T}_*(p) \\
&= (2/3, 1/3)\begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix} \\
&= (2/3, 1/3)
\end{aligned}$$

and commutativity $\hat{T} \circ \phi_{2/3} = \phi_{2/3} \circ T$ in the sense of weak lumpability is proven. Hence, due to the discussion in the introduction, the macroscopic process $\hat{T} : \hat{X} \to \hat{X}$ is a closed description.

Local maxima are localized at the new thresholds $2/5$ and $4/5$ for the partition parameter α, where further forbidden sequences occur, as one can read off the upper table for the joint distribution: If $\alpha < 2/5$ the sequence 010 cannot occur, and for $4/5 < \alpha$ the sequences 110, 101, 011, and 111 vanish. Furthermore, there is a local maximum in 0.628174.

4. MORE TIME STEPS

In this section we want to assemble some general results on the dynamics of the extended states

$$(s_{n+m}, s_{n+m-1}, \ldots, s_n),$$

with

$$s_k(x) = \begin{cases} 0 & \text{if } T^k(x) < \alpha \\ 1 & \text{if } T^k(x) \geq \alpha \end{cases},$$

for $k \in \mathbb{N}$ and a partitioning point $\alpha \in [0, 1]$. In the definition T^k means the k-times iterated tent map which is defined piecewise.

(4.1)
$$T^k(x) = \begin{cases} 2^k x - l & \text{if } x \in \left[\dfrac{l}{2^k}, \dfrac{l+1}{2^k}\right] \\ l + 2 - 2^k x & \text{if } x \in \left[\dfrac{l+1}{2^k}, \dfrac{l+2}{2^k}\right] \end{cases}$$

for $l = 0, 2, 4, \ldots, 2^k - 2$. The k-times iterated tent map T^k consists of k-copies of the ordinary tent map T and has $2^k - 1$ intersection points with the diagonal different from zero. I.e. there are $2^k - 1$ points $0 < p_1 < \ldots < p_{2^k-1} < 1$ such that $T^k(p_i) = p_i$. For the cases $k = 2, 3$ we plotted the graph of T^k. In the sequel we pursue two different goals: first, we investigate for which choices of α forbidden sequences of a certain length appear; second, we determine a countable set of periodic partitioning points whose induced extended state dynamics turns out to be Markovian.
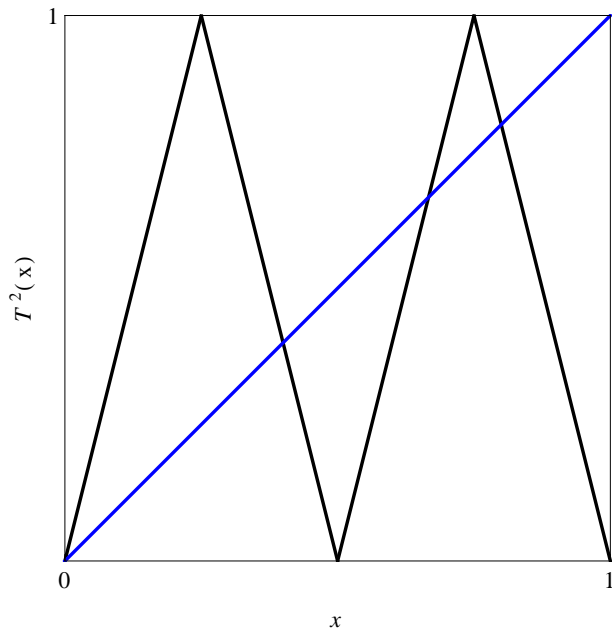
FIGURE 7. The 2-times iterated tent map. The intersections with the diagonal mark the 2-periodic points of the tent map.
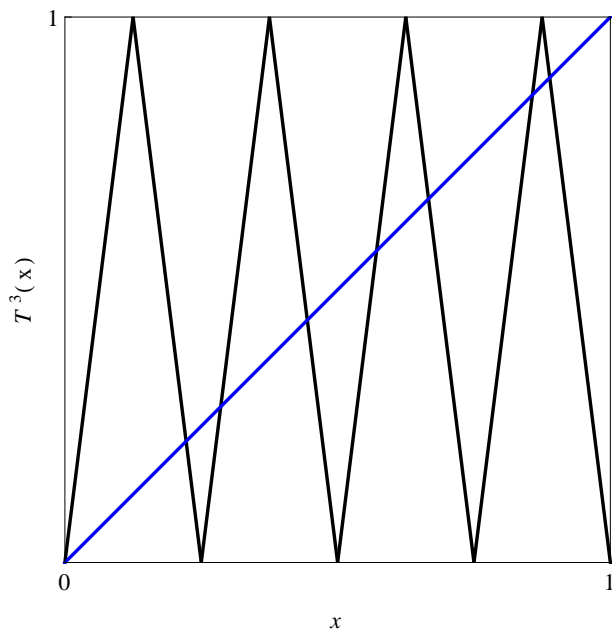

FIGURE 8. The 3-times iterated tent map. The intersections with the diagonal mark the 3-periodic points of the tent map.

4.1. Forbidden sequences.

Previously, we observed that for certain thresholds of the partitioning point α forbidden sequences may occur. If $\alpha \geq 2/3$, the symbol $s_n = 1$ cannot followed by $s_{n+1} = 1$. Hence, the sequence 11 does not occur in the derived symbol dynamics. Analogous results hold true for other

thresholds if one has a look at the extended state dynamics given by two consecutive time steps. One can read off the table in the third section that only for a choice $\alpha \in (2/5, 2/3)$ all sequences of length three have a probability different from zero to occur, i.e. the measure of all sets in Eq. (3.1) is different from zero.

Definition 4.1. Let $\alpha \in [0,1]$, $m \in \mathbb{N}_0$, $a \in \{0,1\}^{m+1}$ be an $m+1$-tupel with entries consisting of 0's or 1's. We define

$$(4.2) \qquad \mathcal{T}^m = \bigcup_{k=0}^{m} T^{-k}(\alpha) \cup \{0,1\} \,.$$

$\mathcal{T}^m$ is the union of all preimages of α under the iterated tent T^k, with $k = 0, \ldots, m$, including $\{0,1\}$, where $T^0 = \mathrm{id}_{[0,1]}$ denotes the identity map on $[0,1]$. Furthermore, we define
(4.3)
$$A_a = \left\{ x \in [0,1] : \phi_\alpha \circ T^k(x) = a_k;\ k = 0, \ldots, m \right\} \backslash \mathcal{T}^m$$

with $a = (a_m, \ldots, a_1, a_0)$.

Lemma 4.1. *The set A_a, for $a \in \{0,1\}^{m+1}$, is the interior of the domain*

$$D_a = \{ x \in [0,1] : (\phi_\alpha \circ T^m(x), \ldots, \phi_\alpha(x)) = a \}$$

of the symbol sequence a – i.e. the biggest open set contained in the domain. Furthermore, A_a is a finite union of open intervals whose boundary points are contained in the set $\mathcal{T}^m$ and we have

$$(4.4) \qquad [0,1] \backslash \mathcal{T}^m = \bigcup_{a \in \{0,1\}^{m+1}} A_a \,.$$

In addition, the family of sets

$$\mathcal{A} = \{ A_a : a \in \{0,1\}^m,\ A_a \neq \emptyset \}$$

is a partition of the unit interval almost sure.

Proof. The preimage $T^{-k}(\alpha)$ consists of 2^k points. We refer to this points with respect to their total order

$$0 < \alpha_1^k < \alpha_2^k < \ldots < \alpha_{2^k-1}^k < \alpha_{2^k}^k < 1 \,.$$

Then

$$0 = \phi_\alpha \circ T^k(x) \Leftrightarrow x \in \bigcup_{j=1}^{2^{k-1}-1} (\alpha_{2j}^k, \alpha_{2j+1}^k)$$
$$\cup \left[0, \alpha_1^k\right) \cup (\alpha_{2^k}^k, 1]$$

$$(4.5) \quad 1 = \phi_\alpha \circ T^k(x) \Leftrightarrow x \in \bigcup_{j=1}^{2^{k-1}} \left[\alpha_{2j-1}^k, \alpha_{2j}^k\right] \,.$$

The domain D_a of the sequence $a = (a_m, \ldots, a_0)$ is the finite section

$$\bigcap_{k=0}^{m} \left(\phi_\alpha \circ T^k\right)^{-1} (a_k)$$

of sets given by Eq. (4.5). Intersecting two intervals is an interval again. Thus, D_a consists of an union of intervals whose boundary points are contained in $\mathcal{T}^m$ and A_a is the interior, that is the biggest open set contained in D_a.

Finally, let $a = (\phi_\alpha \circ T^m(x), \ldots, \phi_\alpha(x)) \in \{0,1\}^m$ for an $x \in [0,1]$. We obtain $x \in D_a$. Clearly, we have $D_a \cap D_b = \emptyset$ for $a \neq b$, with $a, b \in \{0,1\}^m$. Thus

$$[0,1] = \bigcup_{a \in \{0,1\}^{m+1}, D_a \neq \emptyset} D_a$$

is a partition of the unit interval. Since the union of the sets A_a differs from the one of the sets D_a by $\mathcal{T}^m$, the proof is done. $\qquad\square$

Proposition 4.2. *Let be $m \geq 2$ and $\alpha < 1/2$ a partitioning point such that there are no forbidden sequences up to length $m + 1$ then $2^{m-1}/(2^m + 1) < \alpha$.*

Proof. From induction one obtains that

$$A_{\underbrace{(0, \ldots, 0)}_{m-1-\text{times}}} = (0, \alpha/2^{m-2}).$$

From this follows that

$$A_{\underbrace{(0, \ldots, 0, 1)}_{m-\text{times}}} = (1 - \alpha/2^{m-1}, 1),$$

because $1 - \alpha/2^{m-1}$ is the maximal element in $\mathcal{T}^{m-1}$, since it is the maximum of all preimages of α under T^{m-1}. But $1 - \alpha/2^{m-1}$ has the two preimages

$$\eta_0 = \frac{1}{2} - \frac{\alpha}{2^m} \qquad \eta_1 = \frac{1}{2} + \frac{\alpha}{2^m}.$$

If one wants $A_{\underbrace{(0, \ldots, 0, 1, 0)}_{m+1-\text{times}}} \neq \emptyset$ one needs $\eta_0 < \alpha$ which yields

$$\frac{1}{2} - \frac{\alpha}{2^m} < \alpha \Leftrightarrow \frac{2^{m-1}}{2^m + 1} < \alpha.$$

$\qquad\square$

Proposition 4.2 gives a necessary condition for the partitioning point α to be sure that every possible sequence of length $m + 1$ may occur in the symbol dynamics derived from this choice of α. It turns out that this condition is also sufficient. But before we prove this, we need the following lemmata.

Lemma 4.3. *Let be $m \geq 2$. Then*

$$\alpha = \frac{2^{m-1}}{2^m + 1}$$

is the biggest m-periodic point less than $1/2$.

Proof. We need to find the biggest solution x_0 of the equation $T^m(x) = x$ such that $x_0 < 1/2$. Inserting $k = m$ and $l = 2^{m-1} - 2$ into Eq. (4.1) shows that x_0 is contained in the interval $\left[(2^{m-1} - 1)/2^m, 1/2\right]$ and $T^m(x) = 2^{m-1} - 2^m x$. Hence, we need to solve the equation

$$x_0 = 2^{m-1} - 2^m x_0 \Leftrightarrow x_0 = \frac{2^{m-1}}{2^m + 1}$$

and the proof is done. $\qquad\square$

Lemma 4.4. *Let be $m \geq 1$, $2^{m-1}/(2^m + 1) < \alpha < 1/2$, and $0 \leq l \leq m$. Then $T^{-i}(\alpha) \cap T^{-j}(\alpha) = \emptyset$ for all $0 \leq i < j \leq m$, and the cardinality of the set $\mathcal{T}^l$, defined by Eq. (4.2), is $2^{l+1} + 1$.*

Proof. From Eq. (4.1) one obtains that the cardinality of the preimage $T^{-k}(\alpha)$ is 2^k. Furthermore, for all $0 \leq i < j \leq m$ we have $T^{-i}(\alpha) \cap T^{-j}(\alpha) = \emptyset$. Otherwise, there is an $x \in T^{-i}(\alpha) \cap T^{-j}(\alpha)$ and we obtain $T^i(x) = T^j(x) = \alpha$. From this follows $\alpha = T^j(x) = T^{j-i}(T^i(x)) = T^{j-i}(\alpha)$. Therefore, $\alpha < 1/2$ is a periodic point with period $n = j - i \leq m$. Then from lemma 4.3 follows

$$\alpha \leq 2^{n-1}/(2^n + 1) \leq 2^{m-1}/(2^m + 1),$$

a contradiction. Since $|T^{-k}(\alpha)| = 2^k$ for all $1 \leq k \leq m$, the set $\mathcal{T}^l$ consists of $2 + \sum_{k=0}^{l} 2^k = 2^{l+1} + 1$ different points. $\qquad\square$

Lemma 4.5. *Let be $m \geq 1$ and $2^{m-1}/(2^m+1) < \alpha < 1/2$. The maximum of the finite set $\mathcal{T}^{m-1} \setminus \{0,1\}$ is*

$$1 - \alpha/2^{m-1}.$$

Proof. Firstly, we show that $1 - \alpha/2^{m-1} \in \mathcal{T}^{m-1} \setminus \{0,1\}$. We have $T^{m-2}(\alpha/2^{m-2}) = \alpha$ which proves $\alpha/2^{m-2} \in T^{m-2}(\alpha)$. But the two preimages of $\alpha/2^{m-2}$ under the tent map T are $\alpha/2^{m-1}$ and $1 - \alpha/2^{m-1}$ and $1 - \alpha/2^{m-1} \in T^{m-1}(\alpha) \subset \mathcal{T}^{m-1} \setminus \{0,1\}$ is proven. Secondly, we assume the existence of an $x \in \mathcal{T}^{m-1} \setminus \{0,1\}$ such that $1 - \alpha/2^{m-1} < x$. From this and $2^{m-1}/(2^m + 1) < \alpha$ follows $2^m/(2^m + 1) < x$. We have

$$\frac{2^m}{2^m + 1} - \frac{2^k - 1}{2^k} \geq \frac{2^m}{2^m + 1} - \frac{2^m - 1}{2^m}$$

$$= \frac{1}{2^m(2^m + 1)}$$

and therefore $x \in \left(2^k - 1/2^k, 1\right]$ for all $k = 1, \ldots, m$. From Eq. (4.1) follows for $1 \leq k \leq m$

$$T^k(x) = 2^k - 2^k x = 2^k(1 - x)$$

$$\leq 2^m(1 - x) < 2^m\left(1 - \frac{2^m}{2^m + 1}\right)$$

$$= \frac{2^m}{2^m + 1}.$$

But $x \in \mathcal{T}^{m-1}\backslash\{0, 1\}$ and thus there is a $k \in \{1, \ldots, m\}$ such that $T^k(x) = \alpha > 2^{m-1}/(2^m + 1)$ – a contradiction. $\qquad\square$

Corollary 4.6. *For $m \geq 1$, $2^{m-1}/(2^m + 1) < \alpha < 1/2$ and $\eta \in \mathcal{T}^{m-1}\backslash\{0, 1\}$ we have $\eta_0 < \alpha < 1/2 < \eta_1$ with $\{\eta_0, \eta_1\} = T^{-1}(\eta)$.*

Proof. From lemma 4.5 follows $\eta \leq 1 - \alpha/2^{m-1}$. We have

$$\eta_0 = \eta/2 \leq \frac{1}{2} - \frac{\alpha}{2^m}$$

$$< \frac{1}{2} - \frac{1}{2}\frac{1}{2^m + 1}$$

$$= \frac{2^{m-1}}{2^m + 1} < \alpha.$$

Whereas $\eta_1 = 1 - \eta/2 > 1/2$ follows immediately. $\quad\square$

Definition 4.2. Let be $\Lambda \subset [0, 1]$ a discrete (and therefore finite) subset of the unit interval. Suppose $x \in \Lambda$. If the set

$$\{y \in \Lambda : x < y\}$$

is not empty, we call its infimum the *successor* of x in Λ.

Theorem 4.7. *Let be $m \geq 1$, $2^{m-1}/(2^m + 1) < \alpha < 1/2$ and $a \in \{0, 1\}^k$ a k-tupel with $k \leq m + 1$. Then the interior A_a of the domain of the symbol sequence a consists of a single interval (x_1, x_2), with $x_1, x_2 \in \mathcal{T}^{k-1}$, and x_2 is the successor of x_1 in the set $\mathcal{T}^{k-1}$. In particular, the symbol dynamics has no forbidden sequences of length $k \leq m + 1$.*

Proof. Proof by induction over $k \leq m$. For $k = 1$ we have $A_0 = (0, \alpha)$ and $A_1 = (\alpha, 1)$. For $k = 2$ one checks immediately that

$$A_{(0,0)} = (0, \alpha/2), \; A_{(0,1)} = (\alpha/2, \alpha),$$

$$A_{(1,1)} = (\alpha, 1 - \alpha/2), \; A_{(1,0)} = (1 - \alpha, 1).$$

Let the assumption hold true for $k \in \{2, \ldots, m\}$. We prove that for all $a \in \{0, 1\}^k$ the intersection $A_a \cap T^{-k}(\alpha)$ consists only of a single element. Let us assume the opposite. Then there are two different cases: firstly, there is an $a \in \{0, 1\}^k$ such that $A_a \cap T^{-k}(\alpha) = \emptyset$; secondly, it exists a $b \in \{0, 1\}^k$ such

that $A_b \cap T^{-k}(\alpha)$ consists of more than only one element. Let us assume the first case. Due to the induction hypothesis we have 2^k nonempty sets A_a, with $a \in \{0, 1\}^k$, such that $[0, 1]\backslash\mathcal{T}^{k-1} = \bigcup_{a \in \{0,1\}^k} A_a$, see lemma 4.1. We have $|T^{-k}(\alpha)| = 2^k$ and from lemma 4.4 we know $T^{-k}(\alpha) \cap \mathcal{T}^{k-1} = \emptyset$. Hence, by a simple counting argument we obtain that there must be an element $b \in \{0, 1\}^k$ such that the set $A_b \cap T^{-k}(\alpha)$ has more then two elements, and we reduced the first to the second case. Therefore, without loss of generality we can assume that there is a $b \in \{0, 1\}^k$ and $\eta_0, \eta_1 \in T^{-k}(\alpha)$, where η_1 is the successor of η_0 in the set $T^{-k}(\alpha)$, with $\eta_0, \eta_1 \in A_b = (x_1, x_2)$ for $x_i \in \mathcal{T}^{k-1}$. We need to distinguish three different cases.

1. $x_1 < \alpha$: Then we have $T(x_i) = 2x_i$ and $T(\eta_i) = 2\eta_i$. We obtain $T(x_1) < T(\eta_0) < T(\eta_1) < T(x_2)$, with $(T(x_1), T(x_2)) \subset A_{\hat{b}} = (x_3, x_4)$ and the sequence $\hat{b} = (b_{k-1}, \ldots, b_1)$ whenever $b = (b_{k-1}, \ldots, b_1, b_0)$ and $x_3, x_4 \in \mathcal{T}^{k-2}$. Furthermore, $T(\eta_1)$ is the successor of $T(\eta_0)$ in $\mathcal{T}^{k-1}$. Otherwise, there is an $x_0 \in \mathcal{T}^{k-1}$ with $T(\eta_0) < x_0 < T(\eta_1)$, and this implies $\eta_0 < x_0/2 < \eta_1$, with $x_0/2 \in \mathcal{T}^k$ – a contradiction because we assumed that η_1 is the successor of η_0 in $\mathcal{T}^k$. Let denote with x_5 the predecessor of $T(\eta_0)$ in the set $\mathcal{T}^{k-1}$, which is greater or equal x_3, and with x_6 the successor of $T(\eta_1)$ in the set $\mathcal{T}^{k-1}$, which is less or equal x_4. Then for all $x \in (x_5, T(\eta_0)) \cup (T(\eta_1), x_6)$ we have $\phi_\alpha \circ T^{k-1}(x) = \epsilon$, and for all $x \in (T(\eta_0), T(\eta_1))$ we get $\phi_\alpha \circ T^{k-1}(x) = \epsilon + 1 \bmod 2$. We define $b_\epsilon = (\epsilon, b_{k-1}, \ldots, b_1) \in \{0, 1\}^k$. Thus, we get $(x_5, T(\eta_0)) \cup (T(\eta_1), x_6) \subset A_{b_\epsilon}$ and $(T(\eta_0), T(\eta_1)) \cap A_{b_\epsilon} = \emptyset$. Hence, A_{b_ϵ} is not longer an interval. But this is a conflict with the induction hypothesis.

2. $x_1 = \alpha$: From corollary 4.6 we know $1/2 < \eta_0, \eta_1$. Furthermore, the successor x_2 of $x_1 = \alpha$ in $\mathcal{T}^{k-1}$ is equal $\alpha/(2^{k-1}) + 1/2$, and one obtains

$$T(\alpha) - T(x_2) = 2\alpha + 2x_2 - 2 = 2\alpha + \frac{\alpha}{2^{k-2}} + 1$$

$$= \alpha\frac{1 + 2^{k-1}}{2^{k-2}} - 1 > \frac{2^{m-k+1}(1 + 2^{k-1})}{2^m + 1} - 1$$

$$= \frac{2^{m-k+1} + 2^m - 1 - 2^m}{2^m + 1} \geq \frac{1}{2^m + 1}.$$

This proves $T((\alpha, x_2)) = (T(x_2), 1]$, with $T(x_2) < T(\eta_1) < T(\eta_0) < 1$. Then, for all $x \in (T(x_2), T(\eta_1)) \cup (T(\eta_0), 1)$ we have $\phi_\alpha \circ T^{k-1}(x) = \epsilon$, and for all $x \in (T(\eta_0), T(\eta_1))$ we get $\phi_\alpha \circ T^{k-1}(x) = \epsilon + 1 \bmod 2$, and reasoning in the same way as in the previous case yields a contradiction as well.

3. $\alpha < x_1$: Also this case leads to a contradiction in an analogous way as in the first case if one reverses

the inequality signs there.

Induction yields that for every $a = (a_{k-1}, \ldots, a_0) \in \{0,1\}^k$, with $k \leq m$, there is exactly one element x_3 in the preimage $T^{-k}(\alpha)$ contained in $A_a = (x_1, x_2)$. This yields $\phi_\alpha \circ T^k(x) = \epsilon$ for all $x \in (x_1, x_3)$, $\phi_\alpha \circ T^k(x) = \epsilon + 1 \mod 2$ for all $x \in (x_3, x_2)$,

$$A_{a^\epsilon} = (x_1, x_3)$$

with $(\epsilon, a_{k-1}, \ldots, a_0)$, and

$$A_{a^{\epsilon+1}} = (x_3, x_1)$$

with $a^{\epsilon+1} = (\epsilon + 1 \mod 2, a_{k-1}, \ldots, a_0)$. $\qquad \square$

Remark 4.1. Due to a symmetry argument one can repeat all the previous considerations also for the choice

$$\alpha \in \left(\frac{1}{2}, \frac{2^{m-1}}{2^m - 1} \right),$$

obtaining the same result – no forbidden sequences of length less or equal $m + 1$ do occur and vice versa. Furthermore, it is well known that $\alpha = 1/2$ gives us a generating partition. Hence, also for this choice no forbidden sequences of any length can appear.

We can combine proposition 4.2, theorem 4.7 and remark 4.1 to get the following final result of this subsection.

Theorem 4.8. *For $m \geq 2$ the symbol dynamics s_n has no forbidden sequences up to length $m + 1$ iff*

$$\alpha \in \left(\frac{2^{m-1}}{2^m + 1}, \frac{2^{m-1}}{2^m - 1} \right).$$

4.2. **Markovianity.** In the sequel we study the problem for which choices of the partitioning point α the dynamics of the extended states $(s_{n+m}, \ldots, s_n)$ turns out to be Markovian. From lemma 4.1 we know that the family of sets

(4.6) $\qquad \mathcal{A} = \{A_a : a \in \{0,1\}^m, A_a \neq \emptyset\},$

where the sets A_a are defined by Eq. (4.3) for all $a \in \{0,1\}^m$, is a partition of the unit interval almost sure.

Definition 4.3. We call the partition $\mathcal{A}$ of Eq. (4.6) Markovian if for every $a \in \{0,1\}^{m+1}$, with $A_a \neq \emptyset$, there is an index set $I_a \subset \{0,1\}^{m+1}$ such that

$$T(A_a) = \bigcup_{b \in I_a} A_b \quad \text{a.s.}$$

Lemma 4.9. *If the partition $\mathcal{A}$ given by Eq. (4.6) is Markovian, the dynamics of the extended states $(s_{n+m}, \ldots, s_n)$ is Markovian as well.*

Proof. An immediate corollary of theorem 6.5 of Roy L. Adler's work [1]. $\qquad \square$

It turns out that the Markov property of the partition in Eq. (4.6) is closely related to the one of no forbidden sequences.

Lemma 4.10. *Let*

$$\alpha \in \left(\frac{2^{m-1}}{2^m + 1}, \frac{2^{m-1}}{2^m - 1} \right)$$

and $a \in \{0,1\}^{m+1}$ such that we have $A_a = (x_1, x_2)$, with $x_1, x_2 \in \mathcal{T}^m \setminus \{\alpha\}$. Then exists an index set $I_a \subset \{0,1\}^{m+1}$ such that

$$T(A_a) = \bigcup_{b \in I_a} A_b \quad \text{a.s.}$$

Proof. From theorem 4.7 we know that for every $a \in \{0,1\}^{m+1}$ there are $x_1, x_2 \in \mathcal{T}^m$ such that $A_a = (x_1, x_2)$. We assume further that $x_1, x_2 \neq \alpha$. From corollary 4.6 one obtains that in this case either $x_2 < 1/2$ or $1/2 < x_1$. Let us study the first case. The second one can be proven analogously.

The tent map is strictly increasing on $[0, 1/2]$. Hence, $T(A_a) = (T(x_1), T(x_2))$. Since

$$x_1, x_2 \in \bigcup_{k=1}^{m} T^{-k}(\alpha) \cup \{0, 1\}$$

we have

(4.7) $\qquad T(x_1), T(x_2) \in \bigcup_{k=0}^{m-1} T^{-k}(\alpha) \cup \{0, 1\}.$

Define $I_a = \{b \in \{0,1\}^{m+1} : A_b \subset T(A_a)\}$. Then from Eq. (4.7) and theorem 4.7 it follows

$$T(A_a) = \bigcup_{b \in I_a} A_b \setminus \mathcal{T}^m.$$

$\qquad \square$

The crucial point to get Markovianity is $T(\alpha) \in \mathcal{T}^m$, i.e. $T(\alpha) \in T^{-k} \cup \{0, 1\}$ for a $k \in \{0, \ldots, m\}$. But this is equivalent to assume that α is a $k + 1$-periodic point or equal $1/2$. From the proof of lemma 4.4 we know that for all $k < m$ the $k + 1$-periodic points are not in $(2^{m-1}/(2^m + 1), 2^{m-1}/(2^m - 1))$. Hence, only the two choices

$$\alpha = \frac{2^m}{2^{m+1} + 1} \qquad \alpha = \frac{2^m}{2^{m+1} - 1}$$

are left as candidates, which are both $m + 1$-periodic due to lemma 4.3.

Theorem 4.11. *Let be $m \geq 1$ and*

$$\alpha = \frac{2^m}{2^{m+1} + 1} \quad \text{or} \quad \alpha = \frac{2^m}{2^{m+1} - 1}.$$

Then the extended state dynamics $(s_{n+m}, \ldots, s_n)$ is Markovian.

Proof. We only check the case $\alpha = 2^m/(2^{m+1}+1)$. The other one is similar. Suppose $a \in \{0,1\}^{m+1}$. Then from theorem 4.7 follows the existence of two elements $x_1, x_2 \in \mathcal{T}^m$ such that $A_a = (x_1, x_2)$ and x_2 is the successor of x_1 in $\mathcal{T}^m$. We need to prove $T(A_a) = \bigcup_{b \in I_a} A_b$ for an index set $I_a \subset \{0,1\}^{n+1}$. If we have $x_1, x_2 \neq \alpha$, this follows from lemma 4.10. Therefore, we assume $x_1 = \alpha$ or $x_2 = \alpha$.

Suppose $x_1 = \alpha$ and define $y = \alpha/2^m + 1/2$. We get

$$T^m(y) = T^{m-1}\left(1 - \frac{\alpha}{2^{m-1}}\right) = T^{m-2}\left(\frac{\alpha}{2^{m-2}}\right) = \alpha,$$

and $y \in \mathcal{T}^m$ is proven. Assume that there is an element $z \in \mathcal{T}^m \cap (\alpha, y)$. Then

$$\begin{aligned} T(z) &> \min\{T(\alpha), T(y)\} \\ &= \min\left\{\frac{2^{m+1}}{2^{m+1}+1}, 1 - \frac{\alpha}{2^{m-1}}\right\} \\ &= \min\left\{1 - \frac{\alpha}{2^m}, 1 - \frac{\alpha}{2^{m-1}}\right\}. \end{aligned}$$

$T(z) > 1 - \alpha/2^m$ is excluded by lemma 4.5, and we obtain $T(z) \in (1 - \alpha/2^{m-1}, 1 - \alpha/2^m)$ which implies $z \in (1/2, y)$. From this follows

$$\begin{aligned} T(z) &> T(y) \\ T^k(z) &< T^k(y) \quad \text{for all } k = 2, \ldots, m \end{aligned}$$

which yields $T^k(z) \neq \alpha$ for all $k = 0, \ldots, m$ and $z \notin \mathcal{T}^m$ – a contradiction. This proves $x_2 = y = \alpha/2^m + 1/2$, and we obtain

$$T(x_1) = T(\alpha) = \frac{2^{m+1}}{2^{m+1}+1}$$

$$T(x_2) = 2 - 2x_2 = 1 - \frac{\alpha}{2^{m-1}} = \frac{2^{m+1}-1}{2^{m+1}+1} < T(\alpha)$$

We have shown $T(A_a) = (T(x_2), 1]$. Since $T(x_2) \in \mathcal{T}^m$, from theorem 4.7 follows the existence of a set $I_a \subset \{0,1\}^{m+1}$ such that $T(A_a) = \bigcup_{b \in I_a} A_b$.

Conversely, suppose $x_2 = \alpha$. This implies $T(A_a) = (T(x_1), T(\alpha))$, with $T(x_1), T(\alpha) \in \mathcal{T}^m$, because α is assumed to be $m + 1$ periodic. Again, from theorem 4.7 follows the existence of an index set $I_a \subset \{0,1\}^{m+1}$ such that $T(A_a) = \bigcup_{b \in I_a} A_b$. $\square$

Periodicity of the partitioning point α turns out to be crucial for the induced extended symbol dynamics to be Markovian. More precisely, α needs to be an $m + 1$-periodic point if the dynamics of the extended states $(s_{n+m}, \ldots, s_n)$ is Markovian. The following example demonstrates that the opposite does not hold.

Example 4.1. Let be $\alpha = 4/5$. Then we have $T(\alpha) = 2/5$, $T^2(\alpha) = 4/5$ and α is a point with period 2. We

define $\hat{s}_n = (s_{n+1}, s_n)$ and decompose the mutual information in Eq. (1.2)

$$\begin{aligned} I(\hat{s}_n&, \hat{s}_{n+2}|\hat{s}_{n+1}) \\ &= 2H(\hat{s}_{n+1}, \hat{s}_n) - H(\hat{s}_{n+2}, \hat{s}_{n+1}, \hat{s}_n) - H(\hat{s}_n) \\ &= 2H(s_{n+2}, s_{n+1}, s_n) - H(s_{n+3}, s_{n+2}, s_{n+1}, s_n) \\ &\quad - H(s_{n+1}, s_n). \end{aligned}$$

As one can read off, one needs the joint probabilities of one, two, and three consecutive time steps. Table 1 and 2 provide the first two joint probabilities, whereas the third ones are given by

$$\begin{aligned} p(0,0,0,0) &= 3/10, & p(0,0,0,1) &= 1/10, \\ p(0,0,1,0) &= 1/5, & p(1,0,0,0) &= 3/20, \\ p(1,0,0,1) &= 1/20, & p(0,1,0,0) &= 1/5, \end{aligned}$$

and all not listed sequences do not appear at all. Computation yields

$$I(\hat{s}_n, \hat{s}_{n+2}|\hat{s}_{n+1}) \approx 0.0642106.$$

Hence, the dynamics of the extended state (s_{n+1}, s_n) is not Markovian.

4.3. **Aggregation.** Due to theorem 4.11 we obtain for $m \geq 1$ and

$$\alpha = \frac{2^m}{2^{m+1}+1} \quad \text{or} \quad \alpha = \frac{2^m}{2^{m+1}-1}$$

a Markovian dynamics P on a state space V with 2^{m+1} states. M. N. Jacobi, and O. Görnerup [10] developed a method to find all possible strong lumpings $\Pi : V \to \hat{V}$ of the states. A strong lumping aggregates all states such that the new macro process $\hat{P}$ in Fig. (9) is Markovian again, no matter what choice is made for the initial distribution on the state space V, and the diagram Fig. (9) commutes. This method relies on level sets of the eigenvectors derived from the matrix representation of the transition kernel of the process P. We want to apply this technique to come up with further Markovian symbol dynamics for the tent map. Let us start with $\alpha = 2/5$. By

$$\begin{array}{ccc} \hat{V} & \xrightarrow{\hat{P}} & \hat{V} \\ \Pi \uparrow & & \uparrow \Pi \\ V & \xrightarrow{P} & V \end{array}$$

FIGURE 9. A Markovian process $\hat{P}$ obtained from P by a lumping Π.

theorem 4.11 we obtain from this choice a Markovian dynamics on a state space with four elements which we enumerate as follows:
(4.8)
$$(0,0) = 1, \quad (1,0) = 2, \quad (0,1) = 3, \quad (1,1) = 4.$$

With respect to this labeling the matrix representation of the transition kernel of the Markov process P, which we denote also by P, reads as

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

The Jordan decomposition of the matrix P yields a diagonal matrix $D = \mathrm{diag}\,(0, 1, -i/2, i/2)$ with eigenvectors

$$(4.9) \qquad \begin{pmatrix} -1 & 1 & -1-i & -1+i \\ 1 & 1 & 2i & -2i \\ 0 & 1 & -1-i & -1+i \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Due to the eigenvector method described in [10] the lumping of the first and the third state is the only possible. For this lumping we obtain

$$\Pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \hat{P} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \\ 1/2 & 0 & 1/2 \end{pmatrix},$$

and one checks easily that $\Pi\hat{P} = P\Pi$. We want to point out that the new Markovian symbol dynamics does not come from a Markovian partition of the unit interval $[0, 1]$. The initial partition induced by the states Eq. (4.8)

$$\left\{ \left(0, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{2}{5}\right), \left(\frac{4}{5}, 1\right), \left(\frac{2}{5}, \frac{4}{5}\right) \right\}$$

is transformed by the lumping into the one where the intervals $(0, 1/5)$ and $(4/5, 1)$ are unified. But this is not a Markovian partition any longer because $T((2/5, 4/5))$ cannot be represented by a union of the remaining partition elements.

Vice versa, if the aggregation Π lumps the first and the second state together – i.e. we unify the intervals $(0, 1/5)$ and $(1/5, 2/5)$ – one checks immediately that this yields a Markovian partition as well and therefore, see lemma 4.9, a Markovian symbol dynamics with transition kernel

$$\hat{P} = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$$

where we choose the labeling $(0,0) + (1,0) = 1$, $(0,1) = 2$, and $(1,1) = 3$. But this aggregation cannot come from lumping of level sets of the eigenvectors Eq. (4.9). From proposition 4 in [10] it follows that the commutativity relation of Fig. (9), that is $\Pi\hat{P} = P\Pi$, does

not hold. Indeed, verification yields

$$\Pi\hat{P} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = P\Pi.$$

This is a further instance of the same phenomena we studied in example 6.1 of [11]. There we performed a lumping of a Markovian dynamics induced by the Bernoulli shift, from which we obtained a Markovian dynamics as well, but Fig. (9) did not commute at all. From [11] we know that this occurs if and only if the information flow Eq. (1.1) for the aggregation does not vanish. Hence, the aggregation of the first and third state derived from the eigenvector method provides a lumping with vanishing information flow. If we summarize, we can conclude that the lumping of the first and third state, derived from the eigenvector method in [10], leads to a commutative diagram Fig. (9) with vanishing information flow. But the underlying partition is no longer Markovian. While the aggregation of the first and the second state leads to a Markovian partition, but the diagram Fig. (9) does not commute at all because there is some informational flow from the lower to the upper level, i.e with respect to this closure measure this aggregation does not lead to a new closed level.

Similar computations can be performed for the points $\alpha = 4/9$ and $\alpha = 4/7$, respectively. Both points are those with period 3 and by theorem 4.11 the resulting dynamics of the extended state $(s_{n+2}, \ldots, s_n)$ is Markovian. The eigenvector method leads to a single possible aggregation which lumps together the states $(0, 0, 0, 0)$ and $(0, 0, 0, 1)$, which leads again to a non-Markovian partition, but a commutative diagram Fig. (9). A lumping leading to a Markov partitions is also possible. It consists of intervals whose endpoints are in the set $\{0, 1, \alpha, T(\alpha), T(\alpha^2)\}$, see [9]. But again, this lumping does not lead to a commutative diagram Fig. (9).

5. Conclusion

We investigated the relationship between dynamics of a particular time discrete dynamical system, the tent map, and the induced symbolic dynamics using different closure measures discussed in [11]. We answered the question to which extent the symbolic dynamics induced by non-generating partitions provides a self-sufficient level of description.

All illustrated closure measures highlight the choice $\alpha = 2/3$ for the partition parameter. At this scale the symbol dynamics $s_n \to s_{n+1}$ turns out to be Markovian and this Markovian process commutes with the

microscopic one in the sense of weak lumpability. Furthermore, for $\alpha = 2/3$ we have (at least locally) maximal predictability (2.1), maximal mutual information $I(s_{n+1}, s_n)$, and the information flow $I(s_{n+1}, x_n|s_n) = H(s_{n+1}|s_n)$ has a local minimum at $\alpha = 2/3$. Hence, these measures suggest the symbol dynamics derived from the choice $\alpha = 2/3$ as an emergent level of the dynamical system $x_t \rightarrow x_{t+1} = T(x_n)$ defined by the tent map.

A similar result holds true for a whole series of partitioning points given by theorem 4.11, where the dynamics of the extended states turn out to be Markovian. Also these choices for the partition threshold α lead to new closed levels which are multilevel dynamical systems in their own right.

Given these results the tent map turned out to be a rich testbed for the study of closure measures and level identification in multilevel systems.

6. Acknowledgement

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