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Randomness, Heterogeneity and Population Dynamics

by

Jürgen Jost and Wei Li

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Jürgen Jost and Wei Li*
Max Planck Institute for Mathematics in the Sciences,
Inselstr.22, 04103 Leipzig, Germany
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We investigate the emergent population dynamics of an iterated game in which members of two antagonistic populations are randomly paired in each round. The basic setting is symmetric between these two populations, but this symmetry can be broken by giving the members of the two populations access to different strategy spaces or by assigning different values to parameters determining the composition or the evolution of the populations. In particular, we can investigate the effects of population diversity or stochasticity of the individual agents' actions, both for the competition between agents inside a population and for their benefits or disadvantages for the population as a whole.

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I. INTRODUCTION

Classical game theory, based on the von Neumann and Morgenstern model [1], attempts to describe the situation of two interacting, perfectly rational players who both aim at maximizing their gains by accessing and utilizing all relevant information available to them. The framework has been well applied to a large category of social, behavioral and economic systems [2] in which neither player could improve his action if both wish to act on their own favorable interests, which is the key point of, and more elaborated by, the Nash equilibrium theory [3].

On the basis of a crucial paradigm shift in biology called population thinking [4], game theory was also combined with the principles of the Darwinian theory of evolution. Evolutionary game theory, as developed by evolutionary biologists such as Price and Maynard Smith [5–8], deals with the dynamics of the entire population of players who explore a certain strategy space [9]. The composition of the population is updated so that those with higher payoffs will expand their relative frequency within the population at the expense of those with lower payoffs. Although the biologists think in terms of fitness and selection whereas the economists prefer learning and prediction [10], the theory benefits both fields [11, 12]. Evolutionary game theory has also been utilized to understand how cooperation between members of a population can emerge when they play the game repeatedly, instead of only once as in classical game theory. In repeated games, cooperation, that is, apparently altruistic behavior, can be in the interest of selfish players, when they can induce their opponents to also behave cooperatively [13].

Deterministic replicator dynamics has been developed as the mathematical approach to evolutionary dynamics [8, 9]. This replicator dynamics formally assumes infinite populations. However, in real-world systems, the population size is always finite, though possibly large. Therefore, we should also take the resulting effects into account. Stochastic fluctuations caused by finite population size have been investigated, for instance, in [14–16]. Also, the presence of small stochastic noise is normal and sometimes even essential for evolutionary systems. For example, random mutations of genes, that is, occasional wrong coding at a certain position of DNA sequences, or the errant behavior of agents in the market, play crucial roles in the long term. In repeated games, such stochasticity is unavoidable, even when we know exactly the history of players' decision making [17, 18]. Furthermore, the stability of the dynamic equilibrium with respect to stochastic perturbations has been investigated [17, 19–22].

Another issue is heterogeneity of the population. This could refer to different aspects, like age, sex or spatial structure. In particular, heterogeneity of the players' spatial profiles has been considered, for two reasons. Firstly, at the population level, the internal heterogeneity of the interactions may lead to some emergent, collective behavior other than the one displayed in the cases when the interactions are more homogeneous or uniform [12, 23, 24]. Secondly, research on networks [25, 26] has brought up the issue of the topology of the underlying interaction network. Each player does not interact with all other ones, but only with those few with which it is connected in the network. For an overview, see [21] and the references therein.

In this paper, we intend to investigate the effects resulting from stochastic perturbations and heterogeneity in the evolutionary complementarity game introduced in [27]. The noise affects the players' actions; they do not make a

*On leave fromComplexity Science Center, Hua-Zhong Normal University, Wuhan 430079, P.R. China

definite move k, but one drawn from some distribution with mean value k and standard deviation s. One could then check how the magnitude of s will affect both individual performance and population behavior. The heterogeneity is related to the strategy profile of a certain population. This issue has been addressed in [29] in the framework of the prisoner's dilemma. Here, we shall work with some kind of metastrategies among which the players can choose. For example, they could learn from their most recent encounters, imitate their friends or adapt to the distributions of all their previous opponents' offers by using the information more systematically. We then wish to see how such a mixture of strategy profiles will evolve inside the population. In fact, we shall discover the very interesting phenomenon that strategies that perform better when adopted by a uniform population get eliminated inside a heterogeneous population, which will then cause the evolved population to be less successful as a whole.

In our model we have two populations, buyers and sellers, whose members are randomly matched in each round. A buyer bids an offer x, and its co-player at the present round, a seller, asks for y, with both ranging between 0 and K (an integer usually taken as 50 in our simulations). If $x \geq y$ then there is a deal between them, and the seller gains y and the buyer $K - x$, to keep the game symmetric; otherwise no deal is concluded and both gain nothing. Thus, for instance a buyer should ideally offer as little as possible so that the deal is just concluded, that is, he should not offer more than the seller will ask or is expected to ask, and it would good for the buyer to induce the seller to ask little. Of course, the situation for the seller is the reverse. Thus, both players wish to drive as hard a bargain as possible, but if they push too hard the transaction will fail. If neither of them is allowed to learn or coordinate, the game is totally symmetric and might then stay at the symmetric Nash equilibrium $K/2$ forever. We should note, however, that any value between 0 and K is a Nash equilibrium as it will pay off for neither player to deviate when the opponent sticks to that value. In that sense, the game is degenerate. Our key point is to break this degeneracy and the symmetry between the two populations by evolutionary elements. That is, the outcomes of some fixed rounds of interactions will be compared and accordingly the next generation will be constructed by some evolutionary scheme (operated by some evolutionary algorithm or more elaborated genetic algorithm). There is plenty of room to break the symmetry between the two populations. For instance, they can have different random mutation rates or they can be assigned different strategy options, naive or sophisticated. By examining the resulting equilibrium value, one can check which side is more favored: if that value is larger than $K/2$ then the sellers are doing better, otherwise the buyers. In general we find that the simpler and more flexible strategies lead to superior results at the population level because they can process the information in a more efficient way, which will speed up the convergence rate. Also, a population with some bolder players may gain an advantage over one with timid players only.

Before proceeding, we need to introduce some notations for the system parameters and the strategies that will be used in this paper:
First the parameters for the evolutionary scheme of replacing a population of players by a new one composed of possibly mutated members of the present one with a fitness based selection: (1) generation length (time): the number of rounds played (time steps) between two consecutive selections (if applicable);
(2) selection percentage: the percentage of the players who will be chosen as parents to generate the offspring during the evolutionary process;
(3) mutation rate: the rate of random mutation during the evolutionary process.
Next we list the five strategies in the pool, classified on the basis of the types of information they use:

1. average-previous-opponent: the average of one's opponents' bids in the previous, say m (limited and usually much smaller than the generation length), rounds

2. for $m = 1$, that strategy is called 1-round opponent: each player utilizes the offer of his opponent in the most recent round

3. average-friend-opponent: the average of one's friends' opponents' bids in the most recent round (here, each player has a certain (usually small in comparison with the population size) number of friends (usually small in comparison with the population size) within his own population)

4. average-all-friend: the average of one's friends' bids in the most recent round (thus, here, in contrast to the previous strategies, no information about the other population is used during each generation)

5. average-successful-friend: the average of one's friends' successful bids in the most recent round (here, information from the other population is used indirectly, but selectively, because their offers decide which of the friends are successful)

Each strategy can have two variants, either directly employing the value computed according to the chosen strategy as the next own offer, or using that value as the input in a look-up table whose output then is that next offer. The look-up table then is itself an object of evolution. In fact, since the look-up table has K input entries and has to

provide an output for each of them, evolution will take quite some time to test it out thoroughly. To distinguish these two variants, we can simply put "simple" in front of the strategy that is not using look-up tables. For the strategies that involve friends, we will introduce friendship networks of different topologies, with the average degree of each being fixed to, say 4.
To have a stable setting, in our major simulations with evolving the look-up tables, the generation length, the selection percentage and the mutation rate are 1000, 0.5 and 0.01, resp. In this paper, for the "simple" strategies without look-up tables, the generation length has been taken to be 1, 4 or even larger in various simulations. Our population size is always 400 and the maximal offer available, i.e. K, is 50.

II. OPTIMAL RESPONSES TO RANDOM DISTRIBUTIONS

In order to gain some theoretical insights into the simulation results reported below, we analyze the optimal response of players to a fixed probability distribution of the opponent population. While this ignores a crucial aspect of our population game, namely that both populations will adapt, this nevertheless will let us understand the effect of stochastic fluctuations in the behavior of one population onto the adaptation process of the other one.
Since K is relatively large, we shall utilize a continuum approximation. Thus, the offers now vary continuously between 0 and 1 (by rescaling from $[0, K]$ to the unit interval $[0, 1]$). The symmetric equilibrium between sellers and buyers then is at $1/2$. When a seller chooses an offer s with probability density $p(s)$, and the probability that that offer is below the offer of his buyer opponent is $p(s \leq s_b)$, then his expected gain is

$$\int_0^1 sp(s)p(s \leq s_b)ds. \tag{1}$$

When the buyer chooses his offer t with probability density $q(t)$, then

$$p(s \leq s_b) = \int_s^1 q(t)dt. \tag{2}$$

The seller then maximizes his expected gain when he chooses the offer

$$s_0 = \text{argmax}_s s \int_s^1 q(t)dt. \tag{3}$$

This leads to the condition

$$s_0 q(s_0) = \int_{s_0}^1 q(t)dt. \tag{4}$$

When q is the uniform density, i.i., $q(t) = 1$ for all t, then the optimal choice is $s_0 = 1/2$. When q is a delta distribution, i.e., $q(t) = \delta(t - t_0)$ for some t_0, then of course the optimal choice is $s_0 = t_0$. When, more generally, q is an average of delta distributions,

$$q(t) = \frac{1}{m} \sum_{\mu=1}^{m} \delta(t - t_\mu), \text{ with } t_1 \leq t_2 \cdots \leq t_m, \tag{5}$$

then the optimal response is

$$s_0 = \max_\mu \frac{m - \mu + 1}{m} t_\mu. \tag{6}$$

This occurs, for instance, when the player observes m previous values (from his own experience or from observing other players) and considers them all equally likely to recur in the next round.
When we have a nontrivial distribution q with a unique peak at $1/2$, in which case $\int_{t_1}^1 q(t)dt < (1 - t_1)q(t_1) \leq t_1 q(t_1)$ for $t_1 \geq 1/2$, then we get an optimal response

$$s_0 < 1/2. \tag{7}$$

For instance, when $q(t) = 2t$ for $0 \leq t \leq 1/2$ and $q(t) = 2 - 2t$ for $1/2 \leq t \leq 1$, the optimal response is $s_0 = \sqrt{1/8}$. Thus, stochastic fluctuations about the equilibrium value $1/2$ in the buyer population lead to an optimal response of

the sellers that is below 1/2.
By (1), (2), the expected gain of a seller playing s is $s \int_s^1 q(t)dt$. His variance then is

$$V = s^2 \int_s^1 q(t)dt - (s \int_s^1 q(t)dt)^2. \tag{8}$$

For the uniform density $q(t) = 1$, this becomes

$$V = s^3(1-s) \tag{9}$$

the maximum of which is at $s = 3/4$. Thus, the variance increases beyond $s = 1/2$, that is, for achieving a lower variance, it is better to be more cautious and play smaller values.

III. POPULATION INTERACTIONS AND EVOLUTION

As already described, we implement our game with two populations, the members of which are randomly paired in each round. After a fixed number of rounds, the cumulative gains are evaluated, and a new generation of players is created by some fitness based evolutionary scheme. Thus, adaptation operates on two different time scales, for the individual players from round to round, and for the creation of offspring from generation to generation. The first one can be considered as learning, the second one as evolution. Both of them have the potential to exploit information about the behavior of the members of the opposite population. The evolution step always uses some information, in an indirect manner, as those players that have been most successful in their generation could be considered as those that are best adapted to the strategies employed by the members of the opposite population. We can also analyze how efficiently information is used in the learning steps. Ideally, in a learning step, each player would form some estimate $\tilde{q}(t)$ of the distribution of offers from the other population. Here, as in Section II, we shall work with a continuous range of offers $t \in [0, 1]$, to facilitate the analysis.
We should remark at this point that a special (and somewhat different) case of our game has been analyzed by Young [31]. In his setting, players are randomly paired and individually replaced, without any assessment of their fitness. They can observe and remember a certain number m of rounds, observe values $t_1 \leq t_2 \cdots \leq t_m$ and they then optimize and play according to (5), (6). A convention in his terminology occurs when all the observed values coincide, say they are all equal to some t_0, for both of two players when they happen to be randomly matched. They will then continue to play t_0, according to the optimization scheme (5), (6). Since when the range of values is discrete, as in his setting and the remainder of this paper, this happens with positive probability and therefore, eventually the two populations will converge towards such a convention when m is less than half the number of rounds played, and the observed rounds are randomly chosen.
Here, however, we want to systematically explore a wider range of possible strategies and parameter ranges, and also employ the evolutionary setting described above. We shall therefore examine various types of possible strategies. When we have the 1-round opponent strategy, and both populations start with a random distribution of offers, then the players in one population simply copy the random distribution of the players from the other one in the previous round. When both populations do that, nothing changes within a generation, and the distributions of offers stay random, as discussed in [28]. Thus, information is only fed into the population during the evolution step.
For the average-previous opponent strategy, when averaging over $m \geq 2$ rounds, the offers of both populations will converge to the same average value when the generation contains sufficiently many rounds. When both populations play this strategy, the dynamics of the probability distributions p_n, q_n of the sellers and buyers, resp., at time n is easily described:

$$p_n = \frac{1}{m} \sum_{j=1}^{m} q_{n-j} = \frac{1}{m^2} \sum_{j=1}^{m} \sum_{k=1}^{m} p_{n-j-k}. \tag{10}$$

(The reader will easily derive the generalization for the case where sellers and buyers average over different numbers of rounds.) Thus, in the end, each population averages its own distribution.
In the framework of Section II, an estimate leading to this strategy is

$$\tilde{q}(t) = \delta(t - \frac{1}{m} \sum t_j) \tag{11}$$

where the t_j are the values played by the m previous opponents and where δ is the usual delta functional. Thus, the estimate uses only the average of the previous offers, but does not take the variance into account. If another estimate

$\tilde{q}(t)$ were formed, for instance a Gaussian distribution with the same average and variance[32] as the collection of the m values $t_1, \ldots, t_m$ observed in the previous m rounds, the optimal response of a buyer would be smaller than the average $\frac{1}{m}\sum t_j$, as proved in Section II. Correspondingly, the optimal response of a buyer would be higher than the average of the previous m buyer offers. Thus, we see that the average-previous-opponent strategy does not use the information provided in an optimal manner. Of course, the estimate $\tilde{q}(t)$ would always be taken from some fixed class of distributions which should be constrained to avoid overfitting, see [30].

The average-friend-opponent strategy can be discussed in the same manner. The difference between this strategy and the average-previous-opponent is that here only information about the most recent behavior of the opposite population is utilized. Again, this strategy uses its information not in the optimal way.

Coming to the average-all-friend strategy, this strategy does not utilize any information about the opposite population. Therefore, given sufficiently many rounds, and assuming that at least some agents have more than one friend, the behavior of each population will converge to the average of its own initial distribution. Thus, information is only used in the evolution step, and it may therefore be better for the population if that convergence is slower so that some differences exist within each population on which evolution can operate.

Finally, the average-successful friend strategy: By (2), the probability that the offer s_k of a friend k of a seller is smaller than his opponent's offer is $\int_{s_k}^1 q(t)dt$. Therefore, the expected average of the successful friends is

$$\frac{\sum_k s_k \int_{s_k}^1 q(t)dt}{\sum_k \int_{s_k}^1 q(t)dt} \tag{12}$$

where the sum is over all friends. When we also take the average w.r.t. the offer distribution $p(s)$ of the seller population (and thereby suppress all contributions from the particular structure of the friendship network), we obtain the estimate

$$\frac{\int_0^1 sp(s) \int_s^1 q(t)dtds}{\int_0^1 p(s) \int_s^1 q(t)dtds} \tag{13}$$

for the offer of a seller according to that strategy. Again, of course, this assumes fixed distributions p, q for the offers of the two types of players and does not take the adaptation dynamics into account. For that, we should replace p and q by the distributions p_{n-1} and q_{n-1} from the previous round $n-1$ and then utilize (13) for an estimate for p_n, and analogously for q_n. Again, we could simply take the delta distribution at the value encountered whose expectation value is given by (13), or we could also take the variance or even some higher moments into account.

As discussed the simple strategies do not use the available information in an optimal manner as they estimate a distribution by a delta distribution at the observed average. One might then allow the members of the two populations access to some class of distributions inside which they then choose the one that matches the observed opponent or friend behavior best. This is the approach of parametric statistics, and this is mathematically analyzed in [30]. In order to avoid having to work with parametrized classes of distributions, we here give the players the opportunity to evolve their look-up tables.

IV. THE EFFECTS OF RANDOM FLUCTUATIONS

It is of principal interest to understand the effects of stochastic fluctuations at the various levels and scales involved. First, at the individual level, the players can use (partly) random instead of deterministic strategies. For instance, instead of playing a fixed number k, a player can take his offer from a Gaussian distribution centered at k with standard deviation s. Thus, the parameter s here measures the degree of randomness of the individual behavior.

In the simplest case, the same s will be assigned to all the members of a population, so s becomes a parameter for the whole population. We can then compare different populations with different parameters, which allows us to optimize the value of s. In our simulations, we observe that for larger values of s, it takes much longer for the players to optimize their performance. Also, a population with all players subject to stochastic fluctuations performs worse than the one whose members make more definite offers. This observation confirms the insight into the key mechanism at work at the population level that a homogeneous and constant population may evolve more quickly to a stable state by achieving its favorable equilibrium value. The inhomogeneous population is forced to accept that value and trapped into a disadvantageous position.

Second, also at the individual level, s could be a parameter that varies between individuals. This means that each player can choose his own degree of randomness s when making his offer. Our simulations (see one example shown in Fig. 1) indicate that the players with a smaller value of s do better than those with a higher degree of randomness.

Both these observations, namely that a high degree of randomness is a disadvantage both at the population and at the individual level, are not surprising. In fact, according to the analysis of Section II, the important point for every player is to identify the unique optimal response value, and higher variances only distract from that. But here is a more interesting point: We consider a population whose typical members play deterministic strategies but a small fraction of its members is erratic and plays more random strategies. In the simplest case, most members of a population always choose k, whereas the rest makes random bids. Or for the 1-round opponent strategy, when most players of a population make definite offers, the others choose their bids according to a Gaussian or some other distribution. Extensive simulations have shown that the population with some erratic members will gain an advantage over the one with less or no freaks. Fig. 2 shows one of our simulations where all the buyers and 95 percent of the sellers make deterministic offers but 5 percent of the sellers behave randomly. We see that the sellers need more time than the buyers to reach the equilibrium that is, however, more favorable to them.

According to the above points, it is better for each individual to be less random. Collectively, this is also better for a population as a whole when all its members are uniform. However, it is better for a population to have some members with higher randomness. These erratic members may perform less well at the individual level, but it is good for the population to have some around because their actions may seem to confuse their opponents. This issue may be of some interest in theoretical biology, with regard to the issue of group selection. In our example, a group is better when it is more diverse and contains some members that are kept at an individually non-optimal level. Within the group, of course, there then is the evolutionary tendency to eliminate those inferior members, but competition between groups, when sufficiently strong, could work in favor of those groups that can suppress that internal evolutionary pressure. Our framework thus seems to offer some possibility for analyzing this issue in more detail.

Finally, the stochastic effects play a role in the evolutionary scheme, that is, on the longer time scale. In particular, the mutation rate is a randomness parameter that a population can optimize. The results of [27] indicate that the optimum is reached at some particular, non-zero value for the mutation rate, which, again, is not a surprising finding.

V. DYNAMICS OF HETEROGENEOUS POPULATIONS

So far, we have only discussed cases where all the members within the same population adopt the same kind of strategy taken from the (strategy) pool [28]. We now wish to investigate the effects arising from breaking the strategic uniformity. This may add some realism to our game as players have the freedom to choose the best strategies which could, they assume, optimize their profits.

We thus consider a heterogeneous population whose members choose their strategies individually according to a certain distribution. This distribution then changes as the result of evolution, according to the performances (scores or fitness) of strategies accumulated during some fixed rounds of interactions. The more successful strategies will increase in frequency in the next generation and can thus diffuse inside a population. It will then be of interest to see both which strategies will spread inside a population and what strategy mix will be good for the population as a whole. Since both populations evolve their strategy mix, the conclusions derived in Section II for more homogeneous situations no longer strictly apply, and some disadvantages identified there might turn into advantages in this mutually changing setting.

Our strategy pool has been defined in Section I and includes 1-round opponent, average-previous-opponent, average-friend-opponent, average-all-friend and average-successful-friend. Initially the 5 strategies are evenly distributed, that is, each is adopted by 1/5 of the players.

First we consider the cases where the look-up tables are included. Namely the players need to predict their current-round bids on the basis of the previous experiences, of their own or of their friends'. When the members of a heterogeneous population play against those of a homogeneous one, the outcomes are diversified mainly by the strategy type the homogeneous population is taking. If the homogeneous population takes 1-round opponent or average-all-friend, the two opposing sides perform about the same. If the homogeneous side takes average-friend-opponent or average-successful-friend, then it can gain a slight advantage over its heterogeneous opponent. A little surprising finding is that the homogeneous population will, in some cases, lose the competition if the average-previous-opponent has been taken by its members. This seems hard to explain based on our ranking list of different types of strategies made in one of our previous papers [28], in which the average-previous-opponent is consistently superior to any of the other strategies. But the setting here is different and also according to our observations, the average-previous-opponent needs longer time to converge than any other strategy does. Hence the heterogeneous population has been quicker in locating the optimal point, which could push the homogeneous side into a disadvantageous situation (Fig. 3). This is also reflected from another observation that the average-previous-opponent seldom becomes the final dominant one inside the heterogeneous population, mostly due to its relatively slow convergence rate. In any case, the average-previous-opponent strategy employs less recent information than the other ones, and this also might put it at a disadvantage when the two populations rapidly adapt.

Thus in general, the heterogeneous population and the homogeneous one are performing nearly equally well by using the current parameters. Also, the best strategy at the population level is not necessarily the eventually leading one inside a heterogeneous population when the look-up tables are evolved.

In fact, a heterogeneous population does best with a combination of 2 or 3 strategies. But evolution inside the population will eventually eliminate all but one strategy. We find that the eventually dominant strategy could be any of the 5 strategies, with nearly equal chances. Our simulations also indicate that the convergence to the equilibrium is much faster now by only taking 10 generations. Since a fully heterogeneous population explores 5 different types of strategies its members taken together will make full use of all available information. By such an efficient information exchange, the heterogeneous population can quickly identify the optimal strategy distribution. Consequently, the heterogeneous population will be quicker in stabilizing their optimal offers, which can be seen from the jump in Fig. 4. There the buyers have already achieved a value close to the final equilibrium, almost 9,000 steps before that same value is reached by the sellers. The early stabilization of the heterogeneous population is double-edged. On one hand, this is good for the population to spread information and test as many strategies as quickly as possible. On the other hand, this is also sometimes bad since the equilibrium value may not be advantageous, for instance smaller than $K/2$ for the sellers. Once a population is stabilized, the opposite one can then take its time to adapt to the corresponding value.

We now consider the case without look-up tables. That is, the players simply treat the values, calculated according to their strategy, as their offers at the present round. Since the 1-round opponent cannot improve without being combined with evolving the look-up tables, we here only include the remaining four which perform some average and therefore can utilize more information as time evolves, namely simple average-previous-opponent, simple average-friend-opponent, simple average-all-friend and simple average-successful-friend, respectively. As before, the strategy distribution, initially uniform, will change across generations.

Our simulations show that the generation length is crucial for the outcome. For shorter generation length, players have little time to process, and respond to, the information obtained. Hence the players' experience values are almost randomly drawn from a uniform distribution. Whereas when the generation length is longer, the information processing will be more efficient and extensive.

If the generation length is 1, the outcomes of the play will be compared immediately after every round. When the heterogeneous population is playing with the homogeneous one adopting the simple average-previous-opponent, simple average-friend-opponent or simple average-all-friend, the equilibrium value converges to $K/2$. This is expected because the mean value of a uniform distribution with maximal offer K is exactly $K/2$. The final existing strategy is, in most cases, simple average-friend-opponent. We also observed, but only very rarely, that the simple average-all-friend dominates or the simple average-friend-opponent and simple average-all-friend split. As explained in Section II, by itself, average-all-friend does not utilize any information from the opposite population. In a mixed setting, however, some of the friends might play a different strategy, in which case copying them indirectly utilizes information. Simple average-previous-opponent and simple average-successful-friend can never take the leading position. The reason is probably, as discussed in [28], that average-friend-opponent uses both spatial and temporal information, which can expedite the convergence to the final equilibrium. When the heterogeneous population is faced with the homogeneous one having the simple average-successful-friend, the equilibrium value is then $K/4$ or $3K/4$. These results are not difficult to be understood since there is little information exchange due to the short generation length. This is also why it's hard for the simple average-previous-opponent to take the lead because this strategy must utilize all the past information, instead of the zero information conveyed by the random distribution.

If we increase the generation length up to 4, some new phenomena appear. If the heterogeneous population is against the one that plays simple average-previous-opponent or simple average-friend-opponent, the equilibrium can be pushed to K or 0 (Fig. 5). This is of course the extreme case. Consequently, the eventually dominant strategy will be simple average-successful-friend or simple average-all-friend (when the overall success rate is 1, average-successful-friend is equivalent to average-all-friend). If the heterogeneous population opposes the one with simple average-successful-friend, the equilibrium is $3K/4$ or $K/4$ and the final existing strategy shall be the simple average-friend-opponent. When the heterogeneous population is played with the one having the simple average-all-friend, the equilibrium stays at $K/2$ and the simple average-friend-opponent finally dominates. As we see here that when the information exchange is turned on due to longer generation length, totally different schemes emerge. The most interesting finding is that the simple average-successful-friend, the lowest ranked strategy in [28] can take the eventual leading position and forces the equilibrium to its most favorite one (Fig. 6). This phenomenon can be, as well, observed when one plays a heterogeneous population against another heterogeneous one.

To understand the puzzle mentioned in the preceding paragraph we may consult Fig. 6. Its main part displays the generation evolution of the frequencies of different strategies, i.e., the strategy profile. The inset displays the time evolution of the overall success rate of the whole population (defined as the ratio of the number of successful deals per round to population size) and the success rate of the simple average-successful-friend players. The two parts have different time scales because the generation length is 4. We could roughly divide the main curve in Fig. 6 into three different regimes. The first regime is generation 1 where solely the random effects exist. The second one is from generation 2 to generation 5 where the different strategies are competing. The last one is from generation 6 until the end where the simple average-successful-friend already dominates and the other strategies have been completely eliminated. We notice the crucial place is regime 2 where the overall success rate is still as low as 0.6. But for the simple average-successful-friend holders the success rate of their own actions is around 0.9, much higher than 0.6 because they use more selective information. Though their average gain might be lower than the counterparts for the other ones, the total gain (proportional to the product of the success rate and average gain per round) could be higher, in accordance with the analysis of Section II. As shown in the inset of Fig. 5, the difference between the average gains for different strategies is not significant as the one between the success rates. By combining these two factors, the simple average-successful-friend can dominate inside the population, as is exactly shown in Fig. 7 where we plot the strategy percentages versus the generation step. If we remove the simple average-successful-friend from the strategy pool, we find that each of the three remaining strategies has an equal chance to become dominant.

We also extensively explored the situation when the generation length is taken to be even longer, such as 20, 50 or 100. The observed phenomena are similar as for generation length 4. The reason could be that the players are not learning, hence the information exchange has been accomplished within a few (for example 4) steps. The optimal offers for the current round have been nearly identified for the players due to the quick convergence.

VI. CONCLUSION

To summarize, we have investigated the effects of stochasticity and heterogeneity within the framework of our model. The stochastic fluctuations are produced by the random, instead of definite, actions of players. For both individual players and the whole uniform population, it is more beneficial to be less random. But it is good for a more constant population to have a small fraction of errant players when most others play more definite strategies. The more random players are at a disadvantageous position inside the population and thus get eliminated first, which will have some systematic effects on the opposing population. This situation could, in turn, favor the group under consideration in the long term.

The heterogeneity of the population structure considered is strategy related in this paper. Namely, each player inside a population can choose from 4 or 5 different strategies. When all the strategies need to evolve look-up tables, a heterogeneous population generally ties a homogeneous one. But the heterogeneous population converges much more quickly than the homogeneous one, which can be good or bad depending on the equilibrium value reached.
For the "simple" strategies without the look-up tables, the outcomes depend on the generation length used. If the generation length is 1, nobody uses any useful information. Hence, the strategy which needs to adapt to the past information, e.g. average-previous-opponent, can never be the eventual dominant strategy. As generation length is increased up to 4 or even longer, the lowest-ranked strategy for a homogeneous population, the average-successful-friend, can lead inside a heterogeneous one. This leads to the total "coordination" of the game by one side. The reason, as it turns out, is that the average-successful-friend uses more selective information, which could be an advantage when all other strategies are still adapting and testing and thus unable to quickly improve the success rate of deals. Once the advantage is accumulated, the average-successful-friend spreads rapidly inside the population.

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[32] The technical point that such a Gaussian distribution would not be supported on the unit interval $[0, 1]$ is not important for the discussion here.

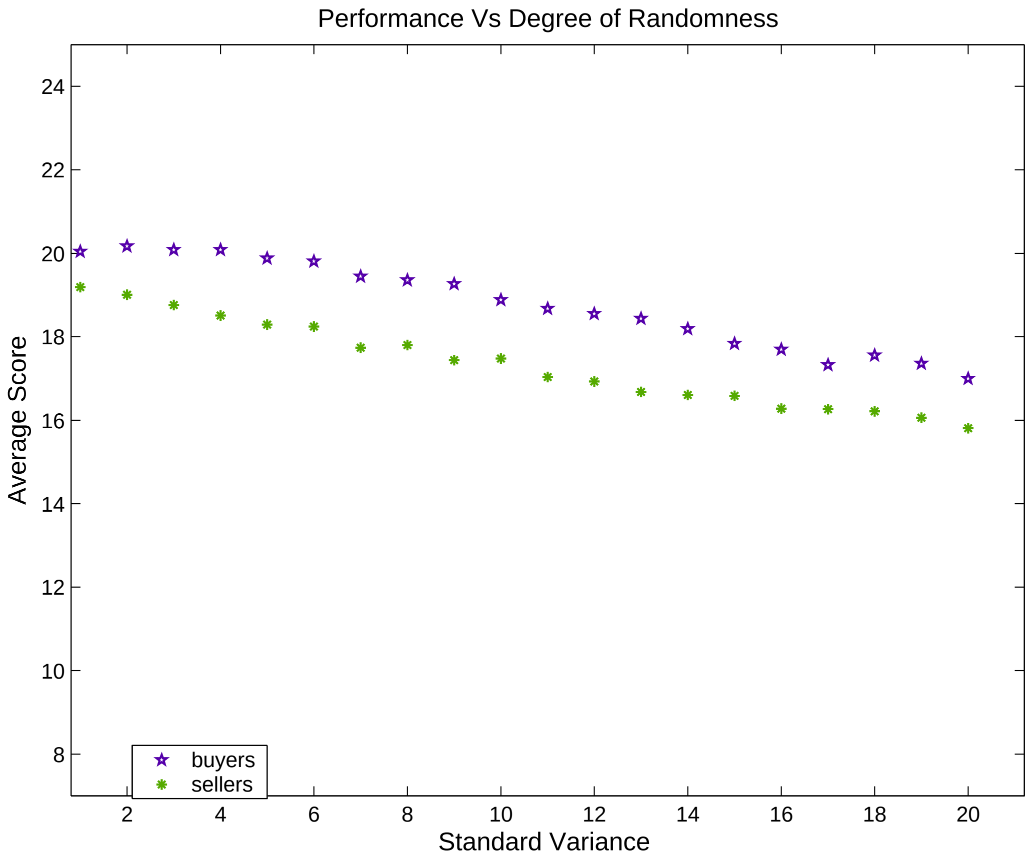

FIG. 1: The performance of individual erratic agents versus the degree of randomness (represented by the standard deviation s of a Gaussian distribution). The performance is evaluated by the average gain (payoff) of the players with the same s.

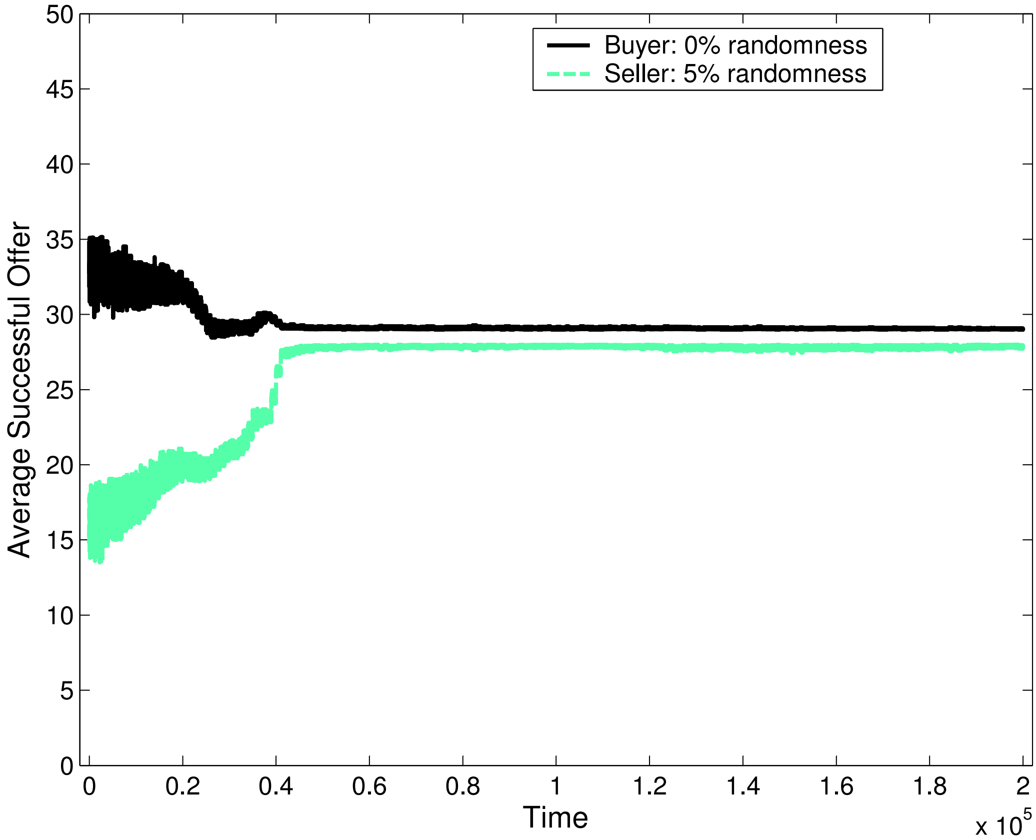

FIG. 2: The performance comparison between the buyer population whose members use 1-round opponent strategy and the seller population most of whose members also use 1-round opponent strategy but 5 percent of them behave randomly. The sellers are performing better.

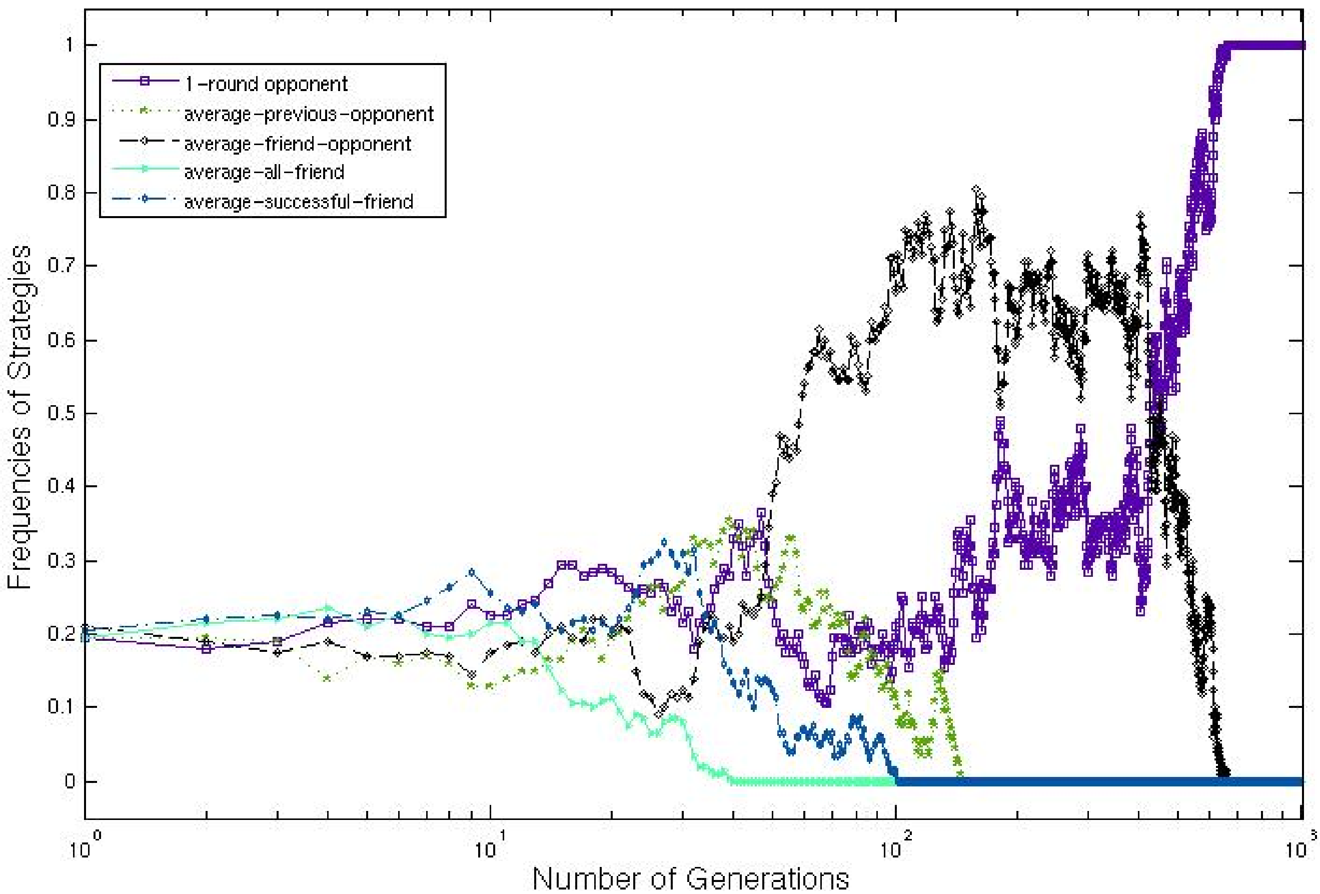

FIG. 3: The generation-scaled evolution of frequencies of different strategies inside a heterogeneous population with each player initially picking one strategy from the uniform strategy pool. The strategy pool consists of 1-round opponent, average-previous-opponent, average-friend-opponent, average-all-friend and average-successful-friend. The strategy profile inside the population is updated every 1000 time steps (one generation), based on the performance (gains) of each strategy, according to a standard evolutionary scheme. Eventually only one strategy, which diffuses inside the whole population, survives.

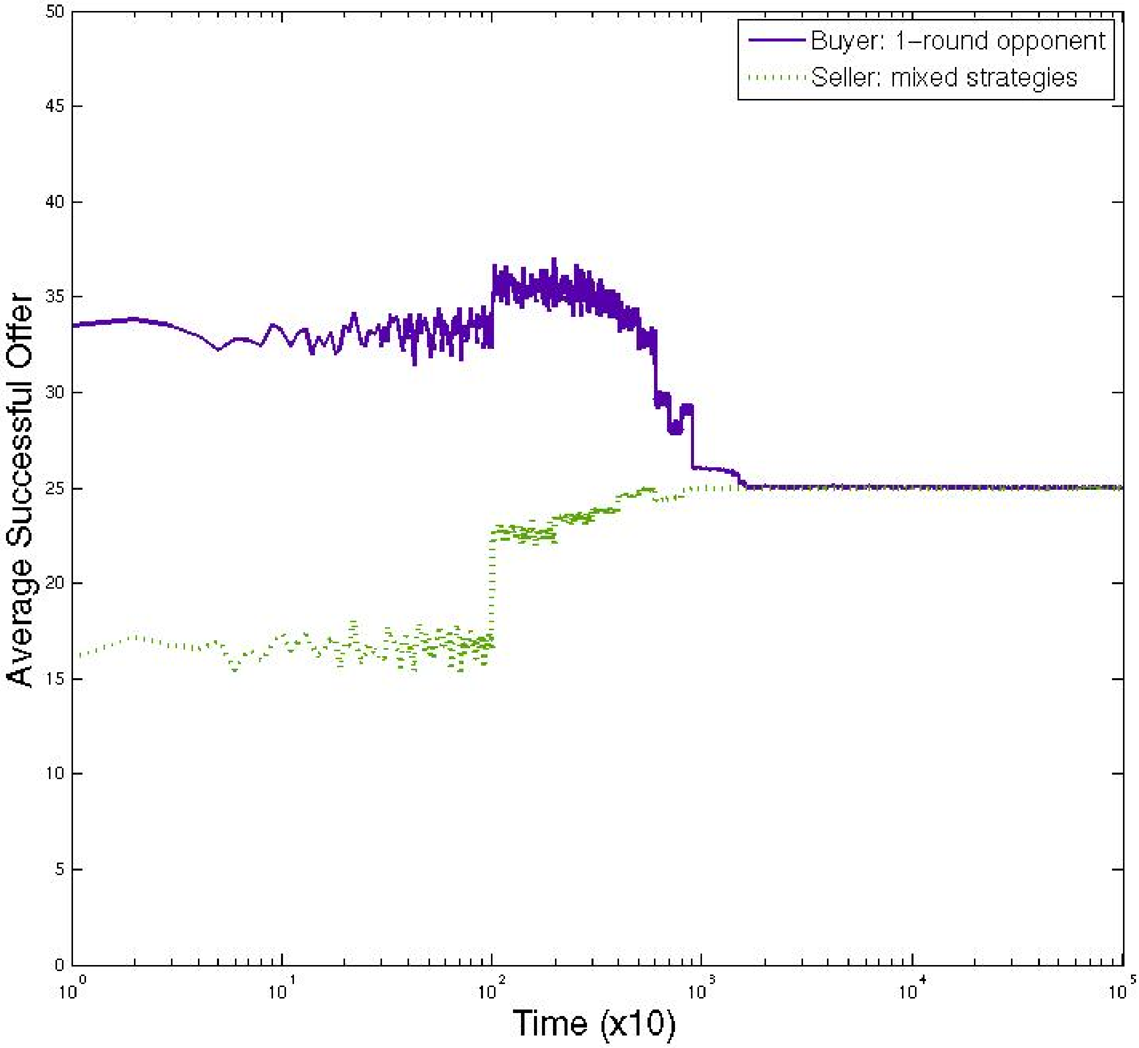

FIG. 4: The heterogeneous population ties the homogeneous one with member players taking average-friend-opponent as their strategy, when the evolutionary scheme is included. The heterogeneous side, however, is converging far more quickly than the homogeneous one. This is mainly due to the extensive and efficient information exchange when fellow players may play different strategies.

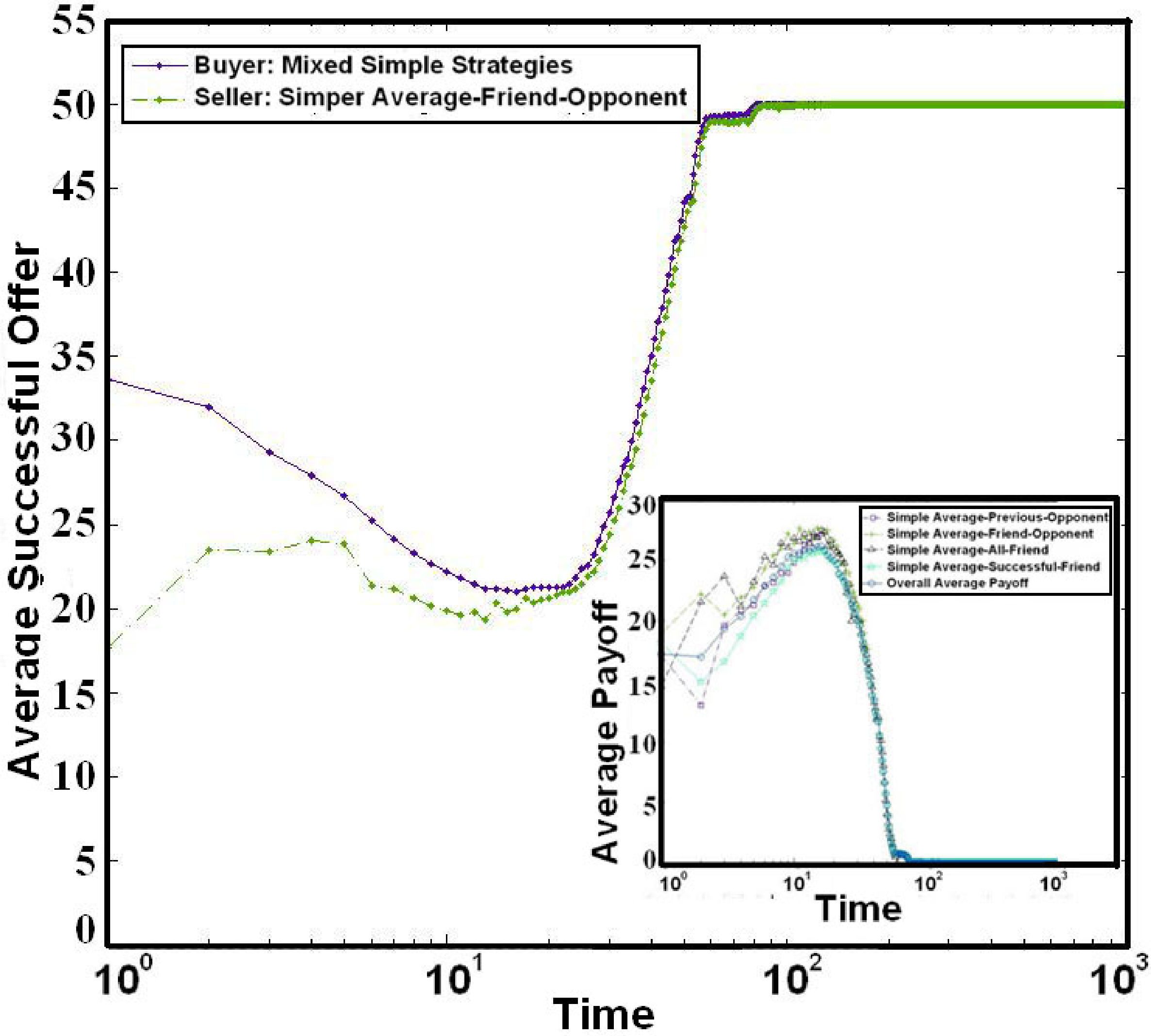

FIG. 5: When the look-up tables are not included and the generation length is 4, the heterogeneous population can achieve the equilibrium most favorable to it. Here, the simple average-successful-friend outperforms the other strategies. This could be fully exploited by the homogeneous population to drive the game to its own favorable direction. The inset displays the overall and respective average payoff for the heterogeneous population. The former is averaged over all members, and the latter, over the fellow members who use the same type of strategy.

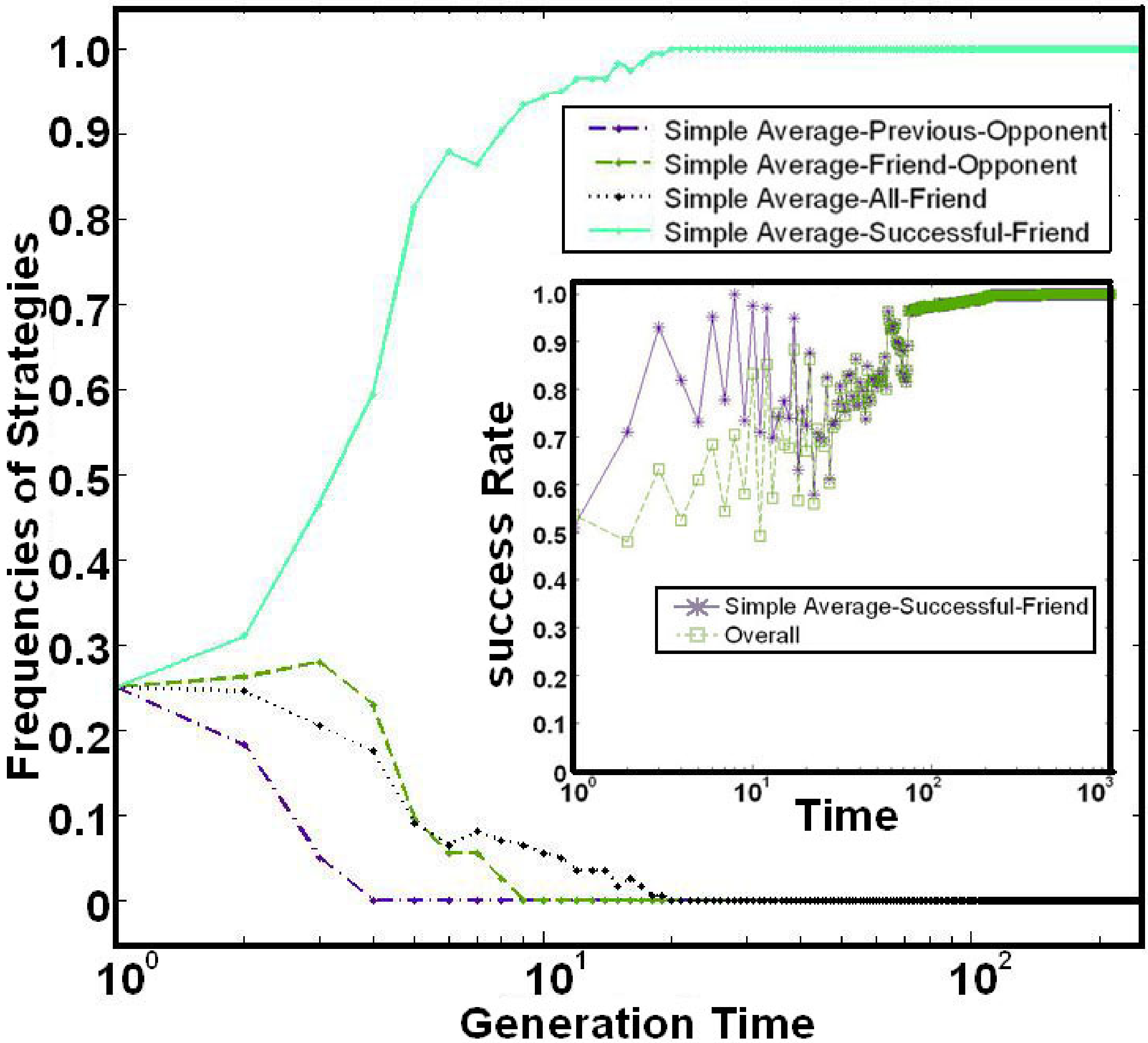

FIG. 6: With the same setting as in Fig. 5, the simple average-successful-friend dominates inside the heterogeneous population. One can roughly divide the plot into three regimes: generation 1, generations 2 to 5, and generations 6 and after. The most crucial place for a strategy to take lead is the second regime. The inset shows the overall success rate for all members and the success rate for simple average-successful-friend players.

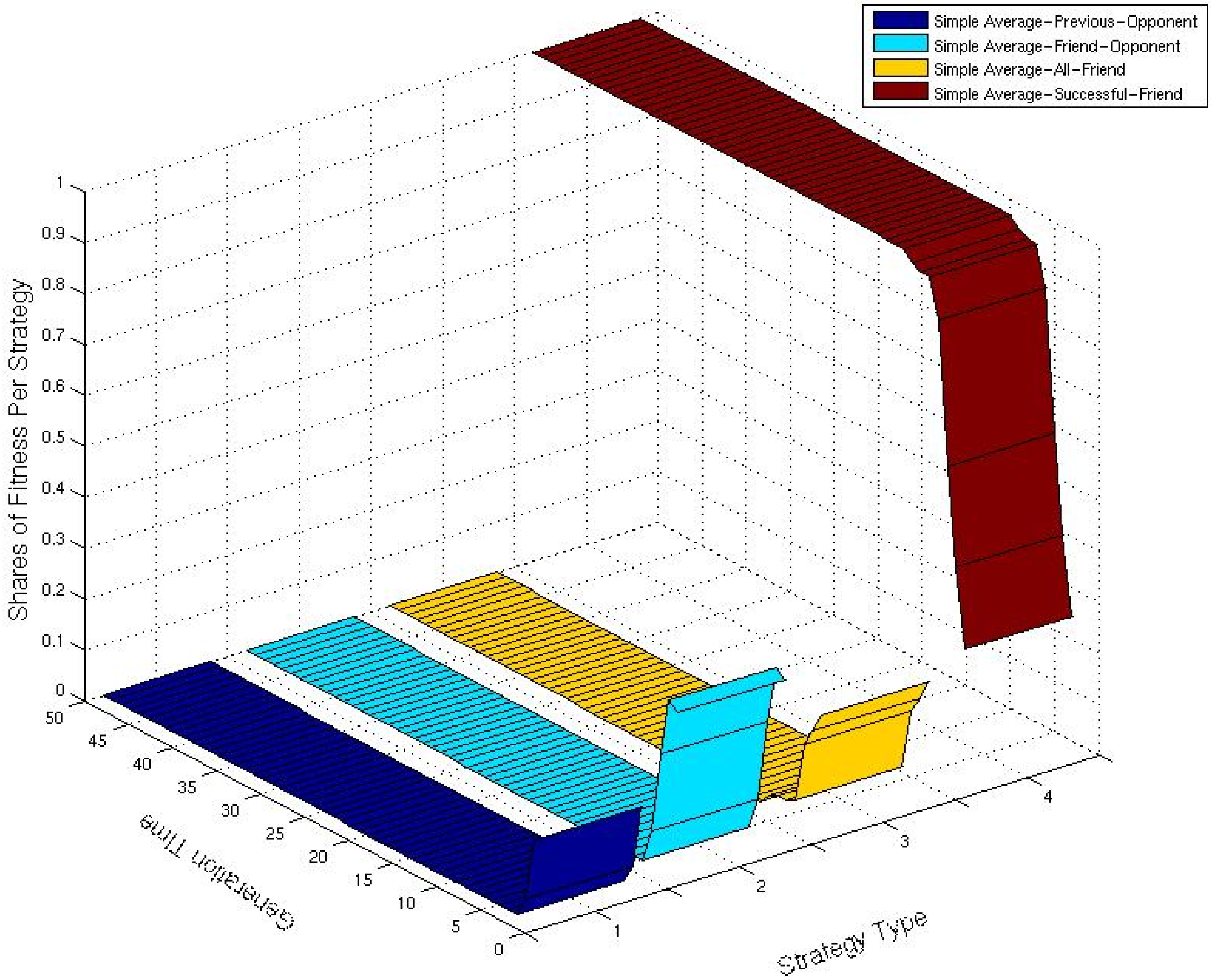

FIG. 7: The generation-scaled evolution of fitness distributed by strategies. While the simple average-successful-friend does not lead initially, its gain grows steadily and is ahead after a few generations.