Homogenization of the nonlinear bending theory for plates

by

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Abstract. We carry out the spatially periodic homogenization of Kirchhoff’s plate theory. The derivation is rigorous in the sense of $\Gamma$-convergence. In contrast to what one naturally would expect, our result shows that the limiting functional is not simply a quadratic functional of the second fundamental form of the deformed plate as it is the case in Kirchhoff’s plate theory. It turns out that the limiting functional discriminates between whether the deformed plate is locally shaped like a “cylinder” or not. For the derivation we investigate the oscillatory behavior of sequences of second fundamental forms associated with isometric immersions of class $W^{2,2}$, using two-scale convergence. This is a non-trivial task, since one has to treat two-scale convergence in connection with a nonlinear differential constraint.

Keywords: homogenization, Kirchhoff plate theory, two-scale convergence, nonlinear differential constraint

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1. Introduction

In this article we study the periodic homogenization of the nonlinear plate model introduced by Kirchhoff in 1850. In that model the elastic behavior of thin plates – undergoing bending only – are described as follows: The reference configuration of the plate in its undeformed, flat state is modeled by a bounded Lipschitz domain $S \subset \mathbb{R}^2$, while bending
deformations are described by isometric immersions $u : S \to \mathbb{R}^3$—differentiable maps that satisfy the isometry constraint

$$\partial_j u \cdot \partial_j u = \delta_{ij},$$

(1)

where $\delta_{ij}$ denotes the Kronecker delta. The elastic bending energy of the deformed plate $u(S)$ is given by the variational integral

$$\int_S Q(\mathbf{II}),$$

(2)

where $\mathbf{II}$ is the second fundamental form associated with $u$ (see (11) below), and $Q$ is the quadratic energy density from linearized elasticity. We are interested in the minimizers of (2), since they are related to equilibrium shapes of thin elastic plates subject to external forces and boundary conditions. Indeed, Friesecke, James, Müller obtained in their celebrated work [FJM02] Kirchhoff’s nonlinear plate model from nonlinear three-dimensional elasticity in the zero-thickness limit. The connection is rigorous in the sense of $\Gamma$-convergence, which roughly speaking means that (almost) minimizers to a large class of minimization problems from three-dimensional nonlinear elasticity converge to solutions to minimization problems associated with the bending energy (2).

The energy density $Q$ encodes the elastic properties of the material and, when the material is heterogeneous, depends on $x \in \mathbb{R}^2$ in addition. In the case of a periodic composite material with small period $\varepsilon \ll 1$, the energy density might be written in the form $Q(y; F)$ where $Q(y; F)$ is periodic in $y$. For definiteness, let $Q$ satisfy the following

**Assumption 1.** Let $Q : \mathbb{R}^2 \times \mathbb{R}^{2 \times 2}_{\text{sym}} \to [0, \infty)$ be

(Q1) measurable and $[0, 1)^2$-periodic in $y \in \mathbb{R}^2$,

(Q2) convex and quadratic in $F \in \mathbb{R}^{2 \times 2}$,

(Q3) bounded and non-degenerate in the sense of

$$\alpha |\text{sym} F|^2 \leq Q(y, F) \leq \frac{1}{\alpha} |\text{sym} F|^2 \quad \text{for all } A \in \mathbb{R}^{2 \times 2} \text{ and a.e. } y \in \mathbb{R}^2$$

(3)

for some constant of ellipticity $\alpha > 0$ which is fixed from now on.

We reformulate the bending energy (4) as the functional $\mathcal{E}^\varepsilon : L^2(\Omega, \mathbb{R}^3) \to [0, \infty]$ given by

$$\mathcal{E}^\varepsilon(u) := \begin{cases} \int_S Q\left(\frac{x}{\varepsilon}, \mathbf{II}(x)\right) \, dx & \text{for } u \in W^{2,2}_{\text{iso}}(S), \\
\infty & \text{else}, \end{cases}$$

(4)

where $W^{2,2}_{\text{iso}}(S)$ denotes the subset of maps $u \in W^{2,2}(S, \mathbb{R}^3)$ that satisfy (11) almost everywhere in $S$.

Our goal is to understand the homogenization limit, $\varepsilon \to 0$, in the spirit of $\Gamma$-convergence. For the description of the limit we need to classify the geometry of surfaces $u(S)$ with $u \in W^{2,2}_{\text{iso}}(S)$. For simplicity, we assume for a moment that $u$ is smooth. Since $u$ is an isometry and $S$ is flat, the Gauss curvature of $u(S)$ vanishes, so that by a classical result, the shape of $u(S)$ is locally either flat, a cylinder or a cone. (With slight abuse of the
standard terminology, we refer to tangent developable surfaces as \textit{cones}). We introduce the sets
\begin{align*}
C_{\nabla u} &= \{ x \in S : u(S) is flat in a neighborhood of u(x) \}, \\
Z_{\nabla u} &= \{ x \in S \setminus \overline{C_{\nabla u}} : u(S) is a cylinder and not flat in a neighborhood of u(x) \}, \\
K_{\nabla u} &= S \setminus (\overline{C_{\nabla u}} \cup Z_{\nabla u}).
\end{align*}
As we explain in Section 2.2 below these definitions extend to $W^{2,2}$-isometric immersions. For the definition of the limiting functional we require averaged and homogenized versions of $Q$. Since the second fundamental form almost surely belongs to the cone of symmetric $2 \times 2$-matrices with rank at most one, it suffices to define the relaxed versions of $Q$ for such matrices: for a unit vector $T \in \mathbb{R}^2$ and $\mu \in \mathbb{R}$ set
\begin{align*}
Q_{\text{av}}(\mu T \otimes T) &:= \mu^2 \int_{(0,1)^2} Q(y, T \otimes T) \, dy, \\
Q_{\text{hom}}(\mu T \otimes T) &:= \mu^2 \min_{\alpha \in W_{2,\text{per}}^1(\mathbb{R})} \left\{ \int_{(0,1)^2} Q\left(y, \left(1 + \alpha'(T \cdot y)\right) T \otimes T\right) \, dy \right\};
\end{align*}
here $W_{T,\text{per}}^1(\mathbb{R})$ denotes the closure w. r. t. the $W^{1,2}$-norm of the set of doubly periodic functions in $C^\infty(\mathbb{R})$ with periods $T \cdot e_1$ and $T \cdot e_2$, see Subsection 4.4 for details. Note that the expression for $Q_{\text{hom}}$ differs from the usual formula used for the homogenization of convex integrands – in fact, as we will see in Subsection 4.4, it can be interpreted as mixture of a one-dimensional averaging and homogenization.

The $\Gamma(L^2)$-limit of $\mathcal{E}^\varepsilon$ is then given by the functional $\mathcal{E}_{\text{hom}} : L^2(\Omega, \mathbb{R}^3) \to [0, \infty),$
\begin{equation*}
\mathcal{E}_{\text{hom}}(u) := \begin{cases} 
\int_S Q_{\text{av}}(\mathbf{II}) + \int_{Z_{\nabla u}} Q_{\text{hom}}(\mathbf{II}) & \text{for } u \in W_{\text{iso}}^{2,2}(S), \\
\infty & \text{else}.
\end{cases}
\end{equation*}

We shall consider boundary conditions of the following form: Let $L$ denote a line segment of the form $L = \{ x_0 + tN : t \in \mathbb{R} \} \cap S$ (for some $x_0 \in \mathbb{R}^2$ and some unit vector $N \in \mathbb{R}^2$). We assume that $\mathcal{H}^1(L) > 0$ and consider the following boundary condition
\begin{equation}
u = \varphi_{BC} \quad \text{and} \quad \nabla u = \nabla \varphi_{BC} \quad \text{on } L,
\end{equation}
where $\varphi_{BC} : \mathbb{R}^2 \to \mathbb{R}^3$ is a fixed rigid isometric immersion, i. e. $\nabla \varphi_{BC}$ is constant and satisfies $\nabla \varphi'_{BC} \nabla \varphi_{BC} = \text{Id}$.

We are now in position to state our main result.

\textbf{Theorem 1.} Let $S \subset \mathbb{R}^2$ be a convex Lipschitz domain and let $Q$ satisfy (Q1) – (Q3).

(a) Consider $u^\varepsilon \in L^2(S, \mathbb{R}^3)$ with finite energy, i. e.
\begin{equation}
\limsup_{\varepsilon \downarrow 0} \mathcal{E}^\varepsilon(u^\varepsilon) < \infty.
\end{equation}

Then there exists $u \in W_{\text{iso}}^{2,2}(S)$ such that $u^\varepsilon - f S u^\varepsilon \to u$ in $L^2(S, \mathbb{R}^3)$ as $\varepsilon \downarrow 0$ (after possibly passing to subsequences).

(b) Let $u^\varepsilon$ converge to some $u$ in $L^2(S, \mathbb{R}^3)$ as $\varepsilon \downarrow 0$. Then
\begin{equation}
\liminf_{\varepsilon \downarrow 0} \mathcal{E}^\varepsilon(u^\varepsilon) \geq \mathcal{E}_{\text{hom}}(u).
\end{equation}
(c) For every $u \in L^2(S, \mathbb{R}^3)$ there exists a sequence $u^\varepsilon \in L^2(S, \mathbb{R}^3)$ that converges to $u$ and
\[ \lim_{\varepsilon \to 0} \mathcal{E}^\varepsilon(u^\varepsilon) = \mathcal{E}^{\text{hom}}(u). \]

Moreover, if $u \in W^{2,2}_{\text{iso}}(S)$ satisfies (\ref{eq:7}), then $u^\varepsilon$ can be chosen such that $u^\varepsilon \in W^{2,2}_{\text{iso}}(S)$ satisfies the boundary condition (\ref{eq:2}) in addition.

The limit $\mathcal{E}^{\text{hom}}$ is not a standard Kirchhoff plate model. In particular, it is not possible to recast $\mathcal{E}^{\text{hom}}$ into the form of (\ref{eq:2}). Still, it is a generalized Kirchhoff plate model in the sense that the energy locally is quadratic in the second fundamental form.

Remark 1. The result also holds true for non-convex Lipschitz domains $S$ with the property that there exists some $\Sigma \subset \partial S$ with $\mathcal{H}^1(\Sigma) = 0$, and the outer normal to $S$ exists and is continuous on $\partial S \setminus \Sigma$. We limit ourselves to the convex case here for the sake of brevity. Our main point is the proof of part (b) of Theorem \ref{thm:1}, which is completely independent of whether $S$ is convex or not. The construction of a recovery sequence in part (c) however becomes somewhat more involved for non-convex domains. It is nevertheless possible by appealing to the results of [Hor11b, Hor11a].

Let us comment on the proof of Theorem \ref{thm:1}. Since $\mathcal{E}^\varepsilon$ is singular and non-convex the derivation of the $\Gamma$-limit is subtle and standard tools, e.g. compactness and representation results for $\Gamma$-limits that rely on integral representations, are not applicable. To overcome these difficulties we take advantage of two observations: First, as a functional of the second fundamental form the mapping
\[ \Pi \mapsto \int_S Q(\frac{x}{\varepsilon}, \Pi) \, dx \tag{8} \]
is convex and quadratic, so that we can pass to the limit $\varepsilon \to 0$ in (\ref{eq:8}) by classical homogenization techniques, in particular two-scale convergence. Secondly, the nonlinear isometry constraint yields a strong rigidity and allows the second fundamental form to oscillate only in a very restricted way.

This second observation is the heart of the matter and requires to describe the structure of two-scale limits of vector fields under a nonlinear differential constraint, cf. Remark \ref{rem} for more details. While the interplay of two-scale convergence and linear differential constraints is reasonably well understood, e.g. see [FK10], in the nonlinear case no systematic approach seems to be available. In fact, to our knowledge our result seems to be the first attempt in that direction in the nonlinear case. Since the main focus of this paper is the derivation of the $\Gamma$-limit to $\mathcal{E}^\varepsilon$, we content ourselves with a partial identification of the two-scale limit which is yet strong enough to treat Theorem \ref{thm:1}. To motivate this in more detail consider a sequence $u^\varepsilon$ that weakly converges in $W^{2,2}_{\text{iso}}(S)$ to some limit $u$. Let $\Pi^\varepsilon$ denote the second fundamental form associated with $u^\varepsilon$. Since $\Pi^\varepsilon$ is bounded in $L^2(S, \mathbb{R}^{2 \times 2})$, we may pass to a weakly two-scale convergent sequence. Since $Q(y, F)$ is convex in $F$, standard results from two-scale convergence, cf. Lemma \ref{lem}, yield the lower bound
\[ \liminf_{\varepsilon \to 0} \int_S Q(\frac{x}{\varepsilon}, \Pi^\varepsilon(x)) \, dx \geq \inf_{H(x,y)} \int_{S} \int_{(0,1)^2} Q(y, H(x,y)) \, dy \, dx, \]
where the infimum is taken over all weak two-scale limits $H(x, y)$ of arbitrary subsequences of $\Pi^\varepsilon$. Seeking for a lower bound that only depends on the limit $u$, we need to identify
the class of limits $H(x, y)$ that might emerge as weak two-scale limits of $II^\varepsilon$. This is done in Section 3. As we shall see in Proposition 2 only certain oscillations on scale $\varepsilon$ are compatible with the nonlinear isometry constraint (II). Loosely speaking, we observe that on cylindrical regions of the limiting plate $u(S)$, only oscillations on scale $\varepsilon$ parallel to the line of curvature are possible, while on conical regions all oscillations on scale $\varepsilon$ are suppressed.

The paper is organized as follows: In Subsection 1.1 we discuss the homogenized quadratic form $Q_{\text{hom}}$ in more detail. In Subsection 1.2 we put our limiting model $\varepsilon^{\text{hom}}$ in relation with models derived from three-dimensional elasticity via simultaneous dimension reduction and homogenization. In Section 4 we recall some basic preliminaries from geometry and two-scale convergence. Section 5 is the core of the paper. There we analyze the structure of oscillations of the second fundamental form. Finally, in the last section we give the proof of Theorem 1.

1.1. Homogenization formula and homogenization effects. Theorem 1 states in particular that locally, there are no homogenization effects if the deformation $u$ is not a cylindrical isometric immersion. On the cylindrical part non-trivial homogenization effects occur and the effective behavior is captured by $Q_{\text{hom}}$ which is defined via (2). The formula involves the space $W^{1,2}_{T\text{-per}}(\mathbb{R})$ which is defined as follows: For any unit vector $T \in \mathbb{R}^2$ we set

$$C_{T\text{-per}}(\mathbb{R}) := \{ \alpha \in C(\mathbb{R}) : \alpha(s + T \cdot k) = \alpha(s) \text{ for all } s \in \mathbb{R}, k \in \mathbb{Z}^2 \},$$

and define $W^{1,2}_{T\text{-per}}(\mathbb{R})$ as the closure of $C_{T\text{-per}}(\mathbb{R})$ w. r. t. the norm

$$\|\alpha\|_{W^{1,2}_{T\text{-per}}(\mathbb{R})}^2 := \int_{\{0, \max\{T \cdot e_1, T \cdot e_2\}\}} \alpha^2(s) + |\alpha'(s)|^2 \, ds.$$

The space $C_{T\text{-per}}(\mathbb{R})$ and thus $W^{1,2}_{T\text{-per}}(\mathbb{R})$ can be characterized as follows: Set

$$r(T) := \begin{cases} 
\sup \{ r \in \mathbb{R} : T \cdot r \mathbb{Z}^d \} & \text{if } T \cdot r \mathbb{Z}^d \text{ for some } r \in \mathbb{R}, \\
+\infty & \text{otherwise.} 
\end{cases}$$

If the ratio of the components of $T$, i. e. $T \cdot e_1$ and $T \cdot e_2$, is irrational, then $r(T) = \infty$ and $C_{T\text{-per}}(\mathbb{R})$ only contains the constant functions. Otherwise $C_{T\text{-per}}(\mathbb{R})$ consists precisely of those functions in $C(\mathbb{R})$ that are periodic with period $r(T)$.

Next, we obtain a more explicit formula for $Q_{\text{hom}}(T \otimes T)$. If $r(T) = \infty$, then we have $Q_{\text{hom}}(T \otimes T) = Q_{\text{av}}(T \otimes T)$. Otherwise, consider for $t \in [0, r(T))$ the finite union of line segments

$$L_t := \{ y \in [0, 1)^2 : T \cdot y - t \in r(T)\mathbb{Z} \},$$

and define $q_{\text{av}, T} : [0, r(T)) \to \mathbb{R}$ by

$$q_{\text{av}, T}(t) = r(T) \int_{L_t} Q(y, T \otimes T) \, d\mathcal{H}^1(y),$$

which in fact is an average since $\mathcal{H}^1(L_t) = r(T)^{-1}$ for all $t \in [0, r(T))$. With this notation, we have by Fubini

$$Q_{\text{hom}}(T \otimes T) = \min \left\{ \int_0^{r(T)} q_{\text{av}, T}(t)(1 + \alpha'(t))^2 \, dt : \alpha \in W^{1,2}_{T\text{-per}}(\mathbb{R}) \right\}.$$

5
The solution of this one-dimensional minimization problem which is obtained by integrating the associated Euler-Lagrange equation is well known. A minimizer \( \alpha_s \) (whose dependency on \( T \) we suppress in the notation) is given by

\[
\alpha_s(t) := \frac{1}{\int_0^{r(T)} ds} \int_0^t ds q_{av,T}(s)
\]

and we obtain

\[
Q_{\text{hom}}(T \otimes T) = \int_0^{r(T)} q_{av,T}(t)(1 + \alpha'_s(t))^2 dt = \frac{1}{\int_0^{r(T)} dt q_{av,T}(t)}.
\]

Thus we have averaging in the direction perpendicular to \( T \) (eq. (11)) and homogenization in the direction of \( T \) (eq. (12)). The averaging takes place over a set of \( H^1 \)-measure \( r(T) \), and the homogenization takes place over a set of \( H^1 \)-measure \( r(T) \). The better \( T \) agrees with the periodic microstructure of the material (which by assumption (Q1) is aligned with the coordinate axes), the smaller is \( r(T) \). Hence, the better \( T \) is chosen to match with the coordinate axes, the more room there is for homogenization effects to make the material softer with respect to bending in this direction.

1.2. Relation to 3d nonlinear elasticity. As mentioned in the introduction Kirchhoff’s plate model can be rigorously derived from nonlinear 3d elasticity. In the following we compare the limit \( \mathcal{E}^{\text{hom}} \) from Theorem 1 to effective models obtained from 3d elasticity via simultaneous dimension reduction and homogenization. To that end we consider the energy functional

\[
\mathcal{E}^{\varepsilon,h}(u) := \frac{1}{h^2} \int \Omega_h W(\frac{x_1}{h}, \frac{x_2}{\varepsilon}, \nabla u(x)) \, dx,
\]

where \( \Omega_h := S \times (-\frac{h}{2}, \frac{h}{2}) \) models the reference domain of a thin, three-dimensional plate with thickness \( h > 0 \), and \( W : \mathbb{R}^2 \times \mathbb{R}^{3 \times 3} \to [0, \infty] \) denotes a stored energy function of an elastic composite material. We assume that \( W(y,F) \) and \( [0,1)^2 \)-periodic in \( y \), and frame-indifferent, non-degenerate, and \( C^2 \) in a neighborhood of the identity in \( F \) (see [FJM02] for details).

The energy \( \mathcal{E}^{\varepsilon,h} \) models a hyperelastic material whose stress free reference state is the thin domain \( \Omega_h \). The described material is a composite that periodically varies in in-plane directions. Note that \( \mathcal{E}^{\varepsilon,h} \) admits two small length scales: the thickness \( h \) and the material fine-scale \( \varepsilon \). The limit \( h \downarrow 0 \) corresponds to dimension reduction, while \( \varepsilon \downarrow 0 \) amounts to homogenization. In [CAMO] it is shown that \( \mathcal{E}^{\varepsilon,h} \) \( \Gamma \)-converges for \( h \downarrow 0 \) (and fixed \( \varepsilon > 0 \)) to the energy \( \mathcal{E}^{\varepsilon} \), cf. (3), where \( Q \) is obtained from the quadratic form \( G \mapsto \frac{\partial^2 W}{\partial F^2}(y, I)(G,G) \) by a relaxation formula, and (by the assumptions on \( W \)) automatically satisfies Assumption 0. Hence, in combination with Theorem 1 we deduce that \( \mathcal{E}^{\text{hom}} \) is the double-limit of the 3d-energy \( \mathcal{E}^{\varepsilon,h} \) that correspond to “homogenization after dimension reduction”; i.e.

\[
\mathcal{E}^{\text{hom}} = \Gamma\text{-lim}_{\varepsilon \downarrow 0} \Gamma\text{-lim}_{h \downarrow 0} \mathcal{E}^{\varepsilon,h}.
\]

We therefore expect \( \mathcal{E}^{\text{hom}} \) to be a good model for the three-dimensional plate in situations where \( h \ll \varepsilon \ll 1 \).
An alternative way to obtain an effective model from $E_{\varepsilon,h}$ is to simultaneously pass to the limit $(\varepsilon, h) \to (0,0)$. In [Nen, Nen12] this has been studied by the first author. In [Nen12] the simpler situation of elastic rods has been analyzed in detail, i.e. when $\Omega_h$ is replaced by a thin rod-like domain of the form $(0,1) \times hB$ where $B$ denotes the two-dimensional cross-section of the rod. As shown in [Nen12] the obtained $\Gamma$-limit depends on the relative scaling between $\varepsilon$ and $h$. More precisely, under the assumption that the ratio $\frac{h}{\varepsilon}$ converges to a prescribed scaling factor $\gamma \in [0, +\infty]$, it is shown that the initial energy $\Gamma$-converges to a bending torsion model for inextensible rods, whose effective energy density continuously depends on the scaling factor $\gamma$. Moreover, it is shown that the model obtained in the case $\gamma = 0$ (which corresponds to simultaneous dimension reduction and homogenization in the regime $h \ll \varepsilon \ll 1$) is equivalent to the model obtained by the sequential limit $"\varepsilon \downarrow 0$ after $h \downarrow 0"$.

For plates, as considered here, this suggests the following: For a given scaling factor $\gamma > 0$ consider the limit $E_\gamma = \Gamma\lim_{h \downarrow 0} E_{\varepsilon,h}$ where we assume that $\frac{h}{\varepsilon} \to \gamma$ as $h \downarrow 0$. This limit corresponds to a simultaneous dimension reduction and homogenization of $E_{\varepsilon,h}$ in the case when the fine-scale $\varepsilon$ and $h$ do not separate. The analysis for rods described above suggests that $E_{\varepsilon,h}^{\text{hom}}$ can be recovered from $E_\gamma$ in the limit $\gamma \downarrow 0$. Surprisingly this is not the case for plates: As shown most recently by Hornung and Velčič and the first author in [hV07, Theorem 2.4], for $\gamma \in (0, \infty)$ the limit $E_\gamma$ takes the form of the plate model (1) with $Q$ replaced by the relaxed and homogenized quadratic form $Q_\gamma$ that depends on the scaling factor $\gamma$. A close look at the relaxation formula defining $Q_\gamma$ shows that typically $\limsup_{\gamma \downarrow 0} Q_\gamma < Q_{av}$. This implies that on the level of the associated energies $E_\gamma$ and $E_{\varepsilon,h}^{\text{hom}}$ we typically have $\limsup_{\gamma \downarrow 0} E_\gamma(u) < E_{\varepsilon,h}^{\text{hom}}(u)$ for conical deformations $u \in W^{2,2}_{iso}(S)$, in contrast to the case of rods, where $\lim_{\gamma \downarrow 0} E_\gamma = E_0 = E_{\varepsilon,h}^{\text{hom}}$.

2. Notation and preliminaries

2.1. Notation. Throughout this article we use the following notation:

- $e_1, e_2$ denotes the standard Euclidean basis of $\mathbb{R}^2$;
- we write $a \cdot b$ for the inner product in $\mathbb{R}^2$, $|.|$ for the induced Euclidean norm, and denote the coefficients of $a \in \mathbb{R}^2$ by $a_i := a \cdot e_i$, $i = 1, 2$;
- for $a = (a_1, a_2) \in \mathbb{R}^2$ we set $a^\perp := (-a_2, a_1)$;
- $S^1 := \{ e \in \mathbb{R}^2 : |e| = 1 \}$;
- for $a = (a_1, a_2), b = (b_1, b_2) \in \mathbb{R}^2$ we denote by $a \otimes b$ the unique $2 \times 2$ matrix characterized by $e_i \cdot (a \otimes b)e_j = a_ib_j$;
- we denote the entries of $A \in \mathbb{R}^{2 \times 2}$ by $A_{ij}$ so that $A = \sum_{i,j=1}^2 A_{ij}(e_i \otimes e_j)$, and we write $A : B := \sum_{i,j=1}^d A_{ij}B_{ij}$ for the inner product in $\mathbb{R}^{2 \times 2}$;
- $A^t$ denotes the transposed of $A \in \mathbb{R}^{2 \times 2}$;
- $B(x, R)$ denotes the unit ball in $\mathbb{R}^2$ with center $x$ and radius $R$;
- $a \times b$ denotes the vector product in $\mathbb{R}^3$.

2.2. Properties of $W^{2,2}$-isometric immersions. From classical geometry it is well known that a smooth surface in $\mathbb{R}^3$ that is isometric to a flat surface is developable — locally it is either flat, a cylinder or a cone. As shown by Kirchheim [Kir11] (see also [PaKr, HorK, Hor]) $W^{2,2}$-isometries share this property. In the following we make this precise. Throughout the paper we use the notation $[x; N] := \{ x + sN : s \in \mathbb{R} \}$ for...
the line through \( x \) parallel to \( N \), and \([x; N]_S\) for the connected component of \([x; N] \cap S\) that contains \( x \). We start our survey with a regularity result on the gradient of isometries:

**Lemma 1** (see [MP85, Proposition 5]). Let \( S \subset \mathbb{R}^2 \) be a Lipschitz domain. Then \( \nabla u \) is continuous for all \( u \in W^{2,2}_{\text{iso}}(S) \).

In the following let \( S \) be a convex Lipschitz domain and \( u \in W^{2,2}_{\text{iso}}(S) \). We shall introduce some objects to describe the geometry of \( u(S) \). We say \( x \in S \) is a flat point of \( \nabla u \), if \( \nabla u \) is constant in some neighborhood of \( x \) and introduce the (open) set

\[
C_{\nabla u} := \{ x \in S : x \text{ is a flat point of } \nabla u \}.
\]

For our purpose it is convenient to describe the geometry of the non-flat part \( S \setminus C_{\nabla u} \) by means of asymptotic lines. We say a unit vector \( N \in \mathbb{R}^2 \) is called an asymptotic direction (for \( \nabla u \) at \( x \in S \) if

\[
\exists s_0 > 0 \text{ such that } \nabla u(x) = \nabla u(x + sN) \text{ for all } s \in (-s_0, s_0).
\]

When \( u \) is a smooth isometry, then it is known from classical geometry that at every non-flat point \( x \) there exists an asymptotic direction \( N(x) \) that is unique up to a sign. In fact, we know more: There exists a mapping \( S \setminus C_{\nabla u} \to S^1 := \{ N \in \mathbb{R}^2 : |N| = 1 \} \) such that for all \( x, y \in S \setminus C_{\nabla u} \)

\[
\begin{align}
\nabla u \text{ is constant on } [x; N(x)]_S, & \quad (14a) \\
[x; N(x)]_S \cap [y; N(y)]_S \neq \emptyset \implies [x; N(x)] = [y; N(y)]. & \quad (14b)
\end{align}
\]

This observation extends to \( W^{2,2}_{\text{iso}} \)-isometries:

**Proposition 1** ([Pak11]). Let \( u \in W^{2,2}_{\text{iso}}(S, \mathbb{R}^3) \). Then there exists a locally Lipschitz continuous vector field \( N : S \setminus C_{\nabla u} \to S^1 \) such that (13a) and (14a) is true for all \( x, y \in S \setminus C_{\nabla u} \). Furthermore, the field \( S \setminus C_{\nabla u} \ni x \mapsto N(x) \times N(x) \) is unique.

For isometries of class \( C^2 \), Proposition 1 is contained in the more general result [HN59]. In the form above, the proposition has been proven in [Pak11], using ideas from [Kni11]. We distinguish two-types of non-flat points: We say \( x_0 \in S \setminus C_{\nabla u} \) is cylindrical if around \( x_0 \) the field of asymptotic directions is constant, i.e.

\[
\exists N \in S^1, B \subset S \text{ ball centered at } x_0 \text{ such that (13) holds for all } x \in B. \quad (15)
\]

Otherwise we say (by slight abuse of the terminology from differential geometry) that a non-flat point is conical. We introduce the sets

\[
Z_{\nabla u} := \{ x \in S \setminus C_{\nabla u} : x \text{ is cylindrical} \}, \quad K_{\nabla u} := S \setminus (\overline{C_{\nabla u} \cup Z_{\nabla u}}).
\]

The second fundamental form associated with \( u \in W^{2,2}_{\text{iso}}(S) \) is given by the matrix field

\[
\Pi : S \to \mathbb{R}^{2 \times 2}
\]

with entries

\[
\Pi_{ij} := -\partial_i n \cdot \partial_j u, \quad (16)
\]

where \( n := \partial_1 u \times \partial_2 u \) denotes the normal field to the surface \( u(S) \). The following elementary property can be found in [BM06, Hor11B]:

\[
}\]
Lemma 2. Let $S \subset \mathbb{R}^2$ be bounded and $u \in W_{iso}^{2,2}(S)$, then almost everywhere on $S$

\begin{align}
\partial_i \partial_j u \cdot n &= \Pi_{ij}, \\
\partial_2 \Pi_{11} &= \partial_1 \Pi_{12}, \\
\partial_2 \Pi_{21} &= \partial_1 \Pi_{22},
\end{align}

and there exists $T : S \to S^1$ and $\mu \in L^2(S)$ such that

$$
\Pi = \mu T \otimes T \quad \text{almost everywhere on } S.
$$

2.3. Two-scale convergence. Let $Y = [0, 1)^2$ denote the unit cell in $\mathbb{R}^2$, and let $\mathcal{Y} := \mathbb{R}^2/\mathbb{Z}^2$ denote the unit torus. We denote by $C(\mathcal{Y})$ (resp. $C^\infty(\mathcal{Y})$) the space of continuous (resp. smooth) functions on the torus. Two-scale convergence allows to conveniently pass to limits in convex functionals with two-scale convergence.

Definition 1. A bounded sequence $w^\varepsilon \in L^2(S)$ is weakly two-scale convergent to $w \in L^2(S \times \mathcal{Y})$ if and only if

$$
\lim_{\varepsilon \downarrow 0} \int_S w^\varepsilon(x) \psi(x, x/\varepsilon) dx = \int_{S \times \mathcal{Y}} w(x, y) \psi(x, y) dx \, dy \quad \forall \psi \in C_0^\infty(S \times \mathcal{Y}).
$$

Then we write $w^\varepsilon \rightharpoonup w$ in $L^2(S \times \mathcal{Y})$. If the sequence satisfies in addition

$$
\lim_{\varepsilon \downarrow 0} \int_S |w^\varepsilon(x)|^2 dx = \int_{S \times \mathcal{Y}} |w(x, y)|^2 dx \, dy
$$

then we say that $w^\varepsilon$ strongly two-scale convergent to $w$ and write $w^\varepsilon \overset{\text{s.t.s.}}{\rightharpoonup} w$. For vector valued functions we define weak and strong two-scale convergence component-wise.

The following result can be found in [Ali92]. It is an elementary but fundamental property of two-scale convergence and allows to pass to the limit in products of weakly convergent sequences.

Lemma 3. Let $S \subset \mathbb{R}^2$ be open and bounded. Consider two sequences $w^\varepsilon$ and $\psi^\varepsilon$ that are bounded in $L^2(S)$, and suppose that $w^\varepsilon \rightharpoonup w$ strongly two-scale and $\psi^\varepsilon \rightharpoonup \psi$ weakly two-scale in $L^2(S \times \mathcal{Y})$. Then

$$
\int_S w^\varepsilon(x) \psi^\varepsilon(x) \, dx \to \int_{S \times \mathcal{Y}} w(x, y) \psi(x, y) \, dy \, dx.
$$

Two-scale convergence allows to conveniently pass to limits in convex functionals with periodic coefficients. The following lemma is a special case of [Vis07, Proposition 1.3]

Lemma 4. Let $A \subset \mathbb{R}^2$ be open and bounded, and let $Q$ satisfy assumption 1.

(a) Suppose that $G^\varepsilon \in L^2(A, \mathbb{R}^{2 \times 2})$ weakly two-scale converges to $G \in L^2(A \times \mathcal{Y}, \mathbb{R}^{2 \times 2})$. Then

$$
\liminf_{\varepsilon \downarrow 0} \int_A Q(\frac{\varepsilon}{x}, G^\varepsilon(x)) \, dx \geq \int_{A \times \mathcal{Y}} Q(y, G(x, y)) \, dy \, dx.
$$

(b) Suppose that $G^\varepsilon \in L^2(A, \mathbb{R}^{2 \times 2})$ strongly two-scale converges to $G \in L^2(A \times \mathcal{Y}, \mathbb{R}^{2 \times 2})$. Then

$$
\lim_{\varepsilon \downarrow 0} \int_A Q(\frac{\varepsilon}{x}, G^\varepsilon(x)) \, dx = \int_{A \times \mathcal{Y}} Q(y, G(x, y)) \, dy \, dx.
$$
3. Two-scale limits of second fundamental forms

In this section we analyze the structure of two-scale limits of second fundamental forms. In particular, we show that oscillations on scale $\varepsilon$ are suppressed in regions where the limiting isometric immersion is neither cylindrical nor flat. Moreover it states that at points where the limiting isometric immersion is cylindrical, oscillations on scale $\varepsilon$ can only emerge perpendicular to asymptotic directions. The upcoming result will be our main tool in proving the lower bound for the $\Gamma$-convergence result.

**Proposition 2.** Consider a sequence $u^\varepsilon$ that weakly converges to $u$ in $W^{2,2}_{\mathrm{iso}}(S)$ and suppose that

$$
\bar{H}^\varepsilon \overset{\ast}{\rightharpoonup} \bar{H} + G \text{ weakly two-scale in } L^2(S \times \mathcal{Y}, \mathbb{R}^{2 \times 2}),
$$

for some $G \in L^2(S \times \mathcal{Y}, \mathbb{R}^{2 \times 2})$. Then the following properties hold:

(a) (conical case). $G = 0$ almost surely in $K_{\nabla u} \times \mathcal{Y}$.

(b) (cylindrical case). For every connected component $Z \subset Z_{\nabla u}$ there exists $T \in S^1$, $\mu \in L^2(\mathcal{Z})$ and $\alpha \in L^2(\mathcal{Z}, W^{1,2}_{1,\text{per}}(\mathbb{R}))$ such that

$$
\bar{H}(x) = \mu(x) T \otimes T,
G(x, y) = \alpha'(x, T : y) T \otimes T
$$

for almost every $x \in Z$ and $y \in \mathcal{Y}$. Here $\alpha'$ denotes the derivative of $\alpha$ w. r. t. its second component.

**Remark 2.** In the proof of Theorem I the preceding proposition is used to establish the lower-bound part of the $\Gamma$-convergence statement. The proposition yields a partial characterization of the possible two-scale limits of $\bar{H}^\varepsilon$. The result is partial; e. g. Proposition 2 does not yield any detailed information on the behavior of $G(x, y)$ in flat regions. Yet, Proposition 2 is sufficient for identifying the $\Gamma$-limit in the proof of Theorem I.

For the proof of Proposition 2 we need the upcoming Lemma 5 which is the heart of the matter.

**Lemma 5.** Consider a sequence $u^\varepsilon$ that weakly converges to $u$ in $W^{2,2}_{\mathrm{iso}}(S)$. Let $B := B(x_0, R) \subset S \setminus C_{\nabla u}$ with $R \leq \text{dist}(x_0, \partial S)/2$, and $\frac{1}{4} B = B(x_0, R/4)$. Then for all $k \in \mathbb{R}^2$ satisfying

$$
\left| \left\{ x \in \frac{1}{4} B : |H : (k^\perp \otimes k^\perp)| = 0 \right\} \right| = 0,
$$

and all $\zeta \in C^\infty_0(\frac{1}{4} B), \ E \in \mathbb{R}^{2 \times 2}$ we have

$$
\lim_{\varepsilon \to 0} \left| \int_S \left( \bar{H}^\varepsilon(x) : E \right) \zeta(x) \exp \left( \frac{2\pi i k \cdot x}{\varepsilon} \right) \, dx \right| = 0.
$$

Here $\bar{H}$ and $\bar{H}^\varepsilon$ denote the second fundamental form associated with $u$ and $u^\varepsilon$, respectively.

**Remark 3.** As a consequence of (12) and (14) we may represent the second fundamental form $\bar{H}$ of an arbitrary $W^{2,2}$-isometry as $\bar{H} = \nabla^2 \varphi$ where $\varphi \in W^{2,2}(S)$ is a scalar function that solves the degenerate Monge-Ampère equation

$$
\det \nabla^2 \varphi = 0.
$$

Above $\nabla^2 \varphi$ denotes the Hessian of $\varphi$. As in [Pak04], Proposition 1 can be reformulated for scalar functions that belong to the non-convex space

$$
\mathcal{A} := \{ \varphi \in W^{2,2}(S) : \det \nabla^2 \varphi = 0 \}.
$$
Without much effort we recover the result of Lemma 4 on the level of the functions \( \varphi \in A \); i.e., the following statement: Consider a sequence \( \varphi^\varepsilon \in W^{2,2}(S) \) of solutions to (22) and assume that \( \varphi^\varepsilon \) weakly converges to some \( \varphi \) in \( W^{2,2}(S) \). Consider concentric ball \( \frac{1}{2}B, B \) and \( 2B \subset S \) with radius \( \frac{1}{4}R, R \) and \( 2R \). Assume that on \( \frac{1}{4}B \) the limit \( \varphi \) is locally not equal to an affine function, i.e., for any open set \( O \subset \frac{1}{4}B \) we have

\[
\min_{A \in \mathbb{R}^2, a \in \mathbb{R}^2} \int_O |\varphi(x) - (A \cdot x + a)|^2 \, dx > 0.
\]

Then for all \( k \in \mathbb{R}^2 \) satisfying

\[
\left\{ x \in \frac{1}{4}B : |\nabla^2 \varphi : (k^\perp \otimes k^\perp)| = 0 \right\} = 0,
\]

and all \( \zeta \in C^\infty_c(\frac{1}{4}B), E \in \mathbb{R}^{2 \times 2} \) we have

\[
\lim_{\varepsilon \to 0} \int_S (\nabla^2 \varphi^\varepsilon(x) : E) \zeta(x) \exp \left( \frac{2\pi ik \cdot x}{\varepsilon} \right) \, dx = 0.
\]

Rephrased in that form, it is apparent that Lemma 4 entails a characterization of two-scale limits under the nonlinear differential constraint (22). Note that the interplay of two-scale convergence and linear differential constraints is reasonably well understood, see e.g., [4] for general results in that direction. In contrast, to our knowledge our result is the first treatment of a nonlinear differential constraint.

Before we turn to the proof of the lemma, we proceed with the proof of Proposition 4, which is a simple consequence of Lemma 5 and soft arguments from two-scale convergence.

Let \( B := B(x_0, R) \) be an arbitrary ball contained in \( K_{\varphi} \) such that \( B(x_0, 2R) \subset S \). Set \( \frac{1}{4}B = B(x_0, R/4) \).

1a. We claim that any wave vector \( k \in \mathbb{N}^2 \setminus \{(0,0)\} \) satisfies (22). Indeed, this can be seen by a contradiction: Suppose that (22) is not valid for some \( k \neq (0,0) \). Then \( |H : (k^\perp \otimes k^\perp)| = 0 \) almost everywhere in an open set \( A \subset B \). Since on \( A \) the second fundamental form \( H \) is symmetric and has rank 1, we deduce that \( H = (k \otimes k) \tilde{\mu} \) for some \( \tilde{\mu} \in L^2(A) \). Hence, \( u \) is cylindrical on \( A \), which contradicts \( A \subset K_{\varphi, u_0} \).

1b. Since \( H^\varepsilon \to H \) in \( L^2(Z) \), we have

\[
\int_{Y} G(x, y) \, dy = 0 \quad \text{for a.e.} \quad x \in Z.
\]

Let \( E \in \mathbb{R}^{2 \times 2}, \zeta \in C^\infty_c(\frac{1}{4}B) \) and \( k \in \mathbb{N}^2 \setminus \{(0,0)\} \). By 1a, the conditions of Lemma 5 are fulfilled for any \( k \in \mathbb{N}^2 \setminus \{(0,0)\} \), and thus

\[
0 = \lim_{\varepsilon \to 0} \int_S (H^\varepsilon(x) : E) \zeta(x) \exp \left( \frac{2\pi ik \cdot x}{\varepsilon} \right) \, dx = \iint_{S \times Y} \left( (H(x) + G(x, y)) : E \right) \zeta(x) \exp (2\pi ik \cdot y) \, dydx.
\]

In the last line we used that \( H^\varepsilon \overset{2}{\to} H(x) + G(x, y) \). Since \( k \neq (0,0) \), we have \( \int_Y \exp(2\pi iy \cdot k) \, dy = 0 \) and thus

\[
0 = \iint_{S \times Y} (G(x, y) : E) \zeta(x) \exp (2\pi ik \cdot y) \, dydx.
\]
By eq. (24), eq. (25) even holds true for any \( k \in \mathbb{N} \), \( \zeta \in C_0^\infty(\frac{1}{4}B) \) and \( E \in \mathbb{R}^{2\times2} \). We deduce that \( G = 0 \) on \( \frac{1}{4}B \times Y \).

**Step 2.** Argument for (b). The existence of \( T \in S^1 \), \( \mu \in L^2(Z) \) such that \( II(x) = \mu(x)T \otimes T \) for a.e. \( x \in Z \) follows from (24) and the definition of \( Z_{\nabla u} \). Let \( N \) be the asymptotic direction of \( Z \), and \( T_T := \{ k \in \mathbb{Z}^2 : k \cdot N = 0 \} \). Arguing as in Step 2, we get
\[
0 = \int_{S \times Y} (G(x,y) : E) \zeta(x) \exp(2\pi ik \cdot y) \; dydx
\]
for all \( k \in \mathbb{Z}^2 \setminus T_T, \zeta \in C_0^\infty(\frac{1}{4}B) \). By periodicity of \( G \) in the second variable, we may write it as a Fourier row. By (24), \( G \cdot N = N \cdot G = 0 \), and thus
\[
G(x,y) = \sum_{k \in \mathbb{Z}^2} a_k(x) \exp(2\pi ik \cdot y) T \otimes T
\]
for some \( a \in L^2(Z, l^2(\mathbb{Z}^2)) \). By eq. (28), \( a_k \equiv 0 \) for \( k \notin T_T \). By eq. (24), we also have \( a_0 \equiv 0 \). Hence \( G(x,y) = \partial_s \alpha(x,T \cdot y) \), where \( \partial_s \) is the derivative with respect to the second variable,
\[
\alpha(x,s) = \sum_{\substack{k \in T_T \\cap k \neq 0}} \frac{a_k(x)}{2\pi i(k \cdot T)} \exp(2\pi i(k \cdot T)s) T \otimes T,
\]
and \( \alpha \in L^2(Z, W^{1,2}(\mathbb{R})) \). Also, for \( k \in T_T, k' \in \mathbb{Z}^2 \), we have
\[(k \cdot T)(s + T \cdot k') = (k \cdot T)s + k \cdot k' \in (k \cdot T)s + \mathbb{Z}\]
and hence by formula (28), \( \alpha(x,s + T \cdot k') = \alpha(x,s) \) for all \( x \in Z, k' \in \mathbb{Z}^2, s \in \mathbb{R} \). This proves \( \alpha \in L^2(Z, W^{1,2}_{T_{\text{per}}}(\mathbb{R})) \) and completes the proof.

In the remainder of this section we prove Lemma 6. The argument is split in several lemmas. As a starting point we need to extend the field \( N \) of asymptotic directions, see Proposition 4 to the non-flat region. We only require a local extension to balls away from the boundary of \( S \). This is the content of the upcoming Lemma 7, which – despite being elementary – plays a crucial role in our analysis.

**Lemma 6.** Let \( u \in W^{2,2}_{loc}(S) \). Consider a ball \( B \) with \( 2B \subset S \). Then there exists a Lipschitz continuous function \( N : B \to S^1 \) such that for all \( x, y \in B \):
\[
\nabla u \text{ is constant on } [x;N(x)]_B,
\]
\[
[x;N(x)]_{2B} \cap [y;N(x)]_{2B} \neq \emptyset \implies [x;N(x)] = [y;N(y)]
\]
Moreover, we have
\[
\text{Lip}(N) \leq \frac{1}{\text{radius}(B)}.
\]
(The proof of Lemma 6 is postponed to the end of this section.)

Combined with the representation (24), we obtain the following observation: In the situation of Lemma 6 we have
\[
II = \mu N \otimes N \quad \text{almost everywhere in } S \setminus C_{\nabla u},
\]
where \( N \) is the vector field of Lemma 6 and some \( \mu \in L^2(S \setminus C_{\nabla u}) \). Since the Lipschitz bound (24) only depends on the radius of \( B \), and in particular not on the isometry \( u \) we get the following compactness result:
Corollary 1. Let $B$ denote a ball with $2B \subset S$. Consider a sequence $u^\varepsilon \in W^{2,2}_\text{loc}(S)$ and let $N_\varepsilon : B \to S^1$ denote the Lipschitz function associated with $u^\varepsilon$ via Lemma 6. Then there exists a Lipschitz function $\bar{N} : B \to S^1$ and $\bar{\mu} \in L^2(B \times Y)$ such that (up to subsequences)

\[
\begin{align*}
N^\varepsilon \otimes N^\varepsilon &\to \bar{N} \otimes \bar{N} \quad \text{uniformly in } B, \\
\mathbf{II}^\varepsilon &\to \bar{\mu}(x,y) \left( \bar{N}^\perp(x) \otimes \bar{N}^\perp(x) \right) \quad \text{two-scale in } L^2(B \times Y).
\end{align*}
\]

Moreover, if $u^\varepsilon \rightharpoonup u$ weakly in $W^{2,2}(S, \mathbb{R}^3)$ and $N : B \to S^1$ is associated with $u$ via Lemma 4, then we have

\[
\bar{N} \otimes \bar{N} = N \otimes N \quad \text{in } B \setminus C_{x_u}.
\]

(The proof of Corollary 1 is postponed to the end of this section.)

Statement (33) partially identifies the structure of possible two-scale limits of second fundamental forms. Indeed, (33) combined with (35) shows that for $B \subset S \setminus C_{x_u}$ the rank-one matrix on the right-hand side of (33) is uniquely determined by $u$. Next, we shall see that also the scalar prefactor $\bar{\mu}$ is not arbitrary in the non-flat region. Lemma 6 is a simple rephrasing of the following statement:

Lemma 7. Let $B = B(x_0, R)$ denote a ball with $2B = B(x_0, 2R) \subset S$. Consider a sequence $u^\varepsilon \in W^{2,2}_\text{loc}(S)$ and let $N_\varepsilon : B \to S^1$ denote the Lipschitz function associated with $u^\varepsilon$ via Lemma 6. Assume that

\[
N^\varepsilon \otimes N^\varepsilon \to N \otimes N \quad \text{uniformly in } B,
\]

and that

\[
\limsup_{\varepsilon \to 0} \|\mathbf{II}^\varepsilon\|_{L^2(S)} < \infty,
\]

where $\mathbf{II}^\varepsilon$ is the second fundamental form of $u^\varepsilon$. Set $\frac{1}{2}B := B(x_0, R/4)$. Then for all $k \in \mathbb{R}^2$ satisfying

\[
\left\{ x \in \frac{1}{2}B : |N(x) \cdot k| = 0 \right\} = 0,
\]

and all $\zeta \in C_0^\infty(\frac{1}{4}B)$, $E \in \mathbb{R}^{2 \times 2}$ we have

\[
\lim_{\varepsilon \to 0} \left| \int_S \mathbf{II}^\varepsilon(x) : E \zeta(x) \exp \left( \frac{2\pi ik \cdot x}{\varepsilon} \right) \, dx \right| = 0.
\]

Proof of Lemma 7. For convenience set $Q := (-\frac{R}{2}, \frac{R}{2})^2$. In the first two steps we recall the construction of line of curvature coordinates; this is standard, see e. g. Horill, Horilla. In Step 3 – Step 5 we prove the lemma using these coordinates.

Step 1. Line of curvature.

We claim that for each $\varepsilon$ there exists a function $\Gamma^\varepsilon \in W^{2,\infty}([-R, R], B)$ with

\[
\Gamma^\varepsilon(0) = x_0, \quad (\Gamma^\varepsilon)'(t) = -(N^\varepsilon)^\perp(\Gamma^\varepsilon(t)) \quad \text{for all } t \in [-R, R],
\]

and additionally

\[
\max_{t \in [-R, R]} |\kappa^\varepsilon(t)| \leq \frac{1}{R}
\]

where $\kappa^\varepsilon(t) := (\Gamma^\varepsilon)'(t) \cdot N^\varepsilon(\Gamma^\varepsilon(t))$. Since the argument does not depend on $\varepsilon$, we drop the superscript $\varepsilon$ from our notation for the remainder of this step. Also, we will write $N(t) := N(\Gamma(t))$.

The existence and regularity of the curve $\Gamma$ follows from a standard fix point argument.

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Since $\Gamma'(t) = -N^\perp(\Gamma(t))$ is a unit vector, we deduce that $\Gamma''(t)$ is orthogonal to $\Gamma'(t)$ and thus parallel to $N(t)$. Hence, there exists an $L^2$ function $\kappa(t)$ such that $\Gamma''(t) = \kappa(t)N(t)$. We have for almost every $t$

$$|\kappa(t)| = |\Gamma''(t)| = |\nabla N(\Gamma(t))\Gamma'(t)| \leq |\nabla N(\Gamma(t))| \leq \text{Lip}(N).$$

The estimate $\text{Lip}(N) \leq \frac{1}{\delta}$ (cf. (31)) completes the argument.

**Step 2.** Line of curvature coordinates.

For $(t, s) \in Q$ define $\Phi^\varepsilon(t, s) := \Gamma^\varepsilon(t) + sN^\varepsilon(t)$. We claim that the map $\Phi^\varepsilon : Q \to \Phi^\varepsilon(Q)$ is one-to-one and Lipschitz continuous with

$$\text{Lip}(\Phi^\varepsilon) \leq 2, \quad \frac{1}{2} \leq \det \nabla \Phi^\varepsilon \leq 2,$$

and satisfies

$$\frac{1}{\varepsilon}B \subset \Phi^\varepsilon(Q).$$

Moreover, we claim that there exists $\kappa^\varepsilon_n \in L^2((-\frac{R}{2}, \frac{R}{2}))$ such that

$$(\mathcal{II}^\varepsilon \circ \Phi^\varepsilon)(t, s) = \frac{\kappa^\varepsilon_n(t)}{1 - s\kappa^\varepsilon_n(t)} (\Gamma^\varepsilon)'(t) \otimes (\Gamma^\varepsilon)'(t),$$

(41)

and

$$(\mathcal{II}^\varepsilon \circ \Phi^\varepsilon)(t, s) \det \nabla \Phi^\varepsilon(t, s) = \kappa^\varepsilon_n(t) (\Gamma^\varepsilon)'(t) \otimes (\Gamma^\varepsilon)'(t),$$

(42)

almost everywhere in $Q$.

Again, we drop the superscript $\varepsilon$ for the rest of this step. Let $(t, s), (t', s') \in Q$. Then

$$|\Phi(t, s) - \Phi(t', s')| \leq |\Phi(t, s) - \Phi(t', s)| + |\Phi(t', s) - \Phi(t', s')|$$

$$\leq |\Gamma(t) - \Gamma(t')| + |N(t) - N(t')||s| + |N(t')||s - s'|$$

$$\leq |t - t'| + R^{-1}|t - t'|R/2 + |s - s'|$$

$$\leq 2|(t, s) - (t', s')|.$$
and write $\chi_{\varepsilon,\delta}$ for the associated characteristic function. Note that the statement of the lemma can be reformulated as:

$$\limsup_{\delta \to 0} \limsup_{\varepsilon \downarrow 0} \left| \int_S \chi_{\varepsilon,\delta}(\Pi^\varepsilon(x) : E)\zeta(x) \exp \left( \frac{2\pi ik \cdot x}{\varepsilon} \right) \, dx \right| = 0, \quad (44)$$

$$\forall \delta > 0 : \limsup_{\varepsilon \downarrow 0} \left| \int_S (1 - \chi_{\varepsilon,\delta})(\Pi^\varepsilon(x) : E)\zeta(x) \exp \left( \frac{2\pi ik \cdot x}{\varepsilon} \right) \, dx \right| = 0. \quad (45)$$

(Note that we effectively integrate over the small ball $\frac{1}{2}B$, since supp($\zeta$) $\subset \frac{1}{2}B$ by assumption.) Before we come to the proof of these estimates, we establish the following two observations:

$$\lim_{\delta \to 0} \|\chi_{0,\delta}\|_{L^2(\frac{1}{4}B)} = \|\chi_{0,0}\|_{L^2(\frac{1}{4}B)} = 0, \quad (46)$$

$$\lim_{\varepsilon \to 0} \|\chi_{\varepsilon,\delta}\|_{L^2(\frac{1}{4}B)} \leq \|\chi_{0,\delta}\|_{L^2(\frac{1}{4}B)}. \quad (47)$$

Convergence (46) follows from condition (17) which can be rewritten as $\|\chi_{0,0}\|_{L^2(\frac{1}{4}B)} = 0$. Since $N^\varepsilon \otimes N^\varepsilon \to N \otimes N$ uniformly as $\varepsilon \downarrow 0$, we have

$$\chi_{\varepsilon,\delta} \leq \chi_{0,\delta} \quad \text{almost everywhere in } \frac{1}{4}B$$

for all $\delta > 0$ and $\varepsilon > 0$ sufficiently small. Combined with (17) this implies (47).

**Step 4.** Proof of (17) – the case $|k \cdot N^\varepsilon| \leq \delta$.

By Hölder’s inequality we have

$$\left| \int_S \chi_{\varepsilon,\delta}(\Pi^\varepsilon(x) : E)\zeta(x) \exp \left( \frac{2\pi ik \cdot x}{\varepsilon} \right) \, dx \right| \leq \|\zeta\|_{L^\infty(\frac{1}{4}B)} \|\Pi^\varepsilon\|_{L^2(\frac{1}{4}B)} \|\chi_{\varepsilon,\delta}\|_{L^2(\frac{1}{4}B)},$$

which, combined with (46) and (17), yields (17).

**Step 5.** Proof of (18) – the case $|k \cdot N^\varepsilon| > \delta$.

This step is the core of the argument. First, we rewrite the integral in (18) by appealing to the change of variables $x = \Phi^\varepsilon(t,s)$. Since $\Phi^\varepsilon$ is a one-to-one Lipschitz map, and $\zeta$ is compactly supported in $\frac{1}{4}B \subset \Phi^\varepsilon(Q)$ (cf. Step 2), we get by the change of variables formula

$$\int_S (1 - \chi_{\varepsilon,\delta})(\Pi^\varepsilon(x) : E)\zeta(x) \exp \left( \frac{2\pi ik \cdot x}{\varepsilon} \right) \, dx \quad (48)$$

$$= \int_Q \left( (1 - \chi_{\varepsilon,\delta})(\Pi^\varepsilon : E)\zeta \right) \circ \Phi^\varepsilon(t,s) \exp \left( \frac{2\pi ik \cdot \Phi^\varepsilon(t,s)}{\varepsilon} \right) \left| \det \nabla \Phi^\varepsilon(t,s) \right| \, dt \, ds$$

We rewrite the right-hand side using that

$$\exp \left( \frac{2\pi ik \cdot \Phi^\varepsilon(t,s)}{\varepsilon} \right) = \exp \left( \frac{2\pi ik \cdot \Gamma^\varepsilon(t)}{\varepsilon} \right) \exp \left( \frac{2\pi ik \cdot N^\varepsilon(t)}{\varepsilon} \right).$$

Define

$$f^\varepsilon(t,s) = (\Pi^\varepsilon(\Phi^\varepsilon(t,s) : E) \exp \left( \frac{2\pi ik \cdot \Gamma^\varepsilon(t)}{\varepsilon} \right) \left| \det \nabla \Phi^\varepsilon(t,s) \right| \zeta(\Phi^\varepsilon(t,s)),$$

and

$$g^\varepsilon(t,s) := \exp \left( \frac{2\pi ik \cdot N^\varepsilon(t)}{\varepsilon} \right) (1 - \chi_{\varepsilon,\delta}(\Phi^\varepsilon(t,s))).$$
Then (18) turns into \( \int_Q f^\varepsilon(t, s) g^\varepsilon(t, s) \, dtds \) and it suffices to prove that for all \( \delta > 0 \)

\[
\limsup_{\varepsilon \downarrow 0} \left| \int_Q f^\varepsilon(t, s) g^\varepsilon(t, s) \, dtds \right| = 0. \tag{49}
\]

We split the argument for (49) into four sub-steps.

**5a.** We claim that the maps \((t, s) \mapsto N^\varepsilon(\Phi^\varepsilon(t, s)) \otimes N^\varepsilon(\Phi^\varepsilon(t, s))\) and \((t, s) \mapsto (1 - \chi_{\varepsilon, \delta}(\Phi^\varepsilon(t, s)))\) are independent of \(s\). Note that the former statement implies the latter, as can be seen from the definition of the set \(A_{\varepsilon, \delta}\). The fact that \(N^\varepsilon \circ \Phi^\varepsilon\) does not depend on \(s\) can be seen as follows: By definition of \(\Phi^\varepsilon\), any two points with the same \(t\)-coordinate, say \(\Phi^\varepsilon(t, s_1)\) and \(\Phi^\varepsilon(t, s_2)\), lie on the same segment \([x, N^\varepsilon(x)]_{\frac{1}{2} B}\) with \(x := \Phi^\varepsilon(t, s_1)\).

Hence, Lemma 4 (31) implies that \(N^\varepsilon(\Phi^\varepsilon(t, s))\) and \(N^\varepsilon(\Phi^\varepsilon(t, s'))\) are parallel, and thus the statement follows.

**5b.** We claim that \(f^\varepsilon\) is compactly supported in \(Q\) and

\[
\limsup_{\varepsilon \downarrow 0} \int_Q |f^\varepsilon|^2 + |\partial_s f^\varepsilon|^2 \, dtds < \infty. \tag{50}
\]

For the argument first notice that by (12) we have

\[
f^\varepsilon(t, s) = \kappa^\varepsilon(t)(\Gamma^\varepsilon)'(t) \otimes (\Gamma^\varepsilon)'(t) : E \exp \left( \frac{2\pi ik \cdot \Gamma^\varepsilon(t)}{\varepsilon} \right) \zeta(\Phi^\varepsilon(t, s)).
\]

Hence, \(f^\varepsilon(t, s)\) is compactly supported in \(Q\), since \(\zeta\) is supported in \((\Phi^\varepsilon)^{-1}(\frac{1}{2} B) \subset Q\). As a consequence of \(\partial_s(N^\varepsilon \circ \Phi^\varepsilon) = 0\), we deduce that

\[
\partial_s f^\varepsilon = (\mathbf{II}^\varepsilon(\Phi^\varepsilon(t, s)) : E) \exp \left( \frac{2\pi ik \cdot \Gamma^\varepsilon(t)}{\varepsilon} \right) \nabla \zeta^\varepsilon(t, s) N^\varepsilon(t) |\det \nabla \Phi^\varepsilon|.
\]

Here and below we write \(N^\varepsilon(t)\) and \(\zeta^\varepsilon(t, s)\) instead of \(N^\varepsilon(\Phi^\varepsilon(t, s))\) and \(\zeta \circ \Phi^\varepsilon(t, s)\) for simplicity. We have (using the fact that \(|\det \nabla \Phi^\varepsilon|^2 \leq 2 |\det \nabla \Phi^\varepsilon|\), cf. (31))

\[
\int_Q |f^\varepsilon|^2 \, dt \, ds \leq 2 \int_Q \left( |(\mathbf{II}^\varepsilon : E)| \zeta^\varepsilon |\det \nabla \Phi^\varepsilon| \, dt \, ds \leq 2 \int_S |(\mathbf{II}^\varepsilon : E)| \zeta^\varepsilon |\det \nabla \Phi^\varepsilon| \, dx.
\]

By (34), the right hand side is bounded as \(\varepsilon \downarrow 0\). Similarly,

\[
\int_Q |\partial_s f^\varepsilon|^2 \, dtds \leq 2 \|\nabla \zeta\|_{L^\infty(\frac{1}{2} B)} \int_S |(\mathbf{II}^\varepsilon : E)|^2 \, dx,
\]

which proves the claim.

**5c.** Define

\[
G^\varepsilon(t, s) := \frac{\delta}{k \cdot N^\varepsilon(t)} \frac{1}{2\pi i} \exp \left( \frac{2\pi ik \cdot N^\varepsilon(t)}{\varepsilon} \right) (1 - \chi_{\varepsilon, \delta}(t)).
\]

Here and below we write \(\chi_{\varepsilon, \delta}(t)\) instead of \(\chi_{\varepsilon, \delta}(\Phi^\varepsilon(t, s))\) for simplicity. We claim that \(G^\varepsilon, \partial_s G^\varepsilon \in L^2(Q)\),

\[
\limsup_{\varepsilon \downarrow 0} \int_Q |G^\varepsilon|^2 \, dtds < \infty, \tag{51}
\]

and

\[
g^\varepsilon(t, s) = \frac{\varepsilon}{\delta} \partial_s G^\varepsilon(t, s). \tag{52}
\]

Indeed, eq. (31) follows from \(\frac{\delta}{k \cdot N^\varepsilon(t)} (1 - \chi_{\varepsilon, \delta}(t)) \leq 1\) on \(Q\), which is a direct consequence of the definition of \(A_{\varepsilon, \delta}\), and eq. (12) is a simple calculation.
Homogenization of the Kirchhoff bending theory for plates

5d. Now we are ready to prove (53). Identity (52), and an integration by parts yields

$$\int_Q f^\varepsilon(t,s)g^\varepsilon(t,s)\,dtds = \frac{\varepsilon}{\delta} \int_Q f^\varepsilon(t,s)\partial_s G^\varepsilon(t,s)\,dtds = -\frac{\varepsilon}{\delta} \int_Q \partial_s f^\varepsilon(t,s)G^\varepsilon(t,s)\,dtds.$$  

Note that the boundary terms vanish since $f^\varepsilon$ is compactly supported in $Q$. Hence, by Cauchy-Schwarz we get

$$\left|\int_Q f^\varepsilon(t,s)g^\varepsilon(t,s)\,dtds\right| \leq \frac{\varepsilon}{\delta} \left(\int_Q |\partial_s f^\varepsilon|^2\,dtds\right)^{\frac{1}{2}} \left(\int_Q |G^\varepsilon|^2\,dtds\right)^{\frac{1}{2}},$$

and (53) follows from (51) and (51).

Lemma 1 directly follows from Lemma 1.

Proof of Lemma 4. Let $N^\varepsilon : B \to S^1$ denote the Lipschitz function associated to $u^\varepsilon$ via Lemma 1, and $N : B \to S^1$ the one associated to $u$. By Corollary 1, $N^\varepsilon \to N$ uniformly in $B$. Note that eq. (57) is equivalent to eq. (22). Hence all conditions of Lemma 4 are fulfilled and Lemma 4 is proved.

Finally, we present the proofs of the auxiliary results.

Proof of Lemma 6. Step 1. We claim that it suffices to construct a vector field $\tilde{N} : B \to S^1$ that satisfies (23) and (24) (with $N$ replaced by $\tilde{N}$) such that $F : B \to \mathbb{R}^{2 \times 2}$, $F(x) := (\tilde{N}(x) \otimes \tilde{N}(x))$ is continuous. Here comes the argument: Since $B$ is simply connected, there exists a continuous vector field $N : B \to S^1$ with $F = N \otimes N$. Hence, it remains to check that $N$ satisfies (61). To that end let $x, y \in B$. We need to show that

$$|N(x) - N(y)| \leq \frac{1}{\text{radius}(B)}|x - y|.$$  

(53)

We distinguish the following cases:

- If either $\{x, N(x)\} = \{y, N(y)\}$ or $\{x, N(x)\} \cap \{y, N(y)\} = \emptyset$, then $N(x)$ and $N(y)$ must be parallel. We argue that $N(x) = N(y)$, which means that (53) is trivially fulfilled. Indeed, if this were not the case, then $N(x)$ and $N(y)$ would be in different connected components of $S^1 \setminus \{\pm(x - y)/|x - y|\}$. By the continuity of $N$ and the fact that $[x, y]$ – the line segment connecting $x$ and $y$ – is contained in $B$, there would have to exist $z \in [x, y] \setminus \{x, y\}$ such that $N(z) \in \{\pm(x - y)/|x - y|\}$, and thus $[z, N(z)]_B \cap [x, N(x)]_B = \{x\} \neq \emptyset$ in contradiction to eq. (61).

- If $\{x, N(x)\} \neq \{y, N(y)\}$ and $\{x, N(x)\} \cap \{y, N(y)\} \neq \emptyset$, then the lines intersect in some point $A \in \mathbb{R}^2$. By elementary geometry and by appealing to the continuity of $N$ as in the argument above, we deduce that

  \[ either \quad N(x) = \frac{x - A}{|x - A|}, \quad N(y) = \frac{y - A}{|y - A|}, \]

  \[ or \quad N(x) = -\frac{x - A}{|x - A|}, \quad N(y) = -\frac{y - A}{|y - A|}. \]
By (30) we necessarily have $A \not\in 2B$, so that (assuming without loss of generality that $|x-A| \leq |y-A|$)

$$|N(x) - N(y)| \leq \left| N(x) - \frac{|y-A|}{|x-A|}N(y) \right|$$

$$\leq \frac{1}{|x-A|}|x-A-y+A|$$

$$\leq \frac{1}{\text{radius}(B)}|x-y|.$$  

**Step 2.** Structure of the connected components of $C_{V_n} \cap B$.

Let $U$ be a connected component of $C_{V_n} \cap B$. We claim that the boundary of $U$ in $B$ can be written as the union of at most 2 disjoint line segments, and the corresponding lines do not intersect in $2B$, that is: there exists $k \in \{0,1,2\}$, and $x_i \in B$, $N_i \in S^1$ for $1 \leq i \leq k$, such that

$$\partial U \cap B = \bigcup_{i=1}^{k} [x_i; N_i]_B,$$

$$[x_i; N_i]_{2B} \cap [x_j; N_j]_{2B} = \emptyset \text{ for } i \neq j. \quad (54)$$

We first define some notation that we are going to use in the argument. For distinct points $A, C \in \mathbb{R}^2$, let $AC$ denote the line $\{A + t(C - A) : t \in \mathbb{R}\}$ and let $AC$ denote the half line $\{A + t(C - A) : t \in [0, \infty)\}$. For pairwise distinct points $A, C, D \in \mathbb{R}^2$, let $\angle ACD$ denote the smaller angle enclosed by the half lines $CA$ and $CD$. We adopt the convention that all such angles are positive. Let the center of $B$ be denoted by $O$.

Now, notice that the boundary of $U$ in $B$ has to be the union of open disjoint line segments since this is true for the boundary of $C_{V_n}$ in $B$ by Proposition 1. Furthermore, the corresponding lines do not intersect in $2B$. This proves eqs. (30) and (30) for some $k \in \mathbb{N}$, and it remains to show that $k \leq 2$.

Assume the contrary. Then there exist three lines $L_1, L_2, L_3$ such that (cf. Figure 1)

- $L_i \cap L_j \cap 2B = \emptyset$ for $i \neq j$
- $L_i \cap B \neq \emptyset$ for $i = 1, 2, 3$
- $\bigcup_{i=1}^{3} L_i \cap B \subset \partial U$

Let $m_i, i \in \{1,2,3\}$ be the midpoints of $L_i \cap B$. Since $U$ is connected, either the $L_i$, $i = 1, 2, 3$ enclose a triangle $\triangle \subset \mathbb{R}^2$ or two of the lines are parallel and the third is not. In the first case, let $A_i$ be the corner of the triangle that is opposite to the side containing $m_i$, see Figure 2. Let $i, j \in \{1,2,3\}, i \neq j$. Since $m_j$ is the midpoint of $L_j \cap B$, the line $\partial Om_j$ is orthogonal to $L_j$, and (see Figure 3)

$$\sin(\angle OA_i m_j) = |m_j - O|/|A_i - O| < 1/2.$$  

The latter estimate holds since $m_j \in B$ and $A_i \not\in 2B$ by assumption. Hence the enclosed angle is less than $\pi/6$. This is true for all pairs $i \neq j$. If $(i,j,k)$ is some permutation of $(1,2,3)$, then

$$\angle m_i A_j m_k \leq \angle O A_i m_j + \angle O A_i m_k.$$  

(Inequality occurs if $O$ is outside $\triangle$.) The contradiction is obtained by using the fact that the sum of the angles in the is equal to $\pi$,

$$\pi = \angle m_1 A_2 m_3 + \angle m_2 A_1 m_3 + \angle m_3 A_2 m_1 < \pi.$$
Figure 1. Three line segments contained in \(\partial U\).

Figure 2. The triangle \(\triangle\) containing the line segments, and the ball \(B\).

Figure 3. The Sine of the angle enclosed by \(\overrightarrow{A_1O}\) and \(\overrightarrow{A_1m_3}\) is given by \(|m_3 - O|/|A_1 - O|\). This ratio is smaller than 1/2 since \(m_3 \in B\) and \(A_1 \not\in 2B\). Thus the angle is smaller than \(\pi/6\).

In the case that two lines, say \(L_1\) and \(L_2\), are parallel, let \(m_i, i \in \{1, 2, 3\}\) be as before, \(A_1\) the point where \(L_2\) and \(L_3\) intersect, and \(A_2\) the point where \(L_1\) and \(L_3\) intersect, see Figure 4.

With the same reasoning as before, the angles \(\angle m_1A_2m_3, \angle m_3A_1m_2\) are both smaller than \(\pi/3\). Since \(L_1\) and \(L_2\) are parallel, the sum of these angles has to be \(\pi\), which produces
the contradiction, and finishes the proof of (24) and (25) with \( k \leq 2 \).

**Step 3.** Conclusion: Construction of \( \tilde{N} \).

By Step 1, to complete the proof we only need to construct a vector field \( \tilde{N} : B \to S^1 \) that satisfies (24) and (25) such that \( F = \tilde{N} \otimes \tilde{N} \) is continuous on \( B \). In the trivial case \( C_{\tilde{V}_u} = B \) we simply set \( \tilde{N} = e_1 \). Suppose now that \( C_{\tilde{V}_u} \neq B \). We define \( \tilde{N} \) on \( B \setminus C_{\tilde{V}_u} \) via Proposition II. The thus defined \( F = \tilde{N} \otimes \tilde{N} \) is continuous on \( B \setminus C_{\tilde{V}_u} \) and \( \tilde{N} \) satisfies (24) and (25) for \( x, y \in B \setminus C_{\tilde{V}_u} \). On the remainder \( B \cap C_{\tilde{V}_u} \) we define \( \tilde{N} \) on each connected component \( U \) separately as described next. Note that on \( U \) (24) is trivially fulfilled. Since \( U \neq B \), by Step 2 the boundary \( \partial U \cap B \) consists of one or two connected components. If \( \partial U \cap B = [x_1; N_1]_B \) for some \( x_1 \in B \) and \( N_1 \in S^1 \), we set \( \tilde{N} = N_1 \) on \( U \).

If \( \partial U \cap B = [x_1; N_1]_B \cup [x_2; N_2]_B \) for some \( x_1, x_2 \in B \) and \( N_1, N_2 \in S^1 \), we distinguish two cases:

- If \( N_1 \) and \( N_2 \) are not parallel, then there exists a unique \( A \in [x_1; N_1] \cap [x_2; N_2] \) and we set \( \tilde{N}(y) := (A - y)/|A - y| \) for \( y \in U \);
- If \( N_1 \) and \( N_2 \) are parallel, then we set \( \tilde{N} = N_1 \).

The thus defined vector field \( \tilde{N} : B \to S^1 \) satisfies (24) and (25) by construction. We argue that \( F = \tilde{N} \otimes \tilde{N} \) is continuous on \( B \). Since we already know that \( F \) is continuous on \( S \setminus C_{\tilde{V}_u} \), it suffices to argue that \( F \) is continuous on \( \partial U \cap B \) for every connected component \( U \) of \( C_{\tilde{V}_u} \cap B \). But this is clear by construction. \( \square \)

**Proof of Corollary III.**

**Step 1.** Argument for (33).

Since \( N^\varepsilon \) is a vector field of unit vectors, and since \( \text{Lip}(N^\varepsilon) \) is bounded uniformly in \( \varepsilon > 0 \), the sequence \( N^\varepsilon \) is bounded in \( W^{1,\infty}(B, \mathbb{R}^2) \). Hence, \( N^\varepsilon \rightharpoonup \tilde{N} \) weakly-star in \( W^{1,\infty} \), up to a subsequence (that we do not relabel), and \( \tilde{N} \in W^{1,\infty}(B, \mathbb{R}^2) \). Since \( W^{1,\infty}(B, \mathbb{R}^2) \) is compactly embedded into the Hölder spaces \( C^{0,\alpha}(B, \mathbb{R}^2), 0 \leq \alpha < 1 \), the convergence holds uniformly and we deduce that \( \tilde{N}(x) \in S^1 \) almost everywhere.

**Step 2.** Argument for (34).

Set \( T^\varepsilon(x) := -(N^\varepsilon(x))^\perp \). By (32) we have

\[ \Pi^\varepsilon(x) = \mu^\varepsilon(x) T^\varepsilon(x) \otimes T^\varepsilon(x) \quad \text{for some } \mu^\varepsilon \in L^2(B). \] (56)

![Figure 4. The case of parallel line segments contained in \( \partial U \).](image-url)
The sequence $\mu^\varepsilon$ is bounded in $L^2(B)$. Hence, we can pass (to a further) subsequence with $\mu^\varepsilon \rightharpoonup \mu(x, y)$ two-scale in $L^2(B \times (0, 1)^2)$. Combined with the uniform convergence $N^\varepsilon \to \bar{N}$, (56) follows.

**Step 3.** Argument for (57).

For convenience set $T := -N^\perp$. Note that (56) remains valid when the superscript $\varepsilon$ is dropped. By assumption we have $u^\varepsilon \rightharpoonup u$ in $W^{2,2}$, and thus $II^\varepsilon \rightharpoonup II$ weakly in $L^2(S, \mathbb{R}^{2 \times 2})$. Since $N^\varepsilon \otimes N^\varepsilon \to \bar{N} \otimes \bar{N}$ strongly in $L^2(S, \mathbb{R}^2)$ we obtain

$$\int_B \langle II : (\bar{N} \otimes \bar{N}) \rangle \varphi \, dx = \lim_{\varepsilon \downarrow 0} \int_B \langle II^\varepsilon : (N^\varepsilon \otimes N^\varepsilon) \rangle \varphi \, dx,$$

for all $\varphi \in L^2(B)$. By orthogonality, the right-hand side vanishes, and thus

$$0 = \int_B \langle II : (\bar{N} \otimes \bar{N}) \rangle \varphi \, dx. \tag{57}$$

The combination of identity (56) (with the superscript $\varepsilon$ dropped) and (57) (with $\varphi = \mu$) yields

$$0 = \int_B |\mu|^2(T \otimes T) : (\bar{N} \otimes \bar{N}) \, dx = \int_B |\mu|^2[T \cdot \bar{N}]^2 \, dx.$$

Since $|\mu|^2 > 0$ almost everywhere in $B \setminus C_{V_u}$, the previous identity implies that $\bar{N}$ and $T$ are orthogonal in that region, and thus, by the continuity of $\bar{N}$ and $N = T^\perp$, we obtain (55).

□

4. **Proof of Theorem II**

4.1. **Proof of Theorem II (a) & (b) – compactness and lower bound.**

**Proof of statement (a) – compactness.** In view of the coercivity assumption (Q3) and Poincaré’s inequality, any sequence $u^\varepsilon$ with finite energy and mean zero is bounded in $W^{2,2}(S, \mathbb{R}^3)$. Hence, the statement follows from the observation that $W^{2,2}_{iso}(S)$ is closed under weak convergence in $W^{2,2}(S, \mathbb{R}^3)$.

**Proof of statement (b) – lower bound.** By the compactness statement (a), we may assume without loss of generality that $u^\varepsilon, u \in W^{2,2}_{iso}(S)$ and

$$u^\varepsilon \rightharpoonup u \quad \text{weakly in } W^{2,2}(S, \mathbb{R}^3), \tag{58}$$

$$II^\varepsilon \rightharpoonup II \quad \text{weakly in } L^2(S, \mathbb{R}^{2 \times 2}), \tag{59}$$

$$II^\varepsilon \rightharpoonup II + G \quad \text{weakly two-scale in } L^2(S \times \mathcal{Y}, \mathbb{R}^{2 \times 2}), \tag{60}$$

where $II^\varepsilon$ and $II$ denote the second fundamental forms associated with $u^\varepsilon$ and $u$, and $G(x, y)$ is a function in $L^2(S \times \mathcal{Y}, \mathbb{R}^{2 \times 2})$. By Lemma II (a) we have

$$\liminf_{\varepsilon \downarrow 0} \int_S Q(\frac{x}{\varepsilon}, II^\varepsilon(x)) \, dx \geq \int_{S \times \mathcal{Y}} Q(y, II(y) + G(x, y)) \, dy \, dx.$$

Since $S = C_{V_u} \cup K_{V_u} \cup Z_{V_u}$, the right-hand side can be written as

$$\int_{S \times \mathcal{Y}} Q(y, II(y) + G(x, y)) \, dy \, dx \geq \int_{K_{V_u} \times \mathcal{Y}} Q(y, II(x) + G(x, y)) \, dy \, dx + \int_{Z_{V_u} \times \mathcal{Y}} Q(y, II(x) + G(x, y)) \, dy \, dx.$$
Hence, it remains to show
\[
\int_{K_{\nabla_u} \times Y} Q(y, \mathbf{II}(x) + G(x, y)) \, dy \, dx = \int_{K_{\nabla_u}} Q_{av}(\mathbf{II}(x)) \, dx, \tag{61}
\]
\[
\int_{Z_{\nabla_u} \times Y} Q(y, \mathbf{II}(x) + G(x, y)) \, dy \, dx \geq \int_{Z_{\nabla_u}} Q_{hom}(\mathbf{II}(x)) \, dx. \tag{62}
\]
An application of Proposition 2 shows that $G = 0$ on $K_{\nabla_u} \times Y$ and thus (61). For (62) it suffices to argue that
\[
\int_{Z \times Y} Q(y, \mathbf{II}(x) + G(x, y)) \, dy \, dx \geq \int_{Z} Q_{hom}(\mathbf{II}(x)) \, dx,
\]
for every open connected component of $Z \subset Z_{\nabla_u}$. Fix such a component $Z$. By Proposition 2 (b) there exists $T \in \mathcal{S}^1$, $\mu \in L^2(Z)$ and $\alpha \in L^2(Z, W_{T,\text{per}}^{1,2}(\mathbb{R}))$ such that
\[
\mathbf{II}(x) = \mu T \otimes T, \quad \mathbf{II}(x) + G(x, y) = (\mu + \alpha(x, T \cdot y)) T \otimes T.
\]
Hence,
\[
\int_{Z_{\nabla_u} \times Y} Q(y, \mathbf{II}(x) + G(x, y)) \, dy \, dx
\]
\[
= \int_{Z_{\nabla_u} \times Y} Q(y, (\mu(x) + \alpha'(x, T \cdot y)) T \otimes T) \, dy \, dx
\]
\[
\geq \int_{Z_{\nabla_u}} \mu^2 Q_{hom}(T \otimes T) \, dx = \int_{Z_{\nabla_u}} Q_{hom}(\mathbf{II}) \, dx.
\]

4.2. Proof of Theorem 1 (c) - construction of recovery sequences. Before we discuss the construction of recovery sequences, we first recall some basic observations and state some auxiliary lemmas. We call a function $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ a rigid motion if it can be written in the form of
\[
\varphi(x) = Rx + c \quad \text{for some } R \in SO(3) \text{ and } c \in \mathbb{R}^3.
\]
We say two $\mathbb{R}^3$-valued functions $u, v$ are equal up to a rigid motion defined on some set $Z \subset \mathbb{R}^2$, and write $u \simeq v$ on $Z$, if there exists a rigid motion $\varphi$ such that $u(x) = \varphi(v(x))$ for all $x \in Z$. As it is easy to check, if $u \simeq v$ on $Z$ then their second fundamental forms are equal on $Z$. Because of the developability of $W^{2,2}$-isometries and the convexity of $S$, the connected components of the cylindrical set $Z_{\nabla_u}$ have a simple geometry, and the deformation on such components is characterized by planar curves:

**Lemma 8.** Let $u \in W_{iso}^{2,2}(S)$ and $Z \subset Z_{\nabla_u}$ a connected component with asymptotic direction $N \in \mathcal{S}^1$. Set $T := -N^{-1}$. Then:

(a) $\partial Z \cap S$ consists of at most two connected components. Each connected component is of the form $[x_0; N]_S$ for some $x_0 \in \partial Z \cap S$.

(b) There exists a unit vector $\nu \in \mathbb{R}^3$, some interval $I \subset \mathbb{R}$ and an arc-length parametrized curve $\gamma \in W^{2,2}(I, \mathbb{R}^3)$ that lies in a plane orthogonal to $\nu$ such that for all $x \in Z$,
\[
u(x) = \gamma(x \cdot T) + (x \cdot N) \nu
\]
\[
\mathbf{II}(x) = \mu(x) T \otimes T,
\]
where $\mathbf{II}$ is the second fundamental form of $u$, and $\mu(x) = (\gamma'(x \cdot T) \times \nu) \cdot \gamma''(x \cdot T)$.
Proof. Step 1. Argument for (a).
For every $x \in S$, $[x; N]_S$ is connected by the convexity of $S$. Thus, $x \in Z$ if and only if $[x; N]_S \subset Z$. This implies $x \in \partial Z \cap S$ if and only if $[x; N]_S \subset \partial Z \cap S$. Hence

$$\partial Z \cap S = \bigcup_{x \in X} [x; N]_S$$

for some set $X \subset S$. It remains to show that $X$ can be chosen such that it only consists of at most two elements. Otherwise, there exist $x_1, i = 1, 2, 3$, such that $[x_i; N]_S \subset \partial Z \cap S$, $x_1 \cdot T < x_2 \cdot T < x_3 \cdot T$. Note that

$$[x; N]_S = \{ y \in S : y \cdot T = x \cdot T \}.$$

Since $Z$ is connected and open, there exists a continuous path $\xi : [0, 1] \to Z$ such that $\xi(0) = x_1$, $\xi(1) = x_3$, and $\xi((0, 1)) \subset Z$. By the mean value theorem, there exists some $s \in (0, 1)$ such that $T \cdot \xi(s) = T \cdot x_2$. Hence $\xi(s) \in Z \cap [x_2; N]_S \subset Z \cap \partial Z$ which is impossible since $Z$ is open.

Step 2. Argument for (b).

By eq. (13), we have

$$\nabla^2 u N = 0$$

on $Z$ and hence $u|_Z$ is an affine function of $x \cdot N$:

$$u(x) = \gamma(x \cdot T) + (x \cdot N)\nu(x \cdot T)$$

for all $x \in Z$, for some $\gamma \in W^{2,2}(I, \mathbb{R}^3)$, $\nu \in W^{1,2}(I, \mathbb{R}^3)$, where $I$ is some interval. By $\nabla u^2 \nabla u = \text{Id}$, we get $\nu' = 0$, $|\gamma'| = |\nu| = 1$, $\gamma' \cdot \nu = 0$. Up to a sign, $\gamma'(x \cdot T) \times \nu$ is the normal of $u$ at $x \in Z$. The formula for $II$ follows by computation after possibly reversing the parametrization of $\gamma$.

The elementary building block in the construction of the recovery sequence is the following:

**Lemma 9.** In the situation of Lemma 8 assume that $\partial Z \cap S \neq \emptyset$ and let $[x_0; N]_S$, $x_0 \in \partial Z$, denote one of the connected components of $\partial Z \cap S$. We claim that there exists a sequence $u^\varepsilon \in W^{2,2}_{iso}(Z)$ such that

$$\lim_{\varepsilon \to 0} \int_Z Q^\varepsilon(x, II(x)) \, dx = \int_Z Q_{\text{hom}}(II(x)) \, dx$$

$$u^\varepsilon \rightharpoonup u \text{ weakly in } W^{2,2}(Z),$$

$$u^\varepsilon = u \text{ and } \nabla u^\varepsilon = \nabla u \text{ on } [x_0; N]_S.$$  

Moreover, if $\partial Z \cap S$ contains a second connected component, say $[x_1; N]_S$ for some $x_1 \in \partial Z \setminus [x_0; N]_S$, then there exist $(R^\varepsilon, c^\varepsilon) \in SO(3) \times \mathbb{R}^3$ converging to $(\text{Id}, 0)$, such that

$$u = R^\varepsilon u^\varepsilon + c^\varepsilon \text{ and } \nabla u = R^\varepsilon \nabla u^\varepsilon \text{ on } [x_1; N]_S.$$  

Proof. Recall that $N$ denotes an asymptotic direction of $u$ on $Z$, and that $T := -N^\perp$. We only need to consider the case when the ratio between the components $T \cdot e_1$ and $T \cdot e_2$ is rational, since otherwise $W^{1,2}_{\text{per}}(\mathbb{R})$ only consists of constant functions and $Q_{\text{hom}}(T \otimes T)$ simplifies to $Q_{\text{av}}(T \otimes T)$, which means that the trivial sequence $u^\varepsilon := u$ already yields an optimal approximation. Hence, for the rest of the argument we assume that $T \cdot e_1$ and
$T \cdot e_2$ have a rational ratio. As shown in Section 11.3 this means that $T \in r(T)\mathbb{Z}^2$ for $r(T)$ defined in (3), and

$$Q_{\text{hom}}(T \otimes T) = \int_0^{r(T)} q_{\text{av}, T}(t)(1 + \alpha'_s(t))^2 dt$$

where $\alpha_s$ is defined in (11).

The construction of $u^\varepsilon$ is based on the representation of $u$ via Lemma 8 (b): there exist an interval $I = (a, b)$, a unit vector $\nu \in \mathbb{R}^2$ and an arc-length parametrized curve $\gamma \in W^{2,2}(I, \mathbb{R}^3)$ that lies in a plane orthogonal to $\nu$ such that

$$u(x) = \gamma(x \cdot T) + (x \cdot N)\nu, \quad II(x) = \mu(x)(T \otimes T), \quad \mu(x) = (\gamma'(x \cdot T) \cdot \nu) \cdot \gamma''(x \cdot T),$$

for all $x \in Z$. W. l. o. g. we may assume that the line segment (on which we fix the boundary condition) is contained in the line $\{x \in \mathbb{R}^2 : T \cdot x = a\}$, i. e. $T \cdot x_0 = a$. Set $\nu_1 := \gamma'(a)$ and chose $\nu_2 \in \mathbb{R}^3$ as the unique unit vector that turns the matrix $(\nu_1, \nu_2, \nu)$ into a rotation. Since $\gamma$ lies in the plane spanned by $\nu_1$ and $\nu_2$, we may identify it with the plane curve $\lambda : I \rightarrow \mathbb{R}^2$ given by

$$\lambda(s) := (\gamma(s) \cdot \nu_1, \gamma(s) \cdot \nu_2).$$

Then it is elementary to check that $\lambda$ is characterized as the unique arc-length parametrized plane curve that satisfies the ODE

$$\lambda(a) = (\gamma(0) \cdot \nu_1, \gamma(0) \cdot \nu_2), \quad \lambda'(a) = (1, 0), \quad (\lambda'(s))' \cdot \lambda''(s) = \kappa(s),$$

where we set $\kappa(s) := -(\gamma'(s) \cdot \nu) \cdot \gamma''(s)$ for brevity. Now let $\lambda^\varepsilon \in W^{2,2}(I, \mathbb{R}^2)$ denote the unique arc-length parametrized curve associated with the ODE

$$\lambda^\varepsilon(a) = (\gamma(0) \cdot \nu_1, \gamma(0) \cdot \nu_2), \quad \lambda'(a) = (1, 0), \quad ((\lambda^\varepsilon)'(s))^\perp \cdot (\lambda^\varepsilon)(s) = \kappa^\varepsilon(s) := \kappa(s)(1 + \alpha'_s(s^\varepsilon)).$$

For simplicity we assume that $\alpha_s$ (which is defined in (11)) is smooth, so that $\kappa^\varepsilon \in L^2(I)$. (Otherwise, we may replace $\alpha_s$ by a sequence of smooth, $r(T)$-periodic functions on $\mathbb{R}$ that converge to $\alpha_s$ sufficiently fast w. r. t. the $L^2_{T, \text{per}}(\mathbb{R})$-norm.) Note that we have $\kappa^\varepsilon \rightharpoonup \kappa$ weakly in $L^2(I)$, and thus $\lambda^\varepsilon \rightharpoonup \lambda$ weakly in $W^{2,2}(I, \mathbb{R}^2)$. Consequently, the arc-length parametrized space curve $\gamma^\varepsilon$ associated with $\lambda^\varepsilon$ via $\gamma^\varepsilon(t) := (\lambda^\varepsilon)(t) \cdot e_1)\nu_1 + (\lambda^\varepsilon(t) \cdot e_2)\nu_2$ converges to $\gamma$ weakly in $W^{2,2}(I, \mathbb{R}^3)$. Now define $u^\varepsilon(x) := \gamma^\varepsilon(x \cdot T) + (x \cdot N)\nu$. From the convergence of $\gamma^\varepsilon$ we deduce that $u^\varepsilon \rightharpoonup u$ weakly in $W^{2,2}(S, \mathbb{R}^3)$. Moreover, because of $(\gamma^\varepsilon(a), (\gamma^\varepsilon)'(a)) = (\gamma(a), (\gamma)'(a))$, we have that $u^\varepsilon = u$ and $\nabla u^\varepsilon = \nabla u$ on $[x_0; N]$. A direct computation shows that $u^\varepsilon$ satisfies the isometry constraint, and that the associated second fundamental form reads

$$II^\varepsilon(x) = \mu(x)(1 + \alpha'_s(T \cdot y)) T \otimes T.$$

Since $\alpha_s$ is $r(T)$-periodic (where $r(T)$ is such that $y \mapsto \alpha'_s(T \cdot y)$ is $Y$-periodic), it follows from the definition of strong two-scale convergence (Definition 11) that

$$II^\varepsilon(x) \rightharpoonup \mu(x)(1 + \alpha'_s(T \cdot y)) T \otimes T \quad \text{strongly two-scale in } L^2(Z \times Y),$$
so that by Lemma 3

$$
\lim_{\varepsilon \to 0} \int_Z Q(\frac{X}{\varepsilon}, X^\varepsilon, (X^\varepsilon)_{X^\varepsilon}) \, dx = \int_{Z \times Y} Q(y, \mu(x) (1 + \alpha_T(T \cdot y)) T \otimes T) \, dy \, dx \\
= \left( \int_Y Q(y, (1 + \alpha_T(T \cdot y)) T \otimes T) \, dy \right) \int_Z \mu^2(x) \, dx \\
= \int_Z Q_{\text{hom}}(\mu(x) T \otimes T) \, dx.
$$

It remains to prove (63). By construction the two unit vectors \((\gamma^\varepsilon)'(b)\) and \(\gamma'(b)\) lie in a plane orthogonal to \(\nu\). Hence, there exists a rotation \(R^\varepsilon \in SO(3)\) about the axis \(\nu\) and with angle less or equal to \(\pi\) such that \(R^\varepsilon(\gamma^\varepsilon)'(b) = \gamma'(b)\). Now define \(c^\varepsilon := \gamma(b) - R^\varepsilon \gamma^\varepsilon(b)\). Then it is easy to check that \(\gamma(b) = R^\varepsilon \gamma^\varepsilon(b) + c^\varepsilon\) and \(\gamma'(b) = R^\varepsilon(\gamma^\varepsilon)'(b)\). In view of (65), the definition of \(u^\varepsilon\), and the identities \(\nabla u = \nu \otimes N + \gamma' \otimes T\) and \(\nabla u^\varepsilon = \nu \otimes N + (\gamma^\varepsilon)' \otimes T\), (63) follows. Finally, we deduce from \(\gamma^\varepsilon \rightarrow \gamma\) in \(W^{2,2}\) that \((\gamma^\varepsilon(b), (\gamma^\varepsilon)'(b)) \rightarrow (\gamma(b), \gamma'(b))\) and thus \((R^\varepsilon, c^\varepsilon) \rightarrow (\text{Id}, 0)\).

**Proof of statement (c) – recovery sequence.** We only consider the case of prescribed boundary conditions. The idea of the construction is the following: We enumerate the connected components \(Z_1, Z_2, \ldots\) of \(Zu\) and iteratively construct a sequence of sequences \(u^\varepsilon_{\text{rec}}\) that approximate \(u\) in such a way that the \(i\)th sequence converges to \(u\) as \(\varepsilon \downarrow 0\) and recovers the limiting energy on the first \(i\) components \(Z_1, \ldots, Z_i\). The recovery sequence is then obtained by selection of a suitable diagonal sequence. The construction of the sequence \(u^\varepsilon_{\text{rec}}\) is done by appealing to Lemma 3, which we use to “update” a given sequence \(u^\varepsilon_{i-1}\) in such a way that the resulting sequence recovers the limiting energy on \(Z_i\), while the behavior on the other components of \(Zu\) and \(S \setminus Zu\) is only changed by superposition with a rigid motion. In Step 1 we explain this “update” mechanism. In Step 2 we present the iterative construction and in Step 3 we draw the conclusion by selecting a diagonal sequence.

**Step 1.** Construction on a single connected component of \(Zu\).

Let \(Z\) be a connected component of \(Zu\). In this step we will modify a given sequence \((\approx u)\) in such a way that the energy on \(Z\) of the resulting sequence converges to the effective energy of \(u\) on \(Z\).

For treating the boundary condition (65) recall that the line segment \(L\) is given as the intersection of a line with \(S\), and that we have \(\mathcal{H}^1(L) > 0\) by assumption. Since \(u\) is affine on \(L\), we deduce that

$$
L \cap Z \neq \emptyset \quad \Rightarrow \quad L \parallel N,
$$

where \(N \in S^1\) is an asymptotic direction of \(u\) on \(Z\).

Now consider a sequence \(u^\varepsilon \in W^{2,2}_{\text{iso}}(S)\) and suppose that

$$
u^\varepsilon \rightarrow u \text{ weakly in } W^{2,2},
$$

$$u^\varepsilon \simeq u \text{ on } Z.
$$
We claim that there exists a sequence \( \tilde{u}^\varepsilon \in W^{2,2}_{\text{iso}}(S) \) that weakly converges to \( u \) on \( W^{2,2} \) such that
\[
\lim_{\varepsilon \to 0} \int_Z Q\left( \frac{X}{\varepsilon}, \tilde{H}^\varepsilon(x) \right) \, dx = \int_Z Q_{\text{hom}}(H(x)) \, dx \tag{69a}
\]
\( \tilde{u}^\varepsilon \simeq u^\varepsilon \) on all connected components \( A \subset S \setminus Z \), \( \tilde{u}^\varepsilon = u^\varepsilon \) and \( \nabla \tilde{u}^\varepsilon = \nabla u^\varepsilon \) on \( L \), \( \tilde{u}^\varepsilon \) on all connected components \( \tilde{u}^\varepsilon \) on all connected components
\[
\tilde{u}^\varepsilon = u^\varepsilon \quad \text{on } L, \quad \tilde{u}^\varepsilon = u^\varepsilon \quad \text{on } L, \tag{69c}
\]where \( \tilde{H}^\varepsilon \) denotes the second fundamental form associated with \( \tilde{u}^\varepsilon \).

Here comes the argument: We first prove the statement of Step 1 in the case when \( L \cap Z = \emptyset \). By Lemma S and the convexity of \( S \), the set \( S \setminus Z \) consists of at most two connected components. We only give the argument in the case that \( S \setminus Z \) consists of exactly two connected components, say \( S_1 \) and \( S_2 \), since the other two cases can be treated by slight variations of the argument below.

W. l. o. g. let \( S_1 \) be the set whose closure intersects \( L \). Due to Lemma S (a) \( \partial Z \cap S \) consists of two disjoint line segments. Hence, \( L_i := (\partial Z \cap S) \cap S_i = [x_i; N]_S \) for some \( x_i \in \partial Z \), \( i = 1, 2 \). Since \( u \) is cylindrical on \( Z \), we may apply Lemma \( \Theta \) to \( u \), \( Z \) and the line segment \( L_1 \). We denote the resulting sequence by \( v^\varepsilon \in W^{2,2}_{\text{iso}}(Z) \). By (II) there exist \( (R_2^\varepsilon, c_2^\varepsilon) \in SO(3) \times \mathbb{R}^3 \) converging to \( (Id, 0) \), such that
\[
v^\varepsilon = R_2^\varepsilon u + c_2^\varepsilon \quad \text{and} \quad \nabla v^\varepsilon = R_2^\varepsilon \nabla u \quad \text{on } L_2. \tag{70}
\]
On the other hand, because of (I) there exist \( (R_1^\varepsilon, c_1^\varepsilon) \in SO(3) \times \mathbb{R}^3 \) such that that \( u^\varepsilon = R_1^\varepsilon u + c_1^\varepsilon \) and \( \nabla u^\varepsilon = R_1^\varepsilon \nabla u \) on \( L_2. \) \( \tag{71} \)

Because of (II) we have \( (R_1^\varepsilon, c_1^\varepsilon) \to (Id, 0) \). Now define the function \( \tilde{u}^\varepsilon \) as follows:
\[
\tilde{u}^\varepsilon(x) := \begin{cases} 
 u^\varepsilon(x) & \text{for } x \in S_1, \\
 R_1^\varepsilon u^\varepsilon(x) + c_1^\varepsilon & \text{for } x \in Z, \\
 R_2^\varepsilon u^\varepsilon(x) + c_2^\varepsilon & \text{for } x \in S_2,
\end{cases}
\]
where \( R_2^\varepsilon := R_2^\varepsilon R_3^\varepsilon (R_2^\varepsilon)^{-1} \) and \( c_2^\varepsilon := -R_2^\varepsilon R_3^\varepsilon c_1^\varepsilon + R_3^\varepsilon c_2^\varepsilon + c_1^\varepsilon \). By construction \( \tilde{u}^\varepsilon \) is an \( W^{2,2} \)-isometric immersion for each of the connected components \( S_1, Z, S_2 \). In order to conclude that \( \tilde{u}^\varepsilon \in W^{2,2}_{\text{iso}}(S) \), we only need to check that \( \tilde{u}^\varepsilon \) and \( \nabla \tilde{u}^\varepsilon \) do not jump across the line segments \( L_1 \) and \( L_2 \). Indeed, this follows a straightforward calculation from (I), (II) and (III).

Since both sequences \( u^\varepsilon \) and \( v^\varepsilon \) converge weakly in \( W^{2,2}(S, \mathbb{R}^3) \) to \( u \), and because \( (R_1^\varepsilon, c_1^\varepsilon) \) and \( (R_2^\varepsilon, c_2^\varepsilon) \) converge to \( (Id, 0) \), we deduce that \( \tilde{u}^\varepsilon \to u \) weakly in \( W^{2,2}(S, \mathbb{R}^3) \). Properties (III) and (IV) are immediate from the definition of \( \tilde{u}^\varepsilon \), (III) and the property that the second fundamental form is invariant under superposition with rigid motions. The boundary condition (I) is satisfied since \( \tilde{u}^\varepsilon = u^\varepsilon \) on \( S_1 \), whose closure contains \( L \). This completes the argument in the case \( L \cap Z = \emptyset \).

Finally we argue that the case \( L \cap Z \neq \emptyset \) can be reduced to the case \( L \cap Z = \emptyset \) discussed above. Indeed, if \( L \cap Z \neq \emptyset \), then \( L \) is parallel to \( N \) by (III), and thus Lemma S (a) yields \( L \subset Z \). In view of the convexity of \( S \), we deduce that each of the sets \( S \setminus L \) and \( Z \setminus L \) consist of two connected components, say \( S_1, S_2 \) and \( Z_1 \subset S_1, Z_2 \subset S_2 \), respectively. Now, we may consider the approximation problem of Step 1 with \( S \) and \( Z \) replaced by \( S_1 \) and \( Z_i \) for \( i = 1, 2 \), separately. Since \( L \cap Z = \emptyset \), the construction described above applies. By
gluing together, which is possible by property (D9c), we obtain the desired sequence for the original problem.

**Step 2.** Iterative construction.

Let \( Z_1, Z_2, \ldots \) be an enumeration of the at most countable set of connected components of \( \partial u \). In the following we iteratively define sequences \( u_i^\varepsilon, i = 1, 2, \ldots \) that approximate \( u \) such that the \( n \)th sequence recovers the effective energy on \( Z_1, \ldots, Z_n \). To that end set \( u_0^\varepsilon := u \) as a starting point, and define iteratively \( u_i^\varepsilon \) based on \( u_{i-1}^\varepsilon \) by appealing to Step 1: We apply Step 1 with \( u^\varepsilon = u_{i-1}^\varepsilon \), \( Z = Z_i \) and obtain \( u_i^\varepsilon := u_{i-1}^\varepsilon + \varepsilon \) – provided assumptions (M) and (DS) are satisfied. Both can be easily checked by induction. We claim that for all \( i \in \mathbb{N} \) we have

\[
\lim_{\varepsilon \downarrow 0} \|u_i^\varepsilon - u\|_{L^2(S)} = 0, \tag{72a}
\]

\[
\lim_{\varepsilon \downarrow 0} E^\varepsilon(u_i^\varepsilon) = \int_{S \setminus (Z_1 \cup \cdots \cup Z_i)} Q_{av}(II(x)) \, dx + \int_{(Z_1 \cup \cdots \cup Z_i)} Q_{hom}(II(x)) \, dx, \tag{72b}
\]

\[
u_i^\varepsilon = u \text{ and } \nabla u_i^\varepsilon = \nabla u \text{ on } L. \tag{72c}
\]

Indeed, (72a) follows from \( u_i^\varepsilon \rightharpoonup u \) weakly in \( W^{2,2} \). Statement (72b) is a direct consequence of property (D9c), the recursive definition of \( u_i^\varepsilon \) and the fact \( u_0^\varepsilon \) satisfies (II) trivially. For the proof of (72c) let \( II_i^\varepsilon \) denote the second fundamental form associated with \( u_i^\varepsilon \). Evidently, (D9d) implies that \( II_i^\varepsilon = II_{i-1}^\varepsilon \) on \( S \setminus Z_i \). Hence,

\[
\int_S Q\left(\frac{\varepsilon}{\varepsilon}, II_i^\varepsilon(x)\right) \, dx = \int_{S \setminus Z_i} Q\left(\frac{\varepsilon}{\varepsilon}, II_{i-1}^\varepsilon(x)\right) \, dx + \int_{Z_i} Q\left(\frac{\varepsilon}{\varepsilon}, II_i^\varepsilon(x)\right) \, dx
\]

\[
= \int_{S \setminus (Z_1 \cup \cdots \cup Z_i)} Q\left(\frac{\varepsilon}{\varepsilon}, II_0^\varepsilon(x)\right) \, dx + \sum_{n=1}^{i} \int_{Z_n} Q\left(\frac{\varepsilon}{\varepsilon}, II_n^\varepsilon(x)\right) \, dx,
\]

by induction. We can pass to the limit on the r. h. s. – indeed, since \( II_0^\varepsilon \) is by construction just the second fundamental form of \( u \) (and thus independent of \( \varepsilon \)), we have

\[
\int_{S \setminus (Z_1 \cup \cdots \cup Z_i)} Q\left(\frac{\varepsilon}{\varepsilon}, II_0^\varepsilon(x)\right) \, dx \to \int_{S \setminus (Z_1 \cup \cdots \cup Z_i)} Q_{av}(II(x)) \, dx.
\]

On the other hand, we can pass to the limit \( \varepsilon \downarrow 0 \) in each of the remaining terms by appealing to (D9a). In conclusion we get (72c).

**Step 3.** Conclusion by selection of a diagonal sequence.

Let \( u_i^\varepsilon \) be defined as in Step 2. Set

\[
c(\varepsilon, i) := \|u_i^\varepsilon - u\|_{L^2(S)} + |E^\varepsilon(u_i^\varepsilon) - E_{\text{hom}}(u)|.
\]

We claim that

\[
\lim_{i \uparrow \infty} \limsup_{\varepsilon \downarrow 0} c(\varepsilon, i) = 0. \tag{73}
\]

Before we give the argument for (73), let us draw the conclusion: By appealing to the diagonalization lemma by Attouch, see [Attouch], there exists a map \( \varepsilon \mapsto i(\varepsilon) \in \mathbb{N} \) such that \( c(\varepsilon, i(\varepsilon)) \to 0 \) as \( \varepsilon \downarrow 0 \). Hence, the diagonal sequence \( u^\varepsilon := u^\varepsilon_{i(\varepsilon)} \) strongly converges in \( L^2(S) \) to \( u \), and its energy satisfies

\[
\lim_{\varepsilon \downarrow 0} E^\varepsilon(u^\varepsilon) = E_{\text{hom}}(u).
\]
Since this especially implies that the associated sequence of fundamental forms \( II^\varepsilon \) is bounded in \( L^2(S) \), we can upgrade the convergence of \( u^\varepsilon \) and deduce that \( u^\varepsilon \rightharpoonup u \) weakly in \( W^{2,2}(S,\mathbb{R}^3) \) as claimed. Moreover, since each \( u_i^\varepsilon \) satisfies the required boundary conditions, also \( u^\varepsilon \) satisfies them.

It remains to prove (\( \mathcal{K} \)). Due to (\( \mathcal{K}_a \)), (\( \mathcal{K}_b \)) and the definition of \( E^{\text{hom}} \) we have

\[
\limsup_{\varepsilon \downarrow 0} c(\varepsilon, i) \leq \limsup_{\varepsilon \downarrow 0} |E^\varepsilon(u_i^\varepsilon) - E^{\text{hom}}(u)| \leq \int_{Z \setminus (Z_1 \cup \ldots \cup Z_i)} |Q_{\text{av}}(II(x)) - Q_{\text{hom}}(II(x))| \, dx.
\]

Hence, in view of the growth assumption (\( \mathcal{G} \)) we get

\[
\limsup_{\varepsilon \downarrow 0} |E^\varepsilon(u) - E^{\text{hom}}(u)| \leq \frac{1}{\alpha} \int_{Z \setminus (Z_1 \cup \ldots \cup Z_i)} |II(x)|^2 \, dx.
\]

Since the measure of \( Z \setminus (Z_1 \cup \ldots \cup Z_i) \) vanishes for \( i \uparrow \infty \), (\( \mathcal{K} \)) follows and the proof is complete.

\[ \square \]

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