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ON THE GENERAL HOMOGENIZATION AND Γ-CLOSURE FOR THE EQUATIONS OF VON KÁRMÁN PLATE FROM $3D$ NONLINEAR ELASTICITY

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Abstract. Starting from $3D$ elasticity equations we derive the model of the homogenized von Kármán plate by means of Γ-convergence. This generalizes the recent results, where the material oscillations were assumed to be periodic. We also prove the locality of Γ-closure i.e. that every energy density obtained in this way by mixing n different materials is at almost every point of domain limit of some sequence of the energy densities obtained by periodic homogenization.

Keywords: elasticity, dimension reduction, homogenization, von Kármán plate model.

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1. Introduction

This paper is about derivation of homogenized von Kármán plate equations, starting from $3D$ elasticity by means of Γ-convergence. We do not presuppose any kind of periodicity, but work in a general framework. There is a vast literature on deriving plate equations from $3D$ elasticity. For the approach using formal asymptotic expansion see [Cia97] and the references therein. The first work of deriving the plate models by means of Γ-convergence was [LDR95] where the authors derived the membrane plate model. It was well known that the obtained models depend on the assumption what is the relation of the external loads (i.e. the energy) with respect to the thickness of the body h. Higher ordered models (such as bending and von Kármán plate models) are also derived by means of Γ-convergence (see [FJM02, FJM06]). The key mathematical ingredient in these cases was the theorem on geometric rigidity.

In [BFF00] (see also [BB06]) the influence of the different inhomogeneities in the combination with dimensional reduction on the limit model was analyzed. These models are obtained in the membrane regime. Recently, the techniques from [FJM02, FJM06] were combined together with two-scale convergence to obtain the models of von Kármán plate (see [Velb, NV]), von Kármán shell (see [HV]) and bending plate (see [HNV, Vela]). These models were derived under the assumption of periodic oscillations where it was assumed that the material oscillates on the scale $\varepsilon(h)$, while the thickness of the body is h. The obtained models depend on the parameter $\gamma = \lim_{h\to 0} \frac{h}{\varepsilon(h)}$. In the case of von Kármán plate the situation $\gamma = 0$ corresponds to the case when dimensional reduction dominates and the obtained model is the model of homogenized von Kármán plate and can be obtained as the limit case when $\gamma \to 0$. Analogously, the situation when $\gamma = \infty$ corresponds to the case when homogenization dominates and can again be obtained as the limit when $\gamma \to \infty$; this is the model of von Kármán plate obtained starting from homogenized energy. In the case of von Kármán shell and bending plate the situation $\gamma = 0$ was more subtle and leaded that the models depend on the further assumption of the relation between $\varepsilon(h)$ and h. We obtained different models for the case $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ and $h \sim \varepsilon(h)^2$.

Here we analyze the case of the simultaneous homogenization and dimensional reduction in the von Kármán regime in the general framework, without any assumption on the periodicity. Simultaneous homogenization and dimensional reduction, without any assumption on periodicity, were also considered in a non-variational framework (see [CM04] for monotone nonlinear elliptic systems and [GM06] for linear elasticity system). In these papers compensated compactness arguments were used and the notion of H-convergence (introduced by Murat and Tartar, see [MT97]) was adapted to the dimensional reduction.

This paper is the first treatment of simultaneous homogenization and dimensional reduction without periodicity assumption by variational techniques in the context of higher ordered models in elasticity, at least to the author's knowledge (membrane case is already analyzed in [BFF00]). However, we restrict ourselves to von Kármán regime where the linearization is already dominated, although the system itself is nonlinear.

We prove the validity of the following asymptotic formula for the energy density

$$Q(x', M_1, M_2) = \lim_{r\to 0} \frac{1}{|B(x',r)|} K\left(M_1 + x_3 M_2, B(x',r)\right), \forall M_1, M_2 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}} \text{ and a.e. } x' \in \omega,$$

where

$$K\left(M_1 + x_3 M_2, B(x',r)\right) = \inf\Big\{ \lim_{h\to 0} \int_{B(x',r)\times I} Q^h\left(x, \iota(M_1 + x_3 M_2) + \nabla_h \psi^h\right) dx : (\psi_1^h, \psi_2^h, h\psi_3^h) \to 0 \text{ strongly in } L^2\left(B(x',r)\times I, \mathbb{R}^3\right) \Big\},$$

$B(x', r)$ is a ball of radius r and center x' in $\mathbb{R}^2$, $\omega \subset \mathbb{R}^2$ is a Lipschitz domain, representing the plate, ι is the natural injection from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{3\times 3}$ (see below) and $(Q^h)_{h>0}$ are quadratic functionals of h problem (see Section 2). We suppose that the limit exists (this is a reasonable assumption, since otherwise the formula is valid on a subsequence). This formula unifies all three regimes obtained in [NV].

We also prove the locality of Γ-closure i.e. that every $Q(x', \cdot, \cdot)$ obtained by the formula above (by mixing n different materials) is, for almost every $x' \in \omega$, pointwise limit of the energy densities obtained by the periodic homogenization i.e. those ones obtained in [NV].

The question of locality of Γ-closure was introduced and proved for the linear equations independently by Tartar in [Tar85] and Lurie and Cherkaev in [LC84]. The corresponding locality property of the G-closure for monotone operators is due to Raitums in [Rai01] (generalizing an unpublished work by Dal Maso and Kohn). Related results of locality of Γ-closure in the class of convex integrands can be found in [BB09]. Yet, the local character of the Γ-closure is an open question in the class of quasiconvex nonconvex integrands satisfying standard growth conditions.

In [GN11] the authors proved the weaker version of the local character of Γ-closure at identity. To show commutability of homogenization and linearization at the identity in the general framework the authors used the higher integrability property of the minimizers of the system of linearized elasticity equations (see [SW94]). This required $C^{1,1}$ regularity of domain. Since it is not clear how to obtain higher integrability result for this case which includes also the dimensional reduction i.e. changing of the domains, we do not use it here. Instead we only use equi-integrability property of the minimizing sequence and show that this is enough to construct the recovery sequence. This does not require $C^{1,1}$ regularity of domain i.e. we work with Lipschitz domain. The other two key points are the characterization of the displacements that have bounded symmetric gradients on thin domains (the result proved in [Gri05, Theorem 2.3]) and the characterization of the displacements that have the energy of order h^4 (see [NV, Proposition 3.1]). The proof of this proposition relied on the theorem on geometric rigidity and similar observations in [FJM06]. The new essential part was to correct the vertical displacements in order to obtain a sequence which is bounded in H^2. This was not done in [FJM06], since the authors did not need more information on the corrector to obtain the lower bound.

Although we start from nonlinear $3D$ equations (the property of objectivity is essential), we are able to prove the locality of Γ-closure in the von Kármán regime. This is because in this regime partial linearization is also done and we are in fact dealing with energies that are convex in strain.

This paper is organized as follows: in Section 2 we give general framework and main result, in Section 3 we give the proofs of the main statements given in Section 2 and in the Appendix we prove some auxiliary claims.

1.1. **Notation.** If $x \in \mathbb{R}^3$ by x' we denote $x' = (x_1, x_2)$. By ∇' we denote the operator $\nabla' u = (\partial_1 u, \partial_2 u)$. The expression $A \lesssim B$ means $A \leq CB$, where the constant $C > 0$ can depend additionally depend on domain ω and on the constants α, β below. By $\bar{\mathbb{R}}$ we denote $\mathbb{R} \cup \{-\infty, +\infty\}$. By $B(x, r)$ in the quadratic norm we denote the ball of radius r with the center x. If $A \subset \mathbb{R}^n$, by $|A|$ we denote the Lebesgue measure of A. If A and B are subsets of $\mathbb{R}^n$, by $A \ll B$ we mean that the closure $\bar{A}$ is contained in the interior $\mathrm{int}(B)$ of B.

ι denotes the natural injection of $\mathbb{R}^{2\times 2}$ into $\mathbb{R}^{3\times 3}$. Denoting the standard basis of $\mathbb{R}^3$ by (e_1, e_2, e_3) it is given by

$$\iota(A) := \sum_{\alpha,\beta=1}^{2} A_{\alpha\beta}(e_\alpha \otimes e_\beta).$$

For $a, b \in \mathbb{R}^3$ by $a \wedge b$ we denote the wedge product of the vectors a and b. We put $Y = [-\frac{1}{2}, \frac{1}{2}]^2$ and $\mathcal{Y} = Y$ with the topology of torrus.

2. General framework and main results

The three-dimensional model. Throughout the paper $\Omega^h := \omega \times (hS)$ denotes the reference configuration of a thin plate with mid-surface $\omega \subset \mathbb{R}^2$ and (rescaled) cross-section $I := (-\frac{1}{2}, \frac{1}{2})$. We suppose that ω is Lipschitz domain i.e. open, bounded and connected set with Lipschitz boundary. We denote by $\Gamma = \partial\omega \times I$. For simplicity we assume that ω is centered, that is

$$\int_\omega \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mathrm{d}x_1 \, \mathrm{d}x_2 = 0. \tag{1}$$

Definition 2.1 (nonlinear material law)**.** Let $0 < \alpha \leq \beta$ and $\rho > 0$. The class $\mathcal{W}(\alpha, \beta, \rho)$ consists of all measurable functions $W : \mathbb{R}^{3\times 3} \to [0, +\infty]$ that satisfy the following properties:

(W1) W is frame indifferent, i.e.

$$W(RF) = W(F) \quad \text{for all } F \in \mathbb{M}^3,\ R \in \mathrm{SO}(3);$$

(W2) W is non degenerate, i.e.

$$W(F) \geq \alpha \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all } F \in \mathbb{M}^3;$$

$$W(F) \leq \beta \operatorname{dist}^2(F, \mathrm{SO}(3)) \quad \text{for all } F \in \mathbb{M}^3 \text{ with } \operatorname{dist}^2(F, \mathrm{SO}(3)) \leq \rho;$$

(W3) W is minimal at I, i.e.

$$W(I) = 0;$$

(W4) W admits a quadratic expansion at I, i.e.

$$W(I + G) = Q(G) + o(|G|^2) \quad \text{for all } G \in \mathbb{M}^3$$

where $Q : \mathbb{M}^3 \to \mathbb{R}$ is a quadratic form.

In the following definition we state our assumptions on the family $(W^h)_{h>0}$

Definition 2.2 (admissible composite material)**.** Let $0 < \alpha \leq \beta$ and $\rho > 0$. We say that a family $(W^h)_{h>0}$

$$W^h : \Omega \times \mathbb{R}^{3\times 3} \to \mathbb{R}^+ \cup \{+\infty\}$$

describes an admissible composite material of class $\mathcal{W}(\alpha, \beta, \rho)$ if

(i) For each $h > 0$, W^h is almost surely equal to a Borel function on $\Omega \times \mathbb{R}^{3\times 3}$,
(ii) $W^h(x, \cdot) \in \mathcal{W}(\alpha, \beta, \rho)$ for every $h > 0$ and almost every $x \in \Omega$.
(iii) the following uniform estimate is valid

$$\lim_{G\to 0} \sup_{h>0} \operatorname*{ess\,sup}_{x\in\Omega} \frac{|W^h(x, I + G) - Q^h(x, G)|}{|G|^2} = 0 \tag{2}$$

Notice that Q^h can be written as the pointwise limit

(3) $$(x,G) \to Q^h(x,G) := \lim_{\varepsilon\to 0} \tfrac{1}{\varepsilon^2} W^h(x, Id+\varepsilon G),$$

and therefore inherits the measurability properties of W^h.

Lemma 2.3. *Let $(W^h)_{h>0}$ be as in Definition 2.2 and let $(Q^h)_{h>0}$ be the quadratic form associated to W through the expansion* (W4). *Then*

(Q1) for all $h>0$ and almost all $x\in\Omega$ the map $Q^h(x,\cdot)$ is quadratic and satisfies

$$\alpha|\operatorname{sym} G|^2 \le Q^h(x,G) = Q^h(x,\operatorname{sym} G) \le \beta|\operatorname{sym} G|^2 \qquad \text{for all } G\in\mathbb{M}^3.$$

Furthermore, there exists a monotone function $r:\mathbb{R}^+\to\mathbb{R}^+\cup\{+\infty\}$, such that $r(\delta)\to 0$ as $\delta\to 0$ and

(4) $$\forall G\in\mathbb{R}^{3\times 3}\ :\ |W^h(x,I+G)-Q^h(x,G)| \le |G|^2 r(|G|)$$

for all $h>0$ and almost all $x\in\Omega$.

Proof. (4) is the direct consequence of (iii) in Definition 2.2, while (Q1) follows from (4) and (W2). □

Remark 1. From (Q1) it follows

(5)
$$\begin{aligned}|Q^h(x,G_1)-Q^h(x,G_2)| &\le \beta|\operatorname{sym} G_1-\operatorname{sym} G_2|\cdot|\operatorname{sym} G_1+\operatorname{sym} G_2|,\\ &\qquad \forall h>0, G_1,G_2\in\mathbb{R}^{3\times 3}.\end{aligned}$$

In the von Kármán regime we look for the energy functionals

$$I^h(y) := \frac{1}{h^4}\int_\Omega W^h(x,\nabla_h y(x))\,\mathrm{d}x,$$

imposing their finiteness. Denote by $e_h(y)$

(6) $$e_h(y) = \tfrac{1}{h^4}\int_\Omega \operatorname{dist}^2(\nabla_h y, \mathrm{SO}(3)).$$

With von Kárman model we associate the triple

$$(\bar R,u,v)\in \mathrm{SO}(3)\times\mathcal{A}(\omega), \qquad \mathcal{A}(\omega) := \left\{\,(u,v)\ :\ u\in H^1(\omega,\mathbb{R}^2),\ v\in H^2(\omega)\,\right\}.$$

The following definition, lemma and proposition can be found in [NV]. The definition is changed in the way that we require less i.e. the strong convergence in L^2 instead of weak convergence in H^1. We will give the proof of the uniqueness for the sake of completeness.

Definition 2.4. We say a sequence $y^h\in L^2(\Omega,\mathbb{R}^3)$ converges to a triple $(\bar R,u,v)\in \mathrm{SO}(3)\times L^2(\omega,\mathbb{R}^2)\times L^2(\omega)$, and write $y^h\to(\bar R,u,v)$, if there exist rotations $\{\bar R^h\}_{h>0}$ and functions $\{u^h\}_{h>0}\subset L^2(\omega,\mathbb{R}^2)$, $\{v^h\}_{h>0}\subset L^2(\omega)$ such that

(7) $$(\bar R^h)^T\left(\int_I y^h(x',x_3)\,dx_3 - \fint_\Omega y^h\,dx\right) = \begin{pmatrix} x'+h^2u^h(x') \\ hv^h(x')\end{pmatrix},$$

(8) $$u^h\to u \text{ in } L^2(\omega,\mathbb{R}^2), \qquad v^h\to v \text{ in } L^2(\omega) \qquad \text{and} \qquad \bar R^h\to\bar R.$$

A limit in the sense of Definition 2.4 is not unique as it stands. However, uniqueness is obtained modulo the following equivalence relation on $L^2(\omega,\mathbb{R}^2)\times L^2(\omega)$:

$$(u_1,v_1)\sim(u_2,v_2) \quad :\Leftrightarrow \quad \begin{cases} u_2(x')=u_1(x')+(A-\frac{1}{2}a\otimes a)x'-v_1(x')a \\ v_2(x')=v_1(\hat{x})+a\cdot x' \end{cases} \quad \text{for some } a\in\mathbb{R}^2, A\in\mathbb{R}^{2\times 2}_{\mathrm{skw}}.$$

The proof of the following lemma is given in the Section 3.

Lemma 2.5 (uniqueness)**.** *Let* $(\bar{R},u,v)$, $(\widetilde{R},\tilde{u},\tilde{v})\in \mathrm{SO}(3)\times L^2(\omega,\mathbb{R}^2)\times L^2(\omega)$ *and consider a sequence* y^h *that converges to* $(\bar{R},u,v)$*. Then*

$$y^h\to(\widetilde{R},\tilde{u},\tilde{v}) \qquad \Leftrightarrow \qquad \widetilde{R}=\bar{R} \text{ and } (u,v)\sim(\tilde{u},\tilde{v}).$$

2.1. **Identification of Γ-limit.** We define for the sequence $(h_n)_{n\in\mathbb{N}}$ which monotonly decreases to zero and arbitrary $A\subset\omega$ open and $M\in L^2(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$

(9)
$$\begin{aligned} K^-_{(h_n)_{n\in\mathbb{N}}}(M,A) &= \inf\Big\{\liminf_{n\to\infty}\int_{A\times I} Q^{h_n}\left(x,\iota(M)+\nabla_{h_n}\psi^{h_n}\right)dx : \\ &\qquad (\psi_1^{h_n},\psi_2^{h_n},h_n\psi_3^{h_n})\to 0 \text{ strongly in } L^2(A\times I)\Big\} \\ &= \sup_{\mathcal{U}\subset\mathcal{N}(0)}\liminf_{n\to\infty}\inf_{\substack{\psi\in H^1(A\times I,\mathbb{R}^3)\\ (\psi_1,\psi_2,h_n\psi_3)\in\mathcal{U}}}\int_{A\times I} Q^{h_n}\left(x,\iota(M)+\nabla_{h_n}\psi\right)dx, \end{aligned}$$

(10)
$$\begin{aligned} K^+_{(h_n)_{n\in\mathbb{N}}}(M,A) &= \inf\Big\{\limsup_{n\to\infty}\int_{A\times I} Q^{h_n}\left(x,\iota(M)+\nabla_{h_n}\psi^{h_n}\right)dx : \\ &\qquad (\psi_1^{h_n},\psi_2^{h_n},h_n\psi_3^{h_n})\to 0 \text{ strongly in } L^2(A\times I)\Big\} \\ &= \sup_{\mathcal{U}\subset\mathcal{N}(0)}\limsup_{n\to\infty}\inf_{\substack{\psi\in H^1(A\times I,\mathbb{R}^3)\\ (\psi_1,\psi_2,h_n\psi_3)\in\mathcal{U}}}\int_{A\times I} Q^{h_n}\left(x,\iota(M)+\nabla_{h_n}\psi\right)dx. \end{aligned}$$

By $\mathcal{N}(0)$ we have denoted the family of all neighborhoods of 0 in the strong L^2 topology.

Remark 2. Since the above expressions are monotonly decreasing in $\mathcal{N}(0)$ it is enough to take the supremum on the monotone sequence of neighborhoods that shrinks to $\{0\}$ e.g. the sequence of (open or closed) balls of radius r, when $r\to 0$.

Remark 3. Notice that instead of $h_n\psi_3^{h_n}$ we could introduce the variable $\tilde{\psi}_3^{h_n}$ and then we could look for Γ-limit in 0 of the changed functional in strong L^2 topology. However the obtained functional is not coercive in gradient and thus we can not use standard abstract theory developed for these kind of functionals (see [DM93]).

Remark 4. By using standard diagonalization argument it can be shown that for any $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 and any $A \subset \omega$ open and $M \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ it holds

$$\begin{aligned} K^-_{(h_n)_{n\in\mathbb{N}}}(M,A) &= \min\Big\{ \liminf_{n\to\infty} \int_{A\times I} Q^{h_n}\left(x, \iota(M) + \nabla_{h_n}\psi^{h_n}\right) dx : \\ &\qquad (\psi_1^{h_n}, \psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0 \text{ strongly in } L^2(A\times I, \mathbb{R}^3)\Big\}, \\ K^+_{(h_n)_{n\in\mathbb{N}}}(M,A) &= \min\Big\{ \limsup_{n\to\infty} \int_{A\times I} Q^{h_n}\left(x, \iota(M) + \nabla_{h_n}\psi^{h_n}\right) dx : \\ &\qquad (\psi_1^{h_n}, \psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0 \text{ strongly in } L^2(A\times I, \mathbb{R}^3)\Big\}. \end{aligned}$$

Let $\mathcal{D}$ denote the countable family of open subsets of ω which is dense (see Definition A.9) and such that every $D \in \mathcal{D}$ is of class $C^{1,1}$. The following lemma uses the standard diagonalization argument and Lemma 3.5.

Lemma 2.6. *For every sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to zero there exists a subsequence, still denoted by $(h_n)_{n\in\mathbb{N}}$, such that*

$$K^+_{(h_n)_{n\in\mathbb{N}}}(M,D) = K^-_{(h_n)_{n\in\mathbb{N}}}(M,D), \quad \forall M \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}),\ \forall D \in \mathcal{D}.$$

Proof. Take a countable family $\{M_n\}_{n\in N} \subset L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ which is dense in $L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. By a diagonal argument it is not difficult to construct a sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 such that

$$K^+_{(h_n)_{n\in\mathbb{N}}}(M_j,D) = K^-_{(h_n)_{n\in\mathbb{N}}}(M_j,D), \quad \forall j \in \mathbb{N},\ \forall D \in \mathcal{D}.$$

From Lemma 3.5 and density we have the claim. □

We will now make an assumption on the family $(Q^h)_{h>0}$.

Assumption 2.7. We suppose that for every $D \in \mathcal{D}$ and every $M \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ there exists $K(M,D)$ such that for every sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 we have

$$K^+_{(h_n)_{n\in\mathbb{N}}}(M,D) = K^-_{(h_n)_{n\in\mathbb{N}}}(M,D) =: K(M,D).$$

The following lemma is easy to prove by a contradiction (see the proof of [Bra02, Proposition 1.44]).

Lemma 2.8. *Suppose that for every $M \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ and every $D \in \mathcal{D}$ there exists $K(M,D)$ such that every sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 has subsequence still denoted by $(h_n)_{n\in\mathbb{N}}$ which satisfies*

$$K(M,D) = K^-_{(h_n)_{n\in\mathbb{N}}}(M,D).$$

Then the assumption 2.7 is satisfied.

Remark 5. From the proof of Lemma 2.6 it can be easily seen that in the Assumption 2.7 it is enough to impose that for each $j \in \mathbb{N}$, $D \in \mathcal{D}$ there exists $K(M_j, D)$ such that for every sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 we have

$$K^+_{(h_n)_{n\in\mathbb{N}}}(M_j,D) = K^-_{(h_n)_{n\in\mathbb{N}}}(M_j,D) = K(M_j,D), \forall j \in \mathbb{N},\ D \in \mathcal{D}.$$

Here $\{M_n\}_{n\in\mathbb{N}}$ is any dense subset of $L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

We introduce the space of matrix fields which appear as limit strains in von Kármán model

$$\mathcal{S}_{vK}(\omega) = \{M_1 + x_3 M_2 :\ M_1, M_2 \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})\}.$$

Remark 6. Starting from Lemma 2.6 and Assumption 2.7 we could state the results, only restricting ourselves on the space $\mathcal{S}_{vK}(\omega)$ instead of $L^2(\Omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}})$. We refrained ourselves from doing so for the sake of generality, when it was meaningfull.

The proofs of the following claims are put in the Section 3.

Proposition 2.9. *There exists a function $Q:\omega\times\mathbb{R}^{2\times 2}_{\mathrm{sym}}\times\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ such that for every $A\subset\omega$ open and every $M\in\mathcal{S}_{vK}(\omega)$*

$$M=M_1+x_3M_2,\ \textit{for some } M_1,\ M_2\in L^2(\omega,\mathbb{R}^{2\times 2}_{\mathrm{sym}}).$$

we have

$$K(M,A)=\int_A Q(x',M_1(x'),M_2(x'))\,dx'. \tag{11}$$

Moreover Q satisfies the following property

(Q'1) *for almost all $x'\in\omega$ the map $Q(x',\cdot,\cdot)$ is a quadratic form and satisfies*

$$\tfrac{\alpha}{12}\left(|G_1|^2+|G_2|^2\right)\le Q(x',G_1,G_2)\le\beta\left(|G_1|^2+|G_2|^2\right)\qquad \textit{for all } G_1,\ G_2\in\mathbb{R}^{2\times 2}_{\mathrm{sym}}.$$

The following lemma gives the alternative to the assumptions in (2.7).

Lemma 2.10. *Assume that for almost every $x'\in\omega$ there exists a sequence $(r_n^{x'})_{n\in\mathbb{N}}$ converging to 0 such that for every $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 we have*

$$K\left(M_1+x_3M_2,B(x',r_n^{x'})\right)=K^-_{(h_n)_{n\in\mathbb{N}}}\left(M,B(x',r_n^{x'})\right)=K^+_{(h_n)_{n\in\mathbb{N}}}\left(M,B(x',r_n^{x'})\right),$$
$$\forall n\in\mathbb{N}, M_1,M_2\in\mathbb{R}^{2\times 2}_{\mathrm{sym}}.$$

Then for every $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0, every $M\in\mathcal{S}_{vK}(\omega)$, $A\subset\omega$ open the following property holds

$$K(M,A)=K^-_{(h_n)_{n\in\mathbb{N}}}(M,A)=K^+_{(h_n)_{n\in\mathbb{N}}}(M,A),\ \forall n\in\mathbb{N}, M\in\mathcal{S}_{vK}(\omega).$$

Denote by $I^0:\mathcal{A}(\omega)\to\mathbb{R}^+_0$ the functional

$$I^0(u,v)=\int_\omega Q(x',\mathrm{sym}\,\nabla u+\tfrac12\nabla v\otimes\nabla v,-\nabla^2 v)\,dx'.$$

Notice that for $(u_1,v_1),(u_2,v_2)\in\mathcal{A}(\omega)$ with $(u_1,v_1)\sim(u_2,v_2)$ we have $I^0(u_1,v_1)=I^0(u_2,v_2)$.

The following two theorems are part of the main results of the paper.

Theorem 2.11. *Let Assumption 2.7 be satisfied.*

(i) (Compactness). Let $y^h\in H^1(\Omega,\mathbb{R}^3)$ be a sequence with equibounded energy, that is $\limsup_{h\to 0} I^h(y^h)<\infty$. Then there exists $(\bar R,u,v)\in\mathrm{SO}(3)\times\mathcal{A}(\omega)$ such that $y^h\to(\bar R,u,v)$ up to a subsequence.

(ii) (Lower bound). Let $y^h\in H^1(\Omega,\mathbb{R}^3)$ be a sequence satisfying $\limsup_{h\to 0} I^h(y^h)<\infty$. Assume that $y^h\to(\bar R,u,v)$. Then

$$\liminf_{h\to 0} I^h(y^h)\ge I^0(u,v).$$

Theorem 2.12. *Let Assumption 2.7 be satisfied. For every $(\bar R,u,v)\in\mathrm{SO}(3)\times\mathcal{A}(\omega)$ and every sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 there exists a subsequence, still denoted by $(h_n)_{n\in\mathbb{N}}$ such that $y^{h_n}\in H^1(\Omega,\mathbb{R}^3)$ with*

$$y^{h_n}\to(\bar R,u,v)\qquad\textit{and}\qquad\lim_{n\to\infty} I^{h_n}(y^{h_n})=I^0(u,v).$$

2.2. **Locality of Γ-closure.**

Definition 2.13. For given $n \in \mathbb{N}$ and $\omega \subset \mathbb{R}^2$ Lipschitz domain we denote $\mathcal{X}^n(\omega)$ the family of functions $(\chi_1, \dots, \chi^n) \in L^\infty(\omega, \{0,1\}^n)$ such that $\sum_{i=1}^n \chi^i(x') = 1$, for a.e. $x' \in \omega$. Equivalently $\chi \in \mathcal{X}^n(\omega)$ if and only if there exists a measurable partition $\{A_i\}_{i=1,\dots,n}$ such that $\chi_i = 1_{A_i}$ for $i = 1, \dots, n$.

We look for the mixtures of n homogeneous materials whose energy densities $W^1, \dots, W^n$ satisfy (W1)-(W4) with quadratic forms $Q^1, \dots, Q^n$. We suppose that allowable mixtures are homogeneous in the variable x_3 i.e. we suppose that the scaled energy functional of the mixture is given by

$$I^h(y) = \tfrac{1}{h^4} \int_\Omega \sum_{i=1}^n W^i(\nabla_h y) \chi_i^h(x')\, dx, \tag{12}$$

where for $i = 1, \dots, n$ and $h > 0$ we have $\chi^h \in \mathcal{X}^n(\omega)$. The following definition is justified by the results in the previous section.

Definition 2.14. We say that a sequence $(\chi^h)_{h>0}$ in $\mathcal{X}^n(\omega)$ has the limit energy density Q if the following is valid

(a) for any $A \subset \omega$ open and any $M \in \mathcal{S}_{vK}(\omega)$ there exists $K_{(\chi^h)_{h>0}}(M, A, \omega)$ such that for any sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to zero we have

$$K_{(\chi^h)_{h>0}}(M, A, \omega) = K^-_{(\chi^{h_n})_{n\in\mathbb{N}}}(M, A, \omega) = K^+_{(\chi^{h_n})_{n\in\mathbb{N}}}(M, A, \omega),$$

where

$$\begin{aligned} &K^-_{(\chi^{h_n})_{n\in\mathbb{N}}}(M, A, \omega) = \\ &\quad \min\Big\{ \liminf_{n\to\infty} \int_{A\times I} \sum_{i=1}^n Q^i\left(x, \iota(M) + \nabla_{h_n}\psi^{h_n}\right) \chi_i^{h_n}(x')\, dx : \\ &\qquad (\psi_1^{h_n}, \psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0 \text{ strongly in } L^2(A\times I) \Big\}, \\ &K^+_{(\chi^{h_n})_{n\in\mathbb{N}}}(M, A, \omega) = \\ &\quad \min\Big\{ \limsup_{n\to\infty} \int_{A\times I} \sum_{i=1}^n Q^i\left(x, \iota(M) + \nabla_{h_n}\psi^{h_n}\right) \chi_i^{h_n}(x')\, dx : \\ &\qquad (\psi_1^{h_n}, \psi_2^{h_n}, h_n\psi_3^{h_n}) \to 0 \text{ strongly in } L^2(A\times I) \Big\} \end{aligned}$$

(b)

$$Q(x', M_1, M_2) = \lim_{r\to 0} \tfrac{1}{|B(x',r)|} K_{(\chi^h)_{h>0}}\left(M_1 + x_3 M_2, B(x', r), \omega\right), \quad \forall M_1, M_2 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \quad \text{for a.e. } x' \in \omega.$$

Remark 7. The assumption (a) in the previous definition can be weakened (see Assumption 2.7, Remark 6 and Lemma 2.10). Also, instead of taking balls around point $x' \in \omega$ in (b) part, we can take any family of sets that shrinks nicely to $x' \in \omega$ (see e.g. [Fol99, Chapter 3]). Notice that we also have for $M \in \mathcal{S}_{vK}(\omega)$ (using Lemma 3.10, see also (9) and (10))

$$\begin{aligned} K_{(\chi^h)_{h>0}}(M, A, \omega) &= \sup_{\mathcal{U}\subset\mathcal{N}(0)} \liminf_{h\to 0} K_{\chi^h}(M, A, \omega, \mathcal{U}) = \sup_{\mathcal{U}\subset\mathcal{N}(0)} \limsup_{h\to 0} K_{\chi^h}(M, A, \omega, \mathcal{U}), \\ &= \sup_{\mathcal{U}\subset\mathcal{N}(0)} \liminf_{h\to 0} K^0_{\chi^h}(M, A, \omega, \mathcal{U}) = \sup_{\mathcal{U}\subset\mathcal{N}(0)} \limsup_{h\to 0} K^0_{\chi^h}(M, A, \omega, \mathcal{U}), \end{aligned}$$

where

$$K_{\chi^h}(M,A,\omega,\mathcal{U}) = \inf_{\substack{\psi\in H^1(A\times I,\mathbb{R}^3)\\ (\psi_1,\psi_2,h\psi_3)\in\mathcal{U}}} \int_{A\times I}\sum_{i=1}^n Q^i\left(x,\iota(M)+\nabla_h\psi\right)\chi_i^h(x')\,dx,$$

$$K^0_{\chi^h}(M,A,\omega,\mathcal{U}) = \inf_{\substack{\psi\in H^1(A\times I,\mathbb{R}^3)\\ \psi=0 \text{ on } \partial A\times I,\ (\psi_1,\psi_2,h\psi_3)\in\mathcal{U}}} \int_{A\times I}\sum_{i=1}^n Q^i\left(x,\iota(M)+\nabla_h\psi\right)\chi_i^h(x')\,dx.$$

Notice that for arbitrary $M\in\mathcal{S}_{vK}$, $A\subset\omega$ open, $h>0$ the sequence $(K_{\chi^h}(M,A,\omega,\mathcal{U})_{\mathcal{U}\subset\mathcal{N}(0)}$ i.e. $(K^0_{\chi^h}(M,A,\omega,\mathcal{U})_{\mathcal{U}\subset\mathcal{N}(0)}$ is monotononly decreasing. Thus sup can be replaced by limit as $n\to\infty$ on any subsequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ monotonly shrinking to 0. If $A\subset\omega$ has Lipshitz boundary and $\mathcal{U}$ is a set which is closed in weak H^1 topology and which guarantees that symmetrized gradients control all H^1 norm (e.g. balls of radius r) then inf in the definition of $K_{\chi^h}(M,A,\omega,\mathcal{U})$ is attained by the direct methods of the calculus of variation.

The goal is to show that for a.e. $x'\in\omega$, $Q(x',\cdot,\cdot)$ can be obtained as a pointwise limit of periodic homogenized energies. We recall some results from [NV]. There it was shown that under the periodicity assumption on W^h i.e. Q^h, when we assume that the energy densities oscillate with the period $\varepsilon(h)\to 0$, we obtain three different regimes, depending on the parameter $\gamma:=\lim_{h\to 0}\frac{h}{\varepsilon(h)}$. As a consequence, if we assume for $i=1,\dots,n$, $A_i^h=\varepsilon(h)\left(A_i+\mathbb{Z}^2\right)\cap\omega$ and $\chi_i^h=1_{A_i^h}$, where $(A_i)_{i=1,\dots,n}$ form the partition of the unit cube $Y=[-\frac12,\frac12)^2$, we obtain that the limit functional has the homogeneous energy density given by the expression

$$Q_\gamma(M_1,M_2):=\inf_{U\in L_\gamma(I\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}})}\sum_{i=1}^n\iint_{A_i\times I}Q^i\left(y,\iota(M_1+x_3M_2)+U\right)\mathrm{d}y\,\mathrm{d}x_3, \tag{13}$$

where $\mathcal{Y}=Y$ with the topology of torus and for $\gamma\in[0,\infty]$ we defined the following function spaces of relaxation fields

$$L_0(S\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}}) := \left\{\begin{pmatrix}\mathrm{sym}\,\hat\nabla_y\zeta+x_3\hat\nabla_y^2\varphi & \begin{matrix}g_1\\ g_2\end{matrix}\\ (g_1,\ g_2) & g_3\end{pmatrix} : \zeta\in H^1(\mathcal{Y},\mathbb{R}^2),\ \varphi\in H^2(\mathcal{Y}),\ g\in L^2(S\times\mathcal{Y},\mathbb{R}^3)\right\}$$

$$L_\infty(S\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}}) := \left\{\begin{pmatrix}\mathrm{sym}\,\hat\nabla_y\zeta & \begin{matrix}\partial_{y_1}\psi+c_1\\ \partial_{y_2}\psi+c_2\end{matrix}\\ \hat\nabla_y\psi+(c_1,c_2) & c_3\end{pmatrix} : \zeta\in L^2(S,H^1(\mathcal{Y},\mathbb{R}^2)),\ \psi\in L^2(S,H^1(\mathcal{Y})),\ c\in L^2(S,\mathbb{R}^3)\right\}$$

$$L_\gamma(S\times\mathcal{Y},\mathbb{M}^3_{\mathrm{sym}}) := \left\{\mathrm{sym}(\hat\nabla_y\phi,\tfrac{1}{\gamma}\partial_3\phi) : \phi\in H^1(S\times\mathcal{Y},\mathbb{R}^3)\right\}\qquad \text{for } \gamma\in(0,\infty).$$

Moreover it was shown that for fixed $M_1,M_2\in\mathbb{R}^{2\times2}_{\mathrm{sym}}$ we have $Q_\gamma(M_1,M_2)\to Q_0(M_1,M_2)$ as $\gamma\to 0$ i.e. $Q_\gamma(M_1,M_2)\to Q_\infty(M_1,M_2)$ as $\gamma\to\infty$. To prove that in almost every point the energy density obtained by mixing n different materials can be obtained as the pointwise limit of the energy densities given by the expression (13) we follow the approach from [BB09].

The following definition characterizes all homogeneous quadratic forms that can be approximated by simultaneous periodic homogenization and dimensional reduction.

Definition 2.15. For $\theta \in [0,1]^n$ such that $\sum_{i=1}^n \theta_i = 1$ we define $\mathcal{P}_\theta$ as the set of all quadratic functions $Q : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}$ which can be obtained as pointwise limits of the quadratic functions Q_{γ_k} of the form (13), for some $\gamma_k \in (0,\infty)$ and A_i^k such that $|A_i^k| = \theta_i$, for all $i = 1, \dots, n$ and $k \in \mathbb{N}$.

Remark 8. The set of all quadratic functions of all quadratic functions of the form (13) is not closed under the pointwise convergence. Sufficiency of the pointwise convergence is justified by Lemma A.14.

The following theorem states the locality of Γ-closure result and is the part of the main result of the paper.

Theorem 2.16. *Let $\omega \subset \mathbb{R}^2$ open, connected set with Lipschitz boundary. The following is satisfied*

(i) Let $(\chi^h)_{h>0}$ be a sequence in $\mathcal{X}^n(\omega)$ which has the limit energy density Q. Then for almost every $x_0' \in \omega$ we have that $Q(x_0', \cdot, \cdot)$ belongs to the set $\mathcal{P}_{\theta(x_0')}$, where θ is the weak star limit of χ^h in L^∞.
(ii) If $Q : \omega \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}$ is such that for a.e. $x_0' \in \omega$ we have that $Q(x_0', \cdot, \cdot)$ belongs to the set $\mathcal{P}_{\theta(x_0')}$ then there exists $(\chi^h)_{h>0}$ a sequence in $\mathcal{X}^n(\omega)$ which has the limit energy density Q.

3. Proofs

3.1. Proof of Lemma 2.5.

Proof. Without loss of generality we can assume that $\bar{R} = I$. Assume that $y^h \to (I, u, v)$ and $y^h \to (\widetilde{R}, \widetilde{u}, \widetilde{v})$. Then, by definition, there exist two sequences $(\bar{R}^h, u^h, v^h)$ and $(\widetilde{R}^h, \tilde{u}^h, \tilde{v}^h)$ with

$$u^h \to u,\ \widetilde{u}^h \to \widetilde{u} \text{ in } L^2, \qquad v^h \to v,\ \widetilde{v}^h \to \widetilde{v} \text{ in } L^2, \tag{14}$$
$$\bar{R}^h \to I,\ \widetilde{R}^h \to \widetilde{R},$$

as $h \to 0$, and

$$\fint_I y^h(x', x_3)\, dx_3 = \bar{R}^h \begin{pmatrix} x' + h^2 u^h \\ h v^h \end{pmatrix} = \widetilde{R}^h \begin{pmatrix} x' + h^2 \widetilde{u}^h \\ h \widetilde{v}^h \end{pmatrix}.$$

Rearranging terms and introducing $\hat{R}^h := (\widetilde{R}^h)^T \bar{R}^h$ yields

$$\begin{aligned} (\hat{R}^h - I) \begin{pmatrix} x' \\ 0 \end{pmatrix} + h\hat{R}^h \begin{pmatrix} 0 \\ v^h(x') \end{pmatrix} + h^2 \hat{R}^h \begin{pmatrix} u^h(x') \\ 0 \end{pmatrix} \\ = h \begin{pmatrix} 0 \\ \tilde{v}^h(x') \end{pmatrix} + h^2 \begin{pmatrix} \tilde{u}^h(x') \\ 0 \end{pmatrix} \end{aligned} \tag{15}$$

for almost every $x' \in \omega$ and all h. In the limit $h \to 0$ we get $(\hat{R} - I) \begin{pmatrix} x' \\ 0 \end{pmatrix} = 0$. Combined with $\hat{R} \in \mathrm{SO}(3)$, and $\hat{R} = \widetilde{R}^T \bar{R} = \widetilde{R}^T$, this implies $\widetilde{R} = I$.

Set $\hat{A}^h := \frac{\hat{R}^h - I}{h}$. We claim that there exists $\hat{A} \in \mathbb{R}^{3\times 3}_{\mathrm{skw}}$ such that

$$\hat{A}^h \to \hat{A} \qquad \text{with } \operatorname{sym} \hat{A} = 0, \tag{16}$$

$$\frac{\operatorname{sym} \hat{A}^h}{h} \to \frac{1}{2} \hat{A}^2. \tag{17}$$

Here comes the argument. Dividing (15) by h, and rearranging terms, yields

$$\hat{A}^h \begin{pmatrix} x' \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v^h(x') \end{pmatrix} + h\hat{A}^h \begin{pmatrix} 0 \\ v^h(x') \end{pmatrix} + h \begin{pmatrix} u^h(x') \\ 0 \end{pmatrix} + h^2 \hat{A}^h \begin{pmatrix} u^h(x') \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \tilde{v}^h(x') \end{pmatrix} + h \begin{pmatrix} \tilde{u}^h(x') \\ 0 \end{pmatrix}. \tag{18}$$

We deduce that the first term of the left hand side converges in $L^2(\omega)$. This implies that $\hat{A}^h e_\alpha$ converges as $h \to 0$. From the identity $\hat{R}^h e_3 = \hat{R}^h e_1 \wedge \hat{R}^h e_2$, we deduce that

$$\hat{A}^h e_3 = (\hat{A}^h e_1 \wedge \hat{R}^h e_2 + e_1 \wedge \hat{A}^h e_2),$$

and thus $\hat{A}^h$ converges to some limit $\hat{A} \in \mathbb{R}^{3\times 3}$. Eventually, the relation $(\hat{A}^h)^T \hat{A}^h = -2\frac{\operatorname{sym} \hat{A}^h}{h}$ yields (16) and (17).

To complete the argument, it remains to prove that

$$\tilde{v}(x') = v(x') + a \cdot x' \qquad \text{where } a := (\hat{A}_{31}, \hat{A}_{32}) \tag{19}$$

$$\tilde{u}(x') = u(x') + (A - \tfrac{1}{2} a \otimes a)x' - v(x')a \tag{20}$$

for some skew symmetric matrix $A \in \mathbb{M}^2_{\text{skw}}$. The first identity appears in the limit $h \to 0$ in the third component of identity (18). For the proof of (20) we introduce the skew-symmetric matrix $A^h \in \mathbb{R}^{2\times 2}_{\text{skw}}$

$$A^h_{\alpha\beta} = \frac{\hat{A}^h_{\alpha\beta}}{h} - \frac{(\operatorname{sym} \hat{A}^h)_{\alpha\beta}}{h}, \text{ for } \alpha, \beta = 1, 2. \tag{21}$$

Going back to (18), after dividing by h, we find that $h^{-1}\hat{A}_{\alpha\beta}$, $\alpha, \beta \in \{1, 2\}$, converges as $h \to 0$. This implies $\hat{A}_{\alpha\beta} = 0$. Combined with (17) we deduce that A^h converges to some $A \in \mathbb{R}^{2\times 2}_{\text{skw}}$. Now, a calculation yields (20) i.e. we divide (18) by h and let $h \to 0$ in the first two components.

Step 2. Argument for "$\Leftarrow$".

Suppose that $y^h \in L^2(\omega; \mathbb{R}^3)$ converges to the triple $(\bar{R}, u, v)$ in the sense of definition (2.4). Let us now take arbitrary $A \in \mathbb{R}^{2\times 2}_{\text{skw}}$ and $a \in \mathbb{R}^2$, and set

$$\tilde{R}^h = \bar{R}^h \exp(-h^2 \iota(A)) \exp(-h a_e), \tag{22}$$

where $a_e \in \mathbb{R}^{3\times 3}$ is defined by

$$a_e := \begin{pmatrix} 0 & -a \\ a^T & 0 \end{pmatrix}. \tag{23}$$

We define $\tilde{u}^h$, $\tilde{v}^h$ via identity (7). From the expansions

$$\exp(h^2 \iota(A)) = I + h^2 \iota(A) + O(h^4), \qquad \exp(h a_e) = I + h a_e + \frac{h^2}{2} a_e^2 + O(h^3), \tag{24}$$

we conclude that

$$\begin{aligned} \tilde{u}^h(x') &= u^h(x') + (A - \tfrac{1}{2} a \otimes a)x' - v^h(x')a + O(h), \\ \tilde{v}^h(x') &= v^h(\hat{x}) + a \cdot x' + O(h), \end{aligned}$$

where $\|O(h)\|_{L^2} \le Ch$, for some $C > 0$. □

3.2. **Proof of Proposition 2.9 and Lemma 2.10.** The following theorem is proved in [Gri05].

Theorem 3.1. *Let $A \subset \omega$ with Lipschitz boundary and $\psi \in H^1(A \times I, \mathbb{R}^3)$ and $h > 0$. Then we have the following decomposition*

$$\psi(x) = \hat{\psi}(x') + r(x') \wedge x_3 e_3 + \bar{\psi}(x) = \begin{cases} \hat{\psi}_1(x') + r_2(x')x_3 + \bar{\psi}_1(x) \\ \hat{\psi}_2(x') - r_1(x')x_3 + \bar{\psi}_2(x) \\ \hat{\psi}_3(x') + \bar{\psi}_3(x) \end{cases},$$

where

$$\hat{\psi} = \int_I \psi \, dx_3, \;\; r = \frac{3}{2}\int_I x_3 e_3 \wedge \psi(x) \, dx_3, \tag{25}$$

and the following estimate is valid

$$\|\operatorname{sym} \nabla_h(\hat{\psi} + r \wedge x_3 e_3)\|^2_{L^2} + \|\nabla_h \bar{\psi}\|^2_{L^2} + \tfrac{1}{h^2}\|\bar{\psi}\|^2_{L^2} \leq C(A)\|\operatorname{sym}\nabla_h \psi\|^2_{L^2}. \tag{26}$$

Remark 9. Notice that

$$\begin{aligned} &\|\operatorname{sym} \nabla_h(\hat{\psi} + r \wedge x_3 e_3)\|^2_{L^2(A\times I)} = \\ &\|\operatorname{sym}\nabla'(\hat{\psi}_1, \hat{\psi}_2)\|^2_{L^2(A)} + \|\operatorname{sym}\nabla'(r_2, -r_1)\|^2_{L^2(A)} \\ &+\tfrac{1}{h^2}\|\partial_1(h\hat{\psi}_3) + r_2\|^2_{L^2(A)} + \tfrac{1}{h^2}\|\partial_2(h\hat{\psi}_3) - r_1\|^2_{L^2(A)}. \end{aligned} \tag{27}$$

Thus from Korn's inequality it follows

$$\begin{aligned} &\|(\hat{\psi}_1, \hat{\psi}_2, h\hat{\psi}_3)\|^2_{H^1(A)} + \|(r_1, r_2)\|^2_{H^1(A)} + \tfrac{1}{h^2}\|\partial_1(h\hat{\psi}_3) + r_2\|^2_{L^2(A)} \\ &+\tfrac{1}{h^2}\|\partial_2(h\hat{\psi}_3) - r_1\|^2_{L^2(A)} \\ &\leq C(A)\left(\|\operatorname{sym}\nabla_h(\hat{\psi} + r \wedge x_3 e_3)\|^2_{L^2(A\times I)} + \|r\|^2_{L^2(A)} + \|(\hat{\psi}_1, \hat{\psi}_2, h\hat{\psi}_3)\|^2_{L^2(A)}\right) \\ &\leq C(A)\left(\|\operatorname{sym}\nabla_h(\hat{\psi} + r \wedge x_3 e_3)\|^2_{L^2(A\times I)} + \|(\psi_1, \psi_2, h\psi_3)\|^2_{L^2(A\times I)}\right). \end{aligned} \tag{28}$$

Corollary 3.2. *If we assume that $(\psi^h)_{h>0} \subset H^1(A\times I, \mathbb{R}^3)$, is such that $\psi^h = 0$ on $\partial A \times I$ for every $h > 0$ and*

$$\limsup_{h\to 0} \|\operatorname{sym}\nabla_h \psi^h\|_{L^2} < \infty,$$

then we have that on a subsequence

$$\operatorname{sym}\nabla_h \psi^h = \iota(-x_3 \nabla'^2 v + \operatorname{sym}\nabla' u) + \operatorname{sym}\nabla_h \bar{\psi}^h,$$

for some $v \in H^2_0(A)$, $u \in H^1_0(A, \mathbb{R}^2)$ and $(\bar{\psi}^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)$ such that $\bar{\psi}^h = 0$ on $\partial A \times I$ and $(\bar{\psi}_1, \bar{\psi}_2, h\bar{\psi}_3) \to 0$ strongly in L^2. Moreover we have

$$\|v\|^2_{L^2} + \|u\|^2_{L^2} \leq \limsup_{h\to 0} \|(\psi^h_1, \psi^h_2, h\psi^h_3)\|^2_{L^2}.$$

Analogously, if we assume that $A = \mathcal{Y}$ and additionally that $\int_{Y\times I} \psi^h = 0$ for every $h > 0$ then we have that on a subsequence

$$\operatorname{sym}\nabla_h \psi^h = \iota(-x_3 \nabla'^2 v + \operatorname{sym}\nabla' u) + \operatorname{sym}\nabla_h \bar{\psi}^h,$$

for some $v \in H^2(\mathcal{Y})$, $u \in H^1(\mathcal{Y}, \mathbb{R}^2)$ and $(\bar{\psi}^h)_{h>0} \subset H^1(\mathcal{Y}\times I, \mathbb{R}^3)$ such that $(\bar{\psi}_1, \bar{\psi}_2, h\bar{\psi}_3) \to 0$ strongly in L^2. In this case we can also demand that $\int_Y v = \int_Y u = \int_{Y\times I} \bar{\psi}^h = 0$. Moreover we have

$$\|v\|^2_{L^2} + \|u\|^2_{L^2} \leq \limsup_{h\to 0} \|(\psi^h_1, \psi^h_2, h\psi^h_3)\|^2_{L^2}.$$

Proof. We prove the first claim. Use the decomposition from Theorem 3.1 and write

$$\psi^h(x) = \hat{\psi}^h(x') + r^h(x') \wedge x_3 e_3 + \bar{\psi}^{h,1}(x). \tag{29}$$

We know that $\|\bar{\psi}^{h,1}\|_{L^2} \to 0$ as $h \to 0$. By using Korn's inequality with boundary condition (see [Cia97]) we conclude from (25) and (27) that $r^h \rightharpoonup r$ weakly in $H^1(A,\mathbb{R}^2)$ and $(\hat{\psi}_1^h, \hat{\psi}_2^h) \rightharpoonup u$ weakly in $H^1(A,\mathbb{R}^2)$, on a subsequence. By the compactness of the trace we see that $r = 0$ on $\partial A \times I$. Using the fact that

$$\tfrac{1}{h^2}\|\partial_1(h\hat{\psi}_3^h) + r_2^h\|^2_{L^2(A)} + \tfrac{1}{h^2}\|\partial_2(h\hat{\psi}_3^h) - r_1^h\|^2_{L^2(A)} \leq C(A)\|\operatorname{sym}\nabla_h \psi^h\|^2_{L^2}, \tag{30}$$

and the compactness of the trace operator we conclude that there exists $v \in H_0^2(A)$ such that

$$h\hat{\psi}_3 \rightharpoonup v \text{ weakly in } H^1(A),\ r_1 = \partial_2 v,\ r_2 = -\partial_1 v.$$

Define l_h as

$$l^h = u + \begin{pmatrix} 0 \\ 0 \\ \frac{v}{h} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 v \\ \partial_2 v \\ 0 \end{pmatrix}.$$

It is easy to see that

$$\operatorname{sym}\nabla_h l^h = \iota(-x_3 \nabla'^2 v + \operatorname{sym}\nabla' u).$$

Define $\bar{\psi}^h = \psi^h - l^h$. It is easy to see from (29) that $(\bar{\psi}_1, \bar{\psi}_2, h\bar{\psi}_3) \to 0$ strongly in L^2. Also we have the estimate

$$\|v\|^2_{L^2} + \|u\|^2_{L^2} \leq \limsup_{h\to 0} \|(\hat{\psi}_1^h, \hat{\psi}_2^h, h\hat{\psi}_3^h)\|^2_{L^2} \leq \limsup_{h\to 0} \|(\psi_1^h, \psi_2^h, h\psi_3^h)\|^2_{L^2}.$$

The second claim follows in the same way by using the Korn's inequality with periodic boundary condition (which can be easily proved by Fourier transform) and noticing that $\int_{Y\times I}\psi^h = 0$ implies $\int_Y \hat{\psi}^h = 0$ and (30) implies that $\int_{\mathcal{Y}} r^h \to 0$. □

Lemma 3.3. *Let $A \subset \omega$ with $C^{1,1}$ boundary. If $r \in H^1(A,\mathbb{R}^2)$ and $\hat{\psi} \in H^1(A,\mathbb{R}^3)$ is such that $\int_{A_i} \hat{\psi}_3\, dx = 0$, for every connected component A_i of A (see Lemma A.13), then there exists $\varphi \in H^2(A)$ and $w \in H^1(A)$ such that $\hat{\psi}_3 = \frac{\varphi}{h} + w$ and*

$$\begin{aligned} &\|\varphi\|^2_{H^2(A)} + \|(\hat{\psi}_1,\hat{\psi}_2)\|^2_{H^1(A)} + \|r\|^2_{H^1(\omega)} + \|w\|^2_{H^1(A)} + \tfrac{1}{h^2}\|\partial_1\varphi + r_2\|^2_{L^2(A)} \\ &+ \tfrac{1}{h^2}\|\partial_2\varphi - r_1\|^2_{L^2(A)} \leq \\ &\quad C(A)\left(\|\operatorname{sym}\nabla_h(\hat{\psi} + r\wedge x_3 e_3)\|^2_{L^2(A\times I)} + \|r\|^2_{L^2(A)} + \|(\hat{\psi}_1,\hat{\psi}_2,h\hat{\psi}_3)\|^2_{L^2(A\times I)}\right). \end{aligned} \tag{31}$$

Proof. We do the regularization of $\hat{\psi}_3$ in the similar way as in [NV, Proposition 3.1] and [HV, Lemma 3.8]. We look for the solution of the problem

$$\min_{\substack{\varphi\in H^1(A) \\ \forall i,\ \int_{A_i}\varphi = 0}} \int_A |\nabla'\varphi + (r_2, -r_1)|^2\, dx'. \tag{32}$$

The associated Euler-Lagrange equation of (32) reads

$$\begin{cases} -\triangle'\varphi = \nabla'\cdot(r_2,-r_1) & \text{in } A \\ \partial_\nu\varphi = -(r_2,-r_1)\cdot\nu & \text{on } \partial A, \end{cases} \tag{33}$$

Since $\nabla' \cdot (r_2, -r_1) \in L^2$, we obtain by standard regularity estimates that $\varphi \in H^2(A)$ and $\|\varphi\|_{H^2(A)} \lesssim \|r\|_{H^1(A)}$, where we need the $C^{1,1}$ regularity of ∂A. The claim follows from (27), (28) and the following inequalities:

$$\|\partial_1 \varphi + r_2\|^2_{L^2(A)} \leq \|\partial_1(h\hat{\psi}_3) + r_2\|^2_{L^2(A)} \tag{34}$$

$$\|\partial_2 \varphi - r_1\|^2_{L^2(A)} \leq \|\partial_2(h\hat{\psi}_3) - r_1\|^2_{L^2(A)} \tag{35}$$

$$\begin{aligned} \left\|\nabla'\left(\hat{\psi}_3 - \tfrac{\varphi}{h}\right)\right\|^2_{L^2(A)} \leq\; & \tfrac{1}{h^2}\|\partial_1(h\hat{\psi}_3) + r_2\|^2_{L^2} + \tfrac{1}{h^2}\|\partial_1\varphi + r_2\|^2_{L^2(A)} \\ & + \tfrac{1}{h^2}\|\partial_2(h\hat{\psi}_3) - r_1\|^2_{L^2(A)} + \tfrac{1}{h^2}\|\partial_2\varphi - r_1\|^2_{L^2(A)}. \end{aligned} \tag{36}$$

□

Remark 10. If we assume that $A = \mathcal{Y}$ and that $\psi \in H^1(\mathcal{Y} \times I, \mathbb{R}^3)$ then the claim of Lemma 3.3 is valid and we can demand that $\varphi \in H^2(\mathcal{Y})$, $w \in H^1(\mathcal{Y})$. To adapt the proof of Lemma 3.3 instead of solving the problem (32), we need to solve

$$\min_{\substack{\varphi \in H^1(\mathcal{Y}),\\ \int_Y \varphi = 0}} \int_Y |\nabla'\varphi + (r_2, -r_1)|^2 \, dx'. \tag{37}$$

Using again the Korn's inequality with periodic boundary condition we can omit $\|r\|_{L^2}$ and $\|(\hat{\psi}_1, \hat{\psi}_2, h\hat{\psi}_3)\|_{L^2}$ on the right hand side of (31), under the additional assumption that $\int_Y \hat{\psi} = 0$.

In the similar way if we assume that $\hat{\psi} = 0$, $r = 0$ on $\partial A \times I$ (without the assumption that $\int_{A_i} \hat{\psi}_3 \, dx = 0$, for every connected component A_i of A) the claim of Lemma 3.3 is valid and we can demand that $\varphi \in H^2(A)$, $\varphi = 0$ on ∂A, $w \in H^1_0(\mathcal{Y})$. Using again the Korn's inequality with boundary conditions we can omit $\|r\|_{L^2}$ and $\|(\hat{\psi}_1, \hat{\psi}_2, h\hat{\psi}_3)\|_{L^2}$ on the right hand side of (31). To adapt the proof of Lemma 3.3 instead of solving the problem (32), we need to solve the problem

$$\min_{\varphi \in H^1_0(A)} \int_A |\nabla'\varphi + (r_2, -r_1)|^2 \, dx'. \tag{38}$$

Proposition 3.4. *Let $A \subset \omega$ with $C^{1,1}$ boundary. Denote by $\{A_i\}_{i=1,\dots,k}$ the connected components of A.*

(a) Let $(\psi^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)$ be such that

$$(\psi^h_1, \psi^h_2, h\psi^h_3) \to 0, \text{ strongly in } L^2, \quad \forall h, i \int_{A_i} \psi^h_3 = 0, \tag{39}$$

$$\limsup_{h \to 0} \|\operatorname{sym} \nabla_h \psi^h\|_{L^2(A)} \leq M < \infty. \tag{40}$$

Then there exist $(\varphi^h)_{h>0} \subset H^2(A)$, $(\tilde{\psi}^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)$ such that

$$\operatorname{sym} \nabla_h \psi^h = -x_3 \iota(\operatorname{sym} \nabla'^2 \varphi^h) + \operatorname{sym} \nabla_h \tilde{\psi}^h + o^h,$$

where $o^h \in L^2(A \times I, \mathbb{R}^{3\times 3})$ is such that $o^h \to 0$, strongly in L^2, and the following properties hold

$$\lim_{h \to 0} \left(\|\varphi^h\|_{H^1(A)} + \|\tilde{\psi}^h\|_{L^2(A\times I)}\right) = 0, \tag{41}$$

$$\limsup_{h \to 0} \left(\|\varphi^h\|_{H^2(A)} + \|\nabla_h \tilde{\psi}^h\|_{L^2(A\times I)}\right) \leq C(A)M. \tag{42}$$

(b) For every $(\varphi^h)_{h>0} \subset H^2(A)$, $(\tilde{\psi}^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)$ *such that*

$$\lim_{h\to 0} \Big(\|\varphi^h\|_{H^1(A)} + \|\tilde{\psi}^h\|_{L^2(A\times I)} \Big) = 0,$$
$$\limsup_{h\to 0} \Big(\|\varphi^h\|_{H^2(A)} + \|\nabla_h \tilde{\psi}^h\|_{L^2(A)} \Big) \le M,$$

there exists a sequence $(\psi^h)_{h>0} \subset H^1(A \times I, \mathbb{R}^3)$ *such that*

$$(\psi_1^h, \psi_2^h, h\psi_3^h) \to 0 \text{ strongly in } L^2,$$
$$\limsup_{h\to 0} \|\operatorname{sym} \nabla_h \psi^h\|_{L^2(A)} \le 2M.$$

Proof. The proof follows immediately from Theorem 3.1 and Lemma 3.3. Namely, if we use the decomposition from Theorem 3.1 and Lemma 3.3 we obtain

$$\begin{aligned} (43)\quad \psi^h &= \hat{\psi}^h + r^h \wedge x_3 e_3 + \bar{\psi}^h \\ &= \begin{pmatrix} \hat{\psi}_1^h \\ \hat{\psi}_2^h \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\varphi^h}{h} + w^h \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi^h \\ \partial_2 \varphi^h \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \partial_1 \varphi^h + r_2^h \\ \partial_2 \varphi^h - r_1^h \\ 0 \end{pmatrix} + \bar{\psi}^h. \end{aligned}$$

From the expressions (25) and (26) it follows $(\hat{\psi}_1, \hat{\psi}_2, h\hat{\psi}_3) \to 0$, $(r_1^h, r_2^h) \to 0$, $\bar{\psi}^h \to 0$ strongly in L^2. This implies $\varphi^h \to 0$ strongly in L^2. Since it holds for every i, $\int_{A_i} \hat{\psi}_3 \, dx = 0$ (see (25)), from (26) and Lemma 3.3 we have that $\limsup_{h\to 0} \|w^h\|_{H^1(A)} \le C(A)M$ and since $(\varphi^h)_{h>0}$ is bounded in H^2 we deduce by compactness that $\varphi^h \to 0$ strongly in H^1. Thus we can find $(\tilde{w}^h)_{h>0} \subset H^2(A)$ such that

$$(44)\qquad \lim_{h\to 0} \|w^h - \tilde{w}^h\|_{L^2(A)} = 0, \ \limsup_{h\to 0} \|\tilde{w}^h\|_{H^1(A)} \le C(A)M, \ \lim_{h\to 0} h\|\tilde{w}^h\|_{H^2(A)} = 0.$$

This can be done by mollifying w^h with the mollifiers of radius $r^h \to 0$, $r^h \gg h$. From (43) we have

$$\begin{aligned} (45)\qquad \psi^h &= \begin{pmatrix} \hat{\psi}_1^h \\ \hat{\psi}_2^h \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{\varphi^h}{h} + \tilde{w}^h \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi^h \\ \partial_2 \varphi^h \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \partial_1 \varphi^h + r_2^h \\ \partial_2 \varphi^h - r_1^h \\ 0 \end{pmatrix} \\ &\quad + \begin{pmatrix} 0 \\ 0 \\ w^h - \tilde{w}^h \end{pmatrix} + \bar{\psi}^h. \end{aligned}$$

Now the claim follows from Lemma 3.3 and (44) by defining

$$\begin{aligned} \tilde{\psi}^h &= \begin{pmatrix} \hat{\psi}_1^h \\ \hat{\psi}_2^h \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} \partial_1 \varphi^h + r_2^h \\ \partial_2 \varphi^h - r_1^h \\ 0 \end{pmatrix} + hx_3 \begin{pmatrix} \partial_1 \tilde{w}^h \\ \partial_2 \tilde{w}^h \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w^h - \tilde{w}^h \end{pmatrix} + \bar{\psi}^h, \\ o^h &= -hx_3 \iota(\nabla'^2 w^h), \end{aligned}$$

after using the identity

$$\operatorname{sym} \nabla_h \begin{pmatrix} 0 \\ 0 \\ \tilde{w}^h \end{pmatrix} = \operatorname{sym} \nabla_h \begin{pmatrix} hx_3 \partial_1 \tilde{w}^h \\ hx_3 \partial_2 \tilde{w}^h \\ 0 \end{pmatrix} - hx_3 \iota(\nabla'^2 \tilde{w}).$$

The second part of corollary is direct by defining

$$\psi^h = \tilde{\psi}^h + \begin{pmatrix} 0 \\ 0 \\ \frac{\varphi^h}{h} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi^h \\ \partial_2 \varphi^h \\ 0 \end{pmatrix}.$$

□

Remark 11. The claim of Proposition 3.4 remains valid for $A = \mathcal{Y}$, the unit torus. In the similar way the part (a) of Proposition 3.4 remains valid without the assumption that $\forall h, i \int_{A_i} \psi_3^h = 0$, under the additional assumption that $\psi^h = 0$ on $\partial A \times I$. Then we can impose that $\varphi^h = 0$ on ∂A, for all $h > 0$. This immediately follows from Remark 10. Part (b) of Proposition 3.4 is true if we additionally assume that $\varphi^h = \nabla \varphi^h = 0$ on ∂A and $\tilde{\psi}^h = 0$ on $\partial A \times I$.

Lemma 3.5. *There exists a constant $C > 0$ dependent only on α, β such that for each sequence $(h_n)_{n \in \mathbb{N}}$ monotonly decreasing to 0 and $A \subset \omega$ open set is valid*

$$\left| K^-_{(h_n)_{n\in\mathbb{N}}}(M_1, A) - K^-_{(h_n)_{n\in\mathbb{N}}}(M_2, A) \right| \leq C \|M_1 - M_2\|_{L^2} \left(\|M_1\|_{L^2} + \|M_2\|_{L^2} \right), \quad \forall M_1, M_2 \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}), \tag{46}$$

and the analogous claim for $K^+_{(h_n)_{n\in\mathbb{N}}}$.

Proof. For fixed $M_1, M_2 \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ take an arbitrary $r, h > 0$ and $\psi_\alpha^{r,h_n} \in H^1(A \times I, \mathbb{R}^3)$ that satisfies for $\alpha = 1, 2$

$$\int_\Omega Q^{h_n}\left(x, \iota(M_\alpha) + \nabla_{h_n} \psi_\alpha^{r,h_n}\right) dx \leq \inf_{\substack{\psi \in H^1(A\times I, \mathbb{R}^3) \\ \|(\psi_1, \psi_2, h_n \psi_3)\|_{L^2} \leq r}} \int_{A\times I} Q^{h_n}\left(x, \iota(M_\alpha) + \nabla_{h_n} \psi\right) dx + h_n \tag{47}$$

$$\|(\psi_{\alpha,1}^{r,h_n}, \psi_{\alpha,2}^{r,h_n}, h_n \psi_{\alpha,3}^{r,h_n})\|_{L^2} \leq r.$$

We want to prove that for every $r > 0$ we have

$$\left| \int_{A\times I} Q^{h_n}\left(x, \iota(M_1) + \nabla_{h_n} \psi_1^{r,h_n}\right) dx - \int_{A\times I} Q^{h_n}\left(x, \iota(M_2) + \nabla_{h_n} \psi_2^{r,h_n}\right) dx \right| \leq C \|M_1 - M_2\|_{L^2} \left(\|M_1\|_{L^2} + \|M_2\|_{L^2} \right) + h_n. \tag{48}$$

From that we would easily obtain (46).

Let us prove (48). From (47) and (Q1), by testing with zero function, we can assume for $\alpha = 1, 2$:

$$\alpha \|M_\alpha + \mathrm{sym}\, \nabla_{h_n} \psi_\alpha^{r,h_n}\|_{L^2}^2 \leq \int_{A\times I} Q^{h_n}\left(x, \iota(M_\alpha) + \nabla_{h_n} \psi_\alpha^{r,h_n}\right) dx \leq \beta \|M_\alpha\|_{L^2}^2.$$

From this we have for $\alpha = 1, 2$

$$\|\mathrm{sym}\, \nabla_{h_n} \psi_\alpha^{r,h_n}\|_{L^2}^2 \leq 2\left(\frac{\beta}{\alpha} - 1\right) \|M_\alpha\|_{L^2}^2. \tag{49}$$

Without any loss of generality we can also assume that

$$\int_{A\times I} Q^{h_n}\left(x, \iota(M_1) + \nabla_{h_n} \psi_1^{r,h_n}\right) dx \geq \int_{A\times I} Q^{h_n}\left(x, \iota(M_2) + \nabla_{h_n} \psi_2^{r,h}\right) dx. \tag{50}$$

We have

$$\begin{aligned}
&\left|\int_{A\times I} Q^{h_n}\left(x,\iota(M_1)+\nabla_{h_n}\psi_1^{r,h}\right)dx-\int_{A\times I} Q^{h_n}\left(x,\iota(M_2)+\nabla_{h_n}\psi_2^{r,h_n}\right)dx\right|\\
&=\int_{A\times I} Q^{h_n}\left(x,\iota(M_1)+\nabla_{h_n}\psi_1^{r,h_n}\right)dx-\int_{A\times I} Q^{h_n}\left(x,\iota(M_2)+\nabla_{h_n}\psi_2^{r,h_n}\right)dx\\
&=\int_{A\times I} Q^{h_n}\left(x,\iota(M_1)+\nabla_{h_n}\psi_1^{r,h_n}\right)dx-\int_{A\times I} Q^{h_n}\left(x,\iota(M_1)+\nabla_{h_n}\psi_2^{r,h_n}\right)dx\\
&+\int_{A\times I} Q^{h_n}\left(x,\iota(M_1)+\nabla_{h_n}\psi_2^{r,h_n}\right)dx-\int_{A\times I} Q^{h}\left(x,\iota(M_2)+\nabla_{h_n}\psi_2^{r,h_n}\right)dx\\
&\le h_n+\beta\sqrt{3+2\left(\tfrac{\beta}{\alpha}-1\right)}\|M_1-M_2\|_{L^2}\left(\|M_1\|_{L^2}+\|M_2\|_{L^2}\right).
\end{aligned}$$

□

Lemma 3.6. *Suppose that for $M\in L^2(\Omega,\mathbb{R}^{2\times2}_{\rm sym})$ and $D\subset\omega$ with $C^{1,1}$ boundary and $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 we have*

$$K(M,D)=\lim_{n\to\infty}\int_{D\times I} Q^{h_n}(x,\iota(M)+\nabla_{h_n}\psi^{h_n})\,dx,$$

for some $(\psi^{h_n})_{n\in\mathbb{N}}$ such that $(\psi_1^{h_n},\psi_2^{h_n},h_n\psi_3^{h_n})\to0$ strongly in L^2. Then there exists a subsequence $(h_{n(k)})_{k\in\mathbb{N}}$ and $(\vartheta_k)_{k\in\mathbb{N}}\subset H^1(D\times I,\mathbb{R}^3)$ such that

(a) $(\vartheta_{k,1},\vartheta_{k,2},h_{n(k)}\vartheta_{k,3})\to0$ strongly in L^2,
(b) $(|\operatorname{sym}\nabla_{h_{n(k)}}\vartheta_k|^2)_{k\in\mathbb{N}}$ is equi-integrable,

$$\operatorname{sym}\nabla_{h_{n(k)}}\vartheta_k=-x_3\iota(\operatorname{sym}\nabla'^2\varphi_k)+\operatorname{sym}\nabla_{h_{n(k)}}\tilde\psi_k,$$

where $\left(|\nabla^2\varphi_k|^2\right)_{k\in\mathbb{N}}$ and $\left(|\nabla_{h_{n(k)}}\tilde\psi_k|^2\right)_{k\in\mathbb{N}}$ are equi-integrable and $\varphi_k\to0$ strongly in H^1 and $\tilde\psi_k\to0$ strongly in L^2. Also the following is valid

$$\limsup_{k\to\infty}\left(\|\varphi_k\|_{H^2(D)}+\|\nabla_{h_{n(k)}}\tilde\psi_k\|_{L^2(D)}\right)\le C(D)\left(\beta\|M\|^2_{L^2}+1\right).$$

(c) there exists $(A_k)_{k\in\mathbb{N}}$ such that for each $k\in\mathbb{N}$, $A_k\subset D\times I$ and $|A_k|\to0$ as $k\to\infty$ and

$$\|\operatorname{sym}\nabla_{h_{n(k)}}\psi^{h_{n(k)}}-\operatorname{sym}\nabla_{h_{n(k)}}\vartheta_k\|_{L^2(A_k)}\to0.$$

(d)

$$K(M,D)=\lim_{k\to\infty}\int_{D\times I} Q^{h_{n(k)}}(x,\iota(M)+\nabla_{h_{n(k)}}\vartheta_k)\,dx.$$

Moreover, one can additionally assume that for each $k\in\mathbb{N}$ we have $\vartheta_k=0$ on $\partial D\times I$ i.e. $\varphi_k=\nabla'\varphi_k=0$, on ∂D and $\tilde\psi_k=0$ on $\partial D\times I$.

Proof. We can without loss of generality assume that $\int_{D_i}\psi_3^{h_n}\,dx=0$, $\forall n\in\mathbb{N}$ and every connected component D_i of D, $i=1,\dots,m$. By comparing with the zero sequence one can additionally assume that

$$\|\operatorname{sym}\nabla_{h_n}\psi^{h_n}\|_{L^2(D)}\le\beta\|M\|^2_{L^2}+1,\quad\forall n\in\mathbb{N}.$$

From Proposition 3.4 we have that there exist $(\tilde\varphi_n)_{n\in\mathbb{N}}\subset H^2(D)$ and $(\tilde{\tilde\psi}_n)_{n\in\mathbb{N}}\subset H^1(D\times I)$ such that

$$\operatorname{sym}\nabla_{h_n}\psi^{h_n}=-x_3\iota(\operatorname{sym}\nabla'^2\tilde\varphi^{h_n})+\operatorname{sym}\nabla_{h_n}\tilde{\tilde\psi}^{h_n}+o^{h_n},\tag{51}$$

where $o^{h_n} \in L^2(D \times I, \mathbb{R}^{3\times 3})$ and the following properties hold

$$\lim_{n\to\infty} \left(\|\tilde{\varphi}^{h_n}\|_{H^1(D)} + \|\tilde{\tilde{\psi}}^{h_n}\|_{L^2(D\times I)} + \|o^{h_n}\|_{L^2(D)} \right) = 0, \tag{52}$$

$$\limsup_{n\to\infty} \left(\|\tilde{\varphi}^{h_n}\|_{H^2(D)} + \|\nabla_{h_n} \tilde{\tilde{\psi}}^{h_n}\|_{L^2(D\times I)} \right) \le C(D) \left(\beta \|M\|_{L^2}^2 + 1 \right). \tag{53}$$

Now we use Proposition A.5 and Theorem A.6 to obtain the sequence $(\varphi_k)_{k\in\mathbb{N}} \subset H^2(D)$ and $(\tilde{\psi}_k)_{k\in\mathbb{N}}$ such that $\left(|\nabla'^2 \varphi_k|^2\right)_{k\in\mathbb{N}}$ and $\left(|\nabla_{h_{n(k)}} \tilde{\psi}_k|^2\right)_{k\in\mathbb{N}}$ are equi-integrable and for A_k defined by

$$A_k \;:=\; \{\tilde{\varphi}_k \neq \varphi^{h_{n(k)}} \text{ or } \tilde{\tilde{\psi}}^{h_{n(k)}} \neq \tilde{\psi}_k\},$$

is valid $|A_k| \to 0$ as $k \to \infty$ and

$$\lim_{k\to\infty} \left(\|\varphi_k\|_{H^1(D)} + \|\tilde{\psi}_k\|_{L^2(D\times I)} \right) = 0.$$

Define ϑ_k as in Proposition 3.4 by

$$\vartheta_k := \tilde{\psi}_k + \begin{pmatrix} 0 \\ 0 \\ \frac{\varphi_k}{h_{n(k)}} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_k \\ \partial_2 \varphi_k \\ 0 \end{pmatrix}.$$

From the property (51), Remark 12, equi-integrability and the fact that $|A_k| \to 0$ as $k \to \infty$ we have the following

$$\begin{aligned} K(M, D) &= \lim_{n\to\infty} \int_{D\times I} Q^{h_n}(x, \iota(M) + \nabla_{h_n} \psi^{h_n})\, dx \\ &= \lim_{n\to\infty} \int_{D\times I} Q^{h_n}(x, \iota(M) - x_3 \nabla'^2 \tilde{\varphi}^{h_n} + \operatorname{sym} \nabla_{h_n} \tilde{\tilde{\psi}}^{h_n})\, dx \\ &\ge \lim_{k\to\infty} \int_{A_k \times I} Q^{h_{n(k)}} \left(x, \iota(M) - x_3 \nabla'^2 \varphi_k + \operatorname{sym} \nabla_{h_{n(k)}} \tilde{\psi}_k \right) dx \\ &= \lim_{k\to\infty} \int_{D\times I} Q^{h_{n(k)}}(x, \iota(M) + \nabla_{h_{n(k)}} \vartheta_k)\, dx \ge K(M, D). \end{aligned}$$

The last inequality follows from the definition of $K(M, D)$ and the fact that $(\vartheta_{k,1}, \vartheta_{k,2}, h_{n(k)} \vartheta_{k,3}) \to 0$ strongly in L^2. The last claim in (b) follows from the equi-integrability property and (53). The last claim in (d) follows from Lemma 3.7. □

Lemma 3.7. *Let $A \subset \omega$ be an open, bounded set. Let $(\vartheta_n)_{n\in\mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ be defined by*

$$\vartheta_n := \psi_n + \begin{pmatrix} 0 \\ 0 \\ \frac{\varphi_n}{h_n} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_n \\ \partial_2 \varphi_n \\ 0 \end{pmatrix},$$

where $(\psi_n)_{n\in\mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ and $(\varphi_n)_{n\in\mathbb{N}} \subset H^2(A)$. Suppose that $\left(|\nabla'^2 \varphi_n|^2\right)_{n\in\mathbb{N}}$ and $\left(|\nabla_{h_n} \psi_n|^2\right)_{n\in\mathbb{N}}$ are equi-integrable and

$$\lim_{n\to\infty} \left(\|\varphi_n\|_{H^1(A)} + \|\psi_n\|_{L^2(A\times I)} \right) = 0. \tag{54}$$

Then there exist sequences $(\tilde{\varphi}_n)_{n\in\mathbb{N}} \subset H^2(A)$, $(\tilde{\psi}_n)_{n\in\mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ and a sequence of sets $(A_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $A_n \ll A_{n+1} \ll A$ and $\cup_{n\in\mathbb{N}} A_n = A$ and

(a) $\tilde{\varphi}_n = 0$, $\nabla' \tilde{\varphi}_n = 0$ in a neighborhood of ∂A, $\tilde{\psi}_n = 0$ in a neighborhood of $\partial A \times I$.
(b) $\tilde{\psi}_n = \psi_n$ on $A_n \times I$, $\tilde{\varphi}_n = \varphi_n$ on A_n,
(c) $\|\tilde{\varphi}_n - \varphi_n\|_{H^2} \to 0$, $\|\tilde{\psi}_n - \psi_n\|_{H^1} \to 0$, $\|\nabla_{h_n} \tilde{\psi}_n - \nabla_{h_n} \psi_n\|_{L^2} \to 0$, as $n \to \infty$.

(d) for $\tilde{\vartheta}_n$ defined by

$$\tilde{\vartheta}_n := \tilde{\psi}_n + \begin{pmatrix} 0 \\ 0 \\ \frac{\tilde{\varphi}_n}{h_n} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \tilde{\varphi}_n \\ \partial_2 \tilde{\varphi}_n \\ 0 \end{pmatrix},$$

we have

$$\lim_{n\to\infty} \left\| \operatorname{sym} \nabla_{h_n} \vartheta_n - \operatorname{sym} \nabla_{h_n} \tilde{\vartheta}_n \right\|_{L^2} = 0.$$

Proof. By $\theta : [0,+\infty) \to [0,+\infty)$ denote the function:

$$\theta(\varepsilon) = \sup_{\substack{n\in\mathbb{N},\ S\subset A \\ \mathrm{meas}(S)\le\varepsilon}} \left(\|\nabla'^2 \varphi_n\|^2_{L^2(S)} + \|\nabla_{h_n} \psi_n\|^2_{L^2(S\times I)} \right).$$

By the equi-integrability property we have $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$. For fixed $k \in \mathbb{N}$ choose $A_k \ll A$ open set with Lipschitz boundary such that $\mathrm{meas}(A\backslash \bar{A}_k) \le \frac{1}{k}$ and smooth cut-off function $\eta_k \in C_0^\infty(A)$ such that $0 \le \eta_k \le 1$ and $\eta_k = 1$ in a neighborhood of $\bar{A}_k$. We can also assume that for every $k \in \mathbb{N}$, $A_k \ll A_{k+1} \ll A$ and that $\cup_{k\in\mathbb{N}} A_k = A$. Define $\tilde{\varphi}_{k,n} := \eta_k \varphi_n$, $\tilde{\psi}_{k,n} := \eta_k \psi_n$. Define $g : \mathbb{N} \times [0,+\infty) \to [0,+\infty)$ by

$$g(k,n) = \|\tilde{\varphi}_{k,n} - \varphi_n\|_{H^2} + \|\tilde{\psi}_{k,n} - \psi_n\|_{H^1} + \|\nabla_{h_n} \tilde{\psi}_{k,n} - \nabla_{h_n} \psi_n\|_{L^2}.$$

Since we have for $\alpha, \beta = 1,2$:

$$\begin{aligned} \partial_{\alpha\beta} \tilde{\varphi}_{k,n} &= \partial_{\alpha\beta} \eta_k \varphi_n + \partial_\alpha \eta_k \partial_\beta \varphi_n + \eta_k \partial_{\alpha\beta} \varphi_n, \\ \partial_\alpha \tilde{\psi}_{k,n} &= \eta_k \partial_\alpha \psi_n + \partial_\alpha \eta_k \psi_n, \\ \partial_3 \tilde{\psi}_{k,n} &= \eta_k \partial_3 \psi_n, \end{aligned}$$

it is easy to conclude that there exists $C > 0$ such that for every $k, n \in \mathbb{N}$ we have

$$g(k,n) \le C \left(\theta(\tfrac{1}{k}) + \|\eta_\varepsilon\|_{C^2} \cdot (\|\varphi_n\|_{H^1} + \|\psi_n\|_{L^2}) \right).$$

Since we also have, by the compactness, that $\varphi_n \to 0$, strongly in H^1 we conclude by the diagonalizing argument that there exists a sequence $k(n)$ monotonly incresing such that $g(k(n),n) \to 0$ as $n \to \infty$. This proves (c). (d) follows directly from (c). □

The following lemma is an easy consequence of Lemma 3.5 and Lemma 3.6.

Lemma 3.8. *The following properties are valid for every $A, A_1, A_2 \subset \omega$ open sets and $M, M_1, M_2 \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$:*

(a) there exists $K(M,A)$ such that for every $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 we have

$$K^+_{(h_n)_{n\in\mathbb{N}}}(M,A) = K^-_{(h_n)_{n\in\mathbb{N}}}(M,A) = K(M,A),$$

(b) if $A_1 \subset A_2$ we have

$$K(M 1_{A_1\times I}, A_2) = K(M, A_1),$$

(c)

$$K(M,A) = \sup_{\substack{D\in\mathcal{D} \\ D\ll A}} K(M,D),$$

(d)

$$K(M,A) \le \beta \|M\|^2_{L^2(A\times I)},$$

(e) if $A_1 \subset A_2$ we have

$$K(M,A_1) \le K(M,A_2).$$

(f) Let $(A_n)_{n\in\mathbb{N}}$ be the family of open subsets of ω such that for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$. Let $A = \cup_{n=1}^\infty A_n$. Then $\lim_{n\to\infty} K(M,A_n) = K(M,A)$,

(g) if $A_1 \cap A_2 = \emptyset$ *then we have*

$$K(M, A_1 \cup A_2) = K(M, A_1) + K(M, A_2),$$

(h)

$$|K(M_1, A) - K(M_2, A)| \leq C\|M_1 - M_2\|_{L^2}\left(\|M_1\|_{L^2} + \|M_2\|_{L^2}\right),$$

(i)

$$K(tM, A) = t^2 K(M, A), \forall t \in \mathbb{R}.$$

(j)

$$K(M_1 + M_2, A) + K(M_1 - M_2, A) \leq 2K(M_1, A) + 2K(M_2, A),$$

(k)

$$K(M, A) \geq \alpha\|M\|^2_{L^2(A\times I)}.$$

Proof. Using Lemma 3.6 it is easy to see that for $A \subset \omega$ open and an arbitrary $D \in \mathcal{D}$, $D \subset A$ we have

$$K^+_{(h_n)_{n\in\mathbb{N}}}(M1_{D\times I}, A) = K^-_{(h_n)_{n\in\mathbb{N}}}(M1_{D\times I}, A) = K(M, D), \tag{55}$$

for an arbitrary $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0. Namely the inequality $K^-_{(h_n)_{n\in\mathbb{N}}}(M1_D, A) \geq K(M, D)$, follows immediately from the definition in Remark 4. To prove the inequality $K^+_{(h_n)_{n\in\mathbb{N}}}(M1_D, A) \leq K(M, D)$ we first take the subsequence $(h_{n(k)})_{k\in\mathbb{N}}$ such that

$$K^+_{(h_{n(k)})_{k\in\mathbb{N}}}(M1_{D\times I}, A) = K^-_{(h_{n(k)})_{k\in\mathbb{N}}}(M1_{D\times I}, A). \tag{56}$$

Then we take, using Lemma 3.6 a further subsequence, still denoted by $(h_{n(k)})_{k\in\mathbb{N}}$, and $(\chi_k)_{k\in\mathbb{N}} \subset H^1(D \times I, \mathbb{R}^3)$ such that $(\chi_{k,1}, \chi_{k,2}, h_{n(k)}\chi_{k,3}) \to 0$, strongly in L^2, for each $k \in \mathbb{N}$, $\chi_k = 0$ on $\partial D \times I$ and

$$K(M, D) = \lim_{k\to\infty} \int_{D\times I} Q^{h_{n(k)}}(x, \iota(M) + \nabla_{h_{n(k)}}\chi_k)\, dx.$$

By extending $\chi_k = 0$ on $(A\backslash D) \times I$ we obtain

$$\begin{aligned} K(M, D) &= \lim_{k\to\infty} \int_{A\times I} Q^{h_{n(k)}}(x, \iota(M1_{D\times I}) + \nabla_{h_{n(k)}}\chi_k)\, dx \\ &\geq K^-_{(h_{n(k)})_{k\in\mathbb{N}}}(M1_{D\times I}, A) = K^+_{(h_n)_{n\in\mathbb{N}}}(M1_{D\times I}, A), \end{aligned}$$

where we have used (56). (a) and (b) follows from (55) and Lemma 3.5 by an approximation argument i.e. by exhausting A with the sets in $\mathcal{D}$. It is easy to notice from the definition in Remark 4 that $K(M, D) \leq K(M, A)$, for $D \in \mathcal{D}$, $D \subset A$. (c) then easily follows from (b) and Lemma 3.5. From the definition in Remark 4, by taking the null sequence, it is easy to see that for every $D \in \mathcal{D}$ we have $K(M, D) \leq \beta\|M\|^2_{L^2(D)}$. (d) now follows from (c). (e) easily follows from (c). (f) is again the direct consequence of (b) and Lemma 3.5. To prove (g) first choose $D_1, D_2 \in \mathcal{D}$, $D_1 \ll A_1$ and $D_2 \ll A_2$. We have that $D_1 \cap D_2 = \emptyset$. From the definition in Remark 4 it is easy to see that

$$K(M, D_1 \cup D_2) = K(M, D_1) + K(M, D_2).$$

(g) now follows from (f). To prove (h) notice that from Lemma 3.5 we can conclude that there exists $C > 0$ dependent only on α, β such that for each $D \in \mathcal{D}$ is valid

$$\begin{aligned} |K(M_1, D) - K(M_2, D)| &\leq C\|M_1 - M_2\|_{L^2}\left(\|M_1\|_{L^2} + \|M_2\|_{L^2}\right), \\ &\qquad \forall M_1, M_2 \in L^2(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}}). \end{aligned} \tag{57}$$

(h) follows from (f) and (57). To prove (i) we can take $D \in \mathcal{D}$ and $M \in C_c^\infty(\omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ and use density argument and (f). Define by

$$K_n\left(M, D, B(r)\right) = \min_{\substack{\psi \in H^1(D\times I,\mathbb{R}^3)\\ \|(\psi_1,\psi_2,h_n\psi_3)\|_{L^2}\le r}} \int_{D\times I} Q^{h_n}\left(x, \iota(M) + \nabla_{h_n}\psi\right) dx. \tag{58}$$

The minimum in the above expression exists by the direct methods of the calculus of variation. Notice that

$$K(M, D) = \sup_{r>0} \lim_{n\to\infty} K_n\left(M, D, B(r)\right),$$

where supremum can be replaced by limit since $K_n\left(M, D, B(r)\right)$ is monotonly increasing in r. Since every Q^{h_n} is quadratic we have

$$t^2 K_n\left(M, D, B(r)\right) = K_n\left(tM, D, B(|t|r)\right).$$

From this identity it follows $K(tM, D) = t^2 K(M, D)$. By approximation we obtain (i). To prove (j) take $M_1, M_2 \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, $D \in \mathcal{D}$ and for $\alpha = 1, 2$, $\psi^{\alpha,r,n} \in H^1(D \times I, \mathbb{R}^3)$ such that

$$K_n\left(M_\alpha, D, B(r)\right) = \int_{D\times I} Q^{h_n}\left(x, \iota(M_\alpha) + \nabla_{h_n}\psi^{\alpha,r,n}\right) dx.$$

and $\|(\psi_1^{\alpha,r,n}, \psi_2^{\alpha,r,n}, h_n\psi_3^{\alpha,r,n})\|_{L^2} \le r$. Notice that:

$$\begin{aligned}
&K_n\left(M_1 + M_2, D, B(2r)\right) + K_n\left(M_1 - M_2, D, B(2r)\right)\\
&\quad\le \int_{D\times I} Q^{h_n}\left(x, \iota(M_1 + M_2) + \nabla_{h_n}\psi^{1,r,n} + \nabla_{h_n}\psi^{2,r,n}\right) dx\\
&\qquad+ \int_{D\times I} Q^{h_n}\left(x, \iota(M_1 - M_2) + \nabla_{h_n}\psi^{1,r,n} - \nabla_{h_n}\psi^{2,r,n}\right) dx\\
&\quad= 2\int_{D\times I} Q^{h_n}\left(x, \iota(M_1) + \nabla_{h_n}\psi^{1,r,n}\right) dx + 2\int_{D\times I} Q^{h_n}\left(x, \iota(M_2) + \nabla_{h_n}\psi^{2,r,n}\right)\\
&\quad= K_n(M_1, D, B(r)) + K_n(M_2, D, B(r)),
\end{aligned}$$

where we have used (e) of Proposition A.7. By letting $n \to \infty$ and then $r \to 0$ we obtain that

$$K(M_1 + M_2, D) + K(M_1 - M_2, D) \le 2K(M_1, D) + 2K(M_2, D).$$

(j) follows by density and (f). To prove (k) take $M \in C^1(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, such that $M = 0$ in a neighborhood of Γ, $D \in \mathcal{D}$ and $(\chi_n)_{n\in\mathbb{N}} \subset H^1(D\times I, \mathbb{R}^3)$ such that $(\chi_{n,1}, \chi_{n,2}, h_n\chi_{n,3}) \to 0$, strongly in L^2 and such that

$$K(M, D) = \lim_{n\to\infty} \int_{D\times I} Q^{h_n}(x, \iota(M) + \nabla_{h_n}\chi_n)\, dx.$$

From (Q1) we conclude

$$\begin{aligned}
\int_{D\times I} Q^{h_n}(x, \iota(M) + \nabla_{h_n}\chi_n)\, dx \;&\ge\; \alpha \int_{D\times I} \left|M + \mathrm{sym}\,\nabla'(\chi_{n,1}\,,\,\chi_{n,2})\right|^2 dx\\
&\ge\; \alpha\|M\|^2_{L^2} - \alpha \int_{D\times I} \mathrm{div}\, M \cdot (\chi_{n,1}\,,\,\chi_{n,2})\, dx.
\end{aligned}$$

By letting $n \to \infty$ and using the fact that $(\chi_{n,1}\,,\,\chi_{n,2}) \to 0$ strongly in L^2 we have that $K(M, D) \ge \alpha\|M\|^2_{L^2}$. (k) follows from (f) and (h). □

Lemma 3.9. *(a) For arbitrary $M \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$ and $A_1, A_2, A_3 \subset \omega$ open, such that $A_1 \subset A_2 \cup A_3$ we have*

$$K(M, A_1) \le K(M, A_2) + K(M, A_3).$$

(b) *For arbitrary* $M \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\rm sym})$ *and* $A_1, A_2, A_3 \subset \omega$ *open, such that* $A_1 \supset A_2 \cup A_3$, $A_2 \cap A_3 = \emptyset$ *we have*

$$K(M, A_1) \geq K(M, A_2) + K(M, A_3).$$

Proof. By using Lemma A.11 and (c) and (e) in Lemma 3.8 it is enough to prove that for arbitrary $D_1, D_2, D_3 \in \mathcal{D}$ such that $D_1 \subset D_2 \cup D_3$ we have

$$K(M, D_1) \leq K(M, D_2) + K(M, D_3).$$

Take $D_3' \in \mathcal{D}$ such that $D_3' \ll D_3 \backslash \bar{D}_2$. From (f) of Lemma 3.8 we have

$$K(M, D_2 \cup D_3') = K(M, D_2) + K(M, D_3').$$

From Lemma 3.5 and Lemma 3.8 we have

$$\begin{aligned}
K(M, D_1) &= K(M 1_{D_1 \times I}, \omega) \\
&\leq K(M 1_{(D_2 \cup D_3') \times I}, \omega) + C\|M\|_{L^2\left((D_3 \backslash (D_2 \cup D_3') \times I\right)}\|M\|_{L^2} \\
&= K(M, D_2 \cup D_3') + C\|M\|_{L^2\left((D_3 \backslash (D_2 \cup D_3'))\right)}\|M\|_{L^2} \\
&= K(M, D_2) + K(M, D_3') + C\|M\|_{L^2\left((D_3 \backslash (D_2 \cup D_3') \times I\right)}\|M\|_{L^2} \\
&\leq K(M, D_2) + K(M, D_3) + C\|M\|_{L^2\left((D_3 \backslash (D_2 \cup D_3') \times I\right)}\|M\|_{L^2}.
\end{aligned}$$

The claim follows by the arbitrariness of D_3'. To prove (b) it is enough to check that for arbitrary $D_1, D_2, D_3 \in \mathcal{D}$ such that $D_1 \supset D_2 \cup D_3$, $\bar{D}_2 \cap \bar{D}_3 = \emptyset$ we have

$$K(M, D_1) \geq K(M, D_2) + K(M, D_3).$$

This easily follows from the definition in Remark 4. □

Now we are ready to improve Lemma 3.6 for arbitrary $A \subset \omega$ open.

Lemma 3.10. *Take* $M \in L^2(\Omega, \mathbb{R}^{2\times 2}_{\rm sym})$ *and* $A \subset \omega$ *open and a sequence* $(h_n)_{n\in\mathbb{N}}$ *monotonly decreasing to* 0. *Then there exists a subsequence* $(h_{n(k)})_{k\in\mathbb{N}}$ *and* $(\vartheta_k)_{k\in\mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ *such that*

(a) $(\vartheta_{k,1}, \vartheta_{k,2}, h_{n(k)}\vartheta_{k,3}) \to 0$ *strongly in* L^2,

(b) $(|\operatorname{sym} \nabla_{h_{n(k)}} \vartheta_k|^2)_{k\in\mathbb{N}}$ *is equi-integrable,*

$$\operatorname{sym} \nabla_{h_{n(k)}} \vartheta_k = -x_3 \iota(\operatorname{sym} \nabla'^2 \varphi_k) + \operatorname{sym} \nabla_{h_{n(k)}} \tilde{\psi}_k,$$

where $\left(|\nabla^2 \varphi_k|^2\right)_{k\in\mathbb{N}}$ *and* $\left(|\nabla_{h_{n(k)}} \tilde{\psi}_k|^2\right)_{k\in\mathbb{N}}$ *are equi-integrable and* $\varphi_k \to 0$ *strongly in* H^1 *and* $\tilde{\psi}_k \to 0$ *strongly in* L^2. *Also the following is valid*

$$\limsup_{k\to\infty} \left(\|\varphi_k\|_{H^2(A)} + \|\nabla_{h_{n(k)}} \tilde{\psi}_k\|_{L^2(A)}\right) \leq C\left(\beta\|M\|^2_{L^2} + 1\right),$$

where C *is independent of the domain* A *and for each* $k \in \mathbb{N}$ *we have* $\vartheta_k = 0$ *in a neighborhood of* $\partial A \times I$ *i.e.* $\varphi_k = \nabla' \varphi_k = 0$, *in a neighborhood of* ∂A *and* $\tilde{\psi}_k = 0$ *in a neighborhood of* $\partial A \times I$.

(c)

$$K(M, A) = \lim_{k\to\infty} \int_{A\times I} Q^{h_{n(k)}}(x, \iota(M) + \nabla_{h_{n(k)}} \vartheta_k)\, dx.$$

Proof. Take $r>0$ such that $B(r)\supset\omega$. Extend Q^h on $(B(r)\backslash\omega)\times I$ by e.g. $Q^h(x,G)=\beta|\operatorname{sym}G|^2$, for all $x\in(B(r)\backslash\omega)\times I$. Apply Lemma 3.6 to $\tilde{M}=M1_A$ and $D=B(r)$ to obtain the sequences $(\tilde{\vartheta}_k)_{k\in\mathbb{N}}\subset H^1(B(r)\times I,\mathbb{R}^3)$, $(\tilde{\varphi}_k)_{k\in N}\subset H^2(B(r))$, $(\tilde{\tilde{\psi}}_k)_{k\in\mathbb{N}}\subset H^1(B(r)\times I,\mathbb{R}^3)$ that satisfy (a), (b), (d) of Lemma 3.6. In the same way as in Lemma 3.7 for each $\varepsilon>0$ we choose $A_\varepsilon\ll A$ with Lipschitz boundary such that $\operatorname{meas}(A\backslash A_\varepsilon)<\varepsilon$ and a cut off function $\eta_\epsilon\in C_0^\infty(A)$ which is 1 on $\bar{A}_\varepsilon$. Again by using the diagonal procedure, one obtains a sequence $(\varphi_k)_{k\in\mathbb{N}}\subset H^2(A)$ and $(\tilde{\psi}_k)_{k\in\mathbb{N}}\subset H^1(A\times I,\mathbb{R}^3)$ such that for each $k\in\mathbb{N}$, $\varphi_k=\nabla\varphi_k=0$ in a neighborhood of $B(r)\backslash A$ i.e. $\tilde{\psi}_k=0$ in a neighborhood of $(B(r)\backslash A)\times I$ and

$$\lim_{k\in\mathbb{N}}\left(\|\tilde{\varphi}_k-\varphi_k\|_{H^2(A)}+\|\tilde{\tilde{\psi}}_k-\tilde{\psi}_k\|_{H^1(A\times I)}\right)=0.$$

Define again

$$\vartheta_k:=\tilde{\psi}_k+\begin{pmatrix}0\\0\\\frac{\varphi_k}{h_{n(k)}}\end{pmatrix}-x_3\begin{pmatrix}\partial_1\varphi_k\\\partial_2\varphi_k\\0\end{pmatrix}.$$

It is easy to see from (b) of Lemma 3.8 that

$$\begin{aligned}K(M,A)&=K(M1_{A\times I},B(r))=\lim_{k\to\infty}\int_{B(r)\times I}Q^{h_{n(k)}}(x,\iota(M1_{A\times I})+\nabla_{h_{n(k)}}\tilde{\vartheta}_k)\,dx\\&\geq\lim_{k\to\infty}\int_{A\times I}Q^{h_{n(k)}}(x,\iota(M)+\nabla_{h_{n(k)}}\vartheta_k)\,dx\geq K(M,A).\end{aligned}$$

From this we have the claim. □

Proof of Proposition 2.9. The proof follows the standard steps in Γ-convergence theory. Notice that

$$\|M\|^2_{L^2(\Omega)}=\|M_1\|^2_{L^2(\omega)}+\tfrac{1}{12}\|M_2\|^2_{L^2(\omega)}. \tag{59}$$

Step 1. Existence of Q.
From Theorem A.12 and Lemma 3.9 as well as the properties (d) and (f) from Lemma 3.8 we conclude from Radon-Nykodim theorem that for an arbitrary $M\in\mathcal{S}_{vK}(\omega)$ there exists $Q_M\in L^1(\omega)$, a positive function, such that

$$K(M,A)=\int_A Q_M(x')\,dx',\forall A\subset\omega\text{ open.} \tag{60}$$

Take a countable dense subset $\mathcal{M}$ of $\mathbb{R}^{2\times2}_{\mathrm{sym}}$ and define

$$E:=\{x'\in\omega:\,x'\text{ is a Lebesgue point for }Q_{M_1+x_3M_2}\text{ for every }M_1,M_2\in\mathcal{M}\}.$$

Notice that $\operatorname{meas}(\omega\backslash E)=0$. Define also

$$Q(x',M_1,M_2)=Q_{M_1+x_3M_2}(x')=\lim_{r\to0}\tfrac{1}{|B(x',r)|}K\left(M_1+x_3M_2,B(x',r)\right), \tag{61}$$
$$\text{for }M_1,M_2\in\mathcal{M}\text{ and }x'\in E.$$

Notice that from the property (h) in Lemma 3.8 we have

$$\begin{aligned}&|Q(x',M_1,M_2)-Q(x',M_1',M_2')|\leq\\&\quad C\left(|M_1-M_1'|+|M_2-M_2'|\right)(|M_1+M_1'|+|M_2+M_2'|),\\&\qquad\text{for all }M_1,M_1',M_2,M_2'\in\mathcal{M},\,x'\in E.\end{aligned}$$

Thus we can extend $Q(\cdot,\cdot)$ by continuity on $E\times\mathbb{R}^{2\times 2}_{\rm sym}\times\mathbb{R}^{2\times 2}_{\rm sym}$. Extend it by 0 on $\omega\times\mathbb{R}^{2\times 2}_{\rm sym}\times\mathbb{R}^{2\times 2}_{\rm sym}$. Notice that such defined Q satisfies

$$\begin{aligned}|Q(x',M_1,M_2)-Q(x',M_1',M_2')| &\leq \\ C\left(|M_1-M_1'|+|M_2-M_2'|\right)&(|M_1+M_1'|+|M_2+M_2'|), \\ \text{for all } M_1,M_1',M_2,M_2' &\in \mathbb{R}^{2\times 2}_{\rm sym},\ x'\in E,\end{aligned} \tag{62}$$

$$|Q(x',M_1,M_2)|\leq \beta\left(|M_1|^2+|M_2|^2\right),\ \text{for all } M_1,M_2\in\mathbb{R}^{2\times 2}_{\rm sym},\ x'\in\omega. \tag{63}$$

Also, from the property (h) in Lemma 3.8 and (62) we conclude that

$$Q(x',M_1,M_2)=\lim_{r\to 0}\tfrac{1}{|B(x',r)|}K\left(M_1+x_3M_2,B(x',r)\right),\forall M_1,M_2\in\mathbb{R}^{2\times 2}_{\rm sym} \text{ and } x'\in E. \tag{64}$$

By approximating $M\in\mathcal{S}_{vK}(\omega)$ by piecewise constant maps with values in the set $\{M_1+x_3M_2 : M_1,M_2\in\mathcal{M}\}$ we conclude from (b), (g), (h) of Lemma 3.8 as well as the properties (62) and (63) that

$$K(M,\omega)=\int_\omega Q(x',M_1(x'),M_2(x'))\,dx,\forall M\in\mathcal{S}_{vK}(\omega). \tag{65}$$

By using (b) of Lemma 3.8 as well as the fact that $Q(x',0,0)=0\ \forall x'\in\omega$ we conclude (11).

Step 2. Quadraticity and coercivity of Q.

To prove that Q is quadratic form we use (64), Proposition A.7 and (i), (j) of Lemma 3.8 as well as the density argument. To prove coercivity we use (59), (64) and property (k) of Lemma 3.8. □

Proof of Lemma 2.10. By Remark 6 and Lemma 2.8 it is enough to see that every sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 has subsequence such $(h_{n(k)})_{k\in\mathbb{N}}$ such that

$$K(M,D)=K^-_{(h_{n(k)})_{k\in\mathbb{N}}}(M,A)=K^+_{(h_{n(k)})_{k\in\mathbb{N}}}(M,A),\ \forall n\in\mathbb{N},M\in\mathcal{S}_{vK}(\omega).$$

This follows from Lemma 2.6, Remark 6 and Proposition 2.9 i.e. (11) and (61). □

3.3. **Proof of Theorem 2.11.** The proof of the following proposition is given in [NV, Proposition 3.1]

Proposition 3.11. *Let $y\in H^1(\Omega,\mathbb{R}^3)$ and $h>0$. There exist $(\bar R,u,v)\in \mathrm{SO}(3)\times H^1(\omega,\mathbb{R}^2)\times H^2_{loc}(\omega)$ and correctors $w\in H^1(\omega)$, $\phi\in H^1(\Omega,\mathbb{R}^3)$ with*

$$\int_\omega w=0,\qquad \int_I \psi(\hat x,x_3)\,dx_3=0\quad \textit{for almost every } x'\in\omega$$

such that

$$\bar R^t\left(y-\fint_\Omega y\,\mathrm{d}x\right)=\begin{pmatrix} x'\\ hx_3\end{pmatrix}+\begin{pmatrix} h^2u\\ h(v+hw)\end{pmatrix}-h^2x_3\begin{pmatrix}\hat\nabla v\\ 0\end{pmatrix}+h^2\psi \tag{66}$$

and

$$\|u\|^2_{H^1(\omega)}+\|v\|^2_{H^1(\omega)}+\|w\|^2_{H^1(\omega)}+\frac{1}{h^2}\|\psi\|^2_{L^2(\Omega)}\lesssim e_h(y)+e_h(y)^2. \tag{67}$$

Here $\lesssim$ means $\leq$ up to a multiplicative constant that only depends on ω. In addition, for all $D\ll\omega$ compactly contained in ω we have

$$\|\nabla^2 v\|^2_{L^2(D)}+\|\nabla_h\psi\|^2_{L^2(D\times I)}\lesssim_D e_h(y)+e_h(y)^2 \tag{68}$$

where $\lesssim_D$ means $\leq$ up to a multiplicative constant that only depends on D next to ω.

If the boundary of ω is of class $C^{1,1}$, then $(u,v) \in \mathcal{A}(\omega)$ and (68) holds for D replaced by ω.

The following lemma is an easy consequence of Taylor expansion.

Lemma 3.12. *Let $G^h, K^h \in L^2(\Omega, \mathbb{R}^{3\times 3})$ be such that*

$$K^h \text{ is skew-symmetric and} \tag{69}$$

$$\limsup_{h\to 0} \left(\|G^h\|_{L^2} + \|K^h\|_{L^4} \right) < \infty, \tag{70}$$

Consider

$$E^h := \frac{\sqrt{(I + hK^h + h^2G^h)^t(I + hK^h + h^2G^h)} - I}{h^2}.$$

Then there exists a sequence $(\chi^h)_{h>0}$ such that, $\chi^h : \Omega \to \{0,1\}$, $\chi^h \to 1$ boundedly in measure and

$$\lim_{h\to 0} \left\| \chi^h \left(E^h - \left(\operatorname{sym} G^h - \frac{1}{2}(K^h)^2 \right) \right) \right\|_{L^2} = 0.$$

Proof. Notice that the following claim is the direct consequence of Taylor expansion: There exists $\delta > 0$ and a monotone increasing function $\eta : (0,\delta) \to (0,\infty)$ such that $\lim_{r\to 0} \eta(r) = 0$ and

$$\left| \sqrt{(I+A)^t(I+A)} - \left(I + \operatorname{sym} A + \tfrac{1}{2}A^tA\right) \right| \leq \eta(|A|) \left(\operatorname{sym} A + \tfrac{1}{2}A^tA\right) \quad \forall A \in \mathbb{R}^{3\times 3}, |A| < \delta. \tag{71}$$

Now we use the truncation argument. Namely let χ^h be the characteristic function of the set S^h, where

$$S^h = \{x \in \Omega : |K^h| \leq \tfrac{1}{\sqrt{h}},\ |G^h| \leq \tfrac{1}{h}\}.$$

The claim follows after putting $A = hK^h + h^2G^h$ into the expression (71), dividing by h^2 and letting $h \to 0$. ☐

We state one simple linearization lemma, which is already given in [NV].

Lemma 3.13 (linearization)**.** *Let $\{\widetilde{E}^h\}_{h>0} \subset L^2(\Omega, \mathbb{M}^3)$ satisfy*

$$\limsup_{h\to 0} \|\widetilde{E}^h\|_{L^2} < \infty \qquad \text{and} \qquad \lim_{h\to 0} h^2 \|\widetilde{E}^h\|_{L^\infty} = 0. \tag{72}$$

Then

$$\lim_{h\to 0} \left| \frac{1}{h^4} \int_\Omega W^h(x, I + h^2\widetilde{E}^h(x))\, dx - \int_\Omega Q^h(x, \widetilde{E}^h(x))\, dx \right| = 0.$$

Proof. We have

$$\begin{aligned}
&\left| \frac{1}{h^4} \int_\Omega W^h(x, I + h^2\widetilde{E}^h(x))\, dx - \int_\Omega Q^h(x, \widetilde{E}^h(x))\, dx \right| \\
&\overset{\triangle\text{-ineq.}}{\leq} \frac{1}{h^4} \int_\Omega \left| W^h(x, I + h^2\widetilde{E}^h(x)) - h^4 Q^h(x, \widetilde{E}^h(x)) \right| dx \\
&\overset{(4)}{\leq} \frac{1}{h^4} \int_\Omega |h^2\widetilde{E}^h(x)|^2\, r(|h^2\widetilde{E}^h(x)|)\, dx \\
&\leq r(h^2\|\widetilde{E}^h\|_{L^\infty}) \int_\Omega |\widetilde{E}^h(x)|^2\, dx,
\end{aligned}$$

where in the last line we used that $r(\cdot)$ is monotonically increasing. By appealing to (72) and $\lim_{\delta\to 0} r(\delta)=0$, we get $\lim_{h\to 0} r(h^2\|\widetilde{E}^h\|_{L^\infty})\int_\Omega |\widetilde{E}^h(x)|^2\,dx = 0$ and the proof is complete. □

Proof of Theorem 2.11. First we assume that ω is of class C^2. Without loss of generality we assume that

$$\liminf_{h\to 0} I^h(y^h) = \limsup_{h\to 0} I^h(y^h) < \infty. \tag{73}$$

Due to the non-degeneracy of W (see (W2)) we have $\limsup_{h\to 0} e^h(y^h) < \infty$. Hence, Proposition 3.11 is applicable, and we easily deduce the part (i), by taking in the expression (66) the integral over the interval I. From the estimate (67) and (68) we conclude that

$$v^h \rightharpoonup v \text{ weakly in } H^2, u^h \rightharpoonup u \text{ weakly in } H^1.$$

Notice that from the expression (66) we have that

$$\nabla_h y^h = I + hK^h + h^2 G^h,$$

where

$$\begin{aligned} K^h &= \begin{pmatrix} 0 & 0 & -\partial_1 v^h \\ 0 & 0 & -\partial_2 v^h \\ \partial_1 v^h & \partial_2 v^h & 0 \end{pmatrix}, \\ G^h &= \iota(-x_3\nabla'^2 v^h + \nabla' u^h) + \nabla_h \psi^h + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 w^h & \partial_2 w^h & 0 \end{pmatrix}. \end{aligned}$$

Notice that K^h, G^h satisfy the hypothesis of Lemma 3.12. Define also

$$E^h = \frac{\sqrt{(\nabla_h y^h)^t \nabla_h y^h} - I}{h^2}.$$

From the inequality valid for any $F\in\mathbb{R}^{3\times 3}$, $|\sqrt{F^tF}-I| \le \operatorname{dist}(F,\mathrm{SO}(3))$, where the equality holds for $F\in\mathbb{R}^{3\times 3}$ such that $\det F > 0$, we conclude that $\limsup_{h\to 0}\|E^h\|_{L^2} < \infty$. We truncate the peaks of E^h and the set of points where $\det \nabla_h y^h$ is negative. Therefore, consider the good set $C^h := \{\, x\in\Omega \,:\, |E^h(x)| \le h^{-1},\ \det\nabla_h y^h(x) > 0 \,\}$ and let χ_1^h denote the indicator function associated with C^h. It is easy to see that $\chi_1^h \to 1$ boundedly in measure. By applying Lemma 3.12 we know that there exists a sequence $(\chi_2^h)_{h>0}$ such that for all h, $\chi_2^h : \Omega \to \{0,1\}$, $\chi_2^h \to 1$ boundedly in measure and

$$\lim_{h\to 0}\left\| \chi_2^h \left(E^h - \left(\operatorname{sym} G^h - \frac{1}{2}(K^h)^2 \right)\right)\right\|_{L^2} = 0. \tag{74}$$

In the same way as in Proposition 3.4 we take $(\tilde{w}^h)_{h>0}$ such that

$$\lim_{h\to 0}\|w^h - \tilde{w}^h\|_{L^2} = 0,\ \limsup_{h\to 0}\|\tilde{w}^h\|_{H^1} \le C(\omega)\limsup_{h\to 0}\|w^h\|_{H^1},\ \lim_{h\to 0} h\|\tilde{w}^h\|_{H^2} = 0. \tag{75}$$

Also we take the sequence $(\tilde{v}^h)_{h>0} \subset H^3(\omega)$ such that

$$\|\tilde{v}^h - v\|_{H^2} \to 0, \quad h\|\tilde{v}^h\|_{C^2} \to 0. \tag{76}$$

This can be done by taking a smooth sequence converging to v and reparametrizing it. Notice that

(77)

$$\begin{aligned} \operatorname{sym} G^h - \tfrac{1}{2}(K^h)^2 &= \iota\,(M_1 + x_3 M_2) - x_3\iota\left(\nabla'^2(v^h - v)\right) \\ &\quad + \operatorname{sym}\nabla_h\tilde{\psi}^h + o^h, \end{aligned}$$

where

$$\begin{aligned} M_1 &= \operatorname{sym}\nabla' u - \tfrac{1}{2}\nabla' v \otimes \nabla' v, \\ M_2 &= -\nabla'^2 v, \\ \tilde{\psi}^h &= \psi^h + \begin{pmatrix} u_1^h - u_1 \\ u_2^h - u_2 \\ w^h - \tilde{w}^h \end{pmatrix} + hx_3 \begin{pmatrix} \partial_1 \tilde{w}^h \\ \partial_2 \tilde{w}^h \\ -\frac{1}{2}\left(|\partial_1 \tilde{v}^h|^2 + |\partial_2 \tilde{v}^h|^2\right) \end{pmatrix}, \\ o^h &= -\tfrac{1}{2}\iota\Big(\nabla' v^h \otimes \nabla' v^h - \nabla' v \otimes \nabla' v\Big) - hx_3 \iota(\nabla'^2 \tilde{w}^h) \\ &\quad + \tfrac{1}{2}\operatorname{sym}\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ hx_3 \nabla'\left(|\partial_1 \tilde{v}^h|^2 + |\partial_2 \tilde{v}^h|^2\right) & |\partial_1 \tilde{v}^h|^2 + |\partial_2 \tilde{v}^h|^2 - |\partial_1 v^h|^2 - |\partial_2 v^h|^2 \end{pmatrix}. \end{aligned}$$

Notice that from (67), (68) as well as (75) and (76) and Sobolev embedding we conclude that

$$\lim_{h\to 0} \|o^h\|_{L^2} = \lim_{h \to 0} \|\tilde{\psi}^h\|_{L^2} = \lim_{h\to 0} \|v^h - v\|_{H^1} = 0, \tag{78}$$

$$\limsup_{h\to 0} \|\tilde{v}^h - v^h\|_{H^2} < \infty, \quad \limsup_{h \to 0} \|\nabla_h \tilde{\psi}^h\|_{L^2} < \infty. \tag{79}$$

By using Proposition A.5 and Theorem A.6 we find a subsequence $(h_{n(k)})$ and sequences $(\varphi_k)_{k\in\mathbb{N}} \subset H^2(\omega)$, $(\tilde{\tilde{\psi}}_k)_{k \in \mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)$, $(\chi_{3,k})_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \|\varphi_k\|_{H^1} = \lim_{k\to\infty} \|\tilde{\tilde{\psi}}_k\|_{L^2} = 0, \tag{80}$$

$$\left(|\nabla'^2 \varphi_k|^2\right)_{k\in\mathbb{N}}, \ \left(|\nabla_{h_{n(k)}} \tilde{\tilde{\psi}}_k|^2\right)_{k\in\mathbb{N}} \text{ are equi-integrable,} \tag{81}$$

$$\chi_{3,k} : \Omega \to \{0,1\}, \forall k, \ \chi_{3,k} \to 1 \text{ boundedly in measure,} \tag{82}$$

$$\{x = (x', x_3) \in \Omega : \varphi_k(x') \neq (v^{h_{n(k)}} - v)(x') \text{ or } \tilde{\tilde{\psi}}_k(x) \neq \tilde{\psi}^{h_{n(k)}}(x)\} = \{\chi_{3,k} = 0\}. \tag{83}$$

Define

$$\vartheta_k = \tilde{\tilde{\psi}}_k + \begin{pmatrix} 0 \\ 0 \\ \frac{\varphi_k}{h_{n(k)}} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 \varphi_n \\ \partial_2 \varphi_k \\ 0 \end{pmatrix},$$

and notice that

$$\operatorname{sym}\nabla_{h(n(k))}\vartheta_k = -x_3 \iota(\nabla'^2 \varphi_k) + \operatorname{sym}\nabla_{h(n(k))} \tilde{\tilde{\psi}}_k.$$

From this and (81) and (83) we conclude that the family $(|\operatorname{sym}\nabla_{h(n(k))}\vartheta_k|^2)_{k\in\mathbb{N}}$ is equi-integrable and

$$\{x \in \Omega : \ \operatorname{sym}\nabla_{h_{n(k)}}\vartheta_k \neq -x_3 \nabla'^2 (v^{h_{n(k)}} - v) + \nabla'_{h_{n(k)}} \tilde{\psi}^{h_{n(k)}}\} = \{\chi_{3,k} = 0\}, \tag{84}$$

up to a set of measure zero (see Remark 12). From (80) we conclude that $(\vartheta_{k,1}, \vartheta_{k,2}, h_{n(k)}\vartheta_{k,3}) \to 0$ strongly in L^2. Now we are ready to prove the lower bound. The key idea is that the replacement by equi-integrable family enables us to establish the lower bound on the whole set. Denote by

$$\chi_k = \chi_1^{h_{n(k)}} \chi_2^{h_{n(k)}} \chi_{3,k}, \quad \widetilde{E}_k = \chi_k E^{h_{n(k)}}.$$

By appealing to the polar factorization for matrices with non-negative determinant, there exists a matrix field $R^h : C^h \to \mathrm{SO}(3)$ such that

$$\forall x \in C^h \ : \ \nabla_h y^h(x) = R^h(x)\sqrt{(\nabla_h y^h(x))^t \nabla_h y^h(x)}.$$

Hence, by frame-indifference (see (W1)), non-negativity (see (W2)) and assumption (W3) we have

$$\begin{aligned} W^{h_{n(k)}}(x,\nabla_{h_{n(k)}}y^{h_{n(k)}}(x)) &\geq \chi^{h_{n(k)}}(x)\chi_1^{h_{n(k)}}(x)\chi_{2,k}(x)W(x,\nabla_h y^h(x)) \\ &= W^{h_{n(k)}}(x, I+h_{n(k)}^2\widetilde{E}_k(x)). \end{aligned}$$

Thus,

$$\begin{aligned} I^{h_{n(k)}}(y^{h_{n(k)}}) &= \frac{1}{h_{n(k)}^4}\int_\Omega W^{h_{n(k)}}(x,\nabla_{h_{n(k)}}y^{h_{n(k)}}(x))\,dx \\ &\geq \frac{1}{h_{n(k)}^4}\int_\Omega W^{h_{n(k)}}(x, I+h_{n(k)}^2\widetilde{E}_k(x)\,)\,dx. \end{aligned}$$

Due to the truncation we have $\lim_{k\to\infty} {h_{n(k)}}^2\|\widetilde{E}_k\|_{L^\infty}=0$. Hence, using (5), Lemma 3.13 and the equi-integrability of $(|\operatorname{sym}\nabla_{h(n(k))}\vartheta_k|^2)_{k\in\mathbb{N}}$ with (Q1) as well as (73), (74), (77), (78), (84) and Proposition 2.9 we get

$$\begin{aligned} \liminf_{h\to 0} I^h(y^h) &= \liminf_{k\to\infty} I^{h_{n(k)}}(y^{h_{n(k)}}) \\ &\geq \liminf_{k\to\infty}\int_\Omega Q^{h_{n(k)}}(x,\widetilde{E}_k(x))\,dx \\ &= \liminf_{k\to\infty}\int_\Omega \chi^k Q^{h_{n(k)}}\left(x, \operatorname{sym} G^{h_{n(k)}} - \tfrac{1}{2}(K^{h_{n(k)}})^2\right)\,dx \\ &= \liminf_{k\to\infty}\int_\Omega \chi^k Q^{h_{n(k)}}\left(x, M_1+x_3M_2+\nabla_{h_{n(k)}}\vartheta_k\right)\,dx \\ &= \liminf_{k\to\infty}\int_\Omega Q^{h_{n(k)}}\left(x, M_1+x_3M_2+\nabla_{h_{n(k)}}\vartheta_k\right)\,dx \\ &\geq \int_\omega Q(x',M_1,M_2)\,dx' = I^0(u,v). \end{aligned}$$

To deal with arbitrary ω Lipschitz one firstly takes $D\ll\omega$ of class C^2 and conclude that $u\in H^1(\omega,\mathbb{R}^2)$, $v\in H^1(\omega)\cap H^2(D)$. In the same way as above we conclude

$$\liminf_{h\to 0} I^h(y^h) \geq \int_D Q(x',M_1,M_2)\,dx' \geq \tfrac{\alpha}{12}\|\nabla'^2 v\|_{L^2(D)}, \tag{85}$$

where we have used (Q'1). Since the left hand side does not depend on D we conclude that $v\in H^2(\omega)$. By exhausting ω with $D\ll\omega$ of regularity C^2 we have the claim. □

3.4. **Proof of Theorem 2.12.**

Proof. As in Lemma 3.10 take $r>0$ such that $B(r)\supset\omega$. Extend Q^h on $(B(r)\backslash\omega)\times I$ by e.g. $Q^h(x,G)=\beta|\operatorname{sym} G|^2$, for all $x\in(B(r)\backslash\omega)\times I$ Without any loss of generality we can assume that $\bar{R}=I$. First we assume that $u\in C^1(\bar\omega,\mathbb{R}^2)$, $v\in C^2(\bar\omega)$. The general claim will follow by density argument and by diagonalization, which is standard in Γ-convergence. Denote by $M_1=\operatorname{sym}\nabla' u-\frac{1}{2}\nabla' v\otimes\nabla' v$, $M_2=-\nabla'^2 v$. By using Lemma 3.10 we find a subsequence, still denoted by $(h_n)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}\subset H^2(B(r))$ and $(\psi_n)_{n\in N}\subset H^1(B(r)\times I)$ such that such that for each $\varphi_n\to 0$ strongly in H^1, $\psi_n\to 0$ strongly in L^2. Moreover the following is valid

(a) $\left(|\nabla'^2\varphi_n|^2\right)_{n\in\mathbb{N}}$ and $\left(|\nabla_{h_n}\psi_n|^2\right)_{n\in\mathbb{N}}$ are equi-integrable,
(b)

$$\limsup_{n\to\infty}\left(\|\varphi_n\|_{H^2}+\|\nabla_{h_n}\psi_n\|_{L^2}\right)\leq C\left(\beta\|M_1+x_3M_2\|_{L^2}^2+1\right).$$

(c) For $(\vartheta_n)_{n\in\mathbb{N}} \subset H^1(\omega \times I)$ defined by

$$\vartheta_n = \psi_n + \begin{pmatrix} 0 \\ 0 \\ \frac{\varphi_n}{h_n} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1\varphi_n \\ \partial_2\varphi_n \\ 0 \end{pmatrix},$$

we have

$$\int_\omega Q(x', M_1, M_2)\, dx' = \lim_{n\to\infty} \int_{B(r)\times I} Q^{h_n}\left(x, \iota\left((M_1 + x_3 M_2)1_\omega\right) + \nabla_{h_n}\vartheta_n\right) dx. \tag{86}$$

We know that $(\vartheta_{n,1}, \vartheta_{n,2}, h_n\vartheta_{n,3}) \to 0$ strongly in L^2 and that

$$\operatorname{sym}\nabla_{h_n}\vartheta_n = -x_3 \operatorname{sym}\hat{\nabla}^2\varphi^n + \operatorname{sym}\nabla_{h_n}\psi_n, \tag{87}$$

is equi-integrable. In the same way as in the proof of Lemma 3.10 we can suppose that $\varphi_n = \nabla'\varphi_n = 0$ in a neighborhood of $B(r)\backslash\omega$ and that $\psi_n = 0$ in a neighborhood of $(B(r)\backslash\omega)\times I$. By using Corollary A.2 and Corollary A.4 we find for each $\lambda > 0$ and $n \in \mathbb{N}$, $\varphi_n^\lambda \in H^2(B(r))$ and $\psi_n^\lambda \in H^1\left(B(r)\times I, \mathbb{R}^3\right)$ such that

$$\|\varphi_n^\lambda\|_{W^{2,\infty}} \leq C(r)\lambda, \tag{88}$$

$$\lim_{\lambda\to\infty}\sup_{n\in\mathbb{N}} \|\varphi_n^\lambda - \varphi_n\|_{H^2} = 0, \tag{89}$$

$$\sup_{\lambda>0}\limsup_{n\to\infty} \|\varphi_n^\lambda\|_{H^2} \leq C(r)\left(\beta\|M_1 + x_3M_2\|^2_{L^2(\omega)} + 1\right). \tag{90}$$

and

$$\|\psi_n^\lambda\|_{L^\infty} + \|\nabla_{h_n}\psi_n^\lambda\|_{L^\infty} \leq C(r)\lambda, \tag{91}$$

$$\lim_{\lambda\to\infty}\sup_{n\in\mathbb{N}}\left(\|\psi_n^\lambda - \psi_n\|_{L^2} + \|\nabla_h\psi_n^\lambda - \nabla_{h_n}\psi_n\|_{L^2}\right) = 0, \tag{92}$$

$$\sup_{\lambda>0}\limsup_{n\to\infty}\left(\|\psi_n^\lambda\|_{L^2} + \|\nabla_{h_n}\psi_n^\lambda\|_{L^2}\right) \leq C(r)\left(\beta\|M_1 + x_3M_2\|^2_{L^2(\omega)} + 1\right). \tag{93}$$

Notice that as the consequence of (89) and (92) we have for all

$$\lim_{\lambda\to\infty}\lim_{n\to\infty}\left(\|\varphi_n^\lambda\|_{H^1} + \|\psi_n^\lambda\|_{L^2}\right) = 0. \tag{94}$$

Define

$$\vartheta_n^\lambda = \psi_n^\lambda + \begin{pmatrix} 0 \\ 0 \\ \frac{\varphi_n^\lambda}{h_n} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1\varphi_n^\lambda \\ \partial_2\varphi_n^\lambda \\ 0 \end{pmatrix}.$$

Again we have

$$\operatorname{sym}\nabla_{h_n}\vartheta_n^\lambda = -x_3\iota(\operatorname{sym}\nabla'^2\varphi_n^\lambda) + \operatorname{sym}\nabla_{h_n}\psi_n^\lambda. \tag{95}$$

Notice that due to (89), (92) we have

$$\lim_{\lambda\to\infty}\sup_{n\in\mathbb{N}} \|\operatorname{sym}\nabla_{h_n}\vartheta_n^\lambda - \operatorname{sym}\nabla_{h_n}\vartheta_n\|_{L^2} = 0. \tag{96}$$

Define also for every n, λ the function $y_n^\lambda : \Omega \to \mathbb{R}^3$ by

$$\begin{aligned} y_n^\lambda(x', x_3) &= \begin{pmatrix} x' \\ h_n x_3 \end{pmatrix} + \begin{pmatrix} h_n^2 u(x') \\ h_n \left(v(x') + \varphi_n^\lambda(x')\right) \end{pmatrix} - h_n^2 x_3 \begin{pmatrix} -\partial_1 \left(v + \varphi_n^\lambda\right)(x') \\ -\partial_2 \left(v + \varphi_n^\lambda\right)(x') \\ 0 \end{pmatrix} \\ &\quad + h_n^2 \psi_n^\lambda(x', x_3) + \tfrac{1}{2} h_n^3 x_3 \begin{pmatrix} 0 \\ 0 \\ |\partial_1 (v + \varphi_n^\lambda)(x')|^2 + |\partial_2 (v + \varphi_n^\lambda)(x')|^2 \end{pmatrix}. \end{aligned}$$

From (94) we have:

$$\lim_{\lambda\to\infty} \lim_{n\to\infty} \left(\left\| \frac{\int_I y_n'^\lambda - x'}{h_n^2} - u \right\|_{L^2} + \left\| \frac{\int_I y_{n,3}^\lambda}{h_n} - v \right\|_{L^2} \right) = 0, \tag{97}$$

where $y_n'^\lambda = (y_{n,1}^\lambda\, , \, y_{n,2}^\lambda)$. Also we easily conclude, by the Taylor expansion, that for all λ

$$\lim_{n\to\infty} \| \frac{\sqrt{(\nabla_{h_n} y_n^\lambda)^t \nabla_{h_n} y_n^\lambda} - I}{h_n^2} - E_n^\lambda \|_{L^\infty} = 0, \tag{98}$$

where

$$E_n^\lambda = \iota \left(\operatorname{sym} \nabla' u - \tfrac{1}{2} \nabla'(v + \varphi_n^\lambda) \otimes \nabla'(v + \varphi_n^\lambda) - x_3 \nabla'^2 v \right) + \operatorname{sym} \nabla_{h_n} \vartheta_n^\lambda. \tag{99}$$

From property (W1), Lemma 3.13 and (5) we conclude that for every λ we have

$$\lim_{n\to\infty} \left| \frac{1}{h_n^4} \int_\Omega W^{h_n}(x, \nabla_{h_n} y_n^\lambda)\, dx - \int_\Omega Q^{h_n}(x, E_n^\lambda(x))\, dx \right| = 0. \tag{100}$$

Notice also that as a consequence of (89), (94), (96) we have

$$\lim_{\lambda\to\infty} \lim_{n\to\infty} \|E_n^\lambda - E_n\|_{L^2} = 0, \tag{101}$$

where

$$E_n = \iota \left(\operatorname{sym} \nabla' u - \tfrac{1}{2} \nabla' v \otimes \nabla' v - x_3 \nabla'^2 v \right) + \operatorname{sym} \nabla_{h_n} \vartheta_n. \tag{102}$$

From (Q1), (5) and (86) we have

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} \left| \int_\Omega Q^{h_n}(x, E_n^\lambda(x))\, dx - \int_\omega Q(x', M_1, M_2)\, dx' \right| = 0. \tag{103}$$

By forming the function

$$\begin{aligned} g(\lambda, n) &= \left\| \frac{\int_I y_n'^\lambda - x'}{h_n^2} - u \right\|_{L^2(\omega)} + \left\| \frac{\int_I y_{n,3}^\lambda}{h_n} - v \right\|_{L^2(\omega)} \\ &\quad + \left| \frac{1}{h^4} \int_\Omega W^{h_n}(x, \nabla_{h_n} y_n^\lambda)\, dx - \int_\omega Q(x', M_1, M_2)\, dx' \right|, \end{aligned}$$

we conclude from (97), (100) and (103) that

$$\lim_{\lambda\to\infty} \limsup_{n\to\infty} g(\lambda, n) = 0.$$

By performing diagonalizing argument we find monotone function $\lambda(n)$, such that $\lim_{n\to\infty} g(\lambda(n), n) = 0$. This gives the desired sequence. To deal with $u \in H^1(\omega, \mathbb{R}^2)$, $v \in H^2(\omega)$ we need to do the further diagonalization. Namely first we choose $u_k \in C^1(\bar{\omega}, \mathbb{R}^2)$, $v_k \in C^2(\bar{\omega})$ such that

$$\|u_k - u\|_{H^1} = 0, \ \lim_{k\to\infty} \|v_k - v\|_{H^2} = 0.$$

Denote by

$$M_{1,k} = \operatorname{sym} \nabla' u - \tfrac{1}{2} \nabla' v \otimes \nabla' v, \quad M_{2,k} = -\nabla'^2 v.$$

We have that $M_{1,k} \to M_1$, $M_{2,k} \to M_2$ strongly in L^2. Then for each $k \in \mathbb{N}$ we choose a sequence of functions $(y_{k,n})_{n\in\mathbb{N}} \subset H^1(\Omega, \mathbb{R}^3)$ such that

$$\lim_{n\to\infty}\left(\left\|\frac{\int_I y'_{k,n}-x'}{h_n^2} - u_k\right\|_{L^2} + \left\|\frac{\int_I y_{k,n,3}}{h_n} - v_k\right\|_{L^2}\right) = 0,$$

and

$$\lim_{n\to\infty}\left|\frac{1}{h_n^4}\int_\Omega W^{h_n}(x, \nabla_{h_n} y_{k,n})\, dx - \int_\omega Q(x', M_{1,k}, M_{2,k})\, dx'\right| = 0.$$

From (Q'1) we see that

$$\lim_{k\to\infty}\left|\int_\omega Q(x', M_{1,k}, M_{2,k})\, dx' - \int_\omega Q(x', M_1, M_2)\, dx'\right| = 0.$$

Thus for the function formed by

$$\begin{aligned} g(k,n) \;=\; & \left\|\frac{\int_I y'_{k,n}-x'}{h_n^2} - u\right\|_{L^2(\omega)} + \left\|\frac{\int_I y_{k,n,3}}{h_n} - v\right\|_{L^2(\omega)} \\ & + \left|\frac{1}{h_n^4}\int_\Omega W^{h_n}(x, \nabla_{h_n} y_{k,n})\, dx - \int_\omega Q(x', M_1, M_2)\, dx'\right|, \end{aligned}$$

we see that $\lim_{k\to\infty}\lim_{n\to\infty} g(k,n) = 0$. Then, by diagonalizing, we obtain the sequence $k(n)$ such that $\lim_{n\to\infty} g(k(n), n) = 0$. □

At the end we prove one lemma that we will need in the next section

Lemma 3.14. *Let $D \subset \mathbb{R}^2$ be the open, bounded set with Lipschitz boundary and let $(Q^h)_{h>0}$, Q be as above and let Assumption 2.7 be valid. Then for every $M = M_1 + x_3 M_2 \in \mathcal{S}_{vK}(\omega)$ we have*

$$\begin{aligned} \lim_{h\to 0} \min_{\substack{\psi\in H^1(D\times I,\mathbb{R}^3),\\ \psi=0 \text{ on } \partial D\times I}} \int_{D\times I} Q^h(x, \iota(M) + \nabla_h \psi)\, dx = \\ \min_{\substack{u\in H_0^1(D,\mathbb{R}^2),\\ v\in H_0^2(D)}} \int_D Q(x', M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}x'. \end{aligned}$$

For every $r > 0$ we have

$$\begin{aligned} \liminf_{h\to 0} \min_{\substack{\psi\in H^1(D\times I,\mathbb{R}^3),\\ \psi=0 \text{ on } \partial D\times I, \|(\psi_1,\psi_2,h\psi_3)\|_{L^2}\le r}} \int_{D\times I} Q^h(x, \iota(M) + \nabla_h \psi)\, dx \ge \\ \min_{\substack{u\in H_0^1(D,\mathbb{R}^2), v\in H_0^2(D)\\ \|u\|_{L^2}^2 + \|v\|_{L^2}^2 \le r^2}} \int_D Q(x', M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}x'. \end{aligned}$$

In the similar way we have

$$\begin{aligned} \lim_{h\to 0} \min_{\psi\in H^1(\mathcal{Y}\times I,\mathbb{R}^3)} \int_{\mathcal{Y}\times I} Q^h(x, \iota(M) + \nabla_h \psi)\, dx = \\ \min_{\substack{u\in H^1(\mathcal{Y},\mathbb{R}^2),\\ v\in H^2(\mathcal{Y})}} \int_{\mathcal{Y}} Q(y, M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}y. \end{aligned}$$

Proof. We shall only prove the first two statements. We prove the claim by using Γ-convergence i.e. compactness, liminf and limsup inequality. Take $\psi^h \in H^1(D \times I, \mathbb{R}^3)$ such that $\psi^h = 0$ on $\partial D \times I$ and such that

$$\min_{\substack{\psi \in H^1(D\times I,\mathbb{R}^3),\\ \psi=0 \text{ on } \partial D\times I}} \int_{D\times I} Q^h(x, \iota(M) + \nabla_h \psi)\, dx = \int_{D\times I} Q^h(x, \iota(M) + \nabla_h \psi^h).$$

By comparing with zero function and using the property (Q1) we obtain

$$\|\operatorname{sym} \nabla_h \psi^h\|_{L^2}^2 \leq 2\left(\frac{\beta}{\alpha} - 1\right) \|M\|_{L^2}^2. \tag{104}$$

By Corollary 3.2 there exist $v \in H_0^2(D)$, $u \in H_0^1(D, \mathbb{R}^2)$ $(\varphi^h)_{h>0} \subset H^2(D)$ such that $\varphi^h = 0$ on ∂D and $(\bar{\psi}^h)_{h>0} \subset H_0^1(D, \mathbb{R}^2)$ with the following property

$$\operatorname{sym} \nabla \psi^h = \iota(-x_3 \nabla'^2 v + \operatorname{sym} \nabla' u) + \operatorname{sym} \nabla_h \bar{\psi}^h + o^h,$$

where $\|o^h\|_{L^2} \to 0$ as $h \to 0$, $(\bar{\psi}_1^h, \bar{\psi}_2^h, h\bar{\psi}_3^h) \to 0$ in L^2.

We define

$$M_1' = M_1 + \operatorname{sym} \nabla' u, \quad M_2' = M_2 - \nabla'^2 v$$

From the the definition of Q we have that

$$\liminf_{h\to 0} \min_{\substack{\psi \in H^1(D\times I,\mathbb{R}^3),\\ \psi=0 \text{ on } \partial D\times I}} \int_{D\times I} Q^h(x, \iota(M) + \nabla_h \psi)\, dx \geq \min_{\substack{u \in H_0^1(D,\mathbb{R}^2),\\ v\in H_0^2(D)}} \int_D Q(x', M_1 + \operatorname{sym} \nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}x'.$$

This is the lower bound. The second statement follows immediatelly from Corollary 3.2 which implies that $\|u\|_{L^2}^2 + \|v\|_{L^2}^2 \leq r^2$. To prove the upper bound for the first statement take $u_0 \in H_0^2(D)$, $v_0 \in H_0^1(D)$ such that

$$\int_D Q(x', M_1 + \operatorname{sym} \nabla' u_0, M_2 - \nabla'^2 v_0)\, \mathrm{d}x' = \min_{\substack{u \in H_0^1(D,\mathbb{R}^2),\\ v\in H_0^2(D)}} \int_D Q(x', M_1 + \operatorname{sym} \nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}x'.$$

Using Lemma 3.10 for arbitrary subsequence $(h_n)_{n\in\mathbb{N}}$ take its subsequence, still denoted by $(h_n)_{n\in\mathbb{N}}$, and $(\psi_n)_{n\in\mathbb{N}}$ such that $\psi_n = 0$ on $\partial D \times I$ and

$$\int_D Q(x', M_1 + \operatorname{sym} \nabla' u_0, M_2 - \nabla'^2 v_0) = \lim_{n\to\infty} \int_D Q^{h_n}\left(\iota(M - x_3 \nabla'^2 v_0 + \operatorname{sym} \nabla' u_0) + \nabla_{h_n} \psi_n\right)\, \mathrm{d}x.$$

Define $(l_n)_{n\in\mathbb{N}} \subset H^1(D, \mathbb{R}^3)$ by

$$l_n = \begin{pmatrix} u_{0,1} \\ u_{0,2} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{v_0}{h_n} \end{pmatrix} - x_3 \begin{pmatrix} \partial_1 v_0 \\ \partial_2 v_0 \\ 0 \end{pmatrix}.$$

Notice that $l_n = 0$ on $\partial D \times I$, for every $n \in \mathbb{N}$ and that

$$\operatorname{sym} \nabla_{h_n} l_n = \iota(-x_3 \nabla'^2 v_0 + \operatorname{sym} \nabla' u_0).$$

From this we have

$$\limsup_{n\to\infty} \min_{\substack{\psi \in H^1(D\times I, \mathbb{R}^3),\\ \psi = 0 \text{ on } \partial D \times I}} \int_{D\times I} Q^h(x, \iota(M) + \nabla_{h_n}\psi)\, dx \leq$$

$$\limsup_{n\to\infty} \int_{D\times I} Q^h\left(x, \iota(M) + \nabla_{h_n}(l_n + \psi_n)\right)\, dx =$$

$$\int_D Q(x', M_1 + \operatorname{sym}\nabla u_0, M_2 - \nabla^2 v_0)\, \mathrm{d}x'.$$

The arbitrariness of the sequence $(h_n)_{n\in\mathbb{N}}$ implies the claim.

The second statement goes in the same way, after noticing that for the minimizers $(\psi^h)_{h>0}$ we can without loss of generality assume that have mean value 0 on $Y \times I$. □

3.5. **Proof of Theorem 2.16.** First we prove one technical lemma.

Lemma 3.15. *Let $(\chi^h)_{h>0}$ be a sequence in $\mathcal{X}^n(\omega)$ that has the limit energy density Q. Assume that for every $i = 1, \dots, n$ the functions χ_i^h converge weakly star $\chi_i^h \overset{*}{\rightharpoonup} \theta_i \in L^\infty(\omega, [0,1])$. Denote by $\bar{\theta}_i := \int_\omega \theta_i$. Then there exists a sequence $(\tilde{\chi}^h)_{h>0}$ in $\mathcal{X}^n(\omega)$ such that $\tilde{\chi}_i^h \overset{*}{\rightharpoonup} \theta_i$ weakly star in $L^\infty(\omega, [0,1])$ for $i = 1, \dots, n$ and*

$$\int_\omega \tilde{\chi}_i^h = \bar{\theta}_i, \quad \forall h > 0,\ i = 1, \dots, n.$$

Moreover $(\tilde{\chi}^h)_{h>0}$ has the same limit energy density Q.

Proof. Notice that $\sum_{i=1}^n \theta_i = 1$ for almost every $x' \in \omega$, which has the consequence that $\sum_{i=1}^n \bar{\theta}_i = 1$. It is not difficult to construct the sequence $(\tilde{\chi}^h)_{h>0}$ which satisfies

$$(105) \qquad \left|\bigcup_{i=1}^n \{\chi_i^h \neq \tilde{\chi}_i^h\}\right| \to 0, \text{ as } h \to 0, \qquad \int_\omega \tilde{\chi}_i^h = \bar{\theta}_i, \quad \forall h > 0,\ i = 1, \dots, n.$$

To make one such possible construction see [BB09, Lemma 3.2]. It is easy to see that we immediately obtain $\tilde{\chi}_i^h \overset{*}{\rightharpoonup} \theta_i$. We now proceed as follows. To prove (a) take the sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to zero and its subsequence still denoted by $(h_n)_{n\in\mathbb{N}}$ such that

$$K^-_{(\tilde{\chi}^{h_n})_{n\in\mathbb{N}}}(M, A, \omega) = K^+_{(\tilde{\chi}^{h_n})_{n\in\mathbb{N}}}(M, A, \omega) =: K_{(\tilde{\chi}^{h_n})_{n\in\mathbb{N}}}(M, A, \omega),$$

for all $A \subset \omega$ open and $M \in \mathcal{S}_{vK}(\omega)$ (see Lemma 2.6 and (a) of Lemma 3.8). By using Lemma 3.10 we find a further subsequence, still denoted by $(h_n)_{n\in\mathbb{N}}$, and a sequence $(\tilde{\vartheta}_n)_{n\in\mathbb{N}} \subset H^1(A\times I, \mathbb{R}^3)$ such that $(\tilde{\vartheta}_{n,1}, \tilde{\vartheta}_{n,2}, h_n\tilde{\vartheta}_{n,3}) \to 0$ strongly in L^2, $(|\operatorname{sym}\nabla_{h_n}\vartheta_{h_n}|^2)_{n\in\mathbb{N}}$ is equi-integrable and

$$K_{(\tilde{\chi}^{h_n})_{n\in\mathbb{N}}}(M, A) = \lim_{n\to\infty} \int_{A\times I} \sum_{i=1}^n Q^i\left(x, \iota(M) + \nabla_{h_n}\vartheta_{h_n}\right) \tilde{\chi}^{h_n}(x')\, \mathrm{d}x.$$

Using the equi-integrability property, $(Q1)$ and (105) it is easy to see that

$$\lim_{n\to\infty} \left| \int_{A\times I} \sum_{i=1}^n Q^i\left(x, \iota(M) + \nabla_{h_n}\vartheta_{h_n}\right) \tilde{\chi}^{h_n}(x')\, \mathrm{d}x - \int_{A\times I} \sum_{i=1}^n Q^i\left(x, \iota(M) + \nabla_{h_n}\vartheta_{h_n}\right) \chi^{h_n}(x')\, \mathrm{d}x \right| = 0.$$

From this we have that

$$K_{(\tilde{\chi}^{h_n})_{n\in\mathbb{N}}}(M,A,\omega) \geq K_{(\chi^h)_{h>0}}(M,A,\omega).$$

In the same way, using Lemma 3.10, we obtain the opposite inequality. The claim now follows from Lemma 2.8. □

Definition 3.16. For $\theta \in [0,1]^n$ such that $\sum_{i=1}^n \theta_i = 1$ and $\omega \subset \mathbb{R}^2$ with Lipschitz boundary we define $\mathcal{G}_\theta(\omega)$ as the set of all homogeneous (independent of $x' \in \omega$) quadratic functions $Q : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}$ for which there exists a sequence $(\chi^h)_{h>0}$ in $\mathcal{X}^n(\omega)$ which has the limit energy density Q and for which is valid $\chi_i^h \overset{*}{\rightharpoonup} \theta_i$ weakly star in $L^\infty(\omega,[0,1])$ for $i=1,\dots,n$ and for all $h>0$ and $i=1,\dots,n$ we have $\int_\omega \chi_i^h = \theta_i$.

Lemma 3.17. *The set $\mathcal{G}_\theta(\omega)$ is independent of $\omega \subset \mathbb{R}^2$.*

Proof. Let $Q \in \mathcal{G}_\theta(\omega)$ be the homogeneous limit energy of the sequence $(\chi^h)_{h>0}$ for which is valid $\chi_i^h \overset{*}{\rightharpoonup} \theta_i$ for $i=1,\dots,n$. Take $\tilde{\omega} \subset \mathbb{R}^2$ with Lipschitz boundary and $x_0' \in \omega$ and $s>0$ such that $x_0' + s\tilde{\omega} \subset \omega$. Define $(\tilde{\chi}^h)_{h>0}$ a sequence in $\mathcal{X}^n(\tilde{\omega})$ in the following way

$$\tilde{\chi}_i^h(x') = \chi_i^{h/s}(x_0' + sx'), \text{ for } i=1,\dots,n.$$

We immediately obtain that $\tilde{\chi}_i^h \overset{*}{\rightharpoonup} \theta_i$ for $i=1,\dots,n$. Notice that for arbitrary $\psi \in H^1\left((x_0'+s\tilde{\omega})\times I,\mathbb{R}^3\right)$ and arbitrary $h>0$ we have

$$\nabla_h \tilde{\psi}(x',x_3) = \nabla_{h/s}\psi(x_0'+sx',x_3), \text{ for } x \in \tilde{\omega},$$

where $\tilde{\psi} \in H^1(\tilde{\omega}\times I,\mathbb{R}^3)$ is defined by $\tilde{\psi}(x',x_3) = \frac{1}{s}\psi(x_0'+sx',x_3)$, for $x \in \tilde{\omega}$. From this, using the homogeneity of Q and the definition of K, we obtain for arbitrary $M_1, M_2 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$

$$\begin{aligned} Q(M_1,M_2) &= \tfrac{1}{s^2|\tilde{\omega}|} K_{(\chi^h)_{h>0}}\left(M_1 + x_3M_2, x_0'+s\tilde{\omega},\omega\right) \\ &= \tfrac{1}{s^2|\tilde{\omega}|} K_{(\chi^{h/s})_{h>0}}\left(M_1 + x_3M_2, x_0'+s\tilde{\omega},\omega\right) \\ &= \tfrac{1}{|\tilde{\omega}|} K_{(\tilde{\chi}^h)_{h>0}}\left(M_1 + x_3M_2, \tilde{\omega},\tilde{\omega}\right). \end{aligned} \tag{106}$$

This together with Lemma 3.15 implies the claim. □

Proposition 3.18. *The set $\mathcal{G}_\theta$ is closed on the pointwise convergence.*

Proof. Take a sequence of homogeneous quadratic forms $Q_k : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}$ such that for each k, $Q_k(\cdot,\cdot)$ is the limit energy density of $(\chi^{h,k})_{h>0}$, a sequence in $\mathcal{X}^n(Y)$. We want to do the diagonalization procedure to obtain a sequence $(\chi^h)_{h>0}$ in $\mathcal{X}^n(Y)$ which has the limit energy density $Q(\cdot,\cdot)$. It is easy to conclude for arbitrary $M_1, M_2 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $A \subset \omega$ with Lipschitz boundary

$$\lim_{k\to\infty} \limsup_{h\to 0} K^0_{\chi^{h,k}}\left(M_1 + x_3M_2, A, \omega, B(r)\right) \leq |A| Q(M_1,M_2),$$

where $B(r)$ of radius r in $L^2(A\times I,\mathbb{R}^3)$. By using Lemma 3.14 and homogeneity we conclude that for every $r>0$

$$\begin{aligned} &\lim_{k\to\infty} \liminf_{h\to 0} K^0_{\chi^{h,k}}\left(M_1 + x_3M_2, A, \omega, B(r)\right) \geq \\ &\qquad \lim_{k\to\infty} \min_{\substack{u\in H_0^1(D,\mathbb{R}^2),\\ v\in H_0^2(D)}} \int_A Q_k(M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\,dx' \geq \\ &\qquad \lim_{k\to\infty} |A| Q_k(M_1,M_2) = |A| Q(M_1,M_2). \end{aligned}$$

Now we proceed as follows. Let $\mathcal{D}=\{D_n\}_{n\in\mathbb{N}}$ be a countable family of open sets with Lipschitz boundary in Y, which is dense in Y. Take $(M_{1,n},M_{2,n})_{n\in\mathbb{N}}\subset\mathbb{R}^{2\times 2}_{\rm sym}\times\mathbb{R}^{2\times 2}_{\rm sym}$ a dense subset and $(r_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0. Define $M_n=M_{1,n}+x_3M_{2,n}$. Take also $\{\varphi_n\}_{n\in\mathbb{N}}\subset L^1(Y)$ which is dense in $L^1(Y)$. Form the function

$$g(k,h)=\sum_{n=1}^{\infty}\tfrac{1}{2^n}\min\left\{\max_{i=1,\dots,n}\left|\int_Y\left(\chi_i^{h.k}(y)-\theta_i\right)\varphi_n(y)\,\mathrm{d}y\right|,1\right\}+$$

$$\sum_{n,m,l=1}^{\infty}\tfrac{1}{2^{n+m+l}}\min\left\{\left|\tfrac{1}{|D_m|}K^0_{\chi^{h,k}}(M_n,D_m,Y,B(r_l))-Q(M_{1,n},M_{2,n})\right|,1\right\}$$

We conclude that $\lim_{k\to\infty}\limsup_{h\to 0}g(k,h)=0$. By using diagonal procedure (see [Att84, Corollary 1.16]) we conclude that there exists a function $h\to k(h)$ such that $k(h)\to\infty$ as $h\to 0$ and $g(k(h),h)\to 0$ as $h\to 0$. This implies that

$$\tfrac{1}{|D_m|}K^0_{\chi^{h,k(h)}}(M_n,D_m,Y,B(r_k))\to Q(M_{1,n},M_{2,n}),\ \forall n,m,k\in\mathbb{N},\tag{107}$$

$$\left|\int_Y\left(\chi_i^{h,k(h)}(y)-\theta_i\right)\varphi_n(y)\,\mathrm{d}y\right|\to 0,\ \forall n\in\mathbb{N}.\tag{108}$$

From (107) we conclude using Lemma 2.6, Lemma 2.8, Proposition 2.9 and the density of $\mathcal{D}$ that $(\chi^{h,k(h)})_{h>0}$ has the limit energy density $Q(\cdot,\cdot)$. From (108) we conclude by density that $\chi_i^{h,k(h)}\overset{*}{\rightharpoonup}\theta_i$. The claim now follows from Lemma 3.15.

□

For $(\chi^h)_{h>0}$ a sequence in $\mathcal{X}^n(\omega)$ and $x_0'\in\omega$, $s>0$ such that $x_0'+sY\subset\omega$ we denote by $\tilde\chi^h(x_0',s)$ a sequence in $\mathcal{X}^n(Y)$ given by $\tilde\chi^h(x_0',s)(y)=\chi^{h/s}(x_0'+sy)$, for any $h>0$ and $y\in Y$. Denote by

$$B(r_1,r_2)=\{\psi\in L^2(x_0'+sD,\mathbb{R}^3):\|(\psi_1,\psi_2)\|_{L^2}\le r_1,\ \|\psi_3\|_{L^2}\le r_2\}.$$

The following Lemma helps us in the proof of Proposition 3.20.

Lemma 3.19. *Let $(\chi^h)_{h>0}$ be a sequence in $\mathcal{X}^n(\omega)$ that has limit energy density Q. Then for almost every $x_0'\in\omega$ and every $D\subset Y$ with the Lipschitz boundary, $r>0$, $M_1,M_2\in\mathbb{R}^{2\times2}_{\rm sym}$ we have*

$$\begin{aligned}|D|Q(x_0,M_1,M_2)&=\lim_{s\to 0}\liminf_{h\to 0}K^0_{\tilde\chi^h(x_0',s)}(M_1+x_3M_2,D,Y,B(r,r))\\&=\lim_{s\to 0}\limsup_{h\to 0}K^0_{\tilde\chi^h(x_0',s)}(M_1+x_3M_2,D,Y,B(r,r))\,.\end{aligned}$$

Proof. For $M_1,M_2\in\mathbb{R}^{2\times2}_{\rm sym}$ denote by $M=M_1+x_3M_2$. Take $(\psi^h_{r_1,r_2,s})_{h>0}\subset H^1(x_0'+sD,\mathbb{R}^3)$ such that (see Remark 7)

$$K^0_{\chi^h}\left(M,x_0'+sD,\omega,B(r_1,r_2)\right)=\int_{(x_0'+sD)\times I}\sum_{i=1}^n Q^i\left(x,\iota(M)+\nabla_h\psi^h_{r_1,r_2,s}\right)\chi_i^h(x')\,dx,$$

$$\psi^h_{r_1,r_2,s}=0\text{ on }\partial(x_0'+sD)\times I,\quad \|(\psi^h_{r_1,r_2,s,1},\psi^h_{r_1,r_2,s,2})\|_{L^2}\le r_1,\|h\psi^h_{r_1,r_2,s,3})\|_{L^2}\le r_2.$$

For $\psi_s\in H^1((x_0'+sD)\times I,\mathbb{R}^3)$, define $\psi_s^d\in H^1(D\times I,\mathbb{R}^3)$ by $\psi_s^d(x',x_3)=\frac{1}{s}\psi_s(x_0'+sx',x_3)$. Using the fact that

$$\nabla_h\psi_s^d(x',x_3)=\nabla_{h/s}\psi_s(x_0'+sx',x_3),\ \text{for }x\in D\times I,$$

we have that

$$\tfrac{1}{s^2} K^0_{\chi^{h/s}}\left(M, x_0' + sD, \omega, B(r, r/s)\right) = K^0_{\tilde{\chi}^h(x_0',s)}\left(M, D, Y, B(r,r)\right). \tag{109}$$

By using the definition of $K^0_{(\chi^h)_{h>0}}$ we easily obtain for every $r, s > 0$

$$\begin{aligned} \limsup_{h\to 0} K^0_{\tilde{\chi}^h(x_0',s)}\left(M, D, Y, B(r,r)\right) &= \limsup_{h\to 0} \tfrac{1}{s^2} K^0_{\chi^{h/s}}\left(M, x_0' + sD, \omega, B(r, r/s)\right) \\ &\leq \tfrac{1}{s^2} K_{(\chi^{h/s})_{h>0}}(M, x_0' + sD, \omega) \\ &= \tfrac{1}{s^2} \int_{x_0'+sD} Q(y, M_1, M_2)\, \mathrm{d}y \end{aligned}$$

By taking x_0' the Lebesgue point of $Q(y, M_1, M_2)$ (this can be done for every M_1, M_2 because of property (Q'1)) we have that

$$\lim_{s\to 0} \limsup_{h\to 0} K^0_{\tilde{\chi}^h(x_0',s)}\left(M, D, Y, B(r,r)\right) \leq |D| Q(x_0, M_1, M_2).$$

To prove the lower bound we use Lemma 3.14. Notice that

$$\begin{aligned} &\liminf_{h\to 0} K^0_{\tilde{\chi}^h(x_0',s)}\left(M, D, Y, B(r,r)\right) = \\ &\quad \tfrac{1}{s^2} \liminf_{h\to 0} K^0_{\chi^{h/s}}\left(M, x_0' + sD, \omega, B(r, r/s)\right) \geq \\ &\tfrac{1}{s^2} \liminf_{h\to 0} K^0_{\chi^{h/s}}\left(M, x_0' + sD, \omega, B(\infty, \infty)\right) = \\ &\quad \tfrac{1}{s^2} \min_{\substack{u\in H_0^1(x_0'+sD,\mathbb{R}^2),\\ v\in H_0^2(x_0'+sD)}} \int_{x_0'+sD} Q(y, M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}y = \\ &\quad \min_{\substack{u\in H_0^1(D,\mathbb{R}^2),\\ v\in H_0^2(D)}} \int_D Q(x_0 + sy, M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}y. \end{aligned}$$

By using Corollary A.15 we obtain that for a.e. $x_0' \in \omega$

$$\begin{aligned} \lim_{s\to 0} &\min_{\substack{u\in H_0^1(D,\mathbb{R}^2),\\ v\in H_0^2(D)}} \int_D Q(x_0 + sy, M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}y = \\ &\min_{\substack{u\in H_0^1(D,\mathbb{R}^2),\\ v\in H_0^2(D)}} \int_D Q(x_0, M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\, \mathrm{d}y. \end{aligned}$$

Since $Q(x_0, \cdot, \cdot)$ is constant and quadratic and since $u = v = \nabla' v = 0$ on ∂D it is easy to see that the minimimum value is $|D| Q(x_0, M_1, M_2)$. This finishes the proof of the lemma.

□

The following proposition is analogous to [BB09, Theorem 3.5]. Here we have to justify diagonalization procedure with Lemma 3.19.

Proposition 3.20. *Q is the limit energy density of the sequence $(\chi^h)_{h>0}$ in $\mathcal{X}^n(\omega)$ if and only if for a.e. $x_0' \in \omega$ we have $Q(x_0', \cdot, \cdot) \in \mathcal{G}_{\theta(x_0')}$. Here θ is the weak star limit of χ^h in $L^\infty(\omega, [0,1])$.*

Proof. We first prove the direction "$\Rightarrow$". For every $\varphi \in C(Y)$ and every $i = 1, \dots, n$ we have:

$$\begin{aligned}
(110) \qquad & \lim_{r\to 0}\lim_{h\to 0}\left|\int_Y \left(\tilde{\chi}^h(x_0', s)_i(y) - \theta_i(x_0')\right)\varphi(y)\,\mathrm{d}y\right| \\
&= \lim_{s\to 0}\lim_{h\to 0}\tfrac{1}{s^2}\left|\int_{x_0'+sY}\left(\chi_i^{h/s}(y) - \theta_i(x_0')\right)\varphi\left(\tfrac{y-x_0}{s}\right)\,\mathrm{d}y\right| \\
&= \lim_{s\to 0}\tfrac{1}{s^2}\left|\int_{x_0'+sY}\left(\theta_i(y) - \theta_i(x_0')\right)\varphi\left(\tfrac{y-x_0}{s}\right)\,\mathrm{d}y\right| \\
&\le \lim_{s\to 0}\|\varphi\|_{L^\infty}\tfrac{1}{s^2}\int_{x_0'+sY}|\theta_i(y) - \theta_i(x_0')|\,\mathrm{d}y = 0.
\end{aligned}$$

Now we proceed in the same way as in Proposition 3.18. Let $\mathcal{D} = \{D_n\}_{n\in\mathbb{N}}$ be a countable family of open sets with Lipshitz boundary in Y, which is dense in Y. Take $(M_{1,n}, M_{2,n})_{n\in\mathbb{N}} \subset \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ a dense subset and $(r_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0. Define $M_n = M_{1,n} + x_3 M_{2,n}$. Take also $(\varphi_n)_{n\in\mathbb{N}} \subset C(Y)$ which is dense in $L^1(Y)$. Form the function

$$\begin{aligned}
g(s,h) = &\sum_{n=1}^{\infty}\tfrac{1}{2^n}\min\left\{\max_{i=1,\dots,n}\left|\int_Y\left(\tilde{\chi}^h(x_0', s)_i(y) - \theta_i(x_0')\right)\varphi_n(y)\,\mathrm{d}y\right|, 1\right\} + \\
&\sum_{n,m,k=1}^{\infty}\tfrac{1}{2^{n+m+l}}\min\left\{\left|\tfrac{1}{|D_m|}K^0_{\tilde{\chi}^h(x_0',s)}(M_n, D_m, Y, B(r_l, r_l)) - Q(x_0', M_{1,n}, M_{2,n})\right|, 1\right\}
\end{aligned}$$

From Lemma 3.19 and (110) we conclude that $\lim_{s\to 0}\limsup_{h\to 0} g(s,h) = 0$. By using diagonal procedure (see [Att84, Corollary 1.16]) we conclude that there exists a function $h \to s(h)$ such that $s(h) \to 0$ as $h \to 0$ and $g(s(h), h) \to 0$ as $h \to 0$. This implies that

$$(111) \qquad \tfrac{1}{|D_m|}K^0_{\tilde{\chi}^h(x_0', s(h))}(M_n, D_m, Y, B(r_k, r_k)) \to Q(x_0', M_{1,n}, M_{2,n}),\ \forall n, m, k \in \mathbb{N},$$

$$(112) \qquad \left|\int_C\left(\tilde{\chi}^h(x_0', s(h))_i(y) - \theta_i(x_0')\right)\varphi_n(y)\,\mathrm{d}y\right| \to 0,\ \forall n \in \mathbb{N}.$$

We conclude the claim as in Proposition 3.18. The proof of "$\Leftarrow$" we divide into several steps.

Step 1. We use the construction as in [BB09, Theorem 3.5]. We can, by diagonal procedure construct for each $k \in \mathbb{N}$ the sequence $(\chi^{h,k})_{h>0}$ in $\mathcal{X}^n(\omega)$ which satisfies

(i)
$$\lim_{k\to\infty}\lim_{h\to 0}\int_\omega \chi_i^{h,k}(x')\varphi(x')\,\mathrm{d}x' = \int_\omega \theta_i(x')\varphi(x')\,\mathrm{d}x';$$
for each $i = 1, \dots, n$ and every $\varphi \in L^1(\omega)$.

(ii) for each $k \in \mathbb{N}$ there exists a partition of ω into a finite number $n(k)$ of Lipschitz subset of ω, $\{U_{l,k}\}_{l=1,\dots,n(k)}$ and for $l = 2, \dots n(k)$ there exists $x_{l,k}' \in U_{l,k}$ such that $(\chi^{h,k}|_{U_{l,k}})_{h>0}$ has the homogeneous limit energy density $Q(x_{l,k}', \cdot, \cdot)$. It is also valid that $|U_{1,k}| \to 0$ as $k \to \infty$;

(iii) for the sequence of quadratic functions defined by

$$Q_k(x', M_1, M_2) = 1_{U_{1,k}}Q(x', M_1, M_2) + \sum_{l=2}^{n(k)} 1_{U_{l,k}}Q(x_{l,k}', M_1, M_2),$$

the following property is satisfied for every $r > 0$

$$\lim_{k\to\infty} \int_\omega \sup_{|M_1|+|M_2|\le r} |Q_k(x', M_1, M_2) - Q(x', M_1, M_2)| \, \mathrm{d}x = 0. \tag{113}$$

This, together with the property (Q'1) implies the pointwise convergence i.e. that $Q_k(x', M_1, M_2) \to Q(x', M_1, M_2)$ for a.e. $x' \in \omega$, for all $M_1, M_2 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$.

Step 2.. Upper bound. It is easy to prove for every $r > 0$, $M_1, M_2 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $A \subset \omega$ open with Lipschitz boundary we have

$$\lim_{k\to\infty} \limsup_{h\to 0} K^0_{\chi^{h,k}} (M_1 + x_3 M_2, A, \omega, B(r)) \le \int_A Q(x', M_1, M_2) \, \mathrm{d}x', \tag{114}$$

where $B(r)$ is a ball of radius r in $L^2(A\times I, \mathbb{R}^3)$. Namely, by using Lemma 3.10 for every $k \in \mathbb{N}$ the following holds: for each sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0 we can take its subsequence, still denoted by $(h_n)_{n\in\mathbb{N}}$, and the sequence $(\vartheta^{l,k}_n)_{n\in\mathbb{N}} \subset H^1(U_{l,k} \times I, \mathbb{R}^3)$ such that $\vartheta^{l,k}_n = 0$ on $\partial U_{l,k} \times I$ for every $n \in \mathbb{N}$, $l = 2, \dots, n(k)$, $\|(\vartheta^{l,k}_{n,1}, \vartheta^{l,k}_{n,2}, h_n \vartheta^{l,k}_{n,3})\|_{L^2} \to 0$ for every $l = 2, \dots, n(k)$ and

$$|A \cap U_{l,k}| Q(x_{l,k}, M_1, M_2) \;=\; \lim_{n\to\infty} \int_{(A\cap U_{l,k})\times I} \sum_{i=1}^{n} Q^i \left(x, \iota(M) + \nabla_{h_n} \vartheta^{l,k}_n \right) \chi^{h,k}_i(x') \, \mathrm{d}x'.$$

By testing with $\vartheta^k_n = \sum_{l=2}^{n(k)} 1_{U_{l,k}} \vartheta^{l,k}_n$ and using (113) we conclude (114).

Step 3.. Lower bound. We want to prove that for every $r > 0$, $M_1, M_2 \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ and $A \subset \omega$ open with Lipschitz boundary we have

$$\lim_{k\to\infty} \liminf_{h\to 0} K^0_{\chi^{h,k}} (M_1 + x_3 M_2, A, \omega, B(r)) \ge \tag{115}$$
$$\min_{\substack{u\in H^1_0(A,\mathbb{R}^2), v\in H^2_0(A)\\ \|u\|^2_{L^2}+\|v\|^2_{L^2}\le r^2}} \int_A Q(x, M_1 + \mathrm{sym}\, \nabla' u, M_2 - \nabla'^2 v) \, \mathrm{d}x' =: m_Q (M_1, M_2, A, \omega, B(r)) .$$

Take an arbitrary sequence $(h_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0. For each $k \in \mathbb{N}$ take a sequence $(\psi^k_n)_{n\in\mathbb{N}} \subset H^1(A \times I, \mathbb{R}^3)$ such that $\psi^k_n = 0$ on $\partial A \times I$ for every $k, n \in \mathbb{N}$ and such that $\|(\psi^k_n, \psi^k_n, h_n \psi^k_n)\|_{L^2} \le r$ and

$$K^0_{\chi^{h_n,k}} (M_1 + x_3 M_2, A, \omega, B(r)) = \int_{A\times I} \sum_{i=1}^{n} Q^i \left(x, \iota(M_1 + x_3 M_2) + \nabla_{h_n} \psi^k_n \right) \chi^{h,k}_i(x') \, dx.$$

By using Corollary 3.2 we conclude that there exists $u \in H^1_0(A, \mathbb{R}^2)$, $v \in H^2_0(A)$, $(\bar\psi^k_n)_{h>0} \subset H^1(A\times I, \mathbb{R}^3)$ such that $\|u\|^2_{L^2} + \|v\|^2_{L^2} \le r^2$, $\bar\psi^k_n = 0$ on $\partial A \times I$ and $(\bar\psi^k_{n,1}, \bar\psi^k_{n,2}, h_n \bar\psi^k_{n,3}) \to 0$ strongly in L^2 and the following identity is valid

$$\mathrm{sym}\, \nabla_{h_n} \psi^k_n = \iota(-x_3 \nabla'^2 v + \mathrm{sym}\, \nabla' u) + \mathrm{sym}\, \nabla_h \bar\psi^k_n. \tag{116}$$

From (116) we conclude that

$$\liminf_{n\to\infty} K^0_{\chi^{h_n,k}} (M_1 + x_3 M_2, A, \omega, B(r)) \ge$$
$$\sum_{l=2}^{n(k)} K_{(\chi^{h_n,k})_{n\in\mathbb{N}}} \left(M_1 + \mathrm{sym}\, \nabla' u + x_3 M_2 - x_3 \nabla'^2 v, A \cap U_{l,k}, U_{l,k} \right) =$$
$$\sum_{l=2}^{n(k)} \int_{(A\cap U_{l,k})\times I} Q(x'_{l,k}, M_1 + \mathrm{sym}\, \nabla' u, M_2 - \nabla'^2 v).$$

Since we have

$$\lim_{k\to\infty}\int_{A\cap U_{1,k}} Q(x',M_1+\operatorname{sym}\nabla' u, M_2-\nabla'^2 v)\,\mathrm{d}x' = 0,$$

we conclude by Lebesgue theorem of dominated convergence that

$$\lim_{k\to\infty}\liminf_{n\to\infty} K^0_{\chi^{h_n,k}}(M_1+x_3M_2,A,\omega,B(r)) \geq \int_A Q(x',M_1+\operatorname{sym}\nabla' u, M_2-\nabla'^2 v)\,\mathrm{d}x'.$$

This implies (115).

Step 5.. Diagonalization. We proceed similar as in the proof of Proposition 3.18. Take $\{\varphi_n\}_{n\in\mathbb{N}}$ a countable dense subset of $L^1(\omega)$, $\mathcal{D}=\{D_n\}_{n\in\mathbb{N}}$ a countable family of open sets with Lipschitz boundary in ω, which is dense in ω. Take $(M_{1,n},M_{2,n})_{n\in\mathbb{N}}\subset\mathbb{R}^{2\times 2}_{\mathrm{sym}}\times\mathbb{R}^{2\times 2}_{\mathrm{sym}}$ a dense subset and $(r_n)_{n\in\mathbb{N}}$ monotonly decreasing to 0. Define $M_n=M_{1,n}+x_3M_{2,n}$. Form the function

$$g(k,h)=\sum_{n=1}^{\infty}\tfrac{1}{2^n}\min\left\{\max_{i=1,\dots,n}\left|\int_Y\left(\chi_i^{h.k}(y)-\theta_i\right)\varphi_n(y)\,\mathrm{d}y\right|,1\right\}+$$
$$\sum_{n,m,l=1}^{\infty}\tfrac{1}{2^{n+m+l}}\min\left\{\left(K^0_{\chi^{h,k}}(M_n,D_m,\omega,B(r_l))-\int_{D_m}Q(x'M_{1,n},M_{2,n})\,\mathrm{d}x'\right)_+,1\right\}+$$
$$\sum_{n,m,l=1}^{\infty}\tfrac{1}{2^{n+m+l}}\min\left\{\left(m_Q(M_{1,n},M_{2,n},D_m,\omega,B(r))-K^0_{\chi^{h,k}}(M_n,D_m,\omega,B(r_l))\right)_+,1\right\}.$$

Here $x_+=\max\{x,0\}$. We conclude that $\lim_{k\to\infty}\limsup_{h\to 0} g(k,h)=0$. By using diagonal procedureb we conclude that there exists a function $h\to k(h)$ such that $k(h)\to\infty$ as $h\to 0$ and

$$\limsup_{h\to 0}\tfrac{1}{|D_m|}K^0_{\chi^{h,k(h)}}(M_n,D_m,\omega,B(r_k))\leq\int_{D_m}Q(x',M_{1,n},M_{2,n})\,\mathrm{d}x',\ \forall n,m,k\in\mathbb{N},$$
$$\liminf_{h\to 0}\tfrac{1}{|D_m|}K^0_{\chi^{h,k(h)}}(M_n,D_m,\omega,B(r_k))\geq m_Q(M_{1,n},M_{2,n},D_m,\omega,B(r_k))\,\mathrm{d}x',\ \forall n,m,k\in\mathbb{N},$$
$$\left|\int_\omega\left(\chi_i^{h,k(h)}(x')-\theta_i(x')\right)\varphi_n(x')\,\mathrm{d}x'\right|\to 0,\ \forall n\in\mathbb{N}.$$

The claim follows in the same way as in Proposition 3.18, after noticing that

$$m_Q(M_{1,n},M_{2,n},A,\omega,B(r_k))\to\int_{D_m}Q(x',M_{1,n},M_{2,n})\,\mathrm{d}x',$$

as $r_k\to 0$. □

Proof of Theorem 2.16. The second claim is the direct consequence of Proposition 3.18 and Proposition 3.20. To prove the first claim from Lemma 3.17 and Proposition 3.20 we conclude that it is enough to prove that for $\theta\in[0,1]^n$ such that $\sum_{i=1}^n\theta_i=1$ we have $\mathcal{G}_\theta(Y)\subset\mathcal{P}_\theta$.

Take a sequence (χ^h) in $\mathcal{X}^n(Y)$ such that $\int_Y\chi_i^h\,\mathrm{d}y=\theta_i$ for all $h>0$, $i=1,\dots,n$ and such that $\chi^h\overset{*}{\rightharpoonup}\theta$ weakly star in $L^\infty(Y,[0,1]^n)$ and $(\chi^h)_{h>0}$ has the homogeneous limit energy density Q. By using Lemma 3.14 and the periodic boundary conditions we have

$$\begin{aligned} Q(M_1, M_2) &= \min_{\substack{u\in H^1(\mathcal{Y},\mathbb{R}^2),\\ v\in H^2(\mathcal{Y})}} \int_{\mathcal{Y}} Q(y, M_1 + \operatorname{sym}\nabla' u, M_2 - \nabla'^2 v)\,\mathrm{d}y \\ &= \lim_{h\to 0} \min_{\psi\in H^1(\mathcal{Y}\times I,\mathbb{R}^3)} \int_{\mathcal{Y}\times I} \sum_{i=1}^{n} Q^i\left(y, x_3, \iota(M_1 + x_3 M_2) + \nabla_h \psi\right) \chi_i^h(y)\,\mathrm{d}y\,\mathrm{d}x_3. \end{aligned}$$

This implies the claim. □

Appendix A. Auxiliary results

Proposition A.1. *Let $1 \le p \le \infty$, $\lambda > 0$. Let A be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary*

(a) *Suppose $u \in W^{1,p}(A)$ Then there exists $u^\lambda \in W^{1,\infty}(A)$ such that*

$$\begin{aligned} \|u^\lambda\|_{W^{1,\infty}} &\le C(n,p,A)\lambda \\ |\{x \in A : u^\lambda(x) \ne u(x)\}| &\le \frac{C(n,p,A)}{\lambda^p} \int_{\{|u|+|\nabla u| \ge \lambda/C(n,p,A)\}} \left(|u| + |\nabla u|\right)^p \mathrm{d}x. \end{aligned}$$

In particular,

$$\lim_{\lambda\to\infty} \left(\lambda^p \left|\{x \in A : \; u^\lambda(x) \ne u(x)\}\right|\right) = 0.$$

If we define Hardy Littlewood maximal function

$$Ma(x) = \sup_{r>0} \fint_{B(x,r)} a(y)dy,$$

where $a = |\tilde{u}| + |\nabla \tilde{u}|$ ($\tilde{u}$ is the extension of u to $W^{2,2}(\mathbb{R}^n)$ which has the compact support) and

$A^\lambda = \{x \in \mathbb{R}^n : \; Ma(x) < \lambda$ *and x is a Lebesgue point of u, ∇u and $\nabla^2 u\}$,*

then we can construct u^λ such that

$$\{u^\lambda \ne u\} = \tilde{A}^\lambda,$$

where $\tilde{A}^\lambda$ is a closed subset of $A^\lambda \cap A$ which satisfies $|A \backslash \tilde{A}^\lambda| \le C|A \backslash A^\lambda|$, for some $C > 1$.

(b) *Assume additionally that A is has the boundary of class $C^{1,1}$ and $u \in W^{2,p}(A)$. Then there exists $u^\lambda \in W^{2,\infty}(A)$ such that*

$$\begin{aligned} \|u^\lambda\|_{W^{2,\infty}} &\le C(n,p,A)\lambda, \\ |\{x \in A : u^\lambda(x) \ne u(x)\}| &\le \\ &\frac{C(n,p,A)}{\lambda^p} \int_{\{(|u|+|\nabla u|+|\nabla^2 u|) \ge \lambda/C(n,p,A)\}} (|u| + |\nabla u| + |\nabla^2 u|)^p \,\mathrm{d}x, \end{aligned}$$

where $a = |u| + |\nabla u| + |\nabla^2 u|$. If we define

$$Ma(x) = \sup_{r>0} \fint_{B(x,r)} a(y)dy,$$

and

$A^\lambda = \{x \in A : \; Ma(x) < \lambda$ *and x is a Lebesgue point of u, ∇u and $\nabla^2 u\}$,*

then we can construct u^λ such that

$$\{u^\lambda \ne u\} = \tilde{A}^\lambda,$$

where $\tilde{A}^\lambda$ is a closed subset of A^λ which satisfies $|A\backslash\tilde{A}^\lambda| \leq C|A\backslash A^\lambda|$, for some $C > 1$.

Proof. See the proof of Proposition A2 in [FJM02]. The condition in (b) that the domain is of class $C^{1,1}$ is not demanded there. The argument is that one can extend $W^{2,p}(A)$ to $W^{2,p}(\mathbb{R}^n)$ when A is only Lipschitz (see e.g. [Ste70]). However, if for $u \in W^{2,p}(A)$ we denote this extension by Eu then it is not clear to the author weather the term

$$\int_{\{(|Eu|+|\nabla Eu|+|\nabla^2 Eu|)\geq\lambda/C_1(n,p,A)\}} (|Eu| + |\nabla Eu| + |\nabla^2 Eu|)^p \,\mathrm{d}x,$$

can be controlled with the term

$$\int_{\{(|u|+|\nabla u|+|\nabla^2 u|)\geq\lambda/C_2(n,p,A)\}} (|u| + |\nabla u| + |\nabla^2 u|)^p \,\mathrm{d}x.$$

For the standard extension operator, constructed using the reflexion, this can be easily proved to be valid. □

Remark 12. Notice that due to [EG92, Theorem 3, Section 6] for $u, v \in W^{1,p}(A)$ we have that

$$\{u = v\} = \{u = v, \nabla u = \nabla v\} \cup N,$$

where N is the set of measure zero. From this it follows that for $u, v \in W^{2,p}(A)$ we have that

$$\{u = v\} = \{u = v, \nabla u = \nabla v, \nabla^2 u = \nabla^2 v\} \cup N,$$

where N is the set of measure zero.

Corollary A.2. *Let $1 \leq p \leq \infty$, $\lambda > 0$ and let A be a open bounded open set in $\mathbb{R}^n$ with Lipschitz boundary.*

(a) Suppose that $(u^h)_{h>0} \subset W^{1,p}(A)$ is a sequence such that $u^h \rightharpoonup u$ weakly in $W^{1,p}$, $u^h = 0$ on Γ and $(|\nabla u^h|^p)_{h>0}$ is equi-integrable. Then there exists $(u^{\lambda,h})_{\lambda,h>0}$ such that

$$\begin{aligned} \|u^{\lambda,h}\|_{W^{1,\infty}} &\leq C(n,p,A)\lambda, \\ \lim_{\lambda\to\infty} \sup_{h>0} \|u^{\lambda,h} - u^h\|_{W^{1,p}} &= 0, \\ \|u^{\lambda,h}\|_{W^{1,p}} &\leq C(n,p,A)\|u^h\|_{W^{1,p}}. \end{aligned}$$

(b) Assume additionally that A has the boundary of class $C^{1,1}$ and that $(u^h)_{h>0} \subset W^{2,p}(A)$ is a sequence such that $u^h \rightharpoonup u$ weakly in $W^{2,p}$ and $(|\nabla^2 u^h|^p)_{h>0}$ is equi-integrable. Then there exists $(u^{\lambda,h})_{\lambda,h>0}$ such that

$$\begin{aligned} \|u^{\lambda,h}\|_{W^{2,\infty}} &\leq C(n,p,A)\lambda, \\ \lim_{\lambda\to\infty} \sup_{h>0} \|u^{\lambda,h} - u^h\|_{W^{2,p}} &= 0, \\ \|u^{\lambda,h}\|_{W^{2,p}} &\leq C(n,p,A)\|u^h\|_{W^{2,p}} \end{aligned}$$

Proof. The proof is the direct consequence of Proposition A.1. We will prove only (a). For each u^h and $\lambda > 0$ we choose $u^{\lambda,h}$ such that

$$\|u^{\lambda,h}\|_{W^{1,\infty}} \leq C(n,p,A)\lambda \tag{117}$$

$$|A^{\lambda,h}| \leq \frac{C(n,p,A)}{\lambda^p} \int_{\{|u^h|+|\nabla u^h|\geq\lambda/C(n,p,A)\}} \left(|u^h| + |\nabla u^h|\right)^p \mathrm{d}x, \tag{118}$$

where $A^{\lambda,h} = \{x \in A : u^\lambda(x) \neq u(x)\}$. Notice that since $u^h \to u$ strongly in L^p and $(|\nabla u^h|^p)_{h>0}$ is equi-integrable we have that

$$\lim_{\lambda\to\infty}\sup_{h>0}\int_{\{|u^h|+|\nabla u^h|\geq \lambda/C(n,p,A)\}}\big(|u^h|+|\nabla u^h|\big)^p\,dx = 0. \tag{119}$$

From this we easily see that $\lim_{\lambda\to\infty}\sup_{h>0}\lambda^p|A^{\lambda,h}| = 0$. Using (117) we conclude that

$$\begin{aligned}\lim_{\lambda\to\infty}\sup_{h>0}\Big(\|u^h\|_{L^p(A^{\lambda,h})}+\|\nabla u^h\|_{L^p(A^{\lambda,h})}\Big) &\to 0,\\ \lim_{\lambda\to\infty}\sup_{h>0}\Big(\|u^{\lambda,h}\|_{L^p(A^{\lambda,h})}+\|\nabla u^{\lambda,h}\|_{L^p(A^{\lambda,h})}\Big) &\to 0.\end{aligned}$$

Notice also simple estimate

$$\|u^{\lambda,h}\|^p_{L^p(A^{\lambda,h})}+\|\nabla u^{\lambda,h}\|^p_{L^p(A^{\lambda,h})} \leq 2C(n,p,A)^2\|u^h\|^p_{W^{1,p}}$$

From this we have the claim since

$$\begin{aligned}\|u^{\lambda,h}-u^h\|_{W^{1,p}} &= \|u^{\lambda,h}-u^h\|_{L^p(A^{\lambda,h})}+\|\nabla u^{\lambda,h}-\nabla u^h\|_{L^p(A^{\lambda,h})}\\ &\leq \|u^{\lambda,h}\|_{L^p(A^{\lambda,h})}+\|\nabla u^{\lambda,h}\|_{L^p(A^{\lambda,h})}+\|u^h\|_{L^p(A^{\lambda,h})}+\|\nabla u^h\|_{L^p(A^{\lambda,h})}.\\ \|u^{\lambda,h}\|_{W^{1,p}} &= \|u^h\|_{L^p\left((A^{\lambda,h})^c\right)}+\|\nabla u^h\|_{L^p\left((A^{\lambda,h})^c\right)}\\ &\quad+\|u^{\lambda,h}\|_{L^p(A^{\lambda,h})}+\|\nabla u^{\lambda,h}\|_{L^p(A^{\lambda,h})}.\end{aligned}$$

□

The following proposition we prove by combining the ideas of extension given in [BF02] and [BZ07] with Proposition A.1.

Proposition A.3. *Let $1 \leq p \leq \infty$ and A be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary. Suppose that $u \in W^{1,p}(A\times I,\mathbb{R}^3)$. Then for every $0 < h < 1$ there exists $u^{\lambda,h} \in W^{1,\infty}(A\times I,\mathbb{R}^3)$ such that*

$$\begin{aligned}\|u^{\lambda,h}\|_{L^\infty}+\|\nabla_h u^{\lambda,h}\|_{L^\infty} &\leq C(n,p,A)\lambda,\\ |\{x\in A : u^{\lambda,h}(x)\neq u(x)\}| &\leq \frac{C(n,p,A)}{\lambda^p}\int_{\{|u|+|\nabla_h u|\geq\lambda/C(n,p,A)\}}\big(|u|+|\nabla_h u|\big)^p\,dx.\end{aligned}$$

Proof. The idea is to look the problem on the physical domain $A \times hI$, extend it by reflection and translation to the domain $A\times I$ and then apply Proposition A.1 and choose the good strip. Define $\tilde u : A\times I \to \mathbb{R}^3$ as $2h$ periodic function in the variable x_3 in the following way

$$\tilde u^h(x',x_3) = \begin{cases} u(x',\frac{x_3}{h}), & \text{if } x_3\in hI,\\ u(x',1-\frac{x_3}{h}), & \text{if } x_3\in[h/2,3h/2],\end{cases}$$

and extend it by periodicity on $A\times I$. This implies that we have $2l+1 = 2\lfloor\frac{1}{2h}-\frac12\rfloor+1$ whole strips and at most 2 strips with the boundary $x_3 = 1/2$ i.e. $x_3 = -1/2$ where the function $\tilde u^h$ does not exhaust the full period $2h$. Denote for $i \in \{-l,\dots,l\}$ the sets $K_i = [(2i-1)h/2,(2i+1)h/2]$ and $L_1 = [(2l+1)h/2, 1/2]$, $L_2 = [-1/2, -(2l+1)h/2]$. Notice that $I = \cup_{i=0}^l K_i \cup L_1\cup L_2$ and

$$\nabla\tilde u^h = \begin{cases}\nabla_h u(x',\frac{x_3}{h}), & \text{if } x_3\in hI,\\ \left(\partial_1 u(x',1-\frac{x_3}{h}),\,\partial_2 u(x',1-\frac{x_3}{h}),\,-\frac1h\partial_3 u(x',1-\frac{x_3}{h})\right), & \text{if } x_3\in[h/2,3h/2].\end{cases}$$

Notice that $\tilde{u}^h \in W^{1,p}(A \times I, \mathbb{R}^3)$. We apply Proposition A.1 on the function $\tilde{u}^h$ to obtain the function $\tilde{u}^{\lambda,h}$ such that

$$\begin{aligned} \|\tilde{u}^{\lambda,h}\|_{W^{1,\infty}} &\leq C(n,p,A)\lambda \\ |\{x \in A : \tilde{u}^{\lambda,h}(x) \neq \tilde{u}(x)\}| &\leq \frac{C(n,p,A)}{\lambda^p} \int_{\{|\tilde{u}^h|+|\nabla \tilde{u}^h| \geq \lambda/C(n,p,A)\}} \left(|\tilde{u}^h| + |\nabla \tilde{u}^h|\right)^p \mathrm{d}x. \end{aligned}$$

We want to show that there exists strip which satisfies the appropriate estimate. Notice that, due to our construction we have for every $i \in \{-l, \dots l\}$ and $j \in \{1,2\}$

$$\begin{aligned} &\int_{\{|\tilde{u}^h|+|\nabla \tilde{u}^h| \geq \lambda/C(n,p,A)\}\cap A\times K_i} \left(|\tilde{u}^h| + |\nabla \tilde{u}^h|\right)^p dx = \\ &\qquad h\int_{\{|u|+|\nabla_h u| \geq \lambda/C(n,p,A)\}} \left(|u| + |\nabla_h u|\right)^p dx, \\ &\int_{\{|\tilde{u}^h|+|\nabla \tilde{u}^h| \geq \lambda/C(n,p,A)\}\cap A\times L_j} \left(|\tilde{u}^h| + |\nabla \tilde{u}^h|\right)^p dx \\ &\qquad \leq h\int_{\{|u|+|\nabla_h u| \geq \lambda/C(n,p,A)\}} \left(|u| + |\nabla_h u|\right)^p dx. \end{aligned} \tag{120}$$

From this we conclude that there exists strip i.e. $i \in \{-l, \dots, l\}$ and the set $A \times K_i$ such that

$$\begin{aligned} &\tfrac{1}{h}|\{x \in A \times K_i : \tilde{u}^{\lambda,h}(x) \neq \tilde{u}(x)\}| \leq \\ &\qquad \frac{3C(n,p,A)}{\lambda^p}\int_{\{|u|+|\nabla_h u| \geq \lambda/C(n,p,A)\}} \left(|u| + |\nabla_h u|\right)^p dx. \end{aligned}$$

To obtain $u^{\lambda,h}$ we take $\tilde{u}^{\lambda,h}|_{A\times K_i}$, translate it to the strip $A \times [-h/2, h/2]$, if necessary reflect it, and then stretch it to the domain $A \times I$. □

The proof of the following corollary goes in the same way as the proof of Corollary A.2, using Proposition A.3 instead of Proposition A.1. We will just state the result.

Corollary A.4. *Let $1 \leq p \leq \infty$ and A be a bounded open set in $\mathbb{R}^2$ with Lipschitz boundary. Suppose that $(u^h)_{h>0} \subset W^{1,p}(A)$ is a sequence such that $u^h \rightharpoonup u$ weakly in $W^{1,p}$ and $(|\nabla_h u^h|^p)_{h>0}$ is equi-integrable. Then there exists $(u^{\lambda,h})_{\lambda,h>0}$ such that*

$$\begin{aligned} \|u^{\lambda,h}\|_{L^\infty} + \|\nabla_h u^{\lambda,h}\|_{L^\infty} &\leq C(n,p,A)\lambda, \\ \lim_{\lambda\to\infty}\sup_{h>0}\left(\|u^{\lambda,h} - u^h\|_{L^p} + \|\nabla_h u^{\lambda,h} - \nabla_h u^h\|_{L^p}\right) &= 0, \\ \|u^{\lambda,h}\|_{L^p} + \|\nabla_h u^{\lambda,h}\|_{L^p} &\leq C(n,p,A)\left(\|u^h\|_{L^p} + \|\nabla_h u^h\|_{L^p}\right). \end{aligned}$$

The following proposition is just simple adaption of [FMP98, Lemma 1.2].

Proposition A.5. *Let $p > 1$. Let $A \subset \mathbb{R}^n$ be a open bounded set.*

(a) Let $(w_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(A)$. Then there exist a subsequence $(w_{n(k)})_{k\in\mathbb{N}}$ and a sequence $(z_k)_{k\in\mathbb{N}} \subset W^{1,p}(A)$ such that

$$|\{z_k \neq w_{n(k)}\}| \to 0, \tag{121}$$

as $k \to \infty$ and $\left(|\nabla z_k|^p\right)_{k\in\mathbb{N}}$ is equi-integrable. Each z_k may be chosen to be Lipschitz function. If $w_n \rightharpoonup w$ weakly in $W^{1,p}$ then $z_k \rightharpoonup w$ weakly in $W^{1,p}$.

(b) Let $(w_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{2,p}(A)$. Then there exist a subsequence $(w_{n(k)})_{k\in\mathbb{N}}$ and a sequence $(z_k)_{k\in\mathbb{N}} \subset W^{2,p}(A)$ such that

$$|\{z_k \neq w_{n(k)}\}| \to 0, \tag{122}$$

as $k \to \infty$ and $\left(|\nabla^2 z_k|^p\right)_{k\in\mathbb{N}}$ is equi-integrable. Each z_k may be chosen such that $z_k \in W^{2,\infty}(S)$. If $w_n \rightharpoonup w$ weakly in $W^{2,p}$ then $z_k \rightharpoonup w$ weakly in $W^{2,p}$.

Proof. Proof of (i) is given in [FMP98, Lemma 1.2]. The proof of (ii) goes in the same way. We can assume that the boundary of A is of class $C^{1,1}$ (to deal with general open bounded set see the proof of Step 2 in [FMP98, Lemma 1.2]. Namely, we extend each w_n on $\mathbb{R}^n$ such that the support of each w_n lies in a fixed compact subset $K \subset \mathbb{R}^n$. We denote this extension also by w_n. Denote by $a_n = |w_n| + |\nabla w_n| + |\nabla^2 w_n|$ and by

$$Ma_n(x) = \sup_{r>0} \fint_{B(x,r)} a_n(y)dy,$$

the Hardy Littlewood maximal function. It is well known that

$$\|M(a_{n(k)})\|_{L^p(\mathbb{R}^n)} \leq C(n,p)\|w_{n(k)}\|_{W^{2,p}(\mathbb{R}^n)} \leq C(n,p,A)\|w_{n(k)}\|_{W^{2,p}(A)} \tag{123}$$

We denote by $\mu = \{\mu_x\}_{x\in\Omega}$ the Young measures associated with the converging subsequence of $(M(\nabla a_n))_{n\in\mathbb{N}}$. We have the following properties:

(a) $\int_\Omega \int_\mathbb{R} |s|^p d\mu_x < +\infty$.
(b) whenever $(f(M(a_n))_{n\in\mathbb{N}}$ converges weakly in $L^1(\Omega)$, its weak limit is given by

$$\bar{f}(x) := \langle \mu_x, f\rangle, \text{ a.e.} x \in \Omega.$$

For $k \in \mathbb{N}$ we consider the truncation map $T_k : \mathbb{R} \to \mathbb{R}$ given by

$$T_k := \begin{cases} x, & |x| \leq k, \\ k\frac{x}{|x|}, & |x| > k \end{cases}$$

In the same way as in proof of Lemma [FMP98, Lemma 1.2] we obtain a subsequence $w_{n(k)}$ such that

$$|T_k(M(a_{n(k)}))|^p \rightharpoonup \overline{f} \text{ weakly in } L^1(A),$$

where

$$\overline{f}(x) = \int_\mathbb{R} |s|^p d\mu_x(s).$$

Set

$$\tilde{R}_k := \{x \in \mathbb{R}^n : M(a_{n(k)}) < k\}.$$

Notice that for k large enough, since the support of $w_{n(k)}$ lies in K, we have that $\tilde{R}_k \subset K_1$ where K_1 is a compact subset of $\mathbb{R}^n$, $K_1 \supset K$. So without the loss of generality we can assume that for each k we have $\tilde{R}_k \subset K_1$. Denote by R_k the closed subset of $\tilde{R}_k \cap A$ such that

$$|A \backslash \tilde{R}_k| \leq 2|A\backslash R_k|,$$

and

$$|\tilde{R}_k \backslash R_k| \leq \frac{1}{k^{p+1}}.$$

By Proposition A.1 (ii) there exists $z_k \in W^{2,\infty}(A)$ such that

$$z_k = w_{n(k)} \text{ a.e. on } R_k, \; \|z_k\|_{W^{2,\infty}} \leq C(n,p,A)k.$$

We have

$$\begin{aligned}|\{x\in\Omega : z_k\neq w_{n(k)}\}| &\leq |\tilde{R}_k|+\frac{1}{k^{p+1}}\\ &\leq \frac{1}{k^p}\|Ma_{n(k)}\|^p_{L^p}+\frac{1}{k^{p+1}},\end{aligned}$$

and this term tends to zero as $k\to\infty$. For a.e. $x\in R_k$ we have

$$|\nabla^2 z_k| = |\nabla^2 w_{n(k)}| \leq \left|M(a_{n(k)})\right| = \left|T_k(M(a_{n(k)}))\right|,$$

while if $x\in A\cap\tilde{R}_k^c$ we have

$$\left|\nabla^2 z_k(x)\right| \leq C(n,p,A)k \leq C(n,p,A)\left|T_k(M(a_{n(k)}))(x)\right|.$$

For $x\in(A\cap\tilde{R}_k)\backslash R_k)$ we can only conclude

$$|\nabla^2 z_k(x)| \leq C(n,p,A)k.$$

Since we have

$$\int_A |\nabla^2 z_k|dx = \int_{R_k}|\nabla^2 z_k|dx + \int_{A\backslash R_k}|\nabla^2 z_k|dx + \int_{(A\cap\tilde{R}_k)\backslash R_k}|\nabla^2 z_k|dx.$$

and

$$\begin{aligned}\int_{R_k}|\nabla^2 z_k|^p dx &\leq \int_{R_k}|T_k(M(a_{n(k)}))|^p,\\ \int_{A\backslash R_k}|\nabla^2 z_k|^p &\leq \int_{A\backslash R_k}|T_k(M(a_{n(k)}))|^p,\\ \int_{(A\cap\tilde{R}_k)\backslash R_k}|\nabla^2 z_k|dx &\leq \frac{C(n,p,A)}{k},\end{aligned}$$

taking into account that $T_k(M(a_{n(k)}))$ is equi-integrable, we have the claim. It is easy to see that from the property (122) it follows that $(z_k)_{k\in\mathbb{N}}$ has the same weak limit as $(w_k)_{k\in\mathbb{N}}$. □

The following proposition can be found in [BF02] (see also [BZ07]).

Proposition A.6. *Let $1<p<+\infty$ and $A\subset\mathbb{R}^2$ be a open bounded set with Lipschitz boundary. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to 0 and let $(w_n)_{n\in\mathbb{N}}$ be a bounded sequence in $W^{1,p}(A\times I,\mathbb{R}^3)$ satisfying:*

$$\limsup_{n\in\mathbb{N}}\int_{A\times I}\left|\begin{pmatrix}\partial_1 w_n & \partial_2 w_n & \frac{1}{h_n}\partial_3 w_n\end{pmatrix}\right|^p dx < +\infty.$$

Suppose further that $w_n \rightharpoonup \psi$ weakly in $W^{1,p}(A\times I,\mathbb{R}^3)$. Then there exists a subsequence $(w_{n(k)})_{k\in\mathbb{N}}$ and a sequence $(z_k)_{k\in\mathbb{N}}$ such that

(a) $\lim_{k\to\infty}|x\in A\times I:\ z_k(x)\neq w_{n(k)}(x)|=0,$
(b) $\left\{\begin{pmatrix}\partial_1 z_k & \partial_2 z_k & \frac{1}{h_k}\partial_3 z_k\end{pmatrix}\right\}$ *is equi-integrable,*
(c) $z_k\rightharpoonup\psi$ *weakly in* $W^{1,p}(A\times I,\mathbb{R}^3)$

The following proposition is the simple case of [DM93, Proposition 11.9.]

Proposition A.7. *Let X be a finite dimensional vector space over the real numbers and $F: X\to[0,+\infty)$ an arbitrary function. If*

a) $F(0)=0,$
b) $F(tx)\leq t^2F(x)$ *for every* $x\in X$ *and for every* $t>0,$

c) $F(x+y)+F(x-y) \leq 2F(x)+2F(y)$ *for every* $x, y \in X$,

then F *is a quadratic form. Conversely if* F *is a quadratic form then (a), (b), (c) are satisfied, and, in addition,*

d) $F(tx) = t^2 F(x)$ *for every* $x \in X$ *and for every* $t \in \mathbb{R}$ *with* $t \neq 0$,
e) $F(x+y)+F(x-y) = 2F(x)+2F(y)$, *for every* $x, y \in X$.

If ω is a Lipschitz domain by $\mathcal{A} = \mathcal{A}(\omega)$ we denote the class of all open subsets of ω; while by $\mathcal{B} = \mathcal{B}(\omega)$ we denote the class of all Borel subsets of ω. By $\mathcal{A}_0$ we denote the class of all open sets of ω that are compactly contained in ω. The following definitions, Lemma and theorem can be found in [DM93, Chapter 14].

Definition A.8. For a function $\alpha : \mathcal{A} \to \overline{\mathbb{R}}$ we say that it is increasing if $\alpha(A) \leq \alpha(B)$, whenever $A, B \in \mathcal{A}$, $A \subset B$. We say that the increasing function $\alpha : \mathcal{A} \to \overline{\mathbb{R}}$ is inner regular if

$$\alpha(A) = \sup\{\alpha(B) : B \in \mathcal{A},\ B \ll A\}.$$

Definition A.9. We say that a subset $\mathcal{D}$ of $\mathcal{A}$ is dense in $\mathcal{A}$ if for every $A, B \in \mathcal{A}$, with $A \ll B$, there exists $D \in \mathcal{D}$, such that $A \ll D \ll B$.

Remark 13. If $\alpha : \mathcal{A} \to \overline{\mathbb{R}}$ is an increasing function and $\mathcal{D}$ is the dense subset of $\mathcal{A}$ then we have that

$$\alpha(A) = \sup\{\alpha(D) : D \in \mathcal{D},\ D \ll A\}.$$

Definition A.10. Let $\alpha : \mathcal{A} \to \overline{\mathbb{R}}$ be non-negative increasing function. We say that

a) α is subadditive on $\mathcal{A}$ if $\alpha(A) \leq \alpha(A_1)+\alpha(A_2)$ for every $A, A_1, A_2 \in \mathcal{A}$ with $A \subset A_1 \cup A_2$;
b) α is superadditive on $\mathcal{A}$ if $\alpha(A) \geq \alpha(A_1)+\alpha(A_2)$ for every $A, A_1, A_2 \in \mathcal{A}$ with $A_1 \cup A_2 \subset A$ and $A_1 \cap A_2 = \emptyset$;
c) α is a measure on $\mathcal{A}$ if there exists a Borel measure $\mu : \mathcal{B} \to [0, +\infty]$ such that $\alpha(A) = \mu(A)$ for every $A \in \mathcal{A}$.

The following is [DM93, Lemma 14.20]

Lemma A.11. *Let* $A, B, C \in \mathcal{A}$ *with* $C \ll A \cup B$. *Then there exist* $A', B' \in \mathcal{A}_0$ *such that* $C \ll A' \cup B'$, $A' \ll A$, $B' \ll B$.

The following is Theorem [DM93, Theorem 14.23].

Theorem A.12. *Let* $\alpha : \mathcal{A} \to [0, +\infty]$ *be a non-negative increasing function such that* $\alpha(\emptyset) = 0$. *The following conditions are equivalent.*

(i) α *is a measure on* $\mathcal{A}$;
(ii) α *is subadditive, superadditive and inner regular on* $\mathcal{A}$.

Remark 14. It can be seen that the measure μ which extends α is given by

$$\mu(E) = \inf\{\alpha(A) : A \in \mathcal{A},\ E \subset A\}.$$

We give some simple lemma about the sets with Lipschitz boundary.

Lemma A.13. *Let* $A \subset \mathbb{R}^n$ *be an open, bounded set with Lipschitz boundary. Then* A *has finite number of connected components.*

Proof. Denote by $\{\Gamma_\alpha\}_{\alpha\in\Lambda}$ the connected components of ∂A. We want to prove that there is only finitely many such components. Suppose that there is infinite many such components. Then for each $n \in \mathbb{N}$ we can find $x_n \in \Gamma_n$, where $\Gamma_i \neq \Gamma_j$, for all $i \neq j$. Since ∂A is compact we have that at least on a subsequence, still denoted by $(x_n)_{n\in\mathbb{N}}$ $x_n \to x \in \partial A$. Since A has Lipschitz boundary we can find a Lipschitz frame around point $x \in \partial A$, with radius $\varepsilon > 0$. This means that there exists a bijective map $f_x : B(x,\varepsilon) \to B(0,1)$ such that f_x and f_x^{-1} are Lipschitz continuous and such that $f_x(\partial A \cap B(x,\varepsilon)) = B(0,1) \cap \{x_n = 0\}$ and $A \cap B(x,\varepsilon) = B(0,1) \cap \{x_n > 0\}$. This contradicts the fact that $x_n \to x$ and that x_n belong to different connected components of ∂A. Thus we have proved that ∂A has finitely many connected components. Take now all the connected components of the set A and denote them by $\{A_\alpha\}_{\alpha\in\Lambda}$. Using that A has Lipschitz boundary it is easy to see that $\partial A \subset \cup_{\alpha\in\Lambda}\partial A_\alpha$. Then it is easy to check that $\partial A = \cup_{\alpha\in\Lambda}\partial A_\alpha$. Moreover it is easy to see that for $\alpha \neq \beta$, $\partial A_\alpha \cap \partial A_\beta = \emptyset$. Namely, if there is $x \in \partial A_\alpha \cap \partial A_\beta$, then by taking Lipschitz frame around x, it can be seen that A_α and A_β would be connected. Also it is easy to see that every connected component of the boundary can be part of the boundary at most one of the connected component of A. This implies that there can be only finitely many connected components of A and that they have disjoint closure. □

We give one lemma about convergence of minimizers for specific elliptic system. This is the well known claim that the pointwise convergence of the coefficients implies the convergence of minimizers. We give the proof for the sake of completeness.

Lemma A.14. *Let $D \subset \mathbb{R}^2$ with Lipschitz boundary. Let $(Q_n)_{n\in\mathbb{N}}, Q : D\times\mathbb{R}^{2\times2}_{\mathrm{sym}}\times\mathbb{R}^{2\times2}_{\mathrm{sym}} \to \mathbb{R}$ satisfy the property (Q'1). Assume that $Q_n(y,M_1,M_2) \to Q(y,M_1,M_2)$ for a.e. $y \in D$, for all $M_1, M_2 \in \mathbb{R}^{2\times2}_{\mathrm{sym}}$. Then we have*

$$\min_{\substack{u\in H^1(D,\mathbb{R}^2),\ v\in H^2(D),\\ (u,v)\in\mathcal{U}(D)}} \int_D Q_n(y, M_1+\nabla u, M_2-\nabla^2 v)\,\mathrm{d}y \to$$
$$\min_{\substack{u\in H^1(D,\mathbb{R}^2),\ v\in H^2(D),\\ (u,v)\in\mathcal{U}(D)}} \int_D Q(y, M_1+\operatorname{sym}\nabla u, M_2-\nabla^2 v)\,\mathrm{d}y,$$

as $n\to\infty$, for arbitrary $M_1, M_2 \in L^2(D,\mathbb{R}^{2\times2}_{\mathrm{sym}})$. Here $\mathcal{U}(D)$ is any closed subset in weak $H^1(D,\mathbb{R}^2)\times H^2(D)$ topology which has the property

$$(u,v)\in\mathcal{U} \implies \|u\|_{H^1}+\|v\|_{H^2} \le C(D)\left(\|\operatorname{sym}\nabla' u\|_{L^2}+\|\nabla'^2 v\|_{L^2}\right), \tag{124}$$

for some $C(D) > 0$ independent of u, v. Moreover, if (u_n, v_n) are the minimizers of the problem

$$\min_{\substack{u\in H^1(D,\mathbb{R}^2),\ v\in H^2(D),\\ (u,v)\in\mathcal{U}(D)}} \int_D Q_n(y, M_1+\nabla' u, M_2-\nabla'^2 v)\,\mathrm{d}y,$$

then we have that $\|\operatorname{sym}\nabla'(u_n-u)\|_{L^2}\to 0$ and $\|\nabla'^2(v_n-v)\|_{L^2}\to 0$ where (u,v) are the minimizers of the problem

$$\int_D Q(y, M_1+\operatorname{sym}\nabla' u, M_2-\nabla'^2 v)\,\mathrm{d}y.$$

Proof. In the proof we will take $M_1 = M_2 = 0$. We will use the arguments from Γ-convergence. The minimums exist, by the direct methods of the calculus of variation, using (Q'1) and (124). Take $u_n \in H^1(D,\mathbb{R}^2)$, $v_n \in H^2(D)$ which satisfy

$$\int_D Q_n(y, \operatorname{sym}\nabla' u_n, \nabla'^2 v_n) = \min_{\substack{u\in H^1(D,\mathbb{R}^2),\ v\in H^2(D),\\ (u,v)\in\mathcal{U}(D)}} \int_D Q_n(y, \operatorname{sym}\nabla' u, \nabla'^2 v)\,\mathrm{d}y,$$
$$(u,v)\in\mathcal{U}(D).$$

Denote by u, v the weak limits (on a subsequence) of $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ in H^1 i.e. H^2 respectively. For $y \in D$, denote by $\mathcal{A}_n(y) : \mathbb{R}^{2\times2}_{\text{sym}} \times \mathbb{R}^{2\times2}_{\text{sym}} \to \mathbb{R}^{2\times2}_{\text{sym}} \times \mathbb{R}^{2\times2}_{\text{sym}}$ the symmetric linear operator which realizes the form $Q_n(y,\cdot,\cdot)$ i.e. for which is valid

$$\mathcal{A}_n(y)(M_1', M_2')\cdot(M_1', M_2') = Q_n(y, M_1', M_2'), \quad \forall M_1', M_2' \in \mathbb{R}^{2\times2}_{\text{sym}},$$

and by $\mathcal{A}(y) : \mathbb{R}^{2\times2}_{\text{sym}} \times \mathbb{R}^{2\times2}_{\text{sym}} \to \mathbb{R}^{2\times2}_{\text{sym}} \times \mathbb{R}^{2\times2}_{\text{sym}}$ the symmetric linear operator which realizes the form $Q(y,\cdot,\cdot)$. Notice that $|\mathcal{A}_n(y)| \leq \beta$, $|\mathcal{A}(y)| \leq \beta$, for any $y \in D$ and $\mathcal{A}_n(y) \to \mathcal{A}(y)$ for a.e. $y \in D$. We obtain

$$\begin{aligned}
(125)\quad & \int_D Q_n(y, \operatorname{sym}\nabla' u_n, \nabla'^2 v_n)\,\mathrm{d}y = \\
& \int_D \mathcal{A}_n(y)(\operatorname{sym}\nabla' u, \nabla'^2 v)\cdot(\operatorname{sym}\nabla' u, \nabla'^2 v)\,\mathrm{d}y + \\
& 2\int_D \mathcal{A}_n(y)(\operatorname{sym}\nabla' u, \nabla'^2 v)\cdot\big(\operatorname{sym}\nabla'(u_n-u), \nabla'^2(v_n-v)\big)\,\mathrm{d}y + \\
& \int_D \mathcal{A}_n(y)\big(\operatorname{sym}\nabla'(u_n-u), \nabla'^2(v_n-v)\big)\cdot\big(\operatorname{sym}\nabla'(u_n-u), \nabla'^2(v_n-v)\big)\,\mathrm{d}y \\
& \geq \int_D \mathcal{A}_n(y)(\operatorname{sym}\nabla' u, \nabla'^2 v)\cdot(\operatorname{sym}\nabla' u, \nabla'^2 v)\,\mathrm{d}y + \\
& \quad 2\int_D \mathcal{A}_n(y)(\operatorname{sym}\nabla' u, \nabla'^2 v)\cdot\big(\operatorname{sym}\nabla'(u_n-u), \nabla'^2(v_n-v)\big)\,\mathrm{d}y + \\
& \quad \tfrac{\alpha}{12}\left(\|\operatorname{sym}\nabla'(u_n-u)\|^2_{L^2} + \|\nabla'^2(v_n-v)\|^2_{L^2}\right).
\end{aligned}$$

By the Lebesgue theorem of the dominated convergence we have

$$\int_D \mathcal{A}_n(y)(\operatorname{sym}\nabla' u, \nabla'^2 v)\cdot(\operatorname{sym}\nabla' u, \nabla'^2 v)\,\mathrm{d}y \to \int_D \mathcal{A}(y)(\operatorname{sym}\nabla' u, \nabla'^2 v)\cdot(\operatorname{sym}\nabla' u, \nabla'^2 v)\,\mathrm{d}y,$$

as $n\to\infty$.

Using the fact that

$$\begin{aligned}
&\left(\int_D \left(\mathcal{A}_n(y)-\mathcal{A}(y)\right)(\operatorname{sym}\nabla' u, \nabla'^2 v)\cdot\big(\operatorname{sym}\nabla'(u-u_n), \nabla'^2(v-v_n)\big)\,\mathrm{d}y\right)^2 \leq \\
&\int_D \left|\left(\mathcal{A}_n(y)-\mathcal{A}(y)\right)(\operatorname{sym}\nabla' u, \nabla'^2 v)\right|^2\mathrm{d}y \int_D \left|\big(\operatorname{sym}\nabla'(u-u_n), \nabla'^2(v-v_n)\big)\right|^2\mathrm{d}y \\
&\to 0,
\end{aligned}$$

by the Lebesgue theorem of the dominated convergence, as well as that

$$\int_D \mathcal{A}(y)(\operatorname{sym}\nabla' u, \nabla'^2 v)\cdot\big(\operatorname{sym}\nabla'(u-u_n), \nabla'^2(v-v_n)\big)\,\mathrm{d}y \to 0,$$

we have that

$$\int_D Q_n(y, \operatorname{sym}\nabla' u_n, \nabla'^2 v_n)\,\mathrm{d}y \geq \int_D Q(y, \operatorname{sym}\nabla' u, \nabla'^2 v)\,\mathrm{d}y,$$

which is the lower bound. The upper bound is trivial, since we can use constant sequence $u_n = u$, $v_n = v$. Argumenting as in Γ-convergence we can prove the claim, using the arbitrariness of the sequence $(s_n)_{n=1}^{\infty}$. Notice that we have also proved that the minimizers (u_n, v_n) weakly converge (on a subsequence) to the minimizer (u, v) in $H^1 \times H^2$. Going back to (125) we also obtain necessary estimate. □

Corollary A.15. *Let $Y = [-\frac{1}{2}, \frac{1}{2}]^2$. Let $Q : \omega \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}$ satisfy the property (Q'1). Then for almost every $x_0' \in \omega$ the following property is valid*

$$\min_{\substack{u\in H^1(D,\mathbb{R}^2),\ v\in H^2(D),\\ (u,v)\in\mathcal{U}(D)}} \int_D Q(x_0' + sy, M_1 + \mathrm{sym}\,\nabla' u, M_2 - \nabla'^2 v)\,\mathrm{d}y \to$$
$$\min_{\substack{u\in H^1(D,\mathbb{R}^2),\ v\in H^2(D),\\ (u,v)\in\mathcal{U}(D)}} \int_D Q(x_0', M_1 + \mathrm{sym}\,\nabla' u, M_2 - \nabla'^2 v)\,\mathrm{d}y,$$

as $s \to 0$, for arbitrary and $D \subset Y$ with Lipschitz boundary and $M_1, M_2 \in L^2(D, \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. Here $\mathcal{U}(D)$ is any closed subset in weak $H^1(D,\mathbb{R}^2)\times H^2(D)$ topology which has the property

$$(u, v) \in \mathcal{U} \implies \|u\|_{H^1} + \|v\|_{H^2} \leq C(D) \left(\|\mathrm{sym}\,\nabla' u\|_{L^2} + \|\nabla'^2 v\|_{L^2}\right),$$

for some $C(D) > 0$ independent of u, v.

Proof. From [AFP00, Lemma 5.38] for almost every $x_0' \in \omega$ the following property is valid: for any sequence $(s_n)_{n=1}^{\infty}$ monotonly decreasing to 0, there exists a subsequence, still denoted by $(s_n)_{n\in\mathbb{N}}$ and $E \subset Y$ of measure 0 such that

$$\lim_{n\to\infty} Q(x_0' + s_{k(n)} y, \cdot, \cdot) = Q(x_0', \cdot, \cdot),$$

locally uniformly in $R^{2\times 2}_{\mathrm{sym}} \times \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ for every $y \in Y \backslash E$. Using Lemma (A.14) we conclude that

$$\min_{\substack{u\in H^1(D,\mathbb{R}^2),\ v\in H^2(D),\\ (u,v)\in\mathcal{U}(D)}} \int_D Q(x_0' + s_n y, M_1 + \mathrm{sym}\,\nabla' u, M_2 - \nabla'^2 v)\,\mathrm{d}y \to$$
$$\min_{\substack{u\in H^1(D,\mathbb{R}^2),\ v\in H^2(D),\\ (u,v)\in\mathcal{U}(D)}} \int_D Q(x_0', M_1 + \mathrm{sym}\,\nabla' u, M_2 - \nabla'^2 v)\,\mathrm{d}y.$$

The arbitrariness of the sequence gives the claim. □

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