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by

Ming Li, Ming-Jing Zhao, Shao-Ming Fei, and Xianqing Li-Jost

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Ming Li[1,3], Ming-Jing Zhao[3], Shao-Ming Fei[2,3] and Zhi-Xi Wang[2]

[1] College of the Science, China University of Petroleum, 266580 Qingdao

[2] School of Mathematical Sciences, Capital Normal University, 100048 Beijing

[3] Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig

Abstract

In this review, we introduce some methods for detecting or measuring entanglement. Several non-linear entanglement witnesses are presented. We derive a series of Bell inequalities whose maximally violations for any multipartite qubit states can be calculated by using our formulas. Both the non-linear entanglement witnesses and the Bell inequalities can be operated experimentally. Thus they supply an effective way for detecting entanglement. We also introduce some experimental methods to measure the entanglement of formation, and the lower bound of the convex-roof extension of negativity.

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I. INTRODUCTION

As one of the most striking feature of quantum systems, quantum entanglement plays crucial roles in quantum information processing [1]. Entangled quantum states are among the most important physical resources in the rapidly expanding field of quantum information science, with various applications such as quantum computation [1, 2], quantum teleportation [3], dense coding [4], quantum cryptographic schemes [5], quantum radar[6], entanglement swapping [7] and remote states preparation (RSP) [8–11]. Nevertheless, there are still many open problems in the theory of quantum entanglement for both bipartite and multipartite quantum systems.

One of the fundamental problems in quantum entanglement theory is to determine which states are entangled and which are not, either theoretically or experimentally. Entanglement witness is traditionally a quantum mechanical observable, whose expectation value is below certain bound for all separable states and exceeds the bound for some entangled states [12, 13]. In recent years there have been considerable efforts in constructing and analyzing the structure of entanglement witness (see [14–17] and the references therein). Generally the Bell inequalities [18–23] can be recast as entanglement witnesses. Better entanglement witnesses can be also constructed from more effective Bell-type inequalities.

On the other side, it is also a significant problem to quantify quantum entanglement in quantum information theory. A number of entanglement measures such as the entanglement of formation (EOF) and distillation [24–26], negativity [27] and relative entropy [26, 28] have been proposed for bipartite systems [25] [28]-[33]. The negativity was derived from the positive partial transposition (PPT) [34]. It bounds two relevant quantities characterizing the entanglement of mixed states: the channel capacity and the distillable entanglement. The convex-roof extension of the negativity (CREN) [35] gives a better characterization of entanglement, which is nonzero for PPT entangled quantum states.

The higher dimensional systems offer advantages such as increased security in a range of quantum information protocols [36], greater channel capacity for quantum communication [37], novel fundamental tests of quantum mechanics [38], and more efficient quantum gates [39]. In particular, hybrid qubit-qutrit system has been extensively studied and already experimentally realized [40]. Entanglement for bipartite quantum systems has been intensively studied with rich understanding. However, for multipartite quantum systems, very few are known for the characterization and quantification.

A method to measure the concurrence of pure sates has been proposed in [41]. The authors have shown that the concurrence $C(|\psi\rangle)$ of an N-partite system pure state $|\psi\rangle \in \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \cdots \otimes \mathcal{H}^N$, can be expressed as the following expectation value with respect to two copies of $|\psi\rangle$,

$$C(|\psi\rangle) = \sqrt{\langle\psi| \otimes \langle\psi|A|\psi\rangle \otimes |\psi\rangle}. \tag{1}$$

Here A is a Hermitian operator acting on $\mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \cdots \otimes \mathcal{H}^N \otimes \mathcal{H}^1 \otimes \mathcal{H}^2 \otimes \cdots \otimes \mathcal{H}^N$. The operator A can be written as:

$$A = 4 \sum_{\{s_{j_i}=\pm\}^+} P^1_{s_{1_i}} \otimes P^2_{s_{2_i}} \otimes \cdots \otimes P^N_{s_{N_i}}, \tag{2}$$

where P^j_+ and P^j_- are the projectors onto the symmetric and antisymmetric subspaces $\mathcal{H}^j \odot \mathcal{H}^j$, and $\mathcal{H}^j \wedge \mathcal{H}^j$ respectively. The set $\{s_{j_i} = \pm\}^+$ is composed of all possible ways of sorting the symbols '$+$' and '$-$' in an N-long string, such that the total amount of '$-$' symbols is an even number, and excluding the completely symmetric case with no '$-$' symbols at all.

The method is further experimentally demonstrated [42, 43]. This protocol needs a twofold copy of the quantum state at every measurement. A way of measuring concurrence for any pure states by using only one copy of the state at each measurement has been presented in [44].

In this review, we introduce several inequalities for experimentally detecting and quantifying entanglement of bipartite quantum systems with arbitrary dimensions in section II. Constructing Bell inequalities for multipartite qubit systems and their maximal violation are shown in section III. Also we derive some measurable lower bounds for entanglement measures in section IV. We give conclusions and remarks in section V.

Throughout this review, $\mathcal{H}_d$ denotes a d-dimensional vector space with computational basis $|0\rangle = (1, 0, ..., 0)^T$, $|1\rangle = (0, 1, ..., 0)^T$, ..., $|d-1\rangle = (0, 0, ..., 1)^T$, where T denotes transpose. Generally an N-partite quantum state is a density matrix in $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \cdots \otimes \mathcal{H}_{d_N}$, with dimensions d_i for the ith system.

II. INEQUALITIES DETECTING ENTANGLEMENT

In this section, we present a series of inequalities as entanglement witnesses for bipartite quantum systems. These inequalities are given by mean values of local observables and give rise to experimental ways for detecting the quantum entanglement of bipartite quantum states.

A. Inequalities detecting bound entanglement for $2 \otimes d$ systems

To construct inequalities that detect entanglement, we first present a lemma that will be used in proving our theorem for $2 \otimes 3$ system.

Lemma 1 *If the inequality*

$$a_i^2 \geq b_i^2 + c_i^2 \tag{3}$$

holds for arbitrary real numbers b_i and c_i, and nonnegative a_i, $i = 1, \cdots, n$, then

$$(\sum_{i=1}^{n} p_i a_i)^2 \geq (\sum_{i=1}^{n} p_i b_i)^2 + (\sum_{i=1}^{n} p_i c_i)^2$$

for $0 \leq p_i \leq 1$ and $\sum_{i=1}^{n} p_i = 1$.

Proof. From Eq. (3), we have $a_i^2 a_j^2 \geq (b_i^2+c_i^2)(b_j^2+c_j^2) \geq (b_ib_j+c_ic_j)^2$. Due to $a_i \geq 0$ for $i=1,\cdots,n$, one gets

$$\begin{aligned}(\sum_{i=1}^n p_ia_i)^2 &= \sum_{i=1}^n p_i^2a_i^2 + 2\sum_{i\neq j} p_ip_ja_ia_j \\ &\geq \sum_{i=1}^n p_i^2(b_i^2+c_i^2) + 2\sum_{i<j} p_ip_j(b_ib_j+c_ic_j) \\ &= (\sum_{i=1}^n p_ib_i)^2 + (\sum_{i=1}^n p_ic_i)^2,\end{aligned}$$

which completes the proof of the lemma. □

Consider bipartite mixed states in $H_2 \otimes H_3$. Let $A_i = U\sigma_iU^\dagger$, $i=1,2,3$, be a set of quantum mechanical observables with U any 2×2 unitary matrix, and $\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_2 = i|0\rangle\langle 1| - i|1\rangle\langle 0|$ and $\sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$ the Pauli matrices, where $|k\rangle \in H_2$, $k=0,1$. Let $B_j = V\lambda_jV^\dagger$, $j=1,2,3,4$, be the observables associated with the space H_3, with V any 3×3 unitary matrix, $\lambda_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$, $\lambda_2 = |0\rangle\langle 0| - |2\rangle\langle 2|$, $\lambda_3 = |0\rangle\langle 1| + |1\rangle\langle 0|$ and $\lambda_4 = i|0\rangle\langle 1| - i|1\rangle\langle 0|$, where $|k\rangle \in H_3$, $k=0,1,2$. According to these observables we can construct inequalities detecting entanglement perfectly for $2\otimes 3$ systems.

Theorem 1 *Any state ρ in $H_2\otimes H_3$ is separable if and only if the following inequality*

$$\begin{aligned}&\langle 2I_2\otimes I_3 - I_2\otimes B_1 + 2I_2\otimes B_2 + 3A_3\otimes B_1\rangle_\rho \\ &\geq (\langle 3I_2\otimes B_1 + 2A_3\otimes I_3 - A_3\otimes B_1 + 2A_3\otimes B_2\rangle_\rho^2 \\ &+9\langle A_1\otimes B_3 + A_2\otimes B_4\rangle_\rho^2)^{\frac{1}{2}}\end{aligned} \quad (4)$$

holds for all set of observables $\{A_i\}_{i=1}^3$ and $\{B_j\}_{j=1}^4$, where I_d denotes the $d\times d$ identity matrix.

Proof. Part 1. First we prove that the state is separable if the inequality (4) holds. Any pure state $|\psi\rangle \in H_2\otimes H_3$ has the Schmidt decomposition:

$$|\psi\rangle = \alpha|00\rangle + \beta|11\rangle, \quad 0\leq\beta\leq\alpha\leq 1. \quad (5)$$

Applying partial transpose with respect to the first space H_2 to $|\psi\rangle\langle\psi|$, we get that the corresponding density matrix $|\psi\rangle\langle\psi|$ becomes

$$|\psi\rangle\langle\psi|^{T_1} = \alpha^2|00\rangle\langle 00| + \beta^2|11\rangle\langle 11| + \alpha\beta(|10\rangle\langle 01| + |01\rangle\langle 10|).$$

By expanding the partial transposed matrix $|\psi\rangle\langle\psi|^{T_1}$ according to the matrices $\{\sigma_i\}_{i=1}^3$ and $\{\lambda_j\}_{j=1}^4$ defined above, we get

$$\begin{aligned} |\psi\rangle\langle\psi|^{T_1} &= \tfrac{1}{6}(I_2\otimes I_3+(-\tfrac{1}{2}+\tfrac{3}{2}\sqrt{1-C^2})I_2\otimes\lambda_1+I_2\otimes\lambda_2 \\ &\quad +\sqrt{1-C^2}\sigma_3\otimes I_3+(\tfrac{3}{2}-\tfrac{1}{2}\sqrt{1-C^2})\sigma_3\otimes\lambda_1 \\ &\quad +\sqrt{1-C^2}\sigma_3\otimes\lambda_2)+\tfrac{1}{4}C(\sigma_1\otimes\lambda_3+\sigma_2\otimes\lambda_4), \end{aligned} \tag{6}$$

where $C=2\alpha\beta$ is just the concurrence of the pure state $|\psi\rangle$, defined by $C(|\psi\rangle)=\sqrt{2(1-Tr\rho_1^2)}$. ρ_1 is the reduced density matrix $\rho_1=Tr_2(|\psi\rangle\langle\psi|)$, where Tr_2 stands for the partial trace with respect to the second space H_3.

Let U be an arbitrary 2×2 unitary matrix and V an arbitrary 3×3 unitary matrix. Then $|\Psi\rangle\equiv U^*\otimes V|\psi\rangle$ represents an arbitrary pure state in $H_2\otimes H_3$. Note that a bipartite state $\rho\in H_2\otimes H_3$ is separable if and only if ρ^{T_1} is positive, that is, $\langle\Psi|\rho^{T_1}|\Psi\rangle\geq 0$ for all $|\Psi\rangle\in H_2\otimes H_3$. Therefore

$$\begin{aligned} 0 &\leq \langle\psi|U^T\otimes V^\dagger\rho^{T_1}U^*\otimes V|\psi\rangle \\ &= Tr(\rho^{T_1}U^*\otimes V|\psi\rangle\langle\psi|U^T\otimes V^\dagger) \\ &= Tr(\rho U\otimes V(|\psi\rangle\langle\psi|)^{T_1}U^\dagger\otimes V^\dagger) \\ &\equiv \langle U\otimes V(|\psi\rangle\langle\psi|)^{T_1}U^\dagger\otimes V^\dagger\rangle_\rho \end{aligned}$$

for all U, V, α and β, where $Tr(A^{T_1}B)=Tr(AB^{T_1})$ has been taken into account and Tr stands for trace. Hence we have

$$\begin{aligned} 12\langle\Psi|\rho^{T_1}|\Psi\rangle &= 12\langle U\otimes V(|\psi\rangle\langle\psi|)^{T_1}U^\dagger\otimes V^\dagger\rangle_\rho \\ &= \langle 2I_2\otimes I_3+(-1+3\sqrt{1-C^2})I_2\otimes B_1+2I_2\otimes B_2 \\ &\quad +2\sqrt{1-C^2}A_3\otimes I_3+(3-\sqrt{1-C^2})A_3\otimes B_1 \\ &\quad +2\sqrt{1-C^2}A_3\otimes B_2\rangle_\rho+3C\langle A_1\otimes B_3+A_2\otimes B_4\rangle_\rho \\ &\geq \langle 2I_2\otimes I_3-I\otimes B_1+2I_2\otimes B_2+3A_3\otimes B_1\rangle_\rho \\ &\quad -|\sqrt{1-C^2}\langle 3I_2\otimes B_1+2A_3\otimes I_3-A_3\otimes B_1 \\ &\quad +2A_3\otimes B_2\rangle_\rho+3C\langle A_1\otimes B_3+A_2\otimes B_4\rangle_\rho| \\ &\geq \langle 2I_2\otimes I_3-I_2\otimes B_1+2I_2\otimes B_2+3A_3\otimes B_1\rangle_\rho \\ &\quad -\{\langle 3I_2\otimes B_1+2A_3\otimes I_3-A_3\otimes B_1+2A_3\otimes B_2\rangle_\rho^2 \\ &\quad +9\langle A_1\otimes B_3+A_2\otimes B_4\rangle_\rho^2\}^{\frac{1}{2}}, \end{aligned} \tag{7}$$

where we have used Eq. (6) and employed the definition of $\{A_i\}$ and $\{B_j\}$ for the second equality. The first inequality is due to $-|x|\leq x$ and the second one is from the Cauchy inequality. Therefore if the inequality (4) holds, the right hand side of inequality (7) is nonnegative. Therefore $\langle\Psi|\rho^{T_1}|\Psi\rangle \geq 0$ for all $|\Psi\rangle \in H_2 \otimes H_3$, and the state is separable according to the PPT criterion.

Part 2. We prove now that if the state is separable, the inequality (4) holds. First we show that inequality (4) holds for all pure separable states, which is equivalent to prove that for arbitrary pure separable state ρ, the following inequality holds:

$$\begin{aligned}&\langle 2I_2 \otimes I_3 - I_2 \otimes \lambda_1 + 2I_2 \otimes \lambda_2 + 3\sigma_3 \otimes \lambda_1\rangle_\rho^2 \\ &\geq \langle 3I_2 \otimes \lambda_1 + 2\sigma_3 \otimes I_3 - \sigma_3 \otimes \lambda_1 + 2\sigma_3 \otimes \lambda_2\rangle_\rho^2 + 9\langle\sigma_1 \otimes \lambda_3 + \sigma_2 \otimes \lambda_4\rangle_\rho^2.\end{aligned} \tag{8}$$

Note that any pure separable state can be written as $|\xi\rangle = (\gamma_1|0\rangle + \gamma_2|1\rangle) \otimes (\phi_0|0\rangle + \phi_1|1\rangle + \phi_2|2\rangle)$ with $|\gamma_1|^2 + |\gamma_2|^2 = 1$ and $|\phi_0|^2 + |\phi_1|^2 + |\phi_2|^2 = 1$. Inserting this separable pure state $|\xi\rangle\langle\xi|$ into Eq. (8), one gets that the square root of the left hand side of (8) becomes

$$\begin{aligned}&\langle 2I_2 \otimes I_3 - I_2 \otimes \lambda_1 + 2I_2 \otimes \lambda_2 + 3\sigma_3 \otimes \lambda_1\rangle_{|\xi\rangle\langle\xi|} \\ &= 6(|\phi_0\gamma_1|^2 + |\phi_1\gamma_2|^2) \geq 0.\end{aligned} \tag{9}$$

While the right hand side of the inequality (8) becomes

$$\begin{aligned}&\langle 3I_2 \otimes \lambda_1 + 2\sigma_3 \otimes I_3 - \sigma_3 \otimes \lambda_1 + 2\sigma_3 \otimes \lambda_2\rangle_{|\xi\rangle\langle\xi|}^2 + 9\langle\sigma_1 \otimes \lambda_3 + \sigma_2 \otimes \lambda_4\rangle_{|\xi\rangle\langle\xi|}^2 \\ &= (6|\gamma_1\phi_0|^2 - 6|\gamma_2\phi_1|^2)^2 + 144(Re(\gamma_1\gamma_2^*)Re(\phi_1^*\phi_0))^2.\end{aligned} \tag{10}$$

The difference between the left and right hand side of (8) is given by

$$\begin{aligned}&\langle 2I_2 \otimes I_3 - I_2 \otimes \lambda_1 + 2I_2 \otimes \lambda_2 + 3\sigma_3 \otimes \lambda_1\rangle_{|\xi\rangle\langle\xi|}^2 \\ &-\langle 3I_2 \otimes \lambda_1 + 2\sigma_3 \otimes I_3 - \sigma_3 \otimes \lambda_1 + 2\sigma_3 \otimes \lambda_2\rangle_{|\xi\rangle\langle\xi|}^2 \\ &-9\langle\sigma_1 \otimes \lambda_3 + \sigma_2 \otimes \lambda_4\rangle_{|\xi\rangle\langle\xi|}^2 \\ &= 144|\gamma_1\gamma_2\phi_0\phi_1|^2 - 144(Re(\gamma_1\gamma_2^*)Re(\phi_1^*\phi_0))^2 \geq 0.\end{aligned} \tag{11}$$

Therefore the inequality (8) holds for any pure separable states.

We now prove that the inequality (4) also holds for general separable mixed states,

$$\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|, \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1,$$

where $|\psi_i\rangle$ are all pure separable states. Set

$$\begin{aligned}a_i &= \langle 2I_2 \otimes I_3 - I_2 \otimes B_1 + 2I_2 \otimes B_2 + 3A_3 \otimes B_1\rangle_{|\psi_i\rangle\langle\psi_i|}, \\ b_i &= \langle 3I_2 \otimes B_1 + 2A_3 \otimes I_3 - A_3 \otimes B_1 + 2A_3 \otimes B_2\rangle_{|\psi_i\rangle\langle\psi_i|}, \\ c_i &= 3\langle A_1 \otimes B_3 + A_2 \otimes B_4\rangle_{|\psi_i\rangle\langle\psi_i|}.\end{aligned}$$

We have

$$(\sum_i p_i a_i)^2 = \langle 2I_2\otimes I_3 - I_2\otimes B_1 + 2I_2\otimes B_2 + 3A_3\otimes B_1\rangle_\rho^2,$$

$$(\sum_i p_i b_i)^2 = \langle 3I_2\otimes B_1 + 2A_3\otimes I_3 - A_3\otimes B_1 + 2A_3\otimes B_2\rangle_\rho^2,$$

$$(\sum_i p_i c_i)^2 = 9\langle A_1\otimes B_3 + A_2\otimes B_4\rangle_\rho^2.$$

Since inequality (8) holds for all pure separable states, $a_i^2 \geq b_i^2 + c_i^2$. Furthermore, from the inequality (9) one gets $a_i \geq 0$. From the lemma one gets $(\sum_i p_i a_i)^2 \geq (\sum_i p_i b_i)^2 + (\sum_i p_i c_i)^2$, which verifies that any mixed separable state ρ obeys the inequality (4). □

We have shown that any state ρ in $H_2\otimes H_3$ is separable if and only if the inequality (4) is satisfied. The inequality (4) gives a necessary and sufficient separability criterion for general qubit-qutrit states. The separability of the state can be determined by experimental measurements on the local observables. For instance, consider the mixed state

$$\rho = p|\psi^+\rangle\langle\psi^+| + \frac{1-p}{6} I_6,$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Let us take $U = I_2$ and $V = |0\rangle\langle 1| + |1\rangle\langle 0| + |2\rangle\langle 2|$. Let $F^{(3)}_{\{U\},\{V\}}(\rho)$ denote the value of violation of the inequality (4),

$$\begin{aligned} F^{(3)}_{\{U\},\{V\}}(\rho) \equiv\ & (\langle 3I_2\otimes B_1 + 2A_3\otimes I_3 - A_3\otimes B_1 + 2A_3\otimes B_2\rangle_\rho^2 \\ & +9\langle A_1\otimes B_3 + A_2\otimes B_4\rangle_\rho^2)^{\frac{1}{2}} \\ & -\langle 2I_2\otimes I_3 - I_2\otimes B_1 + 2I_2\otimes B_2 + 3A_3\otimes B_1\rangle_\rho. \end{aligned}$$

By straightforward calculation we have $F^{(3)}_{U,V}(\rho) = 8p - 2 > 0$ for $p > \frac{1}{4}$. As this state is entangled if and only if $p > \frac{1}{4}$, our inequality (4) detects all the entanglement of the state.

We consider now the maximal violation of the inequality (4). Let $F^{(3)}(\rho) = \max_{\{U\},\{V\}}\{F^{(3)}_{\{U\},\{V\}}(\rho), 0\}$ denote the maximal violation value with respect to a given state ρ, under all $\{U\}$ and $\{V\}$. Obviously, $F^{(3)}(\rho) = 0$ if ρ is separable. For an entangled state ρ, $F^{(3)}(\rho) \geq -12\lambda_{\min}$, where $\lambda_{\min}$ is the minimal eigenvalue of the partial transposed density matrix of ρ, $\lambda_{\min} = \min_{U,V,\alpha,\beta}\langle U\otimes V(|\psi\rangle\langle\psi|)^{T_1} U^\dagger \otimes V^\dagger\rangle_\rho$, where $|\psi\rangle$ is given by Eq. (5). As an example, let us simply take the observables $\{A_i\}_{i=1}^3$ to be $\{\sigma_1, \sigma_2, \sigma_3\}$ and $\{B_j\}_{j=1}^4$ to be $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, i.e. $U = I_2$ and $V = I_3$ in the theorem 1. The violation corresponding to the pure state $\alpha_0|01\rangle + \beta_0|10\rangle$ is $F^{(3)}_{I_2,I_3}(\rho) = 12\alpha_0\beta_0$. For the maximally entangled state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, the corresponding maximal violation value is 6.

For given U and V, inequality (4) also gives rise to a kind of entanglement witness $W_{U,V}$:

$$\begin{aligned} W_{U,V} =\ & \langle 2I_2\otimes I_3 - I_2\otimes B_1 + 2I_2\otimes B_2 + 3A_3\otimes B_1\rangle_\rho \\ & -\ (\langle 3I_2\otimes B_1 + 2A_3\otimes I_3 - A_3\otimes B_1 + 2A_3\otimes B_2\rangle_\rho^2 \\ & +\ 9\langle A_1\otimes B_3 + A_2\otimes B_4\rangle_\rho^2)^{\frac{1}{2}}. \end{aligned}$$

For all separable states σ, $Tr(W_{U,V}\sigma) \geq 0$. If $Tr(W_{U,V}\rho) < 0$ then ρ is entangled. Every entanglement witness $W_{U,V}$ detects a certain set of entangled states. Witnesses $\{W_{U,V}\}$ under all U and V together detect all the entangled states, since all entangled states violate the inequality (4).

Here as in [17], the observables in theorem 1 are not independent. The three observables $\{A_i\}_{i=1}^3$ for the first subsystem and the four observables $\{B_i\}_{j=1}^4$ for the second subsystem fulfill the relations $A_1A_2 = -iA_3$ and $B_3B_4 = -iB_1$ respectively.

Based on the PPT criterion, we have derived the inequality which is both sufficient and necessary for separability of $H_2 \otimes H_3$ system. Our approach can be also applied to other systems such as two-qubit ones, although the approach used in [17] can not be simply applied to $H_2 \otimes H_3$ system. In terms of our approach it is easily to get the following result for two-qubit system: *Any two-qubit state ρ is separable if and only if*

$$\begin{aligned} &\langle I_2 \otimes I_2 + A_3 \otimes B_3\rangle_\rho \\ &\geq (\langle I_2 \otimes B_3 + A_3 \otimes I_2\rangle_\rho^2 + \langle A_1 \otimes B_1 + A_2 \otimes B_2\rangle_\rho^2)^{\frac{1}{2}} \end{aligned} \tag{12}$$

for all set of observables $\{A_i\}_{i=1}^3$ and $\{B_j\}_{j=1}^3$, where $A_i = U\sigma_iU^\dagger$ and $B_j = V\sigma_jV^\dagger$, $i,j = 1,2,3$, U and V are 2×2 unitary matrices. The observables here have the same orientation $\mu = -iA_1A_2A_3 = -iB_1B_2B_3 = 1$. If one replaces σ_3 with $-\sigma_3$, the above inequality still holds. But the orientation becomes $\mu = -iA_1A_2A_3 = -iB_1B_2B_3 = -1$. Namely the inequality (12) is true for all set of observables with the same orientation, which recover the results in [17]. Moreover, one can also obtain that, for a given entangled state the maximal violation of the inequality (12) is $-4\lambda_{\min}$. The possible maximal violation among all states is 3, which is attainable by the maximally entangled states [17].

For higher dimensional bipartite systems, the PPT criterion is only necessary for separability. In the following we study the Bell-type inequalities for $H_2 \otimes H_d$ systems. The quantum states in $H_2 \otimes H_d$ also play important roles in quantum information processing [45–47]. The separability for $H_2 \otimes H_d$ systems could also shed light on the separability of multiqubits systems.

Theorem 2 *(i) Any separable state $\rho \in H_2 \otimes H_d$ obeys the following inequality:*

$$\begin{aligned} &\langle 2I_2 \otimes I_d + (2-d)I_2 \otimes B_1 + 2I_2 \otimes B_2 + \cdots + 2I_2 \otimes B_{d-1} + dA_3 \otimes B_1\rangle_\rho \\ &\geq (\langle dI_2 \otimes B_1 + 2A_3 \otimes I_d + (2-d)A_3 \otimes B_1 + 2A_3 \otimes B_2 \\ &+ \cdots + 2A_3 \otimes B_{d-1}\rangle_\rho^2 + d^2\langle A_1 \otimes B_d + A_2 \otimes B_{d+1}\rangle_\rho^2)^{\frac{1}{2}}, \end{aligned} \tag{13}$$

where the observables $\{A_i\}_{i=1}^3$ are defined as the ones in theorem 1. $B_j = V\lambda_jV^\dagger$, $j = 1,\cdots,d+1$, with V any $d\times d$ unitary matrix, $\lambda_1 = |0\rangle\langle 0| - |1\rangle\langle 1|$, $\lambda_2 = |0\rangle\langle 0| - |2\rangle\langle 2|$, $\cdots$,

$\lambda_{d-1} = |0\rangle\langle 0| - |d-1\rangle\langle d-1|$, $\lambda_d = |0\rangle\langle 1| + |1\rangle\langle 0|$ *and* $\lambda_{d+1} = i|0\rangle\langle 1| - i|1\rangle\langle 0|$, $|j\rangle \in H_d$, $j = 0, ..., d-1$.

(ii) All NPT states in $H_2 \otimes H_d$ *violate the above inequality.*

The proof of (i) is similar to the part 2 in the proof of theorem 1 for necessity of separability. The statement (ii) can be proved analogous to the part 1 in the proof of theorem 1. However as the PPT criterion is no longer both sufficient and necessary for separability of $2 \otimes d$ systems, one has only that all NPT entangled states violate the inequality.

For the cases $d = 2$ and $d = 3$, the inequality (13) reduces to the inequality (12) and (4) respectively.

Let $F^{(d)}(\rho)$ denote the maximal violation value of the inequality (13) for a given state ρ: $F^{(d)}(\rho) = \max_{\{U\},\{V\}}\{F^{(d)}_{\{U\},\{V\}}(\rho), 0\}$, where

$$\begin{aligned} F^{(d)}_{\{U\},\{V\}}(\rho) \;=\; & (\langle dI_2 \otimes B_1 + 2A_3 \otimes I_d + (2-d)A_3 \otimes B_1 + 2A_3 \otimes B_2 \\ & + \cdots + 2A_3 \otimes B_{d-1}\rangle_\rho^2 + d^2\langle A_1 \otimes B_d + A_2 \otimes B_{d+1}\rangle_\rho^2)^{\frac{1}{2}} \\ & -\langle 2I_2 \otimes I_d + (2-d)I_2 \otimes B_1 + 2I_2 \otimes B_2 + \cdots \\ & +2I_2 \otimes B_{d-1} + dA_3 \otimes B_1\rangle_\rho. \end{aligned} \tag{14}$$

Analogously, we have that $F^{(d)}(\rho)$ is invariant under local unitary transformations, $F^{(d)}(\rho) = F^{(d)}(U \otimes V \rho U^\dagger \otimes V^\dagger)$ and $F^{(d)}(\rho) = 0$ if ρ is separable. For any entangled state ρ, we have $F^{(d)}(\rho) \geq -4d\lambda_{\min}$, where $\lambda_{\min}$ is the minimal eigenvalue of the partial transposed density matrix of ρ. Any violation of the inequality (13) implies entanglement. Since all entangled pure states are NPT, Eq. (13) can detect all pure entangled states. Moreover, as all mixed states with rank less than or equal to d are entangled if and only if they are NPT [45], inequality (13) can also detect the entanglement of all such states.

An interesting thing is that although inequality (13) is obtained based on PPT criterion which is no longer sufficient for separability of $2 \otimes d$ systems for $d > 3$, it can still detect the quantum entanglement of some PPT entangled states. Namely, besides all NPT states, some PPT entangled states would also violate the inequality. This can be seen from the proof of the first part of the theorem 1. Any PPT state ρ satisfies $\langle\Psi|\rho^{T_1}|\Psi\rangle \geq 0$ for all pure state $|\Psi\rangle$. From Eq. (7) one can similarly obtain that, for $2 \otimes d$ systems, it is possible that the inequality (13) is violated while $\langle\Psi|\rho^{T_1}|\Psi\rangle \geq 0$ is still satisfied. As an example we consider the family of PPT entangled states in $2 \otimes 4$ systems, introduced in [48]:

$$\sigma_b = \frac{7b}{7b+1}\sigma_{insep} + \frac{1}{7b+1}|\phi_b\rangle\langle\phi_b|, \tag{15}$$

where $0 \le b \le 1$,

$$\sigma_{insep} = \frac{2}{7}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|) + \frac{1}{7}|14\rangle\langle 14|,$$
$$|\phi_b\rangle = |2\rangle \otimes (\sqrt{\tfrac{1+b}{2}}|1\rangle + \sqrt{\tfrac{1-b}{2}}|3\rangle),$$
$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle),$$
$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|12\rangle + |23\rangle),$$
$$|\psi_3\rangle = \frac{1}{\sqrt{2}}(|13\rangle + |24\rangle).$$

The state σ_b is entangled if and only if $0 < b < 1$ [48].

In fact, we can simply choose $U = |0\rangle\langle 1| - |1\rangle\langle 0|$ and $V = I_4$. Then $F^{(4)}_{\{I_2\},\{I_4\}}(\sigma_b) = -8b - 4(1+b) + \sqrt{4096b^2 + (-8b + 4(1+b))^2}$ and $F^{(4)}_{\{I_2\},\{I_4\}}(\sigma_b) > 0$ when $\frac{1}{31} < b < 1$. Therefore, the inequality can detect almost all the entanglement in σ_b (see FIG. 1). In deed our inequality has advantages in detecting entanglement of this PPT entangled state, since the PPT, CCNR, reduction and majorization criteria can not detect all the entanglement of σ_b.

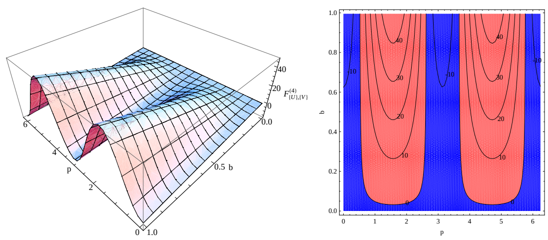

FIG. 1: $U = \cos p(|0\rangle\langle 0| + |1\rangle\langle 1|) + \sin p(|0\rangle\langle 1| - |1\rangle\langle 0|)$, $V = I_4$. Left figure: $F^{(4)}_{\{U\},\{V\}}(\sigma_b)$ with respect to p and b. Right figure: contour plot of the left figure. The dark region: $F^{(4)}_{\{U\},\{V\}}(\sigma_b) < 0$; the gray region: $F^{(4)}_{\{U\},\{V\}}(\sigma_b) > 0$.

The inequality (13) can also detect entanglement for n-qubit pure states. Suppose $|\psi\rangle_{A_1\cdots A_n}$ is an arbitrary n-qubit pure state. If we treat the n-qubit state $|\psi\rangle_{A_1\cdots A_n}$ as a bipartite one with the i-th qubit as one subsystem and the rest qubits as another subsystem, then it is a $2 \otimes 2^{n-1}$ bipartite pure state. $|\psi\rangle_{A_1\cdots A_n}$ is separable under this partition if and only if it fulfills the inequality (13).

B. Inequalities for arbitrary bipartite quantum systems

In this part, we introduce a series of entanglement witnesses that can serve as necessary and sufficient conditions for the separability of bipartite pure quantum states and the isotropic states. These entanglement witnesses are also closely related to the measure of quantum entanglement.

Let us first give a brief review of the 3-setting nonlinear entanglement witnesses enforced by the indeterminacy relation of complementary local observables [17]. For a two-level system there are three mutually complementary observables $A_i = \vec{a}_i \cdot \vec{\sigma}$, where $\vec{a}_i$, $i = 1, 2, 3$, are three normalized vectors that are orthogonal to each other, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. $\mu_A = -iA_1A_2A_3$ is the so called orientation of A_is. μ_A takes values ± 1. Similarly, one can define three mutually complementary observables $B_i = \vec{b}_i \cdot \vec{\sigma}$, $i = 1, 2, 3$, with the corresponding orientations μ_B. It has been shown that [17]: (i) A two-qubit state ρ is separable if and only if the following inequality holds for all sets of observables $\{A_i, B_i\}_{i=1,2,3}$ with the same orientation:

$$\sqrt{\langle A_1B_1 + A_2B_2\rangle_\rho^2 + \langle A_3 + B_3\rangle_\rho^2} - \langle A_3B_3\rangle_\rho \leq 1; \tag{16}$$

(ii) For a given entangled state the maximal violation of the above inequality is $1 - 4\lambda_{\min}$, with $\lambda_{\min}$ being the minimal eigenvalue of the partially transposed density matrix. The maximal possible violation for all states is 3, which is attainable by the maximal entangled states.

The approach in [17] can not be directly generalized to higher dimensional systems, since it is based on the PPT criterion that is both necessary and sufficient only for separability of two-qubit and qubit-qutrit states. For general higher dimensional $M \times N$ bipartite quantum systems a new approach has been employed in [23]. Let $\rho \in \mathcal{H_{AB}}$ be any pure quantum states in vector space $\mathcal{H_{AB}} = \mathcal{H_A} \otimes \mathcal{H_B}$ with dimensions $dim\,\mathcal{H_A} = M$ and $dim\,\mathcal{H_B} = N$ respectively. Assume L_α^A and L_β^B be the generators of special orthogonal groups $SO(M)$ and $SO(N)$ respectively. The $M(M-1)/2$ generators L_α^A are given by $\{|j\rangle\langle k| - |k\rangle\langle j|\}$, $1 \leq j < k \leq M$, where $|i\rangle$, $i = 1, ..., M$, is the usual orthonormal basis of $\mathcal{H_A}$, a column vector with the ith row 1 and the rest zeros. L_β^B can be similarly defined. The matrix operators L_α^A (resp. L_β^B) have $M-2$ (resp. $N-2$) rows and $M-2$ (resp. $N-2$) columns that are identically zero. We define the operators A_i^α (resp. B_j^β) from L_α (resp. L_β) by replacing the four entries in the positions of the two nonzero rows and two nonzero columns of L_α (resp. L_β) with the corresponding four entries of the matrix $\vec{a}_i \cdot \vec{\sigma}$ (resp. $\vec{b}_j \cdot \vec{\sigma}$), and keeping the other entries of A_i^α (resp. B_j^β) zero.

By using L_α^A and L_β^B the pure state ρ can be projected to "two-qubit" ones [23]:

$$\rho_{\alpha\beta} = \frac{L_\alpha^A \otimes L_\beta^B \rho (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger}{\mathrm{Tr}\{L_\alpha^A \otimes L_\beta^B \rho (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger\}}, \tag{17}$$

where $\alpha = 1, 2, \cdots, \frac{M(M-1)}{2}$; $\beta = 1, 2, \cdots, \frac{N(N-1)}{2}$. As the matrix $L_\alpha^A \otimes L_\beta^B$ has $MN-4$ rows and $MN-4$ columns that are identically zero, one can directly verify that there are at most $4 \times 4 = 16$ nonzero elements in each matrix $\rho_{\alpha\beta}$. For every pure state $\rho_{\alpha\beta}$ the corresponding Bell operators are defined by

$$\mathcal{B}_{\alpha\beta} = \tilde{A}_1^\alpha \otimes \tilde{B}_1^\beta + \tilde{A}_1^\alpha \otimes \tilde{B}_2^\beta + \tilde{A}_2^\alpha \otimes \tilde{B}_1^\beta - \tilde{A}_2^\alpha \otimes \tilde{B}_2^\beta, \tag{18}$$

where $\tilde{A}_i^\alpha = L_\alpha^A A_i^\alpha (L_\alpha^A)^\dagger$ and $\tilde{B}_j^\beta = L_\beta^B B_j^\beta (L_\beta^B)^\dagger$ are Hermitian operators. It has been shown that any bipartite pure quantum state is entangled if and only if at least one of the following inequalities is violated [23],

$$|\langle \mathcal{B}_{\alpha\beta} \rangle| \leq 2. \tag{19}$$

Inequalities (19) work only for general high dimensional bipartite pure states. Combining the approaches in [17] and [23], we now define the mean value of nonlinear operators $\mathcal{B}'_{\alpha\beta}$,

$$\langle \mathcal{B}'_{\alpha\beta} \rangle = \sqrt{\langle \tilde{A}_1^\alpha \tilde{B}_1^\beta + \tilde{A}_2^\alpha \tilde{B}_2^\beta \rangle_\rho^2 + \langle \tilde{A}_3^\alpha + \tilde{B}_3^\beta \rangle_\rho^2} - \langle \tilde{A}_3^\alpha \tilde{B}_3^\beta \rangle_\rho, \tag{20}$$

for high dimensional bipartite mixed states.

Theorem 3 *Any bipartite quantum state $\rho \in \mathcal{H}_{\mathcal{AB}}$ is entangled if one of the following inequalities,*

$$\frac{1}{\mathrm{Tr}(L_\alpha \otimes L_\beta \, \rho^{T_A} L_\alpha \otimes L_\beta)} |\langle \mathcal{B}'_{\alpha\beta} \rangle| \leq 1, \tag{21}$$

is violated, where $\alpha = 1, 2, \cdots, \frac{M(M-1)}{2}$, $\beta = 1, 2, \cdots, \frac{N(N-1)}{2}$.

Proof: Assume that ρ is separable (not entangled) quantum state. Since the separability of a state does not change under the local operation $L_{\alpha_0}^A \otimes L_{\beta_0}^B$, one has that for any α and β, $\rho_{\alpha\beta} = \frac{L_\alpha^A \otimes L_\beta^B \rho (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger}{\mathrm{Tr}\{L_\alpha^A \otimes L_\beta^B \rho (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger\}}$, which can be treated as a two qubits state, must be also separable. According to the analysis in [17], a two-qubit state ρ is separable if and only if (16) holds, which contradicts with the condition (21). Thus we have that if any one of the inequalities (21) is violated, ρ must be an entangled quantum state. □

It is obvious that the inequalities (21) must not be weaker than the Bell inequalities given in [23] for detecting entanglement of mixed quantum states, since (21) supplies a sufficient and necessary condition for separability of two qubits (mixed) quantum states, while violating the CHSH inequality is just a sufficient condition for two-qubit entanglement. Actually, (21) is strictly stronger, as seen from the following examples.

Example 1 We consider a 3×3 dimensional state introduced in [49] by Bennett et al. Set $|\xi_0\rangle = \frac{1}{\sqrt{2}}|0\rangle(|0\rangle - |1\rangle)$, $|\xi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|2\rangle$, $|\xi_2\rangle = \frac{1}{\sqrt{2}}|2\rangle(|1\rangle - |2\rangle)$, $|\xi_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)|0\rangle$, $|\xi_4\rangle = \frac{1}{3}(|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)$. Let

$$\rho = \frac{1}{4}(I_9 - \sum_{i=0}^{4} |\xi_i\rangle\langle\xi_i|).$$

This state is entangled according to the realignment criterion [50]. We consider the mixture of ρ and the maximal entangled singlet $P_+ = |\psi_+\rangle\langle\psi_+|$, where $|\psi_+\rangle = \frac{1}{\sqrt{3}}\sum_{i=0}^{2}|ii\rangle$:

$$\rho_p = (1-p)\rho + pP_+. \tag{22}$$

By straightforward computation, the Bell inequalities (19) detect entanglement for $0.57602 \le p \le 1$, while (21) detect entanglement for $0.18221 \le p \le 1$.

Example 2 Consider the state

$$\rho_p(a) = (1-p)\rho(a) + pP_+, \tag{23}$$

where

$$\rho(a) = \frac{1}{8a+1}\begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+a}{2} & 0 & \frac{\sqrt{1-a^2}}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 \\ a & 0 & 0 & 0 & a & 0 & \frac{\sqrt{1-a^2}}{2} & 0 & \frac{1+a}{2} \end{pmatrix},$$

is the weakly inseparable state given in [51], $0 < a < 1$.

Take $a = 0.236$, which is the case that $\rho(a)$ violates the realignment criterion [50] maximally. From Fig.2 we see that the Bell inequalities (19) detect entanglement for $0.26 \le p \le 1$, while (21) detect entanglement for the whole region of $0 < p \le 1$.

Example 3 Isotropic states [52] with dimensions $M = N = d$ can be written as the mixtures of the maximally mixed state and the maximally entangled state $|\psi_+\rangle = \frac{1}{\sqrt{d}}\sum_{i=0}^{d-1}|ii\rangle$,

$$\rho = \frac{1-x}{d^2} I_d \otimes I_d + x|\psi_+\rangle\langle\psi_+|. \tag{24}$$

The inequalities (21) can detect the entanglement for $x \le \frac{1}{d+1}$ which agrees with the result in [52]. Thus (21) serves as a sufficient and necessary condition of separability for isotropic states.

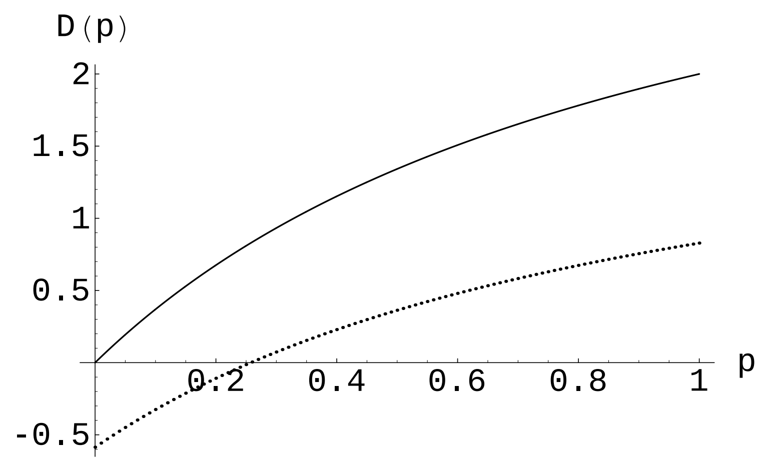

FIG. 2: The differences $D(p)$ between the right and the left sides of the inequalities (21) (solid line) and the Bell inequalities (19) (doted line).

III. BELL INEQUALITIES FOR MULTIPARTITE QUBIT SYSTEMS

In this section, we study Bell inequalities for both pure and mixed multi-qubit systems. We propose a series of Bell inequalities for any N-qubit states ($N \geq 3$), and derive the formulas of the maximal violations of these Bell inequalities. The Bell inequalities are independent of the WWZB inequality and Chen's Bell inequalities constructed in [53], i.e. they can detect some entangled states which fulfill both the WWZB inequality and Chen's Bell inequalities.

Consider an N-qubit quantum system and allow each part to choose independently between two dichotomic observables A_i, $A_i^{'}$ for the ith observer, $i = 1, 2, ..., N$. Each measurement has two possible outcomes 1 and -1. Quantum mechanically these observables can be expressed as $A_i = \vec{a}_i \cdot \vec{\sigma}$, $A_i^{'} = \vec{a}_i^{'} \cdot \vec{\sigma}$, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices and $\vec{a}_i$, $\vec{a}_i^{'}$ are unit vectors, $i = 1, 2, \cdots, N$.

The CHSH Bell inequality for two-qubit systems is given by

$$|\langle B_2 \rangle| \leq 1, \tag{25}$$

where the Bell operator $B_2 = \frac{1}{2}[A_1 A_2 + A_1^{'} A_2 + A_1 A_2^{'} - A_1^{'} A_2^{'}]$. In [54] Horodeckis have derived an elegant formula which serves as a necessary and sufficient condition for violating the CHSH inequality by an arbitrary mixed two qubits state.

The WWZB Bell operator is defined by

$$B_N^{WWZB} = \frac{1}{2^N} \sum_{s_1, s_2, \cdots, s_N = \pm 1} S(s_1, s_2, \cdots, s_N) \sum_{k_1, k_2, \cdots, k_N = \pm 1} s_1^{k_1} s_2^{k_2} \cdots s_N^{k_N} \otimes_{j=1}^{N} O_j(k_j), \tag{26}$$

where $S(s_1, s_2, \cdots, s_N)$ is an arbitrary function of s_i, $i = 1, ..., N$, taking values ± 1, $O_j(1) = A_j$ and $O_j(2) = A_j^{'}$ with $k_j = 1, 2$. It is shown in [55, 56] that local realism requires that $|\langle B_N \rangle| \leq 1$. The MABK inequality is recovered by taking $S(s_1, s_2, \cdots, s_N) = \sqrt{2} \cos[(s_1 + s_2 + \cdots + s_N - N + 1)/\frac{\pi}{4}]$ in (26). In [55, 56] the authors also derived a necessary and sufficient condition of violation of this inequality for an arbitrary N-qubit mixed state, generalizing two-qubit results in [54]. However, when using the results to obtain the maximal violation of the WWZB inequality, one has to select a proper set of unit vectors $\vec{a}_i$ and $\vec{a}_i^{'}$, which makes the approach less operational.

Employing an inductive method from the $(N-1)$-partite WWZB Bell inequality to the N-partite inequality, a family of Bell inequalities was presented in [53]. The Bell operator is defined by

$$B_N = B_{N-1}^{WWZB} \otimes \frac{1}{2}(A_N + A_N^{'}) + \frac{1}{2}(A_N - A_N^{'}), \tag{27}$$

where B_{N-1}^{WWZB} represents the normal WWZB Bell operators defined in (26) for $N-1$ qubits. For simplicity in (27) and in the following we only write the quantum mechanical operators acting on certain qubits, and omit the identity operators acting on the rest qubits. Bell operators (27) yield the violation of the corresponding Bell inequality for the generalized GHZ states, $|\psi\rangle = \cos\alpha|00\cdots 0\rangle + \sin\alpha|11\cdots 1\rangle$, in the whole parameter region of α such that $\cos\alpha \neq 0$ and $\sin\alpha \neq 0$, thus overcoming the drawback of the WWZB inequality. In the three-qubit case, one can construct three different Bell operators from B_2 by using the approach of (27). The corresponding three Bell inequalities can distinguish full separability, detailed partial separability and true entanglement [57]. However, the maximal violation of this Bell inequality is unknown for a generally given three-qubit state.

We start with constructing a set of new Bell inequalities for any N-qubit quantum systems by iteration. First consider the case $N = 3$. As a two-qubit CHSH Bell operator $\mathcal{B}_2$ can act on two of the three qubits in three different ways, we can have three Bell operators,

$$\mathcal{B}_3^i = (\mathcal{B}_2)^i \otimes \frac{1}{2}(A_i + A_i^{'}) + \frac{1}{2}(A_i - A_i^{'}), \quad i = 1, 2, 3, \tag{28}$$

where $(\mathcal{B}_2)^i$ is the two-qubit CHSH Bell operators acting on the two qubits except for the ith one. For $N \geq 4$, the Bell operators can be similarly obtained,

$$\mathcal{B}_N^{(i-1)\frac{(N-1)!}{2}+j} = (\mathcal{B}_{N-1}^j)^i \otimes \frac{1}{2}(A_i + A_i^{'}) + \frac{1}{2}(A_i - A_i^{'}), \tag{29}$$

with $i = 1, 2, \cdots, N$ and $j = 1, 2, \cdots, \frac{(N-1)!}{2}$. Here $(\mathcal{B}_{N-1}^j)^i$ denotes the jth Bell operator acting on the $(N-1)$ qubits except for the ith one. There are totally $\frac{N!}{2}$ Bell operators.

Theorem 4 *If a local realistic description is assumed, the following inequalities must hold,*

$$|\langle \mathcal{B}_N^k \rangle| \leq 1, \tag{30}$$

where $k \in \{1, 2, \cdots, \frac{N!}{2}\}$.

Proof: We prove the theorem by induction. Note that for two qubits systems, local realism requires that $|\langle B_2\rangle| \leq 1$ as shown in (25). Assume that a local realistic model has lead to $|\langle \mathcal{B}^k_{N-1}\rangle| \leq 1$ with $k \in \{1, 2, \cdots, \frac{(N-1)!}{2}\}$. Consider the N-partite systems. If A_i and $A_i^{'}$ are specified by some local parameters each having two possible outcomes -1 and 1, one has either $|A_i + A_i^{'}| = 2$ and $|A_i - A_i^{'}| = 0$, or vice versa. For any $k \in \{1, 2, \cdots, \frac{N!}{2}\}$, from (29) we have that

$$\begin{aligned}
|\langle \mathcal{B}^k_N\rangle| &= |\langle \mathcal{B}_N^{(i-1)\frac{(N-1)!}{2}+j}\rangle| = |\langle (\mathcal{B}^j_{N-1})^i \otimes \frac{1}{2}(A_i + A_i^{'}) + \frac{1}{2}(A_i - A_i^{'})\rangle| \\
&\leq |\langle (\mathcal{B}^j_{N-1})^i\rangle| \otimes \frac{1}{2}|\langle (A_i + A_i^{'})\rangle| + \frac{1}{2}|\langle (A_i - A_i^{'})\rangle| \leq 1.
\end{aligned}$$

□

We remark that any N-qubit fully separable states also satisfy the inequality (30). For $N \geq 4$, the operator $\mathcal{B}^i_{N-1}$ is derived from $\mathcal{B}^i_{N-2}$. Thus $\mathcal{B}^i_N$ are different from the Bell operators in [53] where $\mathcal{B}^i_{N-1}$ is the Bell operator in the WWZB inequality. The following example will show that our Bell inequalities in (30) are independent from the WWZB inequalities and that in [53], and our new Bell inequalities can detect entanglement better.

Example Consider a four-qubit pure state $|\psi\rangle = |\phi\rangle \otimes |0\rangle$, where $|\phi\rangle = \cos\alpha|000\rangle + \sin\alpha|111\rangle$, $\alpha \in [0, \frac{\pi}{12}]$. It has been proved [58] that for $\sin 2\alpha \leq \frac{1}{2}$ (i.e. $\alpha \in [0, \frac{\pi}{12}]$), the WWZB Bell inequalities cannot be violated by the generalized GHZ state $|\phi\rangle$. According to the result in [58], the WWZB inequalities operator B_4^{WWZB} and the Bell operator B_4 in [53] satisfy the following relations,

$$|\langle\psi|B_4^{WWZB}|\psi\rangle| \leq |\langle\phi|B_3^{WWZB}|\phi\rangle| \leq 1, \tag{31}$$

$$|\langle\psi|B_4|\psi\rangle| \leq |\langle\phi|B_3^{WWZB}|\phi\rangle| \leq 1. \tag{32}$$

Therefore both the WWZB Bell inequalities and the inequalities in [53] can not detect entanglement of $|\psi\rangle$.

Nevertheless the mean values of the Bell operator $\mathcal{B}^4_4$ in (30) is $\sqrt{2\sin^2 2\alpha + \cos^2 2\alpha}$ which is always larger than 1 as long as $|\phi\rangle$ is not separable. Therefore the entanglement is detected by our Bell inequality (30).

Now we investigate the maximal violation of the Bell inequalities (30). We first consider the $N = 3$ case. In this situation, (29) gives three operators,

$$\mathcal{B}^1_3 = (\mathcal{B}_2)^1 \otimes \frac{1}{2}(A_1 + A_1^{'}) + \frac{1}{2}(A_1 - A_1^{'}), \tag{33}$$

$$\mathcal{B}^2_3 = (\mathcal{B}_2)^2 \otimes \frac{1}{2}(A_2 + A_2^{'}) + \frac{1}{2}(A_2 - A_2^{'}), \tag{34}$$

$$\mathcal{B}^3_3 = (\mathcal{B}_2)^3 \otimes \frac{1}{2}(A_3 + A_3^{'}) + \frac{1}{2}(A_3 - A_3^{'}), \tag{35}$$

where $(\mathcal{B}_2)^1 = \frac{1}{2}(A_2A_3 + A_2^{'}A_3 + A_2A_3^{'} - A_2^{'}A_3^{'})$, $(\mathcal{B}_2)^2 = \frac{1}{2}(A_1A_3 + A_1^{'}A_3 + A_1A_3^{'} - A_1^{'}A_3^{'})$ and $(\mathcal{B}_2)^3 = \frac{1}{2}(A_1A_2 + A_1^{'}A_2 + A_1A_2^{'} - A_1^{'}A_2^{'})$. Let ρ be a general three-qubit state,

$$\rho = \sum_{i,j,k=0}^{3} T_{ijk}\sigma_i\sigma_j\sigma_k, \tag{36}$$

where $\sigma_0 = I_2$ is the 2×2 identity matrix, σ_i are the Pauli matrices, and

$$T_{ijk} = \frac{1}{8}Tr(\rho\sigma_i\sigma_j\sigma_k). \tag{37}$$

Theorem 5 *The maximum of the mean values of the Bell operators in (33), (34) and (35) satisfy the following relations,*

$$\max|\langle\mathcal{B}_3^1\rangle| = 8\max\{\lambda_1^1(\vec{b}_3) + \lambda_2^1(\vec{b}_3) + ||\vec{T}_{00}^1||^2 - \langle\vec{b}_3, \vec{T}_{00}^1\rangle^2\}^{\frac{1}{2}}, \tag{38}$$

$$\max|\langle\mathcal{B}_3^2\rangle| = 8\max\{\lambda_1^2(\vec{b}_3) + \lambda_2^2(\vec{b}_3) + ||\vec{T}_{00}^2||^2 - \langle\vec{b}_3, \vec{T}_{00}^2\rangle^2\}^{\frac{1}{2}}, \tag{39}$$

$$\max|\langle\mathcal{B}_3^3\rangle| = 8\max\{\lambda_1^3(\vec{b}_3) + \lambda_2^3(\vec{b}_3) + ||\vec{T}_{00}^3||^2 - \langle\vec{b}_3, \vec{T}_{00}^3\rangle^2\}^{\frac{1}{2}}, \tag{40}$$

where $\langle .,.\rangle$ denotes the inner product of two vectors, $||\vec{x}||$ stands for the norm of vector $\vec{x}$. The maximums on the right of (38), (39) and (40) are taken over all the unit vectors $\vec{b}_3$. Given a three-qubit state ρ, one can compute T_{ijk} by using the formula in (37). Then $\lambda_1^i(\vec{b}_3)$ and $\lambda_2^i(\vec{b}_3)$ are defined to be the two greater eigenvalues of the matrix $M_i^\dagger M_i$ with $M_i = \sum_{k=1}^3 b_3^k T_k^i$, $i = 1,2,3$, with respect to the three Bell operators in (33), (34) and (35). Here T_k^l, $l = 1,2,3$, are matrices with entries given by $(T_k^1)_{ij} = T_{kij}$, $(T_k^2)_{ij} = T_{ikj}$ and $(T_k^3)_{ij} = T_{ijk}$. $\vec{T}_{00}^m$, $m = 1,2,3$ are defined to be vectors with entries $(\vec{T}_{00}^1)_k = T_{k00}$, $(\vec{T}_{00}^2)_k = T_{0k0}$ and $(\vec{T}_{00}^3)_k = T_{00k}$.

Proof: We take (35) as an example to show how to calculate the maximal violation. The maximal violation for the Bell operators (33) and (34) can be computed similarly. A direct computation shows that

$$\begin{aligned}
\mathcal{B}_3^3 &= \frac{1}{4}[(A_1 + A_1^{'})A_2 + (A_1 - A_1^{'})A_2^{'}](A_3 + A_3^{'}) + \frac{1}{2}(A_3 - A_3^{'}) \\
&= \frac{1}{4}(A_1 + A_1^{'})A_2(A_3 + A_3^{'}) + \frac{1}{4}(A_1 - A_1^{'})A_2^{'}(A_3 + A_3^{'}) + \frac{1}{2}(A_3 - A_3^{'}) \\
&= \frac{1}{4}\sum_{i,j,k=1}^{3}[a_1^i + (a_1^{'})^i]a_2^j[a_3^k + (a_3^{'})^k]\sigma_i\sigma_j\sigma_k + \frac{1}{4}\sum_{i,j,k=1}^{3}[a_1^i - (a_1^{'})^i](a_2^{'})^j[a_3^k + (a_3^{'})^k]\sigma_i\sigma_j\sigma_k \\
&\quad + \frac{1}{2}\sum_{k=1}^{3}[a_3^k - (a_3^{'})^k]I_4\otimes\sigma_k.
\end{aligned} \tag{41}$$

For any given unit vectors $\vec{a}_1$ and $\vec{a'}_1$, there always exist a pair of unit and mutually orthogonal vectors $\vec{b}_1$, $\vec{b'}_1$ and $\theta \in [0, \frac{\pi}{2}]$ such that

$$\vec{a}_1 + \vec{a'}_1 = 2\cos\theta\,\vec{b}_1, \quad \vec{a}_1 - \vec{a'}_1 = 2\sin\theta\,\vec{b'}_1. \tag{42}$$

Similarly for $\vec{a}_3$ and $\vec{a'}_3$, we have

$$\vec{a}_3 + \vec{a'}_3 = 2\cos\phi\,\vec{b}_3, \quad \vec{a}_3 - \vec{a'}_3 = 2\sin\phi\,\vec{b'}_3, \tag{43}$$

where $\vec{b}_3$ and $\vec{b'}_3$ are orthogonal vectors with unit norm and $\phi \in [0, \frac{\pi}{2}]$.

By inserting (36) into (41) we get the mean value of the Bell operator (35),

$$\begin{aligned}
\langle \mathcal{B}_3^3 \rangle &= Tr(\rho \mathcal{B}_3^3) \\
&= \frac{1}{4} \sum_{i,j,k=1}^{3} [a_1^i + (a_1^{'})^i] a_2^j [a_3^k + (a_3^{'})^k] T_{ijk} Tr(\sigma_i^2 \sigma_j^2 \sigma_k^2) \\
&\quad + \frac{1}{4} \sum_{i,j,k=1}^{3} [a_1^i - (a_1^{'})^i] (a_2^{'})^j [a_3^k + (a_3^{'})^k] T_{ijk} Tr(\sigma_i^2 \sigma_j^2 \sigma_k^2) \\
&\quad + \frac{1}{2} \sum_{k=1}^{3} [a_3^k - (a_3^{'})^k] T_{00k} Tr(I_4^2 \otimes \sigma_k^2) \\
&= 8 \sum_{i,j,k=1}^{3} b_1^i b_3^k a_2^j T_{ijk} \cos\theta \cos\phi + 8 \sum_{i,j,k=1}^{3} (b_1^{'})^i b_3^k (a_2^{'})^j T_{ijk} \sin\theta \cos\phi + 4 \sum_{k=1}^{3} (b_3^{'})^k T_{00k} \sin\phi.
\end{aligned}$$

Let $T_k^3, k = 1, 2, 3$, be the matrix with entries given by $(T_k^3)_{ij} = T_{ijk}$ and $\vec{T}_{00}^3$ a vector with components $(\vec{T}_{00}^3)_k = T_{00k}$. The maximal mean value of the Bell operator (35) can be written as

$$\begin{aligned}
\max\langle \mathcal{B}_3^3 \rangle &= 8 \max[\langle \vec{b}_1, \sum_{k=1}^{3} b_3^k T_k^3 \vec{a}_2 \rangle \cos\theta \cos\phi + \langle \vec{b'}_1, \sum_{k=1}^{3} b_3^k T_k^3 \vec{a}_2^{'} \rangle \sin\theta \cos\phi + \langle \vec{b'}_3, \vec{T}_{00}^3 \rangle \sin\phi] \\
&= 8 \max\{[\langle \vec{b}_1, \sum_{k=1}^{3} b_3^k T_k^3 \vec{a}_2 \rangle \cos\theta + \langle \vec{b'}_1, \sum_{k=1}^{3} b_3^k T_k^3 \vec{a}_2^{'} \rangle \sin\theta]^2 + \langle \vec{b'}_3, \vec{T}_{00}^3 \rangle^2\}^{\frac{1}{2}} \\
&= 8 \max\{[\langle \vec{b}_1, \sum_{k=1}^{3} b_3^k T_k^3 \vec{a}_2 \rangle \cos\theta + \langle \vec{b'}_1, \sum_{k=1}^{3} b_3^k T_k^3 \vec{a}_2^{'} \rangle \sin\theta]^2 + ||\vec{T}_{00}^3||^2 - \langle \vec{b}_3, \vec{T}_{00}^3 \rangle^2\}^{\frac{1}{2}} \\
&= 8 \max\{\lambda_1^3(\vec{b}_3) + \lambda_2^3(\vec{b}_3) + ||\vec{T}_{00}^3||^2 - \langle \vec{b}_3, \vec{T}_{00}^3 \rangle^2\}^{\frac{1}{2}},
\end{aligned} \tag{44}$$

which proves (40). In (44) we have used the fact that the maximum of $x\cos\theta + y\sin\theta$ taking over all θ is $\sqrt{x^2 + y^2}$. Formulae (38) and (39) can be similarly proven. □

Remark: According to the symmetry of the operator $\mathcal{B}_3^3$, the equation (44) also provides the minimum of the operator (35), achieved by $-\mathcal{B}_3^3$.

Since $\vec{b}_3$ is a three dimensional real unit vector, one can always calculate the exact value of the maximum for any given three qubits quantum state. For example, for the generalized three-qubit GHZ state, $|GHZ\rangle = \cos\alpha|000\rangle + \sin\alpha|111\rangle$, by selecting some proper direction of the measurement operators, i.e. $\vec{a}_i$s and $(\vec{a'})_i$s, the maximal mean value of the Bell operator in (35) is shown to be $\sqrt{2\sin^2 2\alpha + \cos^2 2\alpha}$ [53]. From our formulae in Theorem

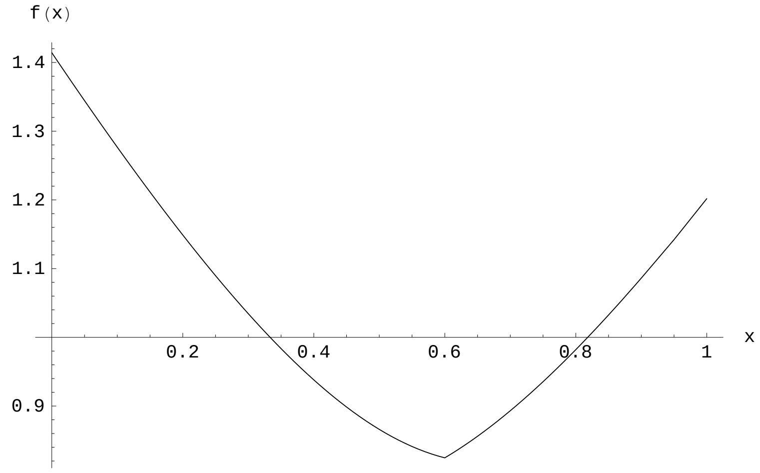

FIG. 3: The maximal mean value of the operator (35) for the mixed state ρ.

2 one can show that the result are in accord with that in [53]. For three-qubit W state, $|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$, our mean value is 1.202, which is also in agreement with that in [53]. However, our method can be also used to calculate the mean value of the Bell operators in (33), (34) and (35) for any three qubits quantum states. For instance, we consider the mixture of $|W\rangle$ and $|GHZ\rangle$, $\rho = x|W\rangle\langle W| + (1-x)|GHZ\rangle\langle GHZ|$, where $0 \leq x \leq 1$. We have the maximal mean value of the Bell operator in (35), see Fig.3, where $f(x)$ stands for the maximal mean value of the operator (35) for the mixed state ρ. For $0 \leq x \leq 0.33$ and $0.82 \leq x \leq 1$, $f(x) > 1$ and ρ is detected to be entangled.

For four-qubit systems, the Bell operators (29) have four different forms. We take $\mathcal{B}_4^{12}$ as an example to investigate the maximal violation of the corresponding Bell inequality. Note that

$$
\begin{aligned}
\mathcal{B}_4^{12} &= (\mathcal{B}_3^3)^4 \otimes \frac{1}{2}(A_4 + A_4') + \frac{1}{2}(A_4 - A_4') \\
&= \frac{1}{8}(A_1 + A_1')A_2(A_3 + A_3')(A_4 + A_4') + \frac{1}{8}(A_1 - A_1')A_2'(A_3 + A_3')(A_4 + A_4') \\
&\quad + \frac{1}{4}I_4 \otimes (A_3 - A_3')(A_4 + A_4') + \frac{1}{2}I_6 \otimes (A_4 - A_4') \\
&= \frac{1}{8}\sum_{i,j,k,l=1}^{3} [a_1^i + (a_1')^i]a_2^j[a_3^k + (a_3')^k][a_4^l + (a_4')^l]\sigma_i\sigma_j\sigma_k\sigma_l \\
&\quad + \frac{1}{8}\sum_{i,j,k,l=1}^{3} [a_1^i - (a_1')^i](a_2')^j[a_3^k + (a_3')^k][a_4^l + (a_4')^l]\sigma_i\sigma_j\sigma_k\sigma_l \\
&\quad + \frac{1}{4}\sum_{k,l=1}^{3}[a_3^k - (a_3')^k][a_4^l + (a_4')^l]I_4 \otimes \sigma_k\sigma_l + \frac{1}{2}\sum_{l=1}^{3}[a_4^l - (a_4')^l]I_6 \otimes \sigma_l. \qquad (45)
\end{aligned}
$$

Let ρ be a general four-qubit quantum state,

$$\rho = \sum_{i,j,k,l=0}^{3} T_{ijkl}\sigma_i\sigma_j\sigma_k\sigma_l, \tag{46}$$

with $T_{ijkl} = \frac{1}{2^4}Tr(\rho\sigma_i\sigma_j\sigma_k\sigma_l)$. We have the mean value of $\mathcal{B}_4^{12}$,

$$\begin{aligned}
\langle\mathcal{B}_4^{12}\rangle &= \frac{1}{8}\sum_{i,j,k,l=1}^{3}[a_1^i + (a_1^{'})^i]a_2^j[a_3^k + (a_3^{'})^k][a_4^l + (a_4^{'})^l]T_{ijkl}Tr(\sigma_i^2\sigma_j^2\sigma_k^2\sigma_l^2) \\
&+\frac{1}{8}\sum_{i,j,k,l=1}^{3}[a_1^i - (a_1^{'})^i](a_2^{'})^j[a_3^k + (a_3^{'})^k][a_4^l + (a_4^{'})^l]T_{ijkl}Tr(\sigma_i^2\sigma_j^2\sigma_k^2\sigma_l^2) \\
&+\frac{1}{4}\sum_{k,l=1}^{3}[a_3^k - (a_3^{'})^k][a_4^l + (a_4^{'})^l]T_{00kl}Tr(I_4\otimes\sigma_k^2\sigma_l^2) \\
&+\frac{1}{2}\sum_{l=1}^{3}[a_4^l - (a_4^{'})^l]T_{000l}Tr(I_6\otimes\sigma_l^2) \\
&= 2^4\sum_{i,j,k,l=1}^{3} b_1^i a_2^j b_3^k b_4^l T_{ijkl}\cos\alpha_1\cos\alpha_3\cos\alpha_4 \\
&+2^4\sum_{i,j,k,l=1}^{3}(b_1^{'})^i(a_2^{'})^j b_3^k b_4^l T_{ijkl}\sin\alpha_1\cos\alpha_3\cos\alpha_4 \\
&+2^4\sum_{k,l=1}^{3}(b_3^{'})^k b_4^l T_{00kl}\sin\alpha_3\cos\alpha_4 + 2^4\sum_{l=1}^{3}(b_4^{'})^l T_{000l}\sin\alpha_4,
\end{aligned}$$

where we have used that $\vec{a}_i + \vec{a'}_i = 2\cos\alpha_i\,\vec{b}_i$, $\vec{a}_i - \vec{a'}_i = 2\sin\alpha_i\,\vec{b'}_i$, $\alpha_i \in [0, \frac{\pi}{2}]$.

The maximum of the mean value can be derived to be

$$\begin{aligned}
\max\langle\mathcal{B}_4^{12}\rangle &= 2^4\max[\langle\vec{b}_1, \sum_{k,l=1}^{3} b_3^k b_4^l T_{kl}^{12}\vec{a}_2\rangle\cos\alpha_1\cos\alpha_3\cos\alpha_4 \\
&+\langle\vec{b}_1^{'}, \sum_{k,l=1}^{3} b_3^k b_4^l T_{kl}^{12}\vec{a}_2^{'}\rangle\sin\alpha_1\cos\alpha_3\cos\alpha_4 \\
&+\langle\vec{b}_3^{'}, T_{00}^{12}\vec{b}_4\rangle\sin\alpha_3\cos\alpha_4] + \langle\vec{b}_4^{'}, \vec{T}_{000}^{12}\rangle\sin\alpha_4] \\
&= 2^4\max\{\langle\vec{b}_1, \sum_{k,l=1}^{3} b_3^k b_4^l T_{kl}^{12}\vec{a}_2\rangle^2 + \langle\vec{b}_1^{'}, \sum_{k,l=1}^{3} b_3^k b_4^l T_{kl}^{12}\vec{a}_2^{'}\rangle^2 + \langle\vec{b}_3^{'}, T_{00}^{12}\vec{b}_4\rangle^2 + \langle\vec{b}_4^{'}, \vec{T}_{000}^{12}\rangle^2\}^{\frac{1}{2}} \\
&= 2^4\max\{\lambda_1^{12}(\vec{b}_3\vec{b}_4) + \lambda_2^{12}(\vec{b}_3\vec{b}_4) + ||T_{00}^{12}\vec{b}_4||^2 - \langle\vec{b}_3, T_{00}^{12}\vec{b}_4\rangle^2 + ||\vec{T}_{000}^{12}||^2 - \langle\vec{b}_4, \vec{T}_{000}^{12}\rangle^2\}^{\frac{1}{2}},
\end{aligned}$$

where $\lambda_1^{12}(\vec{b}_3\vec{b}_4)$ and $\lambda_2^{12}(\vec{b}_3\vec{b}_4)$ are the two greater eigenvalues of the matrix $(M^{12})^\dagger M^{12}$, $M^{12} = \sum_{k,l=1}^{3} b_3^k b_4^l T_{kl}^{12}; T_{kl}^{12}$ stand for the matrices with entries $(T_{kl}^{12})_{ij} = T_{ijkl}$ with $i,j,k,l = 1,2,3$; T_{00}^{12} is a matrix with entries $(T_{00}^{12})_{kl} = T_{00kl}$, and $\vec{T}_{000}^{12}$ is a vector with components

$(T^{12}_{000})_l = T_{000l}$, $l = 1,2,3$. The maximum in the last equation is taken over all the unit vectors $\vec{b}_3$ and $\vec{b}_4$.

In terms of the analysis above, for four-qubit systems we have the following Theorem.

Theorem 6 *The maximum of the mean values of the Bell operators in (29) for four qubits systems are given by the following formula:*

$$\begin{aligned}&\max|\langle\mathcal{B}^m_4\rangle| \\ &= 2^4\max\{\lambda^m_1(\vec{b}_3\vec{b}_4)+\lambda^m_2(\vec{b}_3\vec{b}_4)+||T^m_{00}\vec{b}_4||^2-\langle\vec{b}_3,T^m_{00}\vec{b}_4\rangle^2+||\vec{T}^m_{000}||^2-\langle\vec{b}_4,\vec{T}^m_{000}\rangle^2\}^{\frac{1}{2}}.\end{aligned} \tag{47}$$

The maximums on the right side are taken over all the unit vectors $\vec{b}_3$ and $\vec{b}_4$. Here $\lambda^m_1(\vec{b}_3\vec{b}_4)$ and $\lambda^m_2(\vec{b}_3\vec{b}_4)$ are the two greater eigenvalues of the matrix $(M^m)^\dagger M^m$, $M^m=\sum_{k,l=1}^3 b^k_3 b^l_4 T^m_{kl}$, $m=1,2,\cdots,\frac{N!}{2}$; T^m_{kl} are the matrices with entries $(T^1_{kl})_{ij}=T_{lkij}$, $(T^2_{kl})_{ij}=T_{likj}$, $(T^3_{kl})_{ij}=T_{lijk}$, $(T^4_{kl})_{ij}=T_{klij}$, $(T^5_{kl})_{ij}=T_{ilkj}$, $(T^6_{kl})_{ij}=T_{iljk}$, $(T^7_{kl})_{ij}=T_{kilj}$, $(T^8_{kl})_{ij}=T_{iklj}$, $(T^9_{kl})_{ij}=T_{ijlk}$, $(T^{10}_{kl})_{ij}=T_{kijl}$, $(T^{11}_{kl})_{ij}=T_{ikjl}$ and $(T^{12}_{kl})_{ij}=T_{ijkl}$, $i,j=1,2,3$ and $k,l=0,1,2,3$; $\vec{T}^m_{000}$ stand for the vectors with components $(T^i_{000})_x=T_{x000}$, $(T^j_{000})_x=T_{0x00}$, $(T^k_{000})_x=T_{00x0}$, $(T^l_{000})_x=T_{000x}$, $i=1,2,3$, $j=4,5,6$, $k=7,8,9$, $l=10,11,12$ and $x=1,2,3$.

As an example, consider the 4-qubit W state $|W\rangle=\frac{1}{2}(|1000\rangle+|0100\rangle+|0010\rangle+|0001\rangle)$, by using the formula (47) one gets the maximal mean value $\max|\langle\mathcal{B}^{12}_4\rangle|=1.118$. For the mixed state $\rho=\frac{x}{16}I+(1-x)|W\rangle\langle W|$, entanglement can be detected by (47) for $0\le x\le 0.106$.

Generally, for any N-qubit quantum state, the maximal mean values of the Bell operators in (29) can be calculated similarly by using our approach above. For example, the maximal mean value of $\mathcal{B}^{\frac{N!}{2}}_N$ can be expressed as

$$\begin{aligned}\max|\langle\mathcal{B}^{\frac{N!}{2}}_N\rangle| &= 2^N\max\{\lambda^m_1(\vec{b}_3\cdots\vec{b}_N)+\lambda^m_2(\vec{b}_3\cdots\vec{b}_N)+||\vec{T}_{45\cdots N}||^2-\langle\vec{b}_3,\vec{T}_{45\cdots N}\rangle^2\\ &+\ ||\vec{T}_{5\cdots N}||^2-\langle\vec{b}_4,\vec{T}_{5\cdots N}\rangle^2+\cdots+||\vec{T}_N||^2-\langle\vec{b}_N,\vec{T}_N\rangle^2\}^{\frac{1}{2}},\end{aligned} \tag{48}$$

where $\lambda_1(\vec{b}_3\cdots\vec{b}_N)$ and $\lambda_2(\vec{b}_3\cdots\vec{b}_N)$ are the two greater eigenvalues of the matrix $M^\dagger M$, with $(M)_{ij}=\sum_{i_3,\cdots,i_N=1}^3 b^{i_3}_3\cdots b^{i_N}_N T_{iji_3,\cdots,i_N}$ the entries of matrix M; $\vec{T}_{45\cdots N}$, $\vec{T}_{5\cdots N}$ and $\vec{T}_N$ are vectors with components $(\vec{T}_{45\cdots N})_k=\sum_{i_4,\cdots,i_N=1}^3 b^{i_4}_4\cdots b^{i_N}_N T_{00ki_4,\cdots,i_N}$, $(\vec{T}_{5\cdots N})_k=\sum_{i_5,\cdots,i_N=1}^3 b^{i_5}_5\cdots b^{i_N}_N T_{000ki_5,\cdots,i_N}$ and $(\vec{T}_N)_k=T_{000\cdots 0k}$ respectively. The maximum on the right side is taken over all the unit vectors $\vec{b}_3,\vec{b}_4,\cdots,\vec{b}_N$. The other mean values of the Bell operators in (29) for N-qubit states can be obtained similarly. By expressing the unit vectors $\vec{b}_k$ as $(\cos\theta_k\cos\phi_k,\cos\theta_k\sin\phi_k,\sin\theta_k)$, $k=3,\cdots,N$, our formulas can be used to compute the maximal violation by searching for the maximum over all θ_k and ϕ_k, either analytically or numerically.

IV. MEASUREMENT OF QUANTUM ENTANGLEMENT

In this section, we present an experimental determination of the entanglement of formation for arbitrary dimensional pure quantum states. The measurement only evolves local quantum mechanical observables and the entanglement of formation can be obtained according to the mean values of these observables. We also derive a lower bound based on the inequalities (21) for the convex-roof extension of negativity.

A. Measurement of entanglement of formation

The entanglement of formation (EOF) is defined for bipartite systems. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be m and n ($m \leq n$) dimensional complex Hilbert spaces with orthonormal basis $|i\rangle$, $i = 1, ..., m$, and $|j\rangle$, $i = 1, ..., n$, respectively. A pure quantum state on $\mathcal{H}_A \otimes \mathcal{H}_B$ is generally of the form,

$$|\psi\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} |ij\rangle, \qquad a_{ij} \in \mathbb{C} \tag{49}$$

with normalization

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} a_{ij}^* = 1\,. \tag{50}$$

The entanglement of formation of $|\psi\rangle$ is defined as the partial entropy with respect to the subsystems [59],

$$E(|\psi\rangle) = -\mathrm{Tr}\,(\rho^A \log_2 \rho^A) = -\mathrm{Tr}\,(\rho^B \log_2 \rho^B)\,, \tag{51}$$

where ρ^A (resp. ρ^B) is the reduced density matrix obtained by tracing $|\psi\rangle\langle\psi|$ over the space $\mathcal{H}_B$ (resp. $\mathcal{H}_A$). This definition can be extended to mixed states ρ by the convex roof,

$$E(\rho) \equiv \min_{\{p_i, |\psi_i\rangle\}} \sum_i p_i E(|\psi_i\rangle), \tag{52}$$

where the minimization goes over all possible ensemble realizations of ρ,

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|, \quad p_i \geq 0, \quad \sum_i p_i = 1. \tag{53}$$

A bipartite quantum state $|\psi\rangle$ can be written in the Schmidt form $|\psi\rangle = \sum_{i=1}^{m} \sqrt{\lambda_i}\, |i_A\rangle |i_B\rangle$, $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$, under suitable basis $|i_A\rangle \in \mathcal{H}_A$ and $|i_B\rangle \in \mathcal{H}_B$. λ_i, $i = 1, ..., m$, are in fact the eigenvalues of ρ^A. $E(|\psi\rangle)$ can be further expressed as

$$E(|\psi\rangle) = S(\rho^A) = -\sum_{i=1}^{m} \lambda_i \log \lambda_i. \tag{54}$$

For two-qubit case, $m = n = 2$, $|\psi\rangle = a_{11}|00\rangle + a_{12}|01\rangle + a_{21}|10\rangle + a_{22}|11\rangle$, $|a_{11}|^2 + |a_{12}|^2 + |a_{21}|^2 + |a_{22}|^2 = 1$. (51) can be written as [60],

$$E(|\psi\rangle) = h(\frac{1 + \sqrt{1 - C^2}}{2}), \tag{55}$$

where $h(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$, $C = 2|a_{11}a_{22} - a_{12}a_{21}|$ is the concurrence. In this special case $E(|\psi\rangle)$ is just a monotonically increasing function of the concurrence C. However for $m \geq 3$, there is no such relations like (55) between the entanglement of formation and concurrence in general. Since for the case $m = 2$, due to the normalization condition, $\lambda_1 + \lambda_2 = 1$, only one free parameter is left in the formula (54). For general high dimensional case, $E(|\psi\rangle)$ depends on more free parameters. Nevertheless, if ρ^A has only two non-zero eigenvalues (each of which may be degenerate), the maximal non-zero diagonal determinant D of ρ^A is a generalized concurrence, namely, the corresponding entanglement of formation is again a monotonically increasing function of D [61]. The construction of such kind of states is presented in [62]. In [63], the results are generalized to more general case: relations like (55) holds for states with ρ^A having more non-zero eigenvalues such that all these eigenvalues are functions of two independent parameters.

To measure the quantity (54) experimentally, we first rewrite the expression (54) according to the entanglement of formation of some "two-qubit" states. Let L_α^A and L_β^B be the generators of special unitary groups $SO(m)$ and $SO(n)$, with the $m(m-1)/2$ generators L_α^A given by $\{|i\rangle\langle j| - |j\rangle\langle i|\}$, $1 \leq i < j \leq m$, and the $n(n-1)/2$ generators L_β^B given by $\{|k\rangle\langle l| - |l\rangle\langle k|\}$, $1 \leq k < l \leq n$, respectively. The matrix operators L_α^A (resp. L_β^B) have $m - 2$ (resp. $n - 2$) rows and $m - 2$ (resp. $n - 2$) columns that are identically zero.

Let $\rho = |\psi\rangle\langle\psi|$ be the density matrix with respect to the pure state $|\psi\rangle$. We define

$$\rho_{\alpha\beta} = \frac{L_\alpha^A \otimes L_\beta^B \, \rho \, (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger}{||L_\alpha^A \otimes L_\beta^B \, \rho \, (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger||}, \tag{56}$$

where $\alpha = 1, 2, \cdots, \frac{m(m-1)}{2}$; $\beta = 1, 2, \cdots, \frac{n(n-1)}{2}$, and $||X|| = \sqrt{Tr(XX^\dagger)}$ is the trace norm of matrix X. As the matrix $L_\alpha^A \otimes L_\beta^B$ has $mn-4$ rows and $mn-4$ columns that are identically zero, $\rho_{\alpha\beta}$ has at most $4 \times 4 = 16$ nonzero elements and is called "two-qubit" state. $\rho_{\alpha\beta}$ is still a normalized pure state.

Theorem 7 *For any $m \otimes n$ $(m \leq n)$ pure quantum state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,*

$$E(|\psi\rangle) = \frac{1}{(m-1)^2} \sum_{\alpha\beta} \frac{E(\rho_{\alpha\beta}) + \log(C_{\alpha\beta})}{C_{\alpha\beta}}, \tag{57}$$

where $C_{\alpha\beta} = 1/Tr\{L_\alpha^A \otimes L_\beta^B|\psi\rangle\langle\psi|(L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger\}$.

Proof. To calculate $E(\rho_{\alpha\beta})$ we denote $L_\alpha^A = |a\rangle\langle b| - |b\rangle\langle a|$ and $L_\beta^B = |c\rangle\langle d| - |d\rangle\langle c|$ for convenience, where $1 \leq a < b \leq m$ and $1 \leq c < d \leq m$. Set

$$\rho'_{\alpha\beta} = L_\alpha^A \otimes L_\beta^B |\psi\rangle\langle\psi| (L_\alpha^A)^\dagger \otimes (L_\beta^B)^\dagger. \tag{58}$$

It is direct to verify that

$$\rho'_{\alpha\beta} = |\psi\rangle_{\alpha\beta}\langle\psi|, \tag{59}$$

where $|\psi\rangle_{\alpha\beta} = \lambda_b\delta_{bd}|ac\rangle - \lambda_b\delta_{bc}|ad\rangle - \lambda_a\delta_{ad}|bc\rangle + \lambda_a\delta_{ac}|bd\rangle$.

We now compute the eigenvalues of $\rho'^A_{\alpha\beta} = Tr_B(\rho'_{\alpha\beta})$ according to the values of a, b, c and d:

i). $a \neq b \neq c \neq d$. We have $|\psi\rangle_{\alpha\beta} = 0$.

ii). $b > a = c < d$ and $b \neq d$. We get $|\psi\rangle_{\alpha\beta} = \sqrt{\lambda_a}|bd\rangle$. The eigenvalue of $\rho'^A_{\alpha\beta}$ corresponding to this case is λ_a. As $a = c$ can be chosen to be $1, 2, \cdots, m-2$, b and d have only $m-k$ and $m-k-1$ (corresponding to $a = c = k$, $k = 1, 2, \cdots, m-2$) kinds of choices. Altogether we have $(m-k)(m-k-1)$ eigenvalues of $\rho'^A_{\alpha\beta}$ to be λ_k in this case, with $k = 1, 2, \cdots, m-2$.

iii). $a < b = d > c$ and $a \neq c$. We have $|\psi\rangle_{\alpha\beta} = \sqrt{\lambda_b}|ac\rangle$. The eigenvalue of $\rho'^A_{\alpha\beta}$ is λ_b. In this case $b = d$ can be $3, 4, \cdots, m$. Then a and c have only $k-1$ and $k-2$ (corresponding to $b = d = k$, $k = 3, 4, \cdots, m$) kinds of choices. Hence we have $(k-1)(k-2)$ eigenvalues of $\rho'^A_{\alpha\beta}$ to be λ_k in this case, $k = 3, 4, \cdots, m$.

iv). $b > a = c < d$ and $b = d$. We obtain $|\psi\rangle_{\alpha\beta} = \sqrt{\lambda_b}|ac\rangle + \sqrt{\lambda_a}|bd\rangle$. The eigenvalues of $\rho'^A_{\alpha\beta}$ are λ_a and λ_b, and $a = c$ can be $1, 2, \cdots, m-1$. Then $b = d$ can be $k+1, k+2, \cdots, m$ (corresponding to $a = c = k, k = 1, 2, \cdots, m-1$). We have $(m-1)$ eigenvalues of $\rho'^A_{\alpha\beta}$ that equal to λ_k, $k = 1, 2, \cdots, m$.

v). $a < b = c < d$. We have $|\psi\rangle_{\alpha\beta} = -\sqrt{\lambda_b}|ad\rangle$. The eigenvalue of $\rho'^A_{\alpha\beta}$ is λ_b. $b = c$ can be $2, 3, \cdots, m-1$. Then a and d have only $k-1$ and $m-k$ (corresponding to $b = c = k$, $k = 2, 3, \cdots, m-1$) kinds of choices. We have $(k-1)(m-k)$ eigenvalues of $\rho'^A_{\alpha\beta}$ that equal to λ_k, $k = 2, 3, \cdots, m-1$.

vi). $c < d = a < b$. We have $|\psi\rangle_{\alpha\beta} = -\sqrt{\lambda_a}|bc\rangle$. The eigenvalue of $\rho'^A_{\alpha\beta}$ is λ_a. In this case $a = d$ can be $2, 3, \cdots, m-1$. c and b have only $k-1$ and $m-k$ (corresponding to $b = c = k$, $k = 2, 3, \cdots, m-1$) kinds of choices. Therefore we have $(k-1)(m-k)$ eigenvalues of $\rho'^A_{\alpha\beta}$ that equal to λ_k, with $k = 2, 3, \cdots, m-1$.

Let $\lambda^i_{\alpha\beta}$ stand for the eigenvalues of $\rho^A_{\alpha\beta}$. From the analysis of cases i)-vi) and formula (54), we get

$$E(|\psi\rangle) = -\frac{1}{(m-1)^2} \sum_{\alpha\beta} \sum_{i=1} \lambda^i_{\alpha\beta} \log(\lambda^i_{\alpha\beta}). \tag{60}$$

Since

$$\rho_{\alpha\beta} = \frac{\rho'_{\alpha\beta}}{Tr\{\rho'_{\alpha\beta}\}} = C_{\alpha\beta}\rho'_{\alpha\beta}, \tag{61}$$

we have $\sum_i \lambda^i_{\alpha\beta} C_{\alpha\beta} = 1$ for any α and β. Therefore

$$\begin{aligned} E(\rho_{\alpha\beta}) &= -\sum_{i=1} C_{\alpha\beta}\lambda^i_{\alpha\beta}\log(C_{\alpha\beta}\lambda^i_{\alpha\beta}) \\ &= -\sum_{i=1} C_{\alpha\beta}\lambda^i_{\alpha\beta}\log(C_{\alpha\beta}) - \sum_{i=1} C_{\alpha\beta}\lambda^i_{\alpha\beta}\log(\lambda^i_{\alpha\beta}) \\ &= -\log(C_{\alpha\beta}) - C_{\alpha\beta}\sum_{i=1}\lambda^i_{\alpha\beta}\log(\lambda^i_{\alpha\beta}). \end{aligned}$$

That is

$$-\sum_{i=1}^{m}\lambda^i_{\alpha\beta}\log(\lambda^i_{\alpha\beta}) = \frac{E(\rho_{\alpha\beta}) + \log(C_{\alpha\beta})}{C_{\alpha\beta}}. \tag{62}$$

Substituting (62) into (60), we obtain that

$$E(|\psi\rangle) = \frac{1}{(m-1)^2}\sum_{\alpha\beta}\frac{E(\rho_{\alpha\beta}) + \log(C_{\alpha\beta})}{C_{\alpha\beta}}, \tag{63}$$

which proves the theorem. □

The theorem shows that one can derive the entanglement of formation of a pure quantum state by measuring the values of the entanglement of formation of all the states $\rho_{\alpha\beta}$ and the values of $C_{\alpha\beta}$. Here if $|\psi\rangle_{\alpha\beta} = 0$, then $C_{\alpha\beta}$ goes to infinity and this term does not contribute to the summation in (57). Hence the summation $\sum_{\alpha\beta}$ in (57) simply goes over all the terms such that $Tr\{L^A_\alpha \otimes L^B_\beta|\psi\rangle\langle\psi|(L^A_\alpha)^\dagger \otimes (L^B_\beta)^\dagger\} \neq 0$.

With formula (57), we now show how to get the value of $E(|\psi\rangle)$ experimentally by measuring the quantities on the right hand side of (57).

The quantity $C_{\alpha\beta} = 1/Tr\{\rho'_{\alpha\beta}\}$ can be determined by $Tr\{\rho'_{\alpha\beta}\}$. Since $Tr\{\rho'_{\alpha\beta}\} = \langle\psi|(L^A_\alpha)^\dagger L^A_\alpha \otimes (L^B_\beta)^\dagger L^B_\beta|\psi\rangle$, one can obtain $C_{\alpha\beta}$ by measuring the local Hermitian operator $(L^A_\alpha)^\dagger L^A_\alpha \otimes (L^B_\beta)^\dagger L^B_\beta$ associated with the state $|\psi\rangle$.

To measure $E(\rho_{\alpha\beta})$, we first note that although $\rho_{\alpha\beta}$ are $m \otimes n$ bipartite quantum states, they are basically "two-qubit" ones. For given $L_\alpha = |i\rangle\langle j| - |j\rangle\langle i|$ and $L_\beta = |k\rangle\langle l| - |l\rangle\langle k|$, $i \neq j$, $k \neq l$, the non-zero elements of $\rho_{\alpha\beta}$ are located at the $i*(m-1)+k$th, $i*(m-1)+l$th, $j*(m-1)+k$th, and $j*(m-1)+l$th rows and the $i*(m-1)+k$th, $i*(m-1)+l$th, $j*(m-1)+k$th, and $j*(m-1)+l$th columns. They constitute a 4×4 matrix,

$$\sigma'_{\alpha\beta} = \begin{pmatrix} \rho_{ik,ik} & \rho_{ik,il} & \rho_{ik,jk} & \rho_{ik,jl} \\ \rho_{il,ik} & \rho_{il,il} & \rho_{il,jk} & \rho_{il,jl} \\ \rho_{jk,ik} & \rho_{jk,il} & \rho_{jk,jk} & \rho_{jk,jl} \\ \rho_{jl,ik} & \rho_{jl,il} & \rho_{jl,jk} & \rho_{jl,jl} \end{pmatrix}.$$

Set $\sigma_{\alpha\beta} = \sigma'_{\alpha\beta}/Tr\{\sigma'_{\alpha\beta}\}$. Obviously $E(\rho_{\alpha\beta}) = E(\sigma_{\alpha\beta})$. But $\sigma_{\alpha\beta}$ are actually two-qubit pure states. According to the formula (55), $E(\sigma_{\alpha\beta})$ is determined by the concurrence $C(\sigma_{\alpha\beta}) = C(\rho_{\alpha\beta})$. Therefore if we can measure the quantity $C(\rho_{\alpha\beta})$, we can obtain $E(\rho_{\alpha\beta})$.

The quantity $C(\rho_{\alpha\beta})$ can be measured experimentally in terms of the method introduced in [44], with a few modifications of the measurement operators. Corresponding to the case of $L_\alpha = |i\rangle\langle j| - |j\rangle\langle i|$ and $L_\beta = |k\rangle\langle l| - |l\rangle\langle k|$, we define $m \times m$ matrix operators Σ_s, $s = 0, 1, 2, 3$, such that $(\Sigma_0)_{pq} = \delta_{pi}\delta_{qi} + \delta_{pj}\delta_{qj}$, $(\Sigma_1)_{pq} = \delta_{pi}\delta_{qj} + \delta_{pj}\delta_{qi}$, $(\Sigma_2)_{pq} = I\delta_{pi}\delta_{qj} - I\delta_{pj}\delta_{qi}$, $(\Sigma_3)_{pq} = \delta_{pi}\delta_{qi} - \delta_{pj}\delta_{qj}$, $p, q = 1, ..., m$. Similarly we define $n \times n$ matrix operators $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3 by replacing the indices i, j in $\Sigma_0, \Sigma_1, \Sigma_2$ and Σ_3 with k, l respectively, and setting $p, q = 1, ..., n$. It is straightforward to derive that $C(\rho_{\alpha\beta})$ can be expressed as the mean values of the above local observables,

$$\begin{aligned} C^2(\rho_{\alpha\beta}) \;=\; & \frac{1}{2} + \frac{C^2_{\alpha\beta}}{2}\big(\langle\Sigma_3\otimes\Gamma_3\rangle^2 - \langle\Sigma_3\otimes\Gamma_0\rangle^2 \\ & -\langle\Sigma_0\otimes\Gamma_3\rangle^2 - \langle\Sigma_0\otimes\Gamma_1\rangle^2 + \langle\Sigma_3\otimes\Gamma_1\rangle^2 \\ & -\langle\Sigma_0\otimes\Gamma_2\rangle^2 + \langle\Sigma_3\otimes\Gamma_2\rangle^2\big)\,. \end{aligned} \tag{64}$$

B. Lower bound of entanglement and violation of Bell-type inequalities

In the following we give a relation between the violation of Bell-type inequalities and the lower bound of quantum entanglement, the convex-roof extension of the negativity (CREN).

The negativity of a bipartite quantum states ρ with dimensions $d(H_A) = M$ and $d(H_B) = N$ $(M \le N)$ is defined by [64]

$$\mathcal{N}(\rho) = \frac{||\rho^{T_A}|| - 1}{M-1}, \tag{65}$$

where ρ^{T_A} is the partial transpose of ρ and $||R|| = \mathrm{Tr}\sqrt{RR^\dagger}$ stands for the trace norm of matrix R. The Negativity is defined based on the positive partial transpose criterion(PPT)[34] which can not detect the PPT bound entanglement. Thus it is not sufficient for the negativity to be a good measure of entanglement. Lee et al in [35] introduced the convex-roof extension of the negativity $\mathcal{N}_m(\rho)$. For pure bipartite quantum states $|\psi\rangle$, $\mathcal{N}_m(|\psi\rangle)$ is exactly the negativity $\mathcal{N}(|\psi\rangle)$ defined in (65). For a mixed bipartite quantum state ρ the CREN is defined by

$$\mathcal{N}_m(\rho) = \min \sum_k p_k \mathcal{N}_m(|\psi_k\rangle), \tag{66}$$

where the minimum is taken over all the ensemble decompositions of $\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|$.

The CREN can detect the PPT bound entanglement, since it is zero if and only if the corresponding quantum state is separable. Lee et al also show that $\mathcal{N}_m(\rho)$ does not increase under local quantum operations and classical communication. However, generally it is very difficult to calculate CREN analytically. The inequalities (21) not only can be used to detect entanglement, but also have some direct relations with the CREN. Here we present an experimentally measurable tight lower bound of CREN for arbitrary bipartite quantum states, in terms of the violation of the inequalities (21).

Theorem 8 *For any bipartite quantum states $\rho \in \mathcal{H}_{\mathcal{AB}}$,*

$$\mathcal{N}_m(\rho) \geq \frac{1}{M-1}\sum_{\alpha\beta}|C_{\alpha\beta}|(\frac{X(\rho_{\alpha\beta})}{2}+1)-(M-1), \tag{67}$$

where $C_{\alpha\beta} = \mathrm{Tr}(L_\alpha \otimes L_\beta\, \rho^{T_A} L_\alpha \otimes L_\beta)$, $X(\rho_{\alpha\beta}) = \min\{0, d(\rho_{\alpha\beta})\}$, and $d(\rho_{\alpha\beta}) = |\langle\mathcal{B}'_{\alpha\beta}\rangle| - 1$ stands for the difference of the left and right side of the inequalities (21).

Proof: Let $|\psi\rangle = \sum_i \sqrt{\mu_i}|ii\rangle$ be a bipartite pure state in Schmidt form. One has

$$\mathcal{N}_m(|\psi\rangle) = \frac{2}{M-1}\sum_{i<j}\sqrt{\mu_i\mu_j}. \tag{68}$$

Note that $\sum_i \mu_i = 1$. By calculating the trace norm of $L_\alpha \otimes L_\beta(|\psi\rangle\langle\psi|)^{T_A} L_\alpha \otimes L_\beta$ for each α and β, we derive that

$$\sum_{\alpha\beta}||C^{|\psi\rangle}_{\alpha\beta}\,(|\psi\rangle_{\alpha\beta}\langle\psi|)^{T_A}|| = (M-1)^2 + 2\sum_{i<j}\sqrt{\mu_i\mu_j}, \tag{69}$$

where $|\psi\rangle_{\alpha\beta} = \frac{L_\alpha\otimes L_\beta|\psi\rangle}{\sqrt{C^{|\psi\rangle}_{\alpha\beta}}}$ and $C^{|\psi\rangle}_{\alpha\beta} = \mathrm{Tr}\{L_\alpha \otimes L_\beta|\psi\rangle\langle\psi|L_\alpha \otimes L_\beta\}$.

Let $\rho = \sum_k p_k\rho_k = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ be the optimal decomposition which fulfills that $\mathcal{N}_m(\rho)$ attains its minimum. In terms of (68) and (69) we get

$$\begin{aligned}
\mathcal{N}_m(\rho) &= \sum_k p_k N(\rho_k) \\
&= \frac{1}{M-1}\sum_k p_k \sum_{\alpha\beta}||C^k_{\alpha\beta}\,(\rho^k_{\alpha\beta})^{T_A}|| - (M-1) \\
&\geq \frac{1}{M-1}\sum_{\alpha\beta}||\sum_k p_k\, C^k_{\alpha\beta}\,(\rho^k_{\alpha\beta})^{T_A}|| - (M-1) \\
&= \frac{1}{M-1}\sum_{\alpha\beta}||\sum_k p_k\, L_\alpha \otimes L_\beta \rho_k^{T_A} L_\alpha \otimes L_\beta|| - (M-1) \\
&= \frac{1}{M-1}\sum_{\alpha\beta}||L_\alpha \otimes L_\beta \rho^{T_A} L_\alpha \otimes L_\beta|| - (M-1) \\
&= \frac{1}{M-1}\sum_{\alpha\beta}|C_{\alpha\beta}|\,||\rho^{T_A}_{\alpha\beta}|| - (M-1) \\
&= \frac{1}{M-1}\sum_{\alpha\beta}|C_{\alpha\beta}|(\frac{X(\rho_{\alpha\beta})}{2}+1) - (M-1),
\end{aligned}$$

where we have used that $||\rho_{\alpha\beta}^{T_A}||$ has at most one negative eigenvalue (see [65]) in deriving the last equation. □

Remark: For the isotropic states (24) our lower bound (67) show that $\mathcal{N}_m(\rho) \geq \frac{4x-1}{3}$, which matches with the formula derived in [35]. Thus in this case the lower bound is exact for CREN. Moreover, our lower bound is experimentally measurable, in the sense that $C_{\alpha\beta} = \mathrm{Tr}(L_\alpha \otimes L_\beta\, \rho^{T_A} L_\alpha \otimes L_\beta) = \mathrm{Tr}(L_\alpha \otimes L_\beta\, \rho L_\alpha \otimes L_\beta)$ is the mean value of the Hermitian operator $L_\alpha L_\alpha^\dagger \otimes L_\beta L_\beta^\dagger$, and $X(\rho_{\alpha\beta}) = \min\{0, d(\rho_{\alpha\beta})\}$ can be determined by the mean value of the witness operator $\mathcal{B}'_{\alpha\beta}$. On the other hand, according to the proof of the theorem the lower bound (67) for pure bipartite quantum states is also exact. Thus based on the continuity of the CREN, for weakly mixed quantum state ρ with $\mathrm{Tr}\{\rho^2\} \approx 1$, (67) supplies a good estimation of the CREN.

V. CONCLUSIONS AND REMARKS

It is a fundamental problem to identify and quantify entanglement both theoretically and experimentally. In this review, we have presented several experimental ways to detect and measure entanglement for both bipartite and multipartite quantum systems. The problem is reduced to measure some local quantum mechanical observables.

Recently, people have successfully used high dimensional bipartite systems like in NMR and nitrogen-vacancy defect center in quantum computation and simulation experiments [66]. Our results present a plausible way to detect and to measure entanglement in these systems and to investigate the roles played by quantum entanglement in these quantum information processing.

So far experimental measurement on entanglement of formation and concurrence concerns only pure states. For mixed states, less is known except for experimental determination of separability, both sufficiently and necessary, for two-qubit [17] and qubit-qutrit systems [44]. Generally (52) has only analytical results for some special states [67] and analytical lower bounds [68] which are not experimentally measurable. Recently in [69] we have presented a measurable lower bound of entanglement of formation. The bound is improved in [70]. The formula (57) may also help to study measurable lower bounds of entanglement of formation for mixed states.

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