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Abstract

We derive a set of invariants under local unitary transformations for arbitrary dimensional quantum systems. These invariants are given by hyperdeterminants and independent from the detailed pure state decompositions of a given quantum state. They also give rise to necessary conditions for the equivalence of quantum states under local unitary transformations.

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I. INTRODUCTION

Invariants under local unitary transformations are tightly related to the discussions on nonlocality - a fundamental phenomenon in quantum mechanics, to the quantum entanglement and classification of quantum states under local transformations. In recent years many approaches have been presented to construct invariants of local unitary transformations. One method is developed in terms of polynomial invariants Refs. [1, 2], which allows in principle to compute all the invariants of local unitary transformations, though it is not easy to perform operationally. In Ref. [3], a complete set of 18 polynomial invariants is presented for the locally unitary equivalence of two qubit-mixed states. Partial results have been obtained for three qubits states [4], tripartite pure and mixed states [5], and some generic mixed states [6–8]. Recently the local unitary equivalence problem for multiqubit [9] and general multipartite [10] pure states has been solved.

However, generally one still has no operational ways to judge the equivalence of two arbitrary dimensional bipartite or multipartite mixed states under local unitary transformations. An effective way to deal with the local equivalence of quantum states is to find the complete set of invariants under local unitary transformations. Nevertheless usually these invariants depend on the detailed expressions of pure state decompositions of a state. For a given state, such pure state decompositions are infinitely many. Particularly when the density matrices are degenerate, the problem becomes more complicated. Since in this case even the eigenvector decompositions of a given state are not unique.

In this note, we give a way of constructing invariants under local unitary transformations such that the invariants obtained in this way are independent from the detailed pure state decompositions of a given state. They give rise to operational necessary conditions for the equivalence of quantum states under local unitary transformations. We show that the hyperdeterminants, the generalized determinant for higher dimensional matrices [11], can be used to construct such invariants. The hyperdeterminant is in fact closely related to the entanglement measure like concurrence [12] and 3-tangle [13]. It has also been used in classification of multipartite pure state [14–16]. By employing hyperdeterminants, we construct some trace invariants that are independent of the detailed pure state decompositions of a given state. These trace invariants are a priori invariant under local unitary transformations.

II. STATE DECOMPOSITION INDEPENDENT LOCAL INVARIANT

Let H_1 and H_2 be n and m-dimensional complex Hilbert spaces, with $\{|i\rangle\}_{i=1}^n$ and $\{|j\rangle\}_{j=1}^m$ the orthonormal basis of spaces H_1 and H_2 respectively. Let ρ be an arbitrary mixed state defined on $H_1 \otimes H_2$,

$$\rho = \sum_{i=1}^{I} p_i |v_i\rangle\langle v_i|, \tag{1}$$

where $|v_i\rangle$ is a normalized bipartite pure state of the form:

$$|v_i\rangle = \sum_{k,l=1}^{n,m} a_{kl}^{(i)} |kl\rangle, \quad \sum_{k,l=1}^{n,m} a_{kl}^{(i)} a_{kl}^{(i)*} = 1, \quad a_{kl}^{(i)} \in \mathbb{C},$$

where $*$ denotes complex conjugation. Denote A_i the matrix with entries given by the coefficients of the vector $\sqrt{p_i}|v_i\rangle$, i.e. $(A_i)_{kl} = (\sqrt{p_i} a_{kl}^{(i)})$ for all $i = 1, \cdots, I$. Define $I \times I$ matrix Ω such that

$$(\Omega)_{ij} = tr(A_i A_j^\dagger), \quad i, j = 1, \cdots, I.$$

The pure state decomposition (1) of a given mixed state ρ is not unique. For another decomposition:

$$\rho = \sum_{i=1}^{I} q_i |\psi_i\rangle\langle \psi_i|, \tag{2}$$

with

$$|\psi_i\rangle = \sum_{k,l=1}^{n,m} b_{kl}^{(i)}|kl\rangle, \quad \sum_{k,l=1}^{n,m} b_{kl}^{(i)} b_{kl}^{(i)*} = 1, \quad b_{kl}^{(i)} \in \mathbb{C},$$

one similarly has matrices B_i with entries $(B_i)_{kl} = (\sqrt{q_i} b_{kl}^{(i)})$, $i = 1, \cdots, I$, and $I \times I$ matrix Ω' with entries

$$(\Omega')_{ij} = tr(B_i B_j^\dagger), \quad i, j = 1, \cdots, I.$$

A quantity $F(\rho)$ is said to be invariant under local unitary transformations if $F(\rho) = F((u_1 \otimes u_2)\rho(u_1 \otimes u_2)^\dagger)$ for any unitary operators $u_1 \in SU(n)$ and $u_2 \in SU(m)$, where $\dagger$ stands for transpose and complex conjugation. Generally $F(\rho)$ may depend on the detailed pure state decomposition. We investigate invariants $F(\rho)$ that are independent on the detailed decompositions of ρ. That is, expression in Eq. (1) and expression in Eq. (2) give the same value of $F(\rho)$ for a given state ρ. These kind of invariants are of special significance in determining the equivalence of two density matrices under local unitary transformations.

Two density matrices ρ and $\tilde{\rho}$ are said to be equivalent under local unitary transformations if there exist unitary operators u_1 (resp. u_2) on the first (resp. second) space of $H_1 \otimes H_2$ such that

$$\tilde{\rho} = (u_1 \otimes u_2)\rho(u_1 \otimes u_2)^\dagger. \tag{3}$$

A necessary condition that (3) holds is that the local invariants have the same values $F(\rho) = F(\tilde{\rho})$. Therefore if the expression of the invariants $F(\rho)$ do not depend on the detailed pure state decomposition, one can easily compare the values of the invariants between $F(\rho)$ and $F(\tilde{\rho})$. Otherwise one has to verify $F(\rho) = F(\tilde{\rho})$ by surveying all the possible pure state decompositions of ρ and $\tilde{\rho}$. In particular, when ρ is degenerate, even the eigen-vector decomposition is not unique, which usually gives rise to the main problem in finding an operational criterion for local equivalence of quantum states. In fact, we have presented a complete set of invariants in [18]. However, these invariants depend on the eigenvectors of a state ρ. When the state is degenerate, this set of invariants is no longer efficient as criterion of local equivalence.

We set out to discuss how to find parametrization independent local unitary invariants. First of all we give an elementary result that the determinant can be used to give invariants that are independent from the choice of the pure state decomposition. The idea is much similar to that of cohomological classes such as Chern classes in topology and geometry.

Theorem 1: The coefficients $F_i(\Omega)$, $i = 1, 2, ..., I$, of the characteristic polynomials of the matrix Ω,

$$\det(\Omega - \lambda E) = \lambda^I + \lambda^{I-1} F_1(\Omega) + \cdots + \lambda F_{I-1}(\Omega) + F_I(\Omega) = \Sigma_{i=0}^I \lambda^{I-i} F_i(\Omega), \tag{4}$$

where E is the $I \times I$ unit matrix, $F_0(\Omega) = 1$, det denotes the determinant, have the following properties:

(i) $F_i(\Omega)$ are independent of the pure state decompositions of ρ;

(ii) $F_i(\Omega)$ are invariant under local unitary transformations, $i = 1, \cdots, I$.

Proof: (i) If Eq. (1) and Eq. (2) are two different representations of a given mixed state ρ, we have $B_i = \Sigma_j U_{ij} A_j$ for some unitary operator U[17]. Consequently,

$$\begin{aligned} \Omega'_{ij} &= tr(B_i B_j^\dagger) = tr\left[\sum_{k,l} U_{ik} A_k U_{jl}^* A_l^\dagger\right] \\ &= \sum_{k,l} U_{ik} U_{jl}^* \, tr(A_k A_l^\dagger) = \sum_{k,l} U_{ik} U_{jl}^* \Omega_{kl} = (U \Omega U^\dagger)_{ij}, \end{aligned}$$

i.e. $\Omega' = U\Omega U^\dagger$. Therefore $\det(\Omega' - \lambda E) = \det(U\Omega U^\dagger - \lambda E) = \det(\Omega - \lambda E)$. Thus the matrices Ω and Ω' have the same characteristic polynomials. Namely $F_i(\Omega) = F_i(\Omega')$. Therefore $F_i(\Omega)$ are invariants under the pure state decomposition transformations.

(ii) Let $P \otimes Q \in SU(n) \otimes SU(m)$. Under the local unitary transformations one has

$$\tilde{\rho} = (P \otimes Q)\rho(P \otimes Q)^\dagger = \sum_{i=1}^I p_i (P \otimes Q)|v_i\rangle\langle v_i|(P \otimes Q)^\dagger = \sum_{i=1}^I p_i |w_i\rangle\langle w_i|,$$

with

$$|w_i\rangle = P \otimes Q|v_i\rangle = \sum_{k,l=1}^{n,m} a_{kl}^{(i)\prime} |kl\rangle, \quad \sum_{k,l=1}^{n,m} a_{kl}^{(i)\prime} a_{kl}^{(i)\prime *} = 1, \quad a_{kl}^{(i)\prime} \in \mathbb{C}.$$

Denote $(A'_i)_{kl} = \sqrt{p_i} a_{kl}^{(i)\prime}$. We have

$$A'_i = P A_i Q^t. \tag{5}$$

Therefore $tr(A_i A_j^\dagger) = tr(A'_i A_j'^\dagger)$ and $\Omega(\rho) = \Omega(\tilde{\rho})$. Hence $F_i(\Omega(\rho)) = F_i(\Omega(\tilde{\rho}))$, and $F_i(\Omega)$, $i = 1, \cdots, I$, are invariant under local unitary transformations. $\Box$

In particular, the invariants $F_1 = \sum tr(\sum_i A_i A_i^\dagger)$ and $F_I = \det(\Omega)$. For the case $I = 2$, one has

$$\Omega = \begin{pmatrix} tr(A_1 A_1^\dagger) & tr(A_1 A_2^\dagger) \\ tr(A_2 A_1^\dagger) & tr(A_2 A_2^\dagger) \end{pmatrix}$$

and $F_1 = tr(A_1 A_1^\dagger) + tr(A_2 A_2^\dagger)$, $F_2 = tr(A_1 A_1^\dagger) tr(A_2 A_2^\dagger) - tr(A_1 A_2^\dagger) tr(A_2 A_1^\dagger)$.

Remark: The number of local invariants F_i is uniquely determined by the rank r of the mixed state ρ, i.e. $I = r$. Therefore we only need to calculate the invariants corresponding to the eigenvector decomposition. Because for arbitrary pure state decomposition $\rho = \Sigma_{j=1}^J q_j |\psi_j\rangle\langle\psi_j|$ with $J > r$, the above determinant is the same as that of the eigen-vector decomposition of $\rho = \Sigma_i^r p_i |\phi_i\rangle\langle\phi_i|$ after adding $J-r$ zero vectors. The determinant $\det(\Omega' - \lambda E)$ of the eigen-vector decomposition of ρ after adding $J-r$ zero vectors and $\det(\Omega - \lambda E)$ of $\rho = \Sigma_{j=1}^r q_j |\psi_j\rangle\langle\psi_j|$ without $J-r$ zero vectors have the relation: $\det(\Omega' - \lambda E) = \lambda^{J-r} \det(\Omega - \lambda E)$. This means that the number of independent local invariants given by (4) does not depend on the number of pure states in the ensemble of a given ρ. Therefore if two mixed states ρ and $\tilde{\rho}$ have different ranks, they are not local unitary equivalent. If their ranks are the same, one only needs to calculate the corresponding invariants with respect to the same numbers I of pure states in the pure state decompositions.

In fact for a quantum state ρ in eigenvector decomposition $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$, the corresponding matrix Ω is a diagonal one with ρ's eigenvalues λ_i as the diagonal entries. In this case the local invariants from Theorem 1 are just the coefficients of the characteristic polynomial of the quantum state ρ. Theorem 1 shows that these coefficients are local invariants and pure state decomposition independent. Moreover the approach employed in Theorem 1 can be generalized to construct more local invariants that are independent of the detailed pure state decompositions by using hyperdeterminant [11].

In order to derive more parametrization independent quantities we consider the multilinear form $f_A : \underbrace{V \otimes \cdots \otimes V}_{2s} \mapsto \mathbb{C}$ given by

$$f_A(e_{i_1}, \cdots, e_{i_s}, e_{j_1}, \cdots, e_{j_s}) = tr(A_{i_1} A_{j_1}^\dagger \cdots A_{i_s} A_{j_s}^\dagger), \tag{6}$$

where e_i are standard basis elements in $V = \mathbb{C}^s$. The multilinear form f can also be written as a tensor in $V^* \otimes \cdots \otimes V^*$:

$$f_A = \sum_{\underline{i}, \underline{j}} tr(A_{i_1} A_{j_1}^\dagger \cdots A_{i_s} A_{j_s}^\dagger) e_{i_1}^* \otimes \cdots \otimes e_{i_s}^* \otimes e_{j_1}^* \otimes \cdots \otimes e_{j_s}^*, \tag{7}$$

where e_i^* are standard 1-form on $\mathbb{C}^s$ such that $e_i^*(e_j) = \delta_{ij}$, and $\underline{i} = (i_1, \cdots, i_s), \underline{j} = (j_1, \cdots, j_s)$. In general we call the $2s$-dimensional matrix or hypermatrix $A = (A_{\underline{i}\underline{j}}) = (tr(A_{i_1} A_{j_1}^\dagger \cdots A_{i_s} A_{j_s}^\dagger))$ formed by the coefficients of (7) the hypermatrix of the multilinear form f_A relative to the standard basis.

First we give the following result for coordinate-changes, which generalizes the well-known result in linear algebra.

Lemma 1 *The multilinear form f_A is changed to a new multilinear form f_B under the change of variables $x_i = \sum_j C_{ij}x'_j$, and the hyper-matrix B is given by*

$$B = (C \otimes \cdots \otimes C \otimes C^T \cdots \otimes C^T)A. \tag{8}$$

For an r-dimensional matrix $A = (A_{i_1\cdots i_r})$ where $0 \le i_k \le n_k$ we associate the multilinear form $f_A = \sum_{i_1\cdots i_r} A_{i_1\cdots i_r} e^*_{i_1} \otimes e^*_{i_r}$, and say that A is of *format* $(1+n_1) \times \cdots \times (1+n_r)$. If one uses the standard coordinate $v = \sum_l x_l e_l$ where $x_l = e^*_l(v)$, we can write the multilinear form f_A as

$$f_A(x^{(1)}, \cdots, x^{(r)}) = \sum_{i_1\cdots i_r} A_{i_1\cdots i_r} x^{(1)}_{i_1} \cdots x^{(r)}_{i_r}. \tag{9}$$

The Cayley hyperdeterminant Det(A) [11] is defined to be the resultant of the multilinear form f_A, that is, Det(A) is the normalized integral equation of the hyperplane given by the multilinear form f_A. We recall that the resultant is a polynomial in components of the tensor f which is zero if and only if the map f has a non-trivial point where all partial derivatives with respect to the components of its vector arguments vanish.

For a fixed $l \in \{1, \ldots, 2s\}$, the general linear group $GL(s)$ acts on the ith factor of f as follows:

$$g_{(l)}.f(v_1, \cdots, v_{2s}) = f(v_1, \cdots, g(v_l), \cdots, v_{2s}). \tag{10}$$

The group $GL(I)^{2s} := GL(I) \times \cdots \times GL(I)$ thus acts on the multilinear form f. Suppose $g(x^{(1)}_i) = \sum_j C^j_i x^{(1)}_j$, then the action of g on the first factor is given by

$$g_{(1)}.f(x^{(1)}_{i_1}, \cdots, x^{(2s)}_{i_{2s}}) = \sum_{k=1}^{I} \sum_{j_1\cdots j_{2s}} C^k_{i_1} A^{i_1\cdots i_{2s}}_{j_1\cdots j_{2s}} x^{(1)}_{j_1} \cdots x^{(2s)}_{j_{2s}}. \tag{11}$$

In general the action of $GL(I)$ on the lth component can be expressed as a product on the hypermatrix with respect to the lth index, and we define that

$$(g_{(l)}.A)_{i_1 i_2\cdots i_r} = \sum_{k=1}^{I} C_{i_l k} A_{i_1\cdots k\cdots i_r}, \tag{12}$$

where k is running through the lth index set.

It is known that [11] the hyperdeterminant exists for a given format and is unique up to a scalar factor, if and only if the largest number in the format is less than or equal to the sum of the other numbers in the format. Hyperdeterminants enjoy many of the properties of determinants. One of the most familiar properties of determinants, the multiplication rule $det(AB) = det(A)det(B)$, can be generalized to the situation of hyperdeterminants as follows. Given a multilinear form $f(x^{(1)}, ..., x^{(r)})$ and suppose that a linear transformation acting on one of its components using an $n \times n$ matrix B, $y_r = Bx_r$. Then

$$Det(f.B) = Det(f)det(B)^{N/n}, \tag{13}$$

where N is the degree of the hyperdeterminant. Therefore we have the following result.

Lemma 2 *The hyperdeterminant of format $(k_1, \dots, k_r)$ is an invariant under the action of the group $SL(k_1)\otimes\cdots\otimes SL(k_r)$, and subsequently also invariant under $SU(k_1)\otimes\cdots\otimes SU(k_r)$.*

Proof: For $(A, B, \cdots, C) \in SL(k_1) \otimes \cdots \otimes SL(k_r)$, it follows from Eq. (13) that

$$Det((A_{(1)} \cdot B_{(2)} \cdots C_{(r)} \cdot)f) = Det(f)det(A)^{N/k_1}det(B)^{N/k_2} \cdots det(C)^{N/k_r} = Det(f). \tag{14}$$

The three-dimensional hyperdeterminant of the format $2 \times 2 \times 2$ is known as the Cayley's hyperdeterminant [19]. In this case the hyperdeterminant of a hypermatrix A with components a_{ijk}, $i, j, k \in \{0, 1\}$, is given by

$$\begin{aligned} Det(A) \; = \; & a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2 - 2a_{000}a_{001}a_{110}a_{111} \\ & -2a_{000}a_{010}a_{101}a_{111} - 2a_{000}a_{011}a_{100}a_{111} - 2a_{001}a_{010}a_{101}a_{110} \\ & -2a_{001}a_{011}a_{110}a_{100} - 2a_{010}a_{011}a_{101}a_{100} + 4a_{000}a_{011}a_{101}a_{110} \\ & +4a_{001}a_{010}a_{100}a_{111}. \end{aligned} \tag{15}$$

This hyperdeterminant can be written in a more compact form by using the Einstein convention and the Levi-Civita symbol ε^{ij}, with $\varepsilon^{00} = \varepsilon^{11} = 0, \varepsilon^{01} = -\varepsilon^{10} = 1$; and $b_{kn} = (1/2)\varepsilon^{il}\varepsilon^{jm}a_{ijk}a_{lmn}$, $Det(A) = (1/2)\varepsilon^{il}\varepsilon^{jm}b_{ij}b_{lm}$. The four-dimensional hyperdeterminant of the format $2 \times 2 \times 2 \times 2$ has been calculated in Ref. [15].

For the general mixed state ρ in Eq. (1), we can define a hypermatrix Ω_s with entries

$$(\Omega_s)_{i_1 i_2 \cdots i_s j_1 j_2 \cdots j_s} = tr(A_{i_1}A_{j_1}^\dagger A_{i_2}A_{j_2}^\dagger \cdots A_{i_s}A_{j_s}^\dagger), \tag{16}$$

for $i_k, j_k = 1, \cdots, I$, $s \geq 1$, $0 \leq i_j \leq k_j$. The format of Ω_s is $I \times \cdots \times I$.

Theorem 2: $Det(\Omega_s - \lambda E)$, with $E = (E_{i_1,i_2,\cdots,i_s,j_1,j_2,\cdots,j_s}) = (\delta_{i_1j_1}\delta_{i_2j_2}\cdots\delta_{i_sj_s})$, is independent of the pure state decompositions of ρ. It is also invariant under local unitary transformations of ρ. In particular, all t-coefficient polynomials of $Det(\Omega_s - \lambda E)$ are local invariants independent from the pure state decompositions.

Proof: We first show that it is also independent from the pure state decomposition of ρ. Let Eq. (1) and Eq. (2) be two different representations of a given mixed state ρ. We have

$$\begin{aligned}
\Omega'_{i_1i_2\cdots i_sj_1j_2\cdots j_s} &= tr(B_{i_1}B^\dagger_{j_1}B_{i_2}B^\dagger_{j_2}\cdots B_{i_s}B^\dagger_{j_s}) \\
&= tr\left[\Sigma_{i'_1j'_1,\cdots,i'_sj'_s}U_{i_1i'_1}A_{i'_1}U^*_{j_1j'_1}A^\dagger_{j'_1}\cdots U_{i_si'_s}A_{i'_s}U^*_{j_sj'_s}A^\dagger_{j'_s}\right] \\
&= (U\otimes U\otimes\cdots\otimes U)(\Omega_s)_{i_1i_2\cdots i_sj_1j_2\cdots j_s}(U^\dagger\otimes U^\dagger\otimes\cdots\otimes U^\dagger).
\end{aligned} \tag{17}$$

Therefore $\Omega'_s = (U\otimes U\otimes\cdots\otimes U)\Omega^A_s(U^\dagger\otimes U^\dagger\otimes\cdots\otimes U^\dagger)$. Using the action, the associated multilinear form f_ω is acted upon by $U\otimes U\otimes\cdots\otimes U$ and $U^\dagger\otimes U^\dagger\otimes\cdots\otimes U^\dagger$ as follows:

$$(U_{(1)}\cdot\cdots U_{(s)}\cdot U^\dagger_{(1)}\cdot\cdots U^\dagger_{(s)}\cdot)f_\omega$$

Using the formula under the action (14) we get $Det(\Omega'_s - \lambda E) = Det(\Omega_s - \lambda E)$, and thus $Det(\Omega_s - E\,\lambda)$ does not depend on the detailed pure state decompositions of a given ρ. Note that in general we don't know the exact formula for the hyperdeterminant, but we can still derive its invariance abstractly.

On the other hand, under local unitary transformations $\tilde\rho = (P\otimes Q)\rho(P\otimes Q)^\dagger$ for some local unitary operators $P\otimes Q\in SU(n)\otimes SU(m)$, similar to the proof of the second part of the Theorem 1 and using Lemma 13 in [11], it is easy to get $\Omega_s = \Omega'_s$. Therefore $Det(\Omega_s - \lambda E)$ is invariant under local unitary transformations. $\square$

As an application of our theorems we now give an interesting example. Consider two mixed states $\rho_1 = diag\{1/2, 1/2, 0, 0\}$ and $\rho_2 = diag\{1/2, 0, 1/2, 0\}$. ρ_1 has a pure state decomposition with

$$A_0 = \begin{pmatrix} \frac{1}{\sqrt2} & 0 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \frac{1}{\sqrt2} \\ 0 & 0 \end{pmatrix}.$$

While ρ_2 has a pure state decomposition with

$$B_0 = \begin{pmatrix} \frac{1}{\sqrt2} & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt2} & 0 \end{pmatrix}.$$

We have the corresponding matrices $(\Omega(\rho_1))_{i,j} = tr(A_i A_j^\dagger)$ and $(\Omega(\rho_2))_{i,j} = tr(B_i B_j^\dagger)$, $i, j = 0, 1$. From Theorem 1 one can find that these two states have the same values of the invariants in Eq. (4), $F_i(\Omega(\rho_1)) = F_i(\Omega(\rho_2))$.

We now consider further the four-dimensional hyperdeterminant of the format $2\times2\times2\times2$ [15]. Let $(\Omega(\rho_1))_{ijkl} = tr(A_i A_j^\dagger A_k A_l^\dagger) \equiv a_r$, $r = 0, \cdots, 15$, where $r = 8i + 4j + 2k + l$. From Ref. [15], one invariant with degree 4 is given by

$$N(\rho_1) = det \begin{pmatrix} a_0 & a_1 & a_8 & a_9 \\ a_2 & a_3 & a_{10} & a_{11} \\ a_4 & a_5 & a_{12} & a_{13} \\ a_6 & a_7 & a_{14} & a_{15} \end{pmatrix} = \frac{1}{256}.$$

However for ρ_2 we have $N(\rho_2) = 0$. Therefore ρ_1 and ρ_2 are not equivalent under local unitary transformations.

In Ref. [20], the Ky Fan norm of the realignment matrix of the quantum states $\mathcal{N}(\rho)$ is proved to be invariant under local unitary operations. By calculation we find $\mathcal{N}(\rho_1) = \mathcal{N}(\rho_2) = \frac{1}{\sqrt{2}}$. This means the Ky Fan norm of the realignment matrix can not recognize that ρ_1 and ρ_2 are not equivalent under local unitary transformations. Therefore Theorem 2 has its superiority over it with respect to these two states.

Our results can be generalized to multipartite case. Let $H_1, H_2, \cdots, H_m$ be $n_1, n_2, \cdots, n_m$-dimensional complex Hilbert spaces with $\{|k_1\rangle\}_{k_1=1}^{n_1}$, $\{|k_2\rangle\}_{k_2=1}^{n_2}$, $\cdots$, $\{|k_m\rangle\}_{k_m=1}^{n_m}$ the orthonormal basis of $H_1, H_2, \cdots, H_m$ respectively. Let ρ be an arbitrary mixed state defined on $H_1 \otimes H_2 \otimes \cdots \otimes H_m$, $\rho = \sum_{i=1}^{I} p_i |v_i\rangle\langle v_i|$, where $|v_i\rangle$ is a multipartite pure state of the form: $|v_i\rangle = \sum_{k_1,k_2,\cdots,k_m=1}^{n_1,n_2,\cdots,n_m} a_{k_1k_2\cdots k_m}^{(i)} |k_1 k_2 \cdots k_m\rangle$, $a_{k_1k_2\cdots k_m}^{(i)} \in \mathbb{C}$. Now we view $|v_i\rangle$ as bipartite pure state under the partition between the first l subsystems and the rest, $1 \le l < n$. Then $A_i = (\sqrt{p_i} a_{k_1k_2\cdots k_m}^{(i)})$ can be regarded as the $N_1 \times N_2$ matrix with $N_1 = n_1 \times n_2 \times \cdots \times n_l$ and $N_2 = n_{l+1} \times n_{l+2} \times \cdots \times n_m$ for all $i = 1, \cdots, I$. We define the matrix Ω_s with entries $(\Omega_s)_{i_1i_2\cdots i_s j_1 j_2 \cdots j_s} = tr(A_{i_1} A_{j_1}^\dagger \cdots A_{i_s} A_{j_s}^\dagger)$, for $i_k, j_k = 1, \cdots, I$, $s \ge 1$. Then we have that $Det(\Omega_s - \lambda E)$ does not depend on the pure states decompositions and is invariant under local unitary transformations.

III. CONCLUSION

We have investigated the invariants under local unitary transformations for arbitrary dimensional quantum systems. These invariants are independent of the detailed pure state decompositions of a given state. They give rise to the necessary conditions for the equivalence of quantum states under local unitary transformations. These invariants may be also used in characterizing quantum correlations such as quantum entanglement [21] and quantum discord [22], since all these quantities are just the invariants under local unitary transformations and are independent of the pure state decompositions.

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