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by

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Abstract: Let $X = (X_1, X_2, \cdots, X_m)$ be a system of real smooth vector fields defined in an open domain $\tilde{\Omega} \subset \mathbb{R}^n$, $\Omega \subset\subset \tilde{\Omega}$ be a bounded open subset in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $\triangle_X = \sum_{j=1}^m X_j^2$. In this paper, if λ_j is the j^{th} Dirichlet eigenvalue for the degenerate elliptic operator $-\triangle_X$ (or the degenerate Schrödinger operator $-\triangle_X + V$) on Ω, we deduce respectively that the lower estimates for the sums $\sum_{j=1}^k \lambda_j$ in both cases for the operator $-\triangle_X$ to be finitely degenerate (i.e. the Hörmander condition is satisfied) or infinitely degenerate (i.e. the Hörmander condition is not satisfied).

Key words: Lower bounds for Dirichlet eigenvalues, finite degenerate elliptic operators, infinite degenerate elliptic operators, singular potential term.
MSC(2000): 35P05; 35P20

1 Introduction and Main Results

Let $\tilde{\Omega}$ be an open domain in $\mathbb{R}^n$. For the systems of real smooth vector fields $X = (X_1, X_2, \cdots, X_m)$, we introduce following function space (cf. [16, 22]):

$$H_X^1(\tilde{\Omega}) = \{u \in L^2(\tilde{\Omega}) \mid X_j u \in L^2(\tilde{\Omega}), \quad j = 1, \cdots, m\},$$

which is a Hilbert space with norm $\|u\|^2_{H_X^1} = \|u\|^2_{L^2} + \|Xu\|^2_{L^2}$, $\|Xu\|^2_{L^2} = \sum_{j=1}^n \|X_j u\|^2_{L^2}$. Let $\Omega \subset\subset \tilde{\Omega}$ be a bounded open subset with boundary $\partial\Omega$, here we assume that $\partial\Omega$ is C^∞ smooth and non characteristic for the system of vector fields X. Next, the subspace $H^1_{X,0}(\Omega)$ is defined as a closure of $C_0^\infty(\Omega)$ in $H_X^1(\tilde{\Omega})$, which is also a Hilbert space.

Let $I = (j_1, \cdots, j_k)$ with $1 \le j_i \le m$, we denote $|I| = k$. We say that the vector fields $X = \{X_1, X_2, \cdots, X_m\}$ satisfy the Hörmander's condition (cf. [5]) if X together with their commutators

$$X_I = [X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}] \cdots]]],$$

up to some fixed length $|I| \le Q$ span the tangent space at each point of $\tilde{\Omega}$. Here Q is called the Hörmander index of X on $\tilde{\Omega}$, which is the smallest positive integer for the Hörmander condition above being satisfied.

If $Q > 1$, the operator $\triangle_X = \sum_{j=1}^m X_j^2$ is a degenerate elliptic operator. In this paper we consider the following Dirichlet eigenvalue problems in $H^1_{X,0}(\Omega)$ for the degenerate elliptic operators and the degenerate Schrödinger operators,

$$\begin{cases} -\triangle_X u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\triangle_X u - \varepsilon V(x) u = \mu u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

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where $0 < \varepsilon < 1$, the potential function $V(x) \geq 0$ satisfies the following Hardy type inequality,

$$\int_\Omega Vu^2dx \leq \int_\Omega |Xu|^2dx, \text{ for all } u \in H^1_{X,0}(\Omega). \tag{1.3}$$

In the classical case, $X = \{\partial_{x_1}, \cdots, \partial_{x_n}\}$, $\triangle_X$ is the Laplacian $\triangle$ and Hörmander index $Q = 1$, H. Weyl's asymptotic formula (see [21]) assets that $\lambda_k \sim C_n(\frac{k}{|\Omega|_n})^{\frac{2}{n}}$, where $\{\lambda_k\}_{k\geq 1}$ are Dirichlet eigenvalues for the Laplacian $\triangle$, $|\Omega|_n$ is the n-dimensional Lebesgue measure of Ω and $C_n = (2\pi)^2 B_n^{-\frac{2}{n}}$ with B_n being the volume of the unit ball in $\mathbb{R}^n$. Later in [19], Pólya proved that the above asymptotic relation is in fact a one-sided inequality if Ω is a plane domain which tiles $\mathbb{R}^2$ (and his proof also works in $\mathbb{R}^n$) and he conjectured that, for any domain in $\mathbb{R}^n$, the inequality

$$\lambda_k \geq C_n(\frac{k}{|\Omega|_n})^{\frac{2}{n}}, \text{ for any } k \geq 1, \tag{1.4}$$

holds.

In this direction Lieb [11] proved an inequality like (1.4) for any domain in $\mathbb{R}^n$ but with a constant $\tilde{C_n}$ that differs from the constant C_n by a factor. Later in 1983, by using the Fourier transformation approach, Li and Yau [10] gave a simple proof for the lower bound and obtain

$$\sum_{i=1}^k \lambda_i \geq \frac{nC_n}{n+2} k^{\frac{n+2}{n}} |\Omega|_n^{-\frac{2}{n}}, \text{ for any } k \geq 1. \tag{1.5}$$

If the system of vector fields X with the Hörmander index $1 \leq Q < \infty$, then we know that there is a sequence of discrete eigenvalues for the problem (1.1) (or for the problem (1.2) respectively), which can be ordered, after counting (finite) multiplicity, as $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$, and $\lambda_k \to +\infty$. For each $1 \leq j \leq Q$ and $x \in \tilde{\Omega}$, we denote $V_j(x)$ as the subspace of the tangent space $T_x(\tilde{\Omega})$ which is spanned by the vector fields $\{X_I\}$ with $|I| \leq j$. If the dimension of $V_j(x)$ is constant ν_j in a neighborhood of each $x \in \tilde{\Omega}$, then we say the system of the vector fields X satisfies the so called Métivier's condition. Observe that the Hörmander condition implies that $\nu_Q = n$.

Under the Hörmander condition and the Métivier's condition above, Métivier [12] proved that $\lambda_k \approx k^{\frac{2}{\nu}}$, where $\nu = \sum_{j=1}^Q j(\nu_j - \nu_{j-1})$ (with $\nu_0 = 0$). However, in this case, it seems no result for the lower bound of λ_k. On the other hand, if the Hörmander condition is not satisfied for the system of vector fields X (i.e. $Q = +\infty$, $\triangle_X$ is infinitely degenerate), it seems that there is no any result for the eigenvalues estimates (even for the asymptotic estimate).

In this paper, in case of the operator $\triangle_X$ is degenerate elliptic, we shall use the approach in Li-Yau [10] to give the corresponding estimates for the eigenvalues of the problems (1.1) and (1.2). We have the following main results.

1.1 The case of $\triangle_X$ to be finitely degenerate

First, we give the following well-known result:

Proposition 1.1. *The system of vector fields $X = (X_1, \cdots, X_m)$ satisfies Hörmander's condition, and its Hörmander index is Q, if and only if the following sub-elliptic estimate*

$$\left\||\nabla|^{\frac{1}{Q}}u\right\|^2_{L^2(\Omega)} \leq C(Q)(\|Xu\|^2_{L^2(\Omega)} + \widetilde{C}(Q)\|u\|^2_{L^2(\Omega)}), \tag{1.6}$$

holds for all $u \in C_0^\infty(\Omega)$. Where $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n})$, $C(Q) > 0$ and $\widetilde{C}(Q) \geq 0$ are the best constants for the estimate (1.6) *to be satisfied.*

Proof. See [4], [6] and [18]. $\square$

Then for the problem (1.1), we have

Theorem 1.1. *If the Hörmander condition is satisfied for the vector fields X and Q is the Hörmander index of X. Let λ_j be the j^{th} eigenvalue of the problem* (1.1)*, then for all $k \geq 1$,*

$$\sum_{j=1}^{k} \lambda_j \geq C_1 k^{1+\frac{2}{Qn}} - \widetilde{C}(Q)k, \tag{1.7}$$

where $C_1 = \frac{nQ(2\pi)^{\frac{2}{Q}}}{C(Q)\cdot(nQ+2)(|\Omega|_n B_n)^{\frac{2}{nQ}}}$, $C(Q)$ and $\widetilde{C}(Q)$ are the constants in Proposition 1.1, B_n is the volume of the unit ball in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω.

Since $k\lambda_k \geq \sum_{j=1}^{k} \lambda_j$, then the result of Theorem 1.1 implies that

$$\lambda_k \geq C_1 k^{\frac{2}{Qn}} - \widetilde{C}(Q), \text{ for } k \geq 1.$$

Remark 1.1. *If $X = \{\partial_{x_1}, \partial_{x_2}, \cdots, \partial_{x_n}\}$, then the Hörmander index of X is 1, $C(1) = 1$ and $\widetilde{C}(1) = 0$. Thus for all $k \geq 1$, the lower bound estimate* (1.7) *gives the same result to the estimate* (1.5) *in [10].*

Next, we consider the problem (1.2).

Theorem 1.2. *Under the conditions of Theorem 1.1, let $0 < \varepsilon < 1$ and the potential function $V(x)$ satisfies the Hardy type inequality* (1.3)*. Then if $\widetilde{\lambda}_j$ is the j^{th} eigenvalue of the problem* (1.2)*, we have*

$$\sum_{j=1}^{k} \widetilde{\lambda}_j \geq (1-\varepsilon)\Big(C_1 k^{1+\frac{2}{Qn}} - \widetilde{C}(Q)k\Big), \text{ for all } k \geq 1, \tag{1.8}$$

where C_1 and $\widetilde{C}(Q)$ are the same constants in Theorem 1.1 and Proposition 1.1, Q is the Hörmander index of X.

1.2 The case of $\triangle_X$ to be infinitely degenerate

Here we suppose that X satisfies the finite type of Hörmander's condition on $\tilde{\Omega}$ except a union of smooth surfaces Γ. Then the Hörmander condition is not satisfied for X on Γ and the operator $\triangle_X$ is infinitely degenerate elliptic operator. Also we suppose that the surface Γ is non characteristic for X, and the vector fields X satisfies the following Logarithmic regularity estimate for all $u \in C_0^\infty(\tilde{\Omega})$,

$$\|(\log \Lambda)^s u\|^2_{L^2(\Omega)} \leq C_0 \Big[\int_\Omega |Xu|^2 dx + \|u\|^2_{L^2(\Omega)}\Big], \tag{1.9}$$

where $s > 1$, $C_0 > 0$ and $\Lambda = (e^2 + |\nabla|^2)^{\frac{1}{2}}$. Thus, we know that, cf. [2, 16], the problem (1.1) (or the problem (1.2) respectively) has a sequence of discrete eigenvalues, which can be ordered, after counting (finite) multiplicity, as $0 < \beta_1 \leq \beta_2 \leq \beta_3 \leq \cdots \leq \beta_k \leq \cdots$, and $\beta_k \to +\infty$.

Then we have

Theorem 1.3. *Under the conditions above, let X satisfy the logarithmic regularity estimate* (1.9)*, β_j be the j^{th} eigenvalue of the problem* (1.1)*, then for all $k \geq k_0$,*

$$\sum_{j=1}^{k} \beta_j \geq C_2 k(\log k)^s - k, \tag{1.10}$$

where $k_0 = \max\Big\{\frac{2^{2s}e^n\omega_{n-1}|\Omega|_n}{C_0\pi^n}, 3\Big\}$, $C_2 = n\Big(C_0 2^{n+s}(|\log \frac{|\Omega|_n\omega_{n-1}}{n(2\pi)^n}|^s + n^s)\Big)^{-1}$, ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$, $|\Omega|_n$ is the volume of Ω, s and C_0 are the numbers in (1.9)*.*

Since $k\beta_k \geq \sum_{j=1}^{k} \beta_j$, the the result of Theorem 1.3 gives that

$$\beta_k \geq C_2(\log k)^s - 1, \text{ for } k \geq k_0.$$

Secondly, for the problem (1.2) we have

Theorem 1.4. *Under the conditions above, let $0 < \varepsilon < 1$ and the potential function V satisfies the Hardy type inequality* (1.3). *If X satisfy the logarithmic regularity estimate* (1.9), *and $\widetilde{\beta}_j$ is the j^{th} eigenvalue of the problem* (1.2), *then for all $k \geq k_0$,*

$$\sum_{j=1}^{k} \widetilde{\beta}_j \geq (1-\varepsilon)\Big((C_2 k(\log k)^s - k\Big), \tag{1.11}$$

where k_0 and C_2 are the same constants as given in Theorem 1.3.

Remark 1.2. *More results for the infinitely degenerate operators can be found in [1, 3], [7, 8, 9], [14, 15, 17] and [20].*

In this paper, The proofs of Theorem 1.1 and Theorem 1.2 will be given in Section 2, and in Section 3 we shall prove Theorem 1.3 and Theorem 1.4. In Section 4, we shall give some examples in which the logarithmic regularity estimates (1.9) and the Hardy type inequalities (1.3) will be satisfied.

2 Proofs of Theorem 1.1 and Theorem 1.2

Similar to the approach in [10], we introduce the following lemma.

Lemma 2.1. *Let f be a real-valued function defined on $\mathbb{R}^n$ with $0 \leq f \leq M_1$. For some $s > 0$, if*

$$\int_{\mathbb{R}^n} |z|^s f(z)dz \leq M. \tag{2.1}$$

Then we have the following inequality,

$$(\int_{\mathbb{R}^n} f(z)dz)^{1+\frac{s}{n}} \leq M_2(n,s,M_1)M, \tag{2.2}$$

where $M_2(n,s,M_1) = \left(\frac{1}{n}\right)^{\frac{n+s}{n}}(n+s)(M_1\omega_{n-1})^{\frac{s}{n}}$, ω_{n-1} is the area of the unit $(n-1)$-sphere in $\mathbb{R}^n$.

Proof. First, we choose $R(M)$, such that

$$\int_{\mathbb{R}^n} |z|^s g(z)dz = M, \tag{2.3}$$

where

$$g(z) = \begin{cases} M_1, & |z| < R(M), \\ 0, & |z| \geq R(M). \end{cases}$$

Then $(|z|^s - R^s(M))(f(z) - g(z)) \geq 0$, hence

$$R^s(M)\int (f(z) - g(z))dz \leq \int |z|^s(f(z) - g(z))dz = 0. \tag{2.4}$$

Now we have

$$M = \int_{\mathbb{R}^n} |z|^s g(z)dz = M_1 \int_0^{R(M)} r^{n-1+s}\omega_{n-1}dr = \frac{M_1\omega_{n-1}R^{n+s}(M)}{n+s}. \tag{2.5}$$

From the definition of $g(z)$, we know

$$\int_{\mathbb{R}^n} g(z)dz = M_1 B_n R^n(M), \tag{2.6}$$

where B_n is the volume of the unit n-ball in $\mathbb{R}^n$. Hence by using $nB_n = \omega_{n-1}$ we can deduce that

$$\Big(\int_{\mathbb{R}^n} f(z)dz\Big)^{1+\frac{s}{n}} \le \Big(\int_{\mathbb{R}^n} g(z)dz\Big)^{1+\frac{s}{n}} \le M_2(n,s,M_1)M, \tag{2.7}$$

where

$$M_2(n,s,M_1) = \Big(\frac{1}{n}\Big)^{\frac{n+s}{n}}(n+s)(M_1\omega_{n-1})^{\frac{s}{n}}.$$

□

Proof of Theorem 1.1.

Proof. Let $\{\lambda_k\}_{k\ge1}$ be a sequence of the Dirichlet eigenvalues of the problem (1.1), $\{\psi_k(x)\}_{k\ge1}$ be the corresponding eigenfunctions, then $\{\psi_k(x)\}_{k\ge1}$ constitute an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$. We have

Lemma 2.2. *For the system of vector fields $X = (X_1,\cdots,X_m)$, if $\{\psi_j\}_{j=1}^k$ are the set of orthonormal eigenfunctions which corresponding to the eigenvalues $\{\lambda_j\}_{j=1}^k$. Define*

$$\Psi(x,y) = \sum_{j=1}^k \psi_j(x)\psi_j(y). \tag{2.8}$$

Then for the partial Fourier transformation of $\Psi(x,y)$ in the x-variable, $\hat{\Psi}(z,y) = (2\pi)^{-n/2}\int_{\mathbb{R}^n}\Psi(x,y)e^{-ix\cdot z}dx$, we have

$$\int_\Omega\int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 dzdy = k, \quad \textit{and} \quad \int_\Omega |\hat{\Psi}(z,y)|^2 dy \le (2\pi)^{-n}|\Omega|_n. \tag{2.9}$$

Proof. Since

$$\int_{\mathbb{R}^n} \Psi^2(x,y)dx = \int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 dz. \tag{2.10}$$

Hence by the orthonormality of $\{\psi_j\}_{j=1}^k$, one has

$$\begin{aligned}\int_\Omega\int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 dzdy &= \int_\Omega\int_{\mathbb{R}^n} |\Psi(x,y)|^2 dxdy \\ &= \int_\Omega\int_\Omega |\Psi(x,y)|^2 dxdy = k.\end{aligned} \tag{2.11}$$

On the other hand,

$$\begin{aligned}\int_{\mathbb{R}^n} |\hat{\Psi}(z,y)|^2 dzdy &= \int_\Omega (2\pi)^{-n}|\int_{\mathbb{R}^n}\Psi(x,y)e^{-ix\cdot z}dx|^2 dy \\ &= \int_\Omega (2\pi)^{-n}|\int_\Omega \Psi(x,y)e^{-ix\cdot z}dx|^2 dy.\end{aligned}$$

Using the Fourier expansion for the function $e^{-ix\cdot z}$, i.e.

$$e^{-ix\cdot z} = \sum_{j=1}^\infty a_j(z)\psi_j(x), \text{ with } a_j(z) = \int_\Omega e^{-ix\cdot z}\psi_j(x)dx.$$

Then we know that $\sum_{j=1}^{\infty}|a_j(z)|^2=\int_\Omega|e^{-ix\cdot z}|^2dx=|\Omega|_n$. Thus

$$\begin{aligned}|\int_\Omega \Psi(x,y)e^{-ix\cdot z}dx| &\le |\int_\Omega\sum_{j=1}^{k}\sum_{l=1}^{\infty}a_l(z)\psi_l(x)\psi_j(x)\psi_j(y)dx|\\ &=|\sum_{j=1}^{k}a_j(z)\psi_j(y)|.\end{aligned}$$

Using the estimates above, we have

$$\begin{aligned}\int_\Omega|\hat{\Psi}(z,y)|^2dy &\le (2\pi)^{-n}\int_\Omega|\sum_{j=1}^{k}a_j(z)\psi_j(y)|^2dy\\ &=(2\pi)^{-n}\sum_{j=1}^{k}|a_j(z)|^2\le(2\pi)^{-n}|\Omega|_n.\end{aligned}$$

□

By using the results in Proposition 1.1 and Lemma 2.2, we have

$$\int_{\mathbb{R}^n}\int_\Omega|z|^{\frac{2}{Q}}|\hat{\Psi}(z,y)|^2dydz=\int_{\mathbb{R}^n}\int_\Omega\left||\nabla|^{\frac{1}{Q}}\Psi(x,y)\right|^2dydx, \tag{2.12}$$

and

$$\int_{\mathbb{R}^n}\int_\Omega\left||\nabla|^{\frac{1}{Q}}\Psi(x,y)\right|^2dydx\le C(Q)(\int_\Omega\int_\Omega|X(x)\Psi(x,y)|^2dxdy+\widetilde{C}(Q)\int_\Omega\int_\Omega|\Psi(x,y)|^2dxdy).$$

Next, we can deduce that

$$\begin{aligned}\int_\Omega\int_\Omega|X(x)\Psi(x,y)|^2dxdy &=\int_\Omega\Big(\sum_{l=1}^{m}\int_\Omega|\sum_{j=1}^{k}(X_l(x)\psi_j(x))\psi_j(y)|^2dx\Big)dy\\ &=\sum_{l=1}^{m}\Big(\int_\Omega\sum_{j=1}^{k}|X_l(x)\psi_j(x)|^2\Big)dx \qquad (2.13)\\ &=-\int_\Omega\sum_{j=1}^{k}\psi_j(x)\triangle_X\psi_j(x)dx=\sum_{j=1}^{k}\lambda_j.\end{aligned}$$

Thus from (2.11) and (2.12), we have

$$\int_{\mathbb{R}^n}\int_\Omega|z|^{\frac{2}{Q}}|\hat{\Psi}(z,y)|^2dydz\le C(Q)(\sum_{j=1}^{k}\lambda_j+\widetilde{C}(Q)k).$$

Now we choose $f(z)=\int_\Omega|\hat{\Psi}(z,y)|^2dy$, $M_1=(2\pi)^{-n}|\Omega|_n$, $s=\frac{2}{Q}$ and $M=C(Q)(\sum_{j=1}^{k}\lambda_j+\widetilde{C}(Q)k)$. Then the result of Lemma 2.1 gives that, for any $k\ge1$,

$$k^{1+\frac{2}{nQ}}\le M_2(n,Q,|\Omega|_n)C(Q)\cdot(\sum_{j=1}^{k}\lambda_j+\widetilde{C}(Q)k), \tag{2.14}$$

with $M_2(n,Q,|\Omega|)=((2\pi)^{-n}|\Omega|_nB_n)^{\frac{2}{nQ}}(\frac{n+\frac{2}{Q}}{n})$. That means, for any $k\ge1$,

$$\sum_{j=1}^{k}\lambda_j\ge C_1k^{1+\frac{2}{nQ}}-\widetilde{C}(Q)k, \tag{2.15}$$

with $C_1=\frac{nQ(2\pi)^{\frac{2}{Q}}}{C(Q)\cdot(nQ+2)(|\Omega|B_n)^{\frac{2}{nQ}}}$. Theorem 1.1 is proved. □

Proof of Theorem 1.2.

Proof. From the Hardy type inequality (1.3) we know the operator $-\triangle_X - \varepsilon V(x)$ is a positive operator for $0<\varepsilon<1$. Thus let $\{\tilde{\lambda}_k\}_{k\geq 1}$ be the Dirichlet eigenvalues of the problem (1.2), $\{\varphi_k\}_{k\geq 1}$ be the corresponding eigenfunctions which constitutes an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Observe that

$$(1-\varepsilon)\int_\Omega |X\varphi_j|^2 dx \leq \int_\Omega \varphi_j(x)(-\triangle_X\varphi_j(x) - \varepsilon V\varphi_j(x)) = \tilde{\lambda}_j.$$

If we denote $\tilde{\Psi}(x,y)$ by $\tilde{\Psi}(x,y) = \sum_{j=1}^k \varphi_j(x)\varphi_j(y)$, then similar to the proof of Theorem 1.1,

$$\int_{\mathbb{R}^n}\int_\Omega |\hat{\tilde{\Psi}}(z,y)|^2|z|^{\frac{2}{Q}}dydz = \int_{\mathbb{R}^n}\int_\Omega \left||\nabla|^{\frac{1}{Q}}\tilde{\Psi}(x,y)\right|^2 dydx \leq C(Q)\Big(\frac{\sum_{j=1}^k\tilde{\lambda}_j}{1-\varepsilon} + \widetilde{C}(Q)k\Big).$$

That means, for any $k\geq 1$,

$$\sum_{j=1}^k \tilde{\lambda}_j \geq (1-\varepsilon)C_1 k^{1+\frac{2}{nQ}} - (1-\varepsilon)\widetilde{C}(Q)k, \tag{2.16}$$

where C_1 and $\widetilde{C}(Q)$ are the same constants as that in Theorem 1.1, Q is the Hörmander index of X. The proof of Theorem 1.2 is complete. $\square$

3 The proofs of Theorem 1.3 and Theorem 1.4

First, we have the following extension for the result of Lemma 2.1.

Lemma 3.1. *Let f be a real-valued function defined on $\mathbb{R}^n$ and $0\leq f\leq M_1$. For some $s>0$, if*

$$\int_{\mathbb{R}^n} f(z)dz > e, \ \textit{and} \ \int_{\mathbb{R}^n}(\log(e^2+|z|^2))^s f(z)dz \leq M_3, \tag{3.1}$$

where $M_3 \geq 2^{2s+n}e^n M_1\omega_{n-1}$. Then we have the following inequality,

$$\int_{\mathbb{R}^n} f(z)dz\cdot(\log(\int_{\mathbb{R}^n} f(z)dz))^s \leq M_4(n,s,M_1)M_3, \tag{3.2}$$

where

$$M_4(n,s,M_1) = \frac{2^{n+s}}{n}(|\log(\frac{M_1\omega_{n-1}}{n})|^s + n^s).$$

Proof. Related to M_3, we introduce a constant $R(M_3)>0$, satisfying

$$\int_{\mathbb{R}^n}(\log(e^2+|z|^2))^s g(z)dz = M_3, \tag{3.3}$$

where

$$g(z) = \begin{cases} M_1, & |z|<R(M_3), \\ 0, & |z|\geq R(M_3). \end{cases}$$

Since $M_3 \geq 2^{2s+n}e^n M_1\omega_{n-1}$, that means $R(M_3)\geq 2e$. In fact, if $R(M_3)<2e$, then

$$\begin{aligned} M_3 &= \int_{\mathbb{R}^n}(\log(e^2+|z|^2))^s g(z)dz \\ &= M_1\omega_{n-1}\int_0^{R(M_3)}(\log(e^2+r^2))^s r^{n-1}dr \\ &\leq M_1\omega_{n-1}(\log(5e^2))^s(2e)^n < 2^{2s+n}e^n M_1\omega_{n-1}, \end{aligned}$$

which is incompatible with the condition of M_3.

By $R(M_3)\geq 2e$, one has $\frac{R(M_3)}{2}\geq\sqrt{R(M_3)}$,

$$\begin{aligned}
M_3 &\geq M_1\omega_{n-1}\int_{\frac{R(M_3)}{2}}^{R(M_3)}(\log(e^2+r^2))^s r^{n-1}dr\\
&\geq M_1\omega_{n-1}2^s\int_{\frac{R(M_3)}{2}}^{R(M_3)}(\log r)^s r^{n-1}dr\\
&\geq M_1\omega_{n-1}2^s\Big(\frac{R(M_3)}{2}\Big)^n\Big(\log\frac{R(M_3)}{2}\Big)^s\\
&\geq M_1\omega_{n-1}\frac{R(M_3)^n}{2^n}(\log R(M_3))^s,
\end{aligned}$$

Since $\Big[(\log(e^2+|z|^2))^s-(\log(e^2+|R(M_3)|^2))^s\Big](f(z)-g(z))\geq 0$, we have

$$\int_{\mathbb{R}^n}f(z)dz\leq\int_{\mathbb{R}^n}g(z)dz.$$

Using the inequalities above and the fact $\int_{\mathbb{R}^n}f(z)dz>e$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}f(z)dz\cdot\Big(\log(\int_{\mathbb{R}^n}f(z)dz)\Big)^s &\leq\int_{\mathbb{R}^n}g(z)dz\cdot\Big(\log(\int_{\mathbb{R}^n}g(z)dz)\Big)^s\\
&=M_1B_n(R(M_3))^n\cdot\Big[\log(M_1B_n(R(M_3))^n)\Big]^s\\
&\leq M_1B_n(R(M_3))^n\cdot 2^s(|\log(M_1B_n)|^s+(n\log R(M_3))^s)\\
&=M_1B_n2^s(|\log(M_1B_n)|^s+n^s)(R(M_3))^n(\log R(M_3))^s.
\end{aligned}$$

By using $nB_n=\omega_{n-1}$ and the estimates above, we can deduce that

$$\int_{\mathbb{R}^n}f(z)dz\cdot\Big(\log(\int_{\mathbb{R}^n}f(z)dz)\Big)^s\leq\frac{2^{n+s}}{n}(|\log(M_1B_n)|^s+n^s)M_3.$$

Taking $M_4(n,s,M_1)=\frac{2^{n+s}}{n}(|\log(\frac{M_1\omega_{n-1}}{n})|^s+n^s)$, then

$$\int_{\mathbb{R}^n}f(z)dz\cdot(\log(\int_{\mathbb{R}^n}f(z)dz))^s\leq M_4(n,s,M_1)M_3.$$

□

Since the boundary $\partial\Omega$ and the infinitely degenerate surface Γ are non characteristic for the vector fields X and X satisfies logarithmic regularity estimate (1.9), the potential function V satisfies the Hardy type estimate (1.3). Then from [2, 16] the problem (1.1) (or the problem (1.2) respectively) has a sequence of discrete eigenvalues $\{\beta_k\}_{k\geq 1}$ (or $\{\tilde{\beta}_k\}_{k\geq 1}$) with $\beta_k\to+\infty$ (or $\tilde{\beta}_k\to+\infty$). Also we know that the corresponding eigenfunctions $\{\varphi_k\}_{k\geq 1}$ (or $\{\tilde{\varphi}_k\}_{k\geq 1}$) forms an orthonormal basis of the Sobolev space $H^1_{X,0}(\Omega)$.

Proof of Theorem 1.3.

Proof. Taking $\Phi(x,y)=\sum_{j=1}^k\varphi_j(x)\varphi_j(y)$. Then from the results of Lemma 2.2,

$$\int_\Omega\int_{\mathbb{R}^n}|\hat{\Phi}(z,y)|^2dzdy=k,\quad\text{and}\quad\int_\Omega|\hat{\Phi}(z,y)|^2dy\leq(2\pi)^{-n}|\Omega|_n. \tag{3.4}$$

Thus we have

$$\int_{\mathbb{R}^n}\int_\Omega|\hat{\Phi}(z,y)|^2(\log(e^2+|z|^2))^s dydz=\int_{\mathbb{R}^n}\int_\Omega|(\log(e^2+|\nabla|^2))^{\frac{s}{2}}\Phi(x,y)|^2dydx, \tag{3.5}$$

and the Logarithmic regularity estimate (1.9) gives

$$\int_{\mathbb{R}^n}\int_{\Omega}|(\log(e^2+|\nabla|^2))^{\frac{s}{2}}\Psi(x,y)|^2dydx \le C_0\Big(\int_{\Omega}\int_{\Omega}|X(x)\Phi(x,y)|^2dxdy+\int_{\Omega}\int_{\Omega}|\Phi(x,y)|^2dxdy\Big). \tag{3.6}$$

Similar to the result of (2.13), we have

$$\int_{\Omega}\int_{\Omega}|X(x)\Phi(x,y)|^2dxdy=\sum_{j=1}^{k}\beta_j. \tag{3.7}$$

Thus, from (3.4), (3.5) and (3.6) above, we can deduce that

$$\int_{\mathbb{R}^n}\int_{\Omega}|\hat{\Phi}(z,y)|^2(\log(e^2+|z|^2))^s dydz \le C_0\Big(\sum_{j=1}^{k}\beta_j+k\Big). \tag{3.8}$$

Now we choose

$$f(z)=\int_{\Omega}|\hat{\Phi}(z,y)|^2dy,\ M_1=(2\pi)^{-n}|\Omega|_n,\ M_3=C_0\Big(\sum_{j=1}^{k}\beta_j+k\Big). \tag{3.9}$$

Then we know that $0\le f(z)\le M_1$, and if $k\ge 3$,

$$\int_{\mathbb{R}^n}f(z)dz=k>e.$$

Therefore, if we take $k_0=\max\{\frac{2^{2s}e^n\omega_{n-1}|\Omega|_n}{C_0\pi^n},3\}$, then $M_3\ge 2^{2s}e^n|\Omega|\omega_{n-1}\pi^{-n}=2^{2s+n}e^nM_1\omega_{n-1}$ for any $k\ge k_0$. Thus from the result of Lemma 3.1, we have for any $k\ge k_0$

$$k(\log k)^s \le M_4(n,s,|\Omega|_n)C_0\cdot\Big(\sum_{j=1}^{k}\beta_j+k\Big), \tag{3.10}$$

where $M_4(n,s,|\Omega|_n)=\frac{2^{n+s}}{n}(|\log\frac{|\Omega|_n\omega_{n-1}}{n(2\pi)^n}|^s+n^s)$. That means, for any $k\ge k_0$,

$$\sum_{j=1}^{k}\beta_j \ge C_2k(\log k)^s-k, \tag{3.11}$$

where $C_2=n\Big(C_0 2^{n+s}(|\log\frac{|\Omega|_n\omega_{n-1}}{n(2\pi)^n}|^s+n^s)\Big)^{-1}$. Theorem 1.3 is proved. □

Proof of Theorem 1.4.

Proof. Let $\{\tilde{\beta}_k\}_{k\ge1}$ be the sequence of Dirichlet eigenvalues of the infinitely degenerate Schrödinger operator $-\triangle_X-\varepsilon V(x)$, $\{\tilde{\varphi}_k\}_{k\ge1}$ be the corresponding eigenfunctions. Then the Hardy type estimate (1.3) gives

$$(1-\varepsilon)\int_{\Omega}|X\tilde{\varphi}_j|^2dx \le \int_{\Omega}\tilde{\varphi}_j(x)(-\triangle_X\tilde{\varphi}_j(x)-\varepsilon V\tilde{\varphi}_j(x))=\tilde{\beta}_j.$$

Taking $\tilde{\Phi}(x,y)=\sum_{j=1}^{k}\tilde{\varphi}_j(x)\tilde{\varphi}_j(y)$, then similar to the proof of Theorem 1.3, for $k\ge k_0$, we have

$$\sum_{j=1}^{k}\tilde{\beta}_j \ge (1-\varepsilon)\Big((C_2k(\log k)^s-k\Big), \tag{3.12}$$

where k_0 and C_2 are the same constants in Theorem 1.3. Theorem 1.4 is proved. □

4 Appendix

4.1 Finitely degenerate operators

4.1.1 Examples for degenerate operators

Example 4.1. *Let* $X = (\partial_{x_1}, \partial_{x_2}, \cdots, x_1^k \partial_{x_n})$*, then* $\triangle_X = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + (x_1^k \partial_{x_n})^2$ *with the Hörmander index* $Q = k+1 \geq 1$*. This example can be extended to the case of* $x = (x', x'') \in \mathbb{R}^l \times \mathbb{R}^{n-l}$ $(1 \leq l < n)$*,* $X = (\partial_{x_1}, \partial_{x_2}, \cdots, \phi(x')\partial_{x_n})$*. The function* $\phi(x')$ *is non negative and smooth, and for each* $(x', x'') \in \Omega$ *there exists* $\alpha_0 \in \mathbb{Z}_+^l$ $(|\alpha_0| \leq Q)$*, such that* $\partial_{x'}^{\alpha_0}\phi(x')| \neq 0$*.*

Example 4.2. *Let* $X_j = \partial_{x_j} + 2y_j\partial_t$*,* $Y_j = \partial_{y_j} - 2x_j\partial_t$*, for* $j = 1, \cdots, n$*, be the left invariant vector fields on the Heisenberg group* $\mathbb{H}^n$*,* $X = (X_1, \cdots, X_n, Y_1, \cdots, Y_n)$*. Then the degenerate elliptic operator*

$$\triangle_{\mathbb{H}^n} = \sum_{j=1}^{n} \Big(X_j^2 + Y_j^2 \Big), \tag{4.1}$$

is called Laplacian-Kohn operator on $\mathbb{H}^n$*. In this case the Hörmander index of* X *is* $Q = 2$*.*

4.1.2 Some examples satisfying Hardy type inequality

Proposition 4.1. *If vector fields* $X = (X_1, \cdots, X_n, Y_1, \cdots, Y_n)$*,* $X_j = \partial_{x_j} + 2y_j\partial_t$*,* $Y_j = \partial_{y_j} - 2x_j\partial_t$*, for* $j = 1, \cdots, n$*,* Ω *is the bounded open set of* $\mathbb{H}^n$*. Let singular potential functions*

$$V_1(x, y, t) = \frac{(2n+1)^4}{(2n+2)^2} \frac{1}{((x^2+y^2)^2 + t^2)^{\frac{1}{2}}}, \tag{4.2}$$

and

$$V_2(x, y, t) = (2n+1)^2 \frac{x^2+y^2}{(x^2+y^2)^2 + t^2}. \tag{4.3}$$

Then V_1 *and* V_2 *satisfy the following Hardy type inequality,*

$$\int_\Omega V_i u^2 dxdydt \leq \int_\Omega |Xu|^2 dxdydt, \quad \text{for any } u \in H_{X,0}^1(\Omega) \text{ and } i = 1,\ 2.$$

Proof. See [13], Lemma 2.4 and Lemma 2.5. □

Proposition 4.2. *Let* $V(x) = (\frac{n-3}{2})^2 \frac{1}{|x|^2}$*,* $n \geq 3$*. Then for* $X = (\partial_{x_1}, \partial_{x_2}, \cdots, \phi(x')\partial_{x_n})$ *as given in Example 4.1, we have following the Hardy type inequality,*

$$\int_\Omega V u^2 dx \leq \int_\Omega |Xu|^2 dx, \quad \text{for any } u \in H_{X,0}^1(\Omega). \tag{4.4}$$

Proof. For $n \geq 3$, $C_0^\infty(\Omega \backslash \{0\})$ is dense in $H_{X,0}^1(\Omega)$ (see [14]), then we only need to prove the result for the function $u \in C_0^\infty(\Omega \backslash \{0\})$.

Take a radial vector field R as,

$$R = x_1 \partial x_1 + x_2 \partial x_2 + \cdots + x_{n-1}\partial x_{n-1} + x_n \varphi(x')\partial x_n. \tag{4.5}$$

Then one has $R(V) \geq -2V$ and $div(R) = n - 1 + \varphi(x')$. Thus

$$\int_\Omega -2Vu^2 dx \leq \int_\Omega R(V)u^2 dx = -\int_\Omega div(R)Vu^2 dx - \int_\Omega VR(u^2)dx.$$

This implies

$$\int_\Omega (n - 3 + \varphi(x'))\, Vu^2 dx \leq -\int_\Omega VR(u^2)dx.$$

On the other hand,

$$\begin{aligned}
&-\int_\Omega VR(u^2)dx = -2\int_\Omega VuR(u)dx \\
&= -\int_\Omega V(2ux_1\partial_{x_1}u + 2ux_2\partial_{x_2}u + \cdots + 2ux_{n-1}\partial_{x_{n-1}}u + 2ux_n\varphi(x')\partial_{x_n}u)dx \\
&\leq 2\Big(\int_\Omega V^2(x_1^2 + x_2^2 + \cdots + x_n^2)u^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega \big(\sum_{i=1}^{n-1}(\partial_{x_i}u)^2 + (\varphi(x')\partial_{x_n}u)^2\big)dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Observe that $V(x_1^2 + x_2^2 + \cdots + x_n^2) = (\frac{n-3}{2})^2$, and $n - 3 + \varphi(x') \geq n - 3$. Then we obtain,

$$\int_\Omega Vu^2dx \leq \Big(\int_\Omega Vu^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |Xu|^2dx\Big)^{\frac{1}{2}},$$

which means

$$\int_\Omega Vu^2dx \leq \int_\Omega |Xu|^2dx. \tag{4.6}$$

□

4.2 Infinitely degenerate operator

4.2.1 Some examples satisfying Logarithmic regularity estimates

Let us introduce a sufficient condition for the infinitely degenerate vector fields X, in which the logarithmic regularity estimate (1.9) will be satisfied. Let $X = (X_1, X_2, \cdots, X_m)$, a infinitely degenerate vector fields on $\tilde{\Omega} \subset \mathbb{R}^n$, X_J denote the repeated commutator

$$[X_{j_1}, [X_{j_2}, [X_{j_3}, \cdots [X_{j_{k-1}}, X_{j_k}] \cdots]]], \tag{4.7}$$

for $J = (j_1, \cdots, j_k)$, $j_i \in 1, \cdots, m$, and $|J| = k$. For $k \geq 1$, we take

$$G(x,k) = \min_{\xi\in\mathbb{S}^{n-1}} \sum_{|J|\leq k} |X_J(x,\xi)|^2, \qquad g(t,j,k,x_0) = G((\exp tX_j)(x_0), k),$$

where $(\exp tX_j)(x_0)$ denotes the integral curve of X_j starting from $x_0 \in \Gamma$. Here $\Gamma = \{x \in \tilde{\Omega}; \exists\ \xi \in \mathbb{S}^{n-1}, X_J(x,\xi) = 0,$ for any $J\}$, and $g_I^{j,k}(x_0) = \frac{1}{|I|}\int_I g(t,j,k,x_0)dt$ is the mean value of $g(t,j,k,x_0)$ on the interval I. One has the following result:

Proposition 4.3. *If $s > 0$, and there exists $\varepsilon_1 > 0$ such that*

$$\inf_{\delta>0,k\in\mathbb{N},\mu>0,1\leq j\leq m}\Big\{\sup\big(|I|^{\frac{1}{s}}|\log g_I^{j,k}(x_0)|\big); I \subset (-\mu,\mu), g_I^{j,k}(x_0) < \delta)\Big\} < \varepsilon_1, \tag{4.8}$$

for any $x_0 \in \Gamma$, then there exist constants $C_0 > 0$ which is independent with ε_1 and C_{ε_1} such that

$$\|(\log\Lambda)^s u\|^2_{L^2(\Omega)} \leq C_0\varepsilon_1^{2s}\int_\Omega |Xu|^2dx + C_{\varepsilon_1}\|u\|^2_{L^2(\Omega)},$$

for any $u \in C_0^\infty(\tilde{\Omega})$.

Proof. See [16], Proposition 5.2. □

For infinitely degenerate operators, we have following examples.

Example 4.3. *Let $s > 0$, and*

$$\varphi(x_1) = \begin{cases} e^{-\frac{1}{|x_1|^{1/s}}}, & x_1 \neq 0\ , \\ 0, & x_1 = 0. \end{cases} \tag{4.9}$$

Then $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n})$ with infinitely degenerate surface $\Gamma = \{x_1 = 0\}$. We can prove that the vector fields X will satisfy the logarithmic regularity estimate (1.9) *(cf. [2, 14]).*

Actually, in this case, we know that $G(x,1) = \min_{\xi\in\mathbb{S}^{n-1}} \sum_{|J|\le 1} |X_J(x,\xi)|^2 = \min_{\xi\in\mathbb{S}^{n-1}}(\sum_{j=1}^{n-1}\xi_j^2 + \varphi^2(x')\xi_n^2 + (\frac{\partial\varphi(x')}{\partial x_i}\xi_i)^2)$, $\Gamma = \{x = (0, x_2, \cdots, x_n) \in \tilde{\Omega}\}$. Thus for $x_0 = (x', x_n) = (0, x_2, \cdots, x_{n-1}, x_n) \in \Gamma$, and $\xi_0 = (0, \cdots, 0, 1) \in \mathbb{S}^{n-1}$, the unit normal vector on Γ, we have $G(x_0, 1) = \varphi^2(x')|_{x_1=0} = 0$. Also from

$$[X_i, X_j] = \begin{cases} \frac{\partial\varphi(x')}{\partial x_1}\partial_{x_n} & i = 1,\ j = n, \\ 0 & \text{else}, \end{cases}$$

we can deduce that, for $x_0 \in \Gamma$, $G(x_0, k) = 0$ for $k \ge 2$.

For t is small and let $y = \exp(tX_j)(x_0) = (y', y_n) = (y_1, y_2, \cdots, y_n)$, then $g(t, j, 1, x_0) = G((\exp tX_j)(x_0), 1) = G(y, 1) = \varphi^2(y')$, let $t' = (t, 0, \cdots, 0) \in \mathbb{R}^{n-1}$, which means

$$g(t, j, 1, x_0) = \begin{cases} \varphi^2(t'), & j = 1, \\ 0, & j \ne 1. \end{cases}$$

Then

$$g_I^{1,1}(x_0) = \frac{1}{|I|}\int_I g(t, 1, 1, x_0)dt = \frac{1}{|I|}\int_I e^{-\frac{2}{|t|^{1/s}}}dt.$$

We consider $|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)|$, $I \subset (-\mu, \mu)$. If $0 \notin I$, then as $|I| \to 0$, $|\log g_I^{1,1}(x_0)| \le M$, one has $|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)| \to 0$. If $0 \in I$, by the symmetry of $\varphi(t)$, we suppose that $I = (0, a)$, then

$$|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)| = -a^{\frac{1}{s}}\log\left(\frac{1}{a}\int_0^a e^{-\frac{2}{|t|^{1/s}}}dt\right) \le -a^{\frac{1}{s}}\log\left(\frac{1}{2}e^{-\frac{2}{|\frac{a}{2}|^{1/s}}}\right) \le 2^{\frac{1}{s}+1} + a^{\frac{1}{s}}\log 2.$$

So we choose $\varepsilon_0 = 2^{\frac{1}{s}+1} + \log 2$, then as $a \to 0$, $|I|^{\frac{1}{s}}|\log g_I^{1,1}(x_0)| < \varepsilon_0$. By using Proposition 4.3, we can deduce that $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x')\partial_{x_n})$ satisfies the logarithmic regularity estimate (1.9).

The examples below give more infinitely degenerate vector fields X in which the logarithmic regularity estimate (1.9) will be satisfied (cf. [2, 16]).

Example 4.4. *The system of vector fields $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x_1)\partial_{x_n})$, where*

$$\varphi(x_1) = \begin{cases} e^{-\frac{1}{|x_1\sin(\frac{\pi}{x_1})|^{1/s}}}, & x_1 \ne 0\ , \\ 0, & x_1 = 0. \end{cases} \tag{4.10}$$

. Then X is infinitely degenerate on $\Gamma = \bigcup_{j\in\mathbb{Z}_+}\Gamma_j$, for $\Gamma_j = \{x_1 = \frac{1}{j}\}$, $j \ge 1$, and $\Gamma_0 = \{x_1 = 0\}$.

Example 4.5. *The system of vector fields $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x')\partial_{x_n})$, for $k \ge 1$ and $n \ge 3$, where*

$$\varphi(x') = \begin{cases} e^{-\frac{1}{|x_1|^{1/s}}}x_2{}^k, & x_1 \ne 0, \\ 0, & x_1 = 0. \end{cases} \tag{4.11}$$

The the infinitely degenerate surface for X is $\Gamma = \{x_1 = 0\}$.

4.2.2 Some examples satisfying Hardy type inequalities

Let the vector fields $X = (\partial_{x_1}, \cdots, \partial_{x_{n-1}}, \varphi(x')\partial_{x_n})$ is defined on an open domain $\tilde{\Omega} \subset \mathbb{R}^n$ for $n \ge 3$. Ω is a bounded open subset in $\tilde{\Omega}$ which contains the origin, and $\partial\Omega$ is smooth. Denote $x' = (x_1, \cdots, x_l)$ for $1 \le l < n$, $x = (x', x'')$, and $\varphi(x')$ is a non-negative C^∞-smooth function in x', which satisfies, for any $\alpha \in \mathbb{Z}_+^l$, that $\partial_{x'}^\alpha\varphi(x')|_{x'=0} = 0$. The infinitely degenerate surface of X is $\Gamma = \{x' = 0\}$.

Proposition 4.4. i). *If $V_1(x) = (\frac{n-3}{2})^2\frac{1}{|x|^2}$, then $V_1(x) \in C^\infty(\Omega\backslash\{0\})$ (for $n \ge 3$), and*

$$\int_\Omega V_1 u^2 dx \le \int_\Omega |Xu|^2 dx, \quad \text{for any } u \in H_{X,0}^1(\Omega). \tag{4.12}$$

ii). *If* $V_2(x) = (\frac{n-2}{2})^2 \frac{x_1^{-2}\exp(-\frac{1}{|x_1|^2})}{\exp(-\frac{1}{|x_1|^2})+\sum_{i=2}^n x_i^2}$, $x = (x_1, x'') = (x_1, x_2, \cdots, x_n)$, *then* $V_2(x) \in C^\infty(\Omega\backslash\{0\})$ *(for* $n \geq 3$*), and when* $x_1 \to 0$ *we have* $V_2(x_1, x'') \to 0$ *if* $x'' \neq 0$ *and* $V_2(x_1, x'') \to +\infty$ *if* $x'' = 0$*. Thus for* $\Omega \subset \Big\{x = (x_1, x'') \in \tilde{\Omega} \mid |x_1| \leq \sqrt{\frac{1}{5}}\Big\}$*, there holds*

$$\int_\Omega V_2 u^2 dx \leq \int_\Omega |Xu|^2 dx, \quad \text{for any } u \in H^1_{X,0}(\Omega). \tag{4.13}$$

Proof. The proof of (4.12) is similar to the proof of the result in Proposition 4.2. Here we prove only the inequality (4.13). Let us take the following radial vector field R_2,

$$R_2 = x_1^3 \partial x_1 + x_2 \partial x_2 + \cdots + x_{n-1}\partial x_{n-1} + x_n \varphi(x')\partial x_n. \tag{4.14}$$

Then $R_2(V_2) \geq -2x_1^2 V_2$ and $div(R_2) = 3x_1^2 + n - 2 + \varphi(x')$, which means

$$\int_\Omega -2x_1^2 V_2 u^2 dx \leq \int_\Omega R_2(V_2)u^2 dx = -\int_\Omega div(R_2)V_2 u^2 dx - \int_\Omega V_2 R_2(u^2)dx.$$

Thus we have

$$\int_\Omega (x_1^2 + n - 2 + \varphi(x'))V_2 u^2 dx \leq -\int_\Omega V_2 R_2(u^2)dx, \tag{4.15}$$

and

$$\begin{aligned}
&-\int_\Omega V_2 R_2(u^2)dx = -2\int_\Omega V_2 u R_2(u)dx \\
&= -\int_\Omega V_2(2ux_1^3\partial_{x_1}u + 2ux_2\partial_{x_2}u + \cdots + 2ux_{n-1}\partial_{x_{n-1}}u + 2ux_n\varphi(x')\partial_{x_n}u)dx \\
&\leq 2\Big(\int_\Omega V_2^2(x_1^6 + x_2^2 + \cdots + x_n^2)u^2dx\Big)^{\frac{1}{2}}\Big(\int_\Omega (\Sigma_{i=1}^{n-1}(\partial_{x_i}u)^2 + (\varphi(x')\partial_{x_n}u)^2)dx\Big)^{\frac{1}{2}}.
\end{aligned}$$

Since $x_1^6 \geq \exp\big\{-\frac{1}{|x_1|^2}\big\}$ for $|x_1| \leq \sqrt{\frac{1}{5}}$, then

$$V_2(x_1^6 + x_2^2 + \cdots + x_n^2) \leq x_1^4\Big(\frac{n-2}{2}\Big)^2 \leq \Big(\frac{n-2}{2}\Big)^2,$$

and

$$x_1^2 + n - 2 + \varphi(x') \geq n - 2.$$

Thus we have from (4.15),

$$\int_\Omega V_2 u^2 dx \leq \Big(\int_\Omega V_2 u^2 dx\Big)^{\frac{1}{2}}\Big(\int_\Omega |Xu|^2 dx\Big)^{\frac{1}{2}},$$

which implies

$$\int_\Omega V_2 u^2 dx \leq \int_\Omega |Xu|^2 dx. \tag{4.16}$$

The Hardy type inequality (4.13) is proved. □

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