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Multiple Solutions for Semi-linear Corner Degenerate Elliptic Equations

by

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Multiple Solutions for Semi-linear Corner Degenerate Elliptic Equations ☆

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Abstract

The present paper is concerned with the existence of multiple solutions for semi-linear corner-degenerate elliptic equations with subcritical conditions. First, we introduce the corner type weighted p-Sobolev spaces and discuss the properties of continuous embedding, compactness and spectrum. Then, we prove the corner type Sobolev inequality and Poincaré inequality, which are important in the proof of the main result.

Keywords: Dirichlet problem, multiple solutions, corner-degenerate elliptic operators, corner type weighted p-Sobolev spaces, corner type Sobolev inequality.

1. Introduction

Write $\mathbb{M} = [0,1)\times X\times[0,1)$ as a local model of stretched corner-manifolds (i.e. manifolds with corner singularities) with dimension $N = n+2 \geq 3$. Here X is a closed compact sub-manifold of dimension n embedded in the unit sphere of $\mathbb{R}^{n+1}$. Let $\mathbb{M}_0$ denote the interior of $\mathbb{M}$ and $\partial\mathbb{M} = \{0\}\times X\times\{0\}$ denote the boundary of $\mathbb{M}$. The so-called corner-Laplacian is defined as

$$\Delta_{\mathbb{M}} = (r\partial_r)^2 + (\partial_{x_1})^2 + \cdots + (\partial_{x_n})^2 + (rt\partial_t)^2,$$

which is a degenerate elliptic operator on the boundary $\partial\mathbb{M}$. The present paper is concerned with the existence of multiple weak solutions for the

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following Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{M}} u = g(z,u) & z := (r,x,t) \in \mathbb{M}_0 \\ u = 0 & \text{on } \partial\mathbb{M}. \end{cases} \tag{1.1}$$

Our main result can be stated as follows.

Theorem 1.1. *Let $g(z,u) : \mathbb{M} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the following assumptions*

(H-1) *let $g(z,u)$ be odd, i.e $g(z,-u) = -g(z,u)$;*

(H-2) $2 < p < 2^* = \frac{2N}{N-2}$ *and there exists a constant $C_0 > 0$ such that the following estimate holds almost everywhere*

$$|g(z,u)| \le C_0(1 + |u|^{p-1});$$

(H-3) *For the primitive $G(\cdot,u) = \int_0^u g(\cdot,v)dv$, there exist $q > 2$ and a constant R_0 such that for almost every $z \in \mathbb{M}$ and $|u| \ge R_0$ we have*

$$0 < qG(z,u) \le g(z,u)u.$$

Then the Dirichlet problem (1.1) *admits infinity many weak solutions in the corner type weighted Sobolev space $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$.*

To show this result, methods of variational theory are employed, which can be trace back to Ambrosetti and Rabinowitz [1] in 1973, and Rabinowitz [11] in 1974. In [2], Bartolo, Benci and Fortunato proved optimal multiplicity results in the case of degenerate critical values. All these results also can be found in book [15]. Authors studied the Dirichlet problem of semi-linear elliptic equations on stretched cone in [3] and [4]. The corresponding cone Laplacian $\Delta_{\mathbb{B}} = (x_1\partial_{x_1})^2 + (\partial_{x_2})^2 + \cdots + (\partial_{x_n})^2$, which is degenerate at $x_1 = 0$. This kind of operator is a simple example of conical differential operators. Also the authors studied similar nonlinear problem in [5] for the edge Laplacian $\triangle_{\mathbb{E}} = (x_1\partial_{x_1})^2 + (\partial_{x_2})^2 + \cdots + (\partial_{x_n})^2 + (x_1\partial_{y_1})^2 + \cdots + (x_1\partial_{y_q})^2$ with edge singularity at $x_1 = 0$. On the other hand, the pseudo-differential operators with conical singularities and edge singularities have been wildly studied from various motivations by Egorov and Schulze [6], Schulze [13], Schrohe and Seiler [12], Melrose and Mendoza [9] and Mazzeo [8]. In this paper, we pursue further study for the existence of solutions to semi-linear degenerate elliptic equations on manifold with corner singularities. Here the so called corner Laplacian $\Delta_{\mathbb{M}} = (r\partial_r)^2 + (\partial_{x_1})^2 + \cdots + (\partial_{x_n})^2 + (rt\partial_t)^2$ is degenerate at both $r = 0$ and $t = 0$, which is named after the local structure of manifold with corner singularities. R. Melrose and P. Piazza studied the structure of manifolds with corners in [10]. Schulze discussed the calculus of corner degenerate pseudo-differential operators in [14].

This paper is organized as follows. The motivation of corner degenerate Laplacian $\Delta_{\mathbb{M}}$ will be given as first. Then, in section 2, we introduce the corner type weighted p-Sobolev spaces $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ with $1 \leq p < +\infty$, the smoothness $m \in \mathbb{N}$ and the double weight data γ_1, $\gamma_2 \in \mathbb{R}$. The continuous embedding, compactness and spectral property of $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ will be also given in section 2. Further, we extend the classic Sobolev inequality and Poincaré inequality to the corner type weighted Sobolev spaces in section 3, which are crucial for the proof of Theorem 1.1. Finally, we give the proof for the existence of multiple solutions for the Dirichlet problem 1.1 in section 4.

2. Corner type weighted *p*-Sobolev spaces

Let $X \subset S^n$ be a bounded open set in the unit sphere of $\mathbb{R}^{n+1}_{\tilde{x}}$, then the straight cone is defined as

$$X^{\Delta} = \left\{ \tilde{x} \in \mathbb{R}^{n+1} \mid \tilde{x} = 0 \text{ or } \frac{\tilde{x}}{|\tilde{x}|} \in X \right\}.$$

In general, we can define an infinite cone in $\mathbb{R}^{n+1}$ as a quotient space

$$X^{\Delta} = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X),$$

with base X. By using the cylindrical coordinates in $\mathbb{R}^{n+1} \setminus \{0\}$, the coordinates $(r, \varphi) \in X^{\Delta} \setminus \{0\}$ are the standard coordinates. This gives us the description of $X^{\Delta} \setminus \{0\}$ in the form $\mathbb{R}_+ \times X$. The stretched cone is defined as

$$X^{\wedge} = \overline{\mathbb{R}}_+ \times X.$$

Set $(r, x) \in X^{\wedge}$. It is sufficient to consider the case for $0 \leq r < 1$, which gives us a finite cone

$$E = ([0, 1) \times X)/(\{0\} \times X). \tag{2.1}$$

The finite stretched cone to E is

$$\mathbb{E} = [0, 1) \times X,$$

with a smooth boundary $\partial\mathbb{E} = \{0\} \times X$.

An infinite corner can be defined as

$$E^{\Delta} = (E \times \overline{\mathbb{R}}_{+})/(E \times \{0\}),$$

where the base E is a finite cone defined in (2.1). The stretched corner is

$$E^{\wedge} = \mathbb{E} \times \overline{\mathbb{R}}_{+}.$$

Let $(r, x, t) \in E^{\wedge}$, we focus, in this paper, on the case of $0 \leq t < 1$, then the finite corner is

$$M = (E \times [0, 1))/(E \times \{0\}).$$

Thus

$$\mathbb{M} = \mathbb{E} \times [0, 1) = [0, 1) \times X \times [0, 1), \tag{2.2}$$

is a finite stretched corner with the smooth boundary $\partial\mathbb{M} = \partial\mathbb{E} \times \{0\} = \{0\} \times X \times \{0\}$ (see figure 1 and figure 2 below).

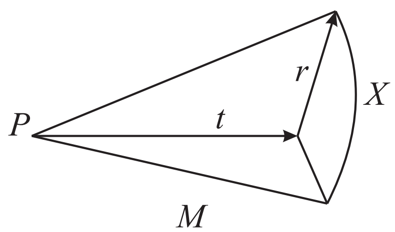

Figure 1

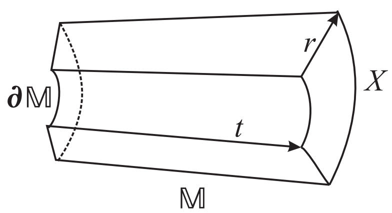

Figure 2

The typical degenerate differential operator A on the stretched cone $\mathbb{E}$ is as follows,

$$A = r^{-\mu} \sum_{j \leq \mu} a_j(r)(r\partial_r)^j = r^{-\mu} A_{\mathbb{E}},$$

with coefficients $a_j(r) \in C^{\infty}(\overline{\mathbb{R}}_{+}, \mathrm{Diff}^{\mu-j}(X))$. Here $A_{\mathbb{E}}$ is degenerate cone operator. Denote $\mathrm{Diff}^{\mu}_{\deg}(\mathbb{E})$ for the set of cone differential operators as A. The typical differential operator B on the stretched corner $\mathbb{M}$ is then of the following form

$$B = t^{-\nu} \sum_{l \leq \nu} b_l(t)(t\partial_t)^l,$$

where the coefficients $b_l(t) \in C^{\infty}(\overline{\mathbb{R}}_{+}, \mathrm{Diff}^{\nu-l}_{\deg}(\mathbb{E}))$, i.e.

$$b_l(t) = r^{-(\nu-l)} \sum_{j \leq (\nu-l)} a_{jl}(r, t)(r\partial_r)^j,$$

with $a_{jl}(r,t) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\nu-l-j}(X))$. It implies that

$$B = (rt)^{-\nu} \sum_{j+l\le\nu} \tilde{a}_{jl}(r,t)(r\partial_r)^j (rt\partial_t)^l = (rt)^{-\nu} B_{\mathbb{M}},$$

where $\tilde{a}_{jl}(r,t) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\nu-l-j}(X))$ and $B_{\mathbb{M}}$ is called as a degenerate corner operator. In fact we have following Riemannian metric on the corner M

$$dt^2 + t^2(dr^2 + r^2 g_X),$$

where g_X is a Riemannian metric on X. Then the corresponding gradient operator with corner degeneracy is

$$\nabla_{\mathbb{M}} = (r\partial_r, \partial_{x_1}, ..., \partial_{x_n}, rt\partial_t).$$

Now we define the weighted $L_p^{\gamma_1,\gamma_2}$ space on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$ as follows.

Definition 2.1. *Let $(r,x,t) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$, weight datas $\gamma_1 \in \mathbb{R}$, $\gamma_2 \in \mathbb{R}$ and $1 \le p < +\infty$. Then $L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+, \frac{dr}{r} dx \frac{dt}{rt})$ denotes the space of all $u(r,x,t) \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$ such that*

$$\|u\|_{L_p^{\gamma_1,\gamma_2}} = \Big(\int_{\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+} |r^{\frac{N}{p}-\gamma_1} t^{\frac{N}{p}-\gamma_2} u(r,x,t)|^p \frac{dr}{r} dx \frac{dt}{rt}\Big)^{1/p} < +\infty.$$

By the above weighted $L_p^{\gamma_1,\gamma_2}$ space, we can define the following weighted p-Sobolev spaces on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+$ with natural scale for all $1 \le p < \infty$.

Definition 2.2. *Let $m \in \mathbb{N}$, $\gamma_1, \gamma_2 \in \mathbb{R}$, and set $N = n+2$, the weighted Sobolev space*

$$\begin{aligned}\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+) = \{&u \in \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+) | \\ &(r\partial_r)^l \partial_x^\alpha (rt\partial_t)^k u(r,x,t) \in L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+, \frac{dr}{r} dx \frac{dt}{rt})\},\end{aligned}$$

for $k, l \in \mathbb{N}$ and the multi-index $\alpha \in \mathbb{N}^n$, with $k + |\alpha| + l \le m$. Moreover, the closure of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$ is denoted by $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$.

Similarly, we can define the following weighted p-Sobolev spaces on an open stretched corner $\mathbb{R}_+ \times X \times \mathbb{R}_+$,

$$\begin{aligned}\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+) = \Big\{&u \in \mathcal{D}'(\mathbb{R}_+ \times X \times \mathbb{R}_+) | \\ &(r\partial_r)^l \partial_x^\alpha (rt\partial_t)^k u(r,x,t) \in L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+ \times X \times \mathbb{R}_+, \frac{dr}{r} dx \frac{dt}{rt})\Big\},\end{aligned}$$

for $k,\ l \in \mathbb{N}$ and the multi-index $\alpha \in \mathbb{N}^n$, with $k + |\alpha| + l \leq m$, which is a Banach space with the following norm,

$$\|u\|_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}} = \Big\{ \sum_{l+|\alpha|+k\leq m} \int_{\mathbb{R}_+\times X\times\mathbb{R}_+} |r^{\frac{N}{p}-\gamma_1} t^{\frac{N}{p}-\gamma_2} \\ (r\partial_r)^l \partial_x^\alpha (rt\partial_t)^k u(r,x,t)|^p \frac{dr}{r} dx \frac{dt}{rt} \Big\}^{1/p}.$$

Moreover, the subspace $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$ denotes as the closure of C_0^∞ functions in $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$.

Now we can introduce the following weighted p-Sobolev space on the finite stretched corner $\mathbb{M}$ defined in (2.2).

Definition 2.3. *Let $m \in \mathbb{N}$, $1 \leq p < \infty$, and $\gamma_1,\ \gamma_2 \in \mathbb{R}$, $W_{\rm loc}^{m,p}(\mathbb{M}_0)$ is the classical local Sobolev space. Then we define*

$$\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M}) = \Big\{ u(r,x,t) \in W_{\rm loc}^{m,p}(\mathbb{M}_0) \mid \\ (\omega\sigma)u \in \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+) \Big\},$$

for any cut-off functions $\omega = \omega(r,x)$ and $\sigma = \sigma(t,x)$, supported by a collar neighborhoods of $(0,1) \times \partial\mathbb{M}$ and $\partial\mathbb{M} \times (0,1)$ respectively.

It can be deduced from Definition 2.3 that $\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ is a Banach space for $1 \leq p < \infty$, and is a Hilbert space for $p = 2$. Also we have that $r^{\gamma_1'} t^{\gamma_2'} \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{M}) = \mathcal{H}_p^{m,(\gamma_1+\gamma_1',\gamma_2+\gamma_2')}(\mathbb{M})$.

Here $\omega(r,x)$ and $\sigma(t,x)$ can be simply denoted by $\omega(r)$ and $\sigma(t)$ respectively. Observe that there exist $\varepsilon_1 \in (0,1)$ and $\varepsilon_2 \in (0,1)$, depending on $\omega(r)$ and $\sigma(t)$ respectively, such that $\omega(r) = 1$ for $r \in \operatorname{supp}\omega \cap \{0 < r \leq \varepsilon_1\}$ and $\sigma(t) = 1$ for $t \in \operatorname{supp}\sigma \cap \{0 < t \leq \varepsilon_2\}$. Thus

$$\begin{aligned} \mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{M}) =& [\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+) \\ &+ [1-\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m,\gamma_2}(\Omega_{\varepsilon_1} \times X \times \mathbb{R}_+) \\ &+ [\omega(r)][1-\sigma(t)]\mathcal{H}_{p,0}^{m,\gamma_1}(\mathbb{R}_+ \times X \times \Omega_{\varepsilon_2}) \\ &+ [1-\omega(r)][1-\sigma(t)]W_0^{m,p}(\Omega_{\varepsilon_1} \times X \times \Omega_{\varepsilon_2}), \end{aligned} \tag{2.3}$$

where $\Omega_{\varepsilon_1} = (\varepsilon_1, 1)$ and $\Omega_{\varepsilon_2} = (\varepsilon_2, 1)$, and the weighted p-Sobolev spaces $\mathcal{H}_{p,0}^{m,\gamma_1}(\mathbb{R}_+ \times X \times \Omega_{\varepsilon_2})$ and $\mathcal{H}_{p,0}^{m,\gamma_2}(\Omega_{\varepsilon_1} \times X \times \mathbb{R}_+)$ are the closures of C_0^∞

functions in the following weighted edge p-Sobolev spaces (cf. [5])

$$\mathcal{H}_p^{m,\gamma_1}(\mathbb{R}_+ \times X \times \Omega_{\varepsilon_2}) := \{u(r,x,t) \in W_{\mathrm{loc}}^{m,p}(\mathbb{R}_+ \times X \times \Omega_{\varepsilon_2}) \mid r^{\frac{N}{p}-\gamma_1}(r\partial_r)^k \partial_x^\alpha (r\partial_t)^l u \in L_p(\mathbb{R}_+ \times X \times \Omega_{\varepsilon_2}, \frac{dr}{r} dx \frac{dt}{r})\}$$

for k, $l \in \mathbb{N}$ and multi-index $\alpha \in \mathbb{N}^n$, with $k + |\alpha| + l \leq m$, and

$$\mathcal{H}_p^{m,\gamma_2}(\Omega_{\varepsilon_1} \times X \times \mathbb{R}_+) := \{u(r,x,t) \in W_{\mathrm{loc}}^{m,p}(\Omega_{\varepsilon_1} \times X \times \mathbb{R}_+) | t^{\frac{N}{p}-\gamma_2}(t\partial_r)^k \partial_x^\alpha (t\partial_t)^l u \in L_p(\Omega_{\varepsilon_1} \times X \times \mathbb{R}_+, \frac{dr}{t} dx \frac{dt}{t})\},$$

for k, $l \in \mathbb{R}$ and multi-index $\alpha \in \mathbb{R}^n$, with $k + |\alpha| + l \leq m$.

We have the following embedding theorem:

Proposition 2.4. *The embedding $\mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{M}) \hookrightarrow \mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ is continuous for $m' \geq m$, $\gamma_1' \geq \gamma_1$, $\gamma_2' \geq \gamma_2$ and is compact for $m' > m$, $\gamma_1' \geq \gamma_1$, $\gamma_2' \geq \gamma_2$.*

Proof. The weighted corner Sobolev spaces $\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ are in the form of non-direct sum as (2.3). Then, for classical Sobolev spaces $W_0^{m,p}(\Omega_{\varepsilon_1} \times X \times \Omega_{\varepsilon_2})$, it is well known that the embedding

$$W_0^{m',p}(\Omega_{\varepsilon_1} \times X \times \Omega_{\varepsilon_2}) \hookrightarrow W_0^{m,p}(\Omega_{\varepsilon_1} \times X \times \Omega_{\varepsilon_2}) \tag{2.4}$$

is continuous for $m' \geq m$ and is compact for $m' > m$. According to Proposition 2.6 in [5], we know that the following embedding

$$[1-\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m',\gamma_2'}(\Omega_{\varepsilon_1} \times X \times \mathbb{R}_+) \hookrightarrow [1-\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m,\gamma_2}(\Omega_{\varepsilon_1} \times X \times \mathbb{R}_+)$$

is continuous for $m' \geq m$, $\gamma_2' \geq \gamma_2$ and is compact for $m' > m$, $\gamma_2' \geq \gamma_2$; and the embedding

$$[\omega(r)][1-\sigma(t)]\mathcal{H}_{p,0}^{m',\gamma_1'}(\mathbb{R}_+ \times X \times \Omega_{\varepsilon_2}) \hookrightarrow [\omega(r)][1-\sigma(t)]\mathcal{H}_{p,0}^{m,\gamma_1}(\mathbb{R}_+ \times X \times \Omega_{\varepsilon_2})$$

is continuous for $m' \geq m$, $\gamma_1' \geq \gamma_1$ and is compact for $m' > m$, $\gamma_1' \geq \gamma_1$.

By (2.3), it is sufficient to prove that the embedding

$$[\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times \mathbb{R}_+) \hookrightarrow [\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$$

is continuous for $m' \geq m$, $\gamma_1' \geq \gamma_1$, $\gamma_2' \geq \gamma_2$ and is compact for $m' > m$, $\gamma_1' \geq \gamma_1$, $\gamma_2' \geq \gamma_2$, for any cut-off functions $\omega(r)$ and $\sigma(t)$ with support in a collar neighborhoods of $(0,1) \times \partial\mathbb{M}$ and $\partial\mathbb{M} \times (0,1)$ respectively.

For $\tilde{u}(r,x,t) \in \mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$, we set $u(r,x,t) = \omega(r)\sigma(t)\tilde{u}(r,x,t)$ and the following mappings

$$S_{p,\gamma_2} : u(r,x,t) \to e^{-\tau(\frac{N}{p}-\gamma_2)} u(r,x,e^{-\tau}) := v(r,x,\tau) \tag{2.5}$$

with $\tau = -\ln t$, and

$$S_{p,\gamma_1} : v(r,x,\tau) \to e^{-\rho(\frac{N}{p}-\gamma_1)} v(e^{-\rho},x,e^{-\rho}\zeta) := w(\rho,x,\zeta) \tag{2.6}$$

with $\rho = -\ln r$, $\zeta = \frac{-\ln t}{r}$. Thus we have following transform:

$$(r,x,t) \to (\rho,x,\zeta) = (-\ln r, x, \frac{-\ln t}{r}), \tag{2.7}$$

and $(\rho,x,\zeta) \in \tilde{\Omega}_{\rho,x,\zeta}$ iff $(r,x,t) \in \Omega_{r,x,t} = \{(r,x,t) \mid (r,x) \in \operatorname{supp} \omega, \text{ and } (x,t) \in \operatorname{supp} \sigma\}$. Then the mapping

$$\begin{aligned} S_{p,(\gamma_1,\gamma_2)} &:= S_{p,\gamma_1} \circ S_{p,\gamma_2} : u(r,x,t) \to \\ & w(\rho,x,\zeta) = e^{-\rho(\frac{N}{p}-\gamma_1)} e^{-(e^{-\rho}\zeta)(\frac{N}{p}-\gamma_2)} u(e^{-\rho},x,e^{-(e^{-\rho}\zeta)}) \end{aligned} \tag{2.8}$$

gives an isomorphism as follows

$$S_{p,(\gamma_1,\gamma_2)} : [\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+) \to [\tilde{\omega}(\rho)][\tilde{\sigma}(\zeta)] W_0^{m,p}(\mathbb{R} \times X \times \mathbb{R}) \tag{2.9}$$

where $W^{m,p}(\cdot)$ denotes the classical Sobolev spaces, $\tilde{\omega}(\rho) = \omega(e^{-\rho})$, $\tilde{\sigma}(\zeta)$ is a cut-off function in $\zeta = \frac{-\ln t}{r}$ for $t \in \operatorname{supp} \sigma(t)$ and $r \in \operatorname{supp} \omega(r)$.

From the transform (2.7), we have $\partial_\rho = r\partial_r - t\ln t\partial_t$, $\partial_\zeta = rt\partial_t$, and the determinant of Jacobian is $\frac{1}{r^2 t}$. Then for $\tilde{\Omega}_{\rho,x,\zeta} = \tilde{\Omega}$ and $\Omega_{r,x,t} = \Omega$,

$$\begin{aligned} \|w(\rho,x,\zeta)\|^p_{W^{m,p}(\tilde{\Omega})} &= \sum_{k+|\alpha|+l \le m} \int_{\tilde{\Omega}} |\partial_\rho^k \partial_x^\alpha \partial_\zeta^l w(\rho,x,\zeta)|^p d\rho dx d\zeta \\ &= \sum_{k+|\alpha|+l \le m} \int_{\tilde{\Omega}} |\partial_\rho^k \partial_x^\alpha \partial_\zeta^l e^{-\rho(\frac{N}{p}-\gamma_1)} e^{-(e^{-\rho}\zeta)(\frac{N}{p}-\gamma_2)} u(e^{-\rho},x,e^{-(e^{-\rho}\zeta)})|^p d\rho dx d\zeta \\ &\sim \sum_{k+|\alpha|+l \le m} \int_{\tilde{\Omega}} |e^{-\rho(\frac{N}{p}-\gamma_1)} e^{-(e^{-\rho}\zeta)(\frac{N}{p}-\gamma_2)} \partial_\rho^k \partial_x^\alpha \partial_\zeta^l u(e^{-\rho},x,e^{-(e^{-\rho}\zeta)})|^p d\rho dx d\zeta \\ &\sim \sum_{k+|\alpha|+l \le m} \int_{\Omega} |r^{\frac{N}{p}-\gamma_1} t^{\frac{N}{p}-\gamma_2} (r\partial_r)^k \partial_x^\alpha (rt\partial_t)^l u(r,x,t)|^p \frac{dr}{r} dx \frac{dt}{rt} \\ &= \|u(r,x,t)\|^p_{\mathcal{H}_p^{m,(\gamma_1,\gamma_2)}(\Omega)}. \end{aligned}$$

Similarly, if $\tilde{u}(r,x,t) \in \mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times \mathbb{R}_+)$, then for $u(r,x,t) = \omega(r)\sigma(t)\tilde{u}(r,x,t)$, we have

$$\begin{aligned} S_{p,(\gamma_1,\gamma_2)}u(r,x,t) &= e^{-\rho(\frac{N}{p}-\gamma_1)}e^{-(e^{-\rho}\zeta)(\frac{N}{p}-\gamma_2)}u(e^{-\rho},x,e^{-(e^{-\rho}\zeta)}) \\ &= e^{-\rho(\gamma_1'-\gamma_1)}e^{-(e^{-\rho}\zeta)(\gamma_2'-\gamma_2)}a(\rho,x,\zeta) \end{aligned}$$

where $a(\rho,x,\zeta) = e^{-\rho(\frac{N}{p}-\gamma_1')}e^{-(e^{-\rho}\zeta)(\frac{N}{p}-\gamma_2')}u(e^{-\rho},x,e^{-(e^{-\rho}\zeta)}) \in W_0^{m',p}(\mathbb{R} \times X \times \mathbb{R})$. Then $S_{p,(\gamma_1,\gamma_2)}$ induces another isomorphism

$$\begin{aligned} S_{p,(\gamma_1,\gamma_2)} &: [\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times \mathbb{R}_+) \to \\ &[\tilde{\omega}(\rho)][\tilde{\sigma}(\zeta)]e^{-\rho(\gamma_1'-\gamma_1)}e^{-(e^{-\rho}\zeta)(\gamma_2'-\gamma_2)}W_0^{m',p}(\mathbb{R} \times X \times \mathbb{R}). \quad (2.10) \end{aligned}$$

From the isomorphisms (2.9) and (2.10), since, for $\rho \in \operatorname{supp}\tilde{\omega}(\rho)$ and $\zeta \in \operatorname{supp}\tilde{\sigma}(\zeta)$, the embedding

$$\begin{aligned} &[\tilde{\omega}(\rho)][\tilde{\sigma}(\zeta)]e^{-\rho(\gamma_1'-\gamma_1)}e^{-(e^{-\rho}\zeta)(\gamma_2'-\gamma_2)}W_0^{m',p}(\mathbb{R} \times X \times \mathbb{R}) \\ &\qquad \hookrightarrow [\tilde{\omega}(\rho)][\tilde{\sigma}(\zeta)]W_0^{m,p}(\mathbb{R} \times X \times \mathbb{R}) \end{aligned}$$

is continuous for $m' \geq m$, $\gamma_1' \geq \gamma_1$, $\gamma_2' \geq \gamma_2$ and is compact for $m' > m$, $\gamma_1' \geq \gamma_1$, $\gamma_2' \geq \gamma_2$. Thus we prove the result for the embedding

$$[\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m',(\gamma_1',\gamma_2')}(\mathbb{R}_+ \times X \times \mathbb{R}_+) \hookrightarrow [\omega(r)][\sigma(t)]\mathcal{H}_{p,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times X \times \mathbb{R}_+),$$

as required. □

Let $\mathcal{H}_2^{-m,(-\gamma_1,-\gamma_2)}(\mathbb{M})$ denote the dual space of $\mathcal{H}_{2,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{M})$ with the following norm

$$\|g\|_{\mathcal{H}_2^{-m,(-\gamma_1,-\gamma_2)}(\mathbb{M})} = \sup_{\psi \in C_0^\infty(\mathbb{M}),\psi \neq 0} \frac{|\langle g,\psi\rangle|}{\|\psi\|_{\mathcal{H}_{2,0}^{m,(\gamma_1,\gamma_2)}(\mathbb{M})}}.$$

We have

Proposition 2.5. *There exist the eigenvalues*

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$$

of the elliptic operator $-\Delta_{\mathbb{M}}$ with homogeneous Dirichlet data, such that the corresponding eigenfunctions $\{\varphi_k\}_{k\geq 1}$ constitute the orthonormal basis of $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$.

Proof. For any $u, v \in \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ and $u \neq 0$, we denote $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{L_2^{\frac{N-1}{2},\frac{N}{2}}}$,

$$\langle -\Delta_{\mathbb{M}} u, u \rangle = \|\nabla_{\mathbb{M}} u\|^2_{L_2^{\frac{N-1}{2},\frac{N}{2}}} > 0,$$

and

$$\langle -\Delta_{\mathbb{M}} u, v \rangle = \langle u, -\Delta_{\mathbb{M}} v \rangle.$$

This implies that the operator $-\Delta_{\mathbb{M}}$ is positive and self-adjoint in $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Then Lax-Milgram theorem gives that for any $f \in \mathcal{H}_2^{-1,(-\frac{N-1}{2},-\frac{N}{2})}(\mathbb{M})$, the following Dirichlet problem

$$\begin{cases} -\Delta_{\mathbb{M}} u = f, & z \in \mathbb{M}_0, \\ u = 0 & \text{on } \partial\mathbb{M}, \end{cases} \tag{2.11}$$

admits a unique solution in $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Therefore the mapping

$$-\Delta_{\mathbb{M}} : \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}) \to \mathcal{H}_2^{-1,(-\frac{N-1}{2},-\frac{N}{2})}(\mathbb{M})$$

is continuous. Furthermore, the inverse operator $(-\Delta_{\mathbb{M}})^{-1}$ is well-defined and a continuous map as follows

$$(-\Delta_{\mathbb{M}})^{-1} : \mathcal{H}_2^{-1,(-\frac{N-1}{2},-\frac{N}{2})}(\mathbb{M}) \to \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}).$$

By Proposition 2.4, the embedding

$$i : \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}) \to L_2^{(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$$

is compact, and then the embedding

$$i^* : L_2^{(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}) \to \mathcal{H}_{2,0}^{-1,(-\frac{N-1}{2},-\frac{N}{2})}(\mathbb{M})$$

is continuous. Thus the operator

$$K := (-\Delta_{\mathbb{M}})^{-1} \circ i^* \circ i : \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}) \to \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$$

is compact and self-adjoint. Then there exist eigenvalues $\{\eta_k\}_{k\geq 1}$ of K, such that $\eta_k > 0$ and $\eta_k \to 0$ as $k \to +\infty$. The corresponding normal eigenfunctions $\{\varphi_k\}_{k\geq 1}$ form a complete basis of $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ with

$$K\varphi_k = \eta_k \varphi_k \quad \text{for} \quad k \geq 1.$$

This completes the proof. ∎

3. Some important inequalities on corner type weighted p-Sobolev Spaces

In this section, we shall prove the following Sobolev inequality and Poincarè inequality on corner type weighted p-Sobolev spaces. These inequalities will be important in the proof of the main result. Similar inequalities on doubling spaces were discussed in [7]. However from the example in [5], we know that the corner type weighted p-Sobolev spaces in this paper will be not the doubling spaces.

Proposition 3.1 (Corner Sobolev Inequality). *Assume* $1 \le p < N$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}$, $N = 1 + n + 1$ *and* $\gamma_1, \gamma_2 \in \mathbb{R}$. *For* $u(r,x,t) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$, *the following estimate holds*

$$\begin{aligned}
\|u\|_{L_{p^*}^{\gamma_1^*,\gamma_2^*}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \le{}& \alpha(c_3+c_4)\|r\partial_r u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \\
&+ \alpha(c_1+c_2+c_3+c_4)\sum_{i=1}^n \|\partial_{x_i} u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \\
&+ \alpha(c_2+c_4)\|rt\partial_t u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \\
&+ (c_1+c_2)\|u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \\
&+ (c_1+c_3)\|u\|_{L_p^{\gamma_1-1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \qquad (3.1)
\end{aligned}$$

where $\gamma_1^* = \gamma_1 - 1$, $\gamma_2^* = \gamma_2 - 1$, *and* $\alpha = \frac{(N-1)p}{N-p}$ *with constants* $c_1 = \frac{1}{N}\left|\frac{(N-1)(N-\gamma_1 p)}{N-p}\right|^{\frac{1}{N}}\left|\frac{(N-1)(N-\gamma_2 p)}{N-p}\right|^{\frac{1}{N}}$, $c_2 = \frac{1}{N}\left|\frac{(N-1)(N-\gamma_1 p)}{N-p}\right|^{\frac{1}{N}}$, $c_3 = \frac{1}{N}\left|\frac{(N-1)(N-\gamma_2 p)}{N-p}\right|^{\frac{1}{N}}$, *and* $c_4 = \frac{1}{N}$.

Proof. First we consider the case of $p = 1$ with $p^* = \frac{N}{N-1}$. Let $\gamma_1', \gamma_2' \in \mathbb{R}$. For $u(r,x,t) \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$, we have

$$\begin{aligned}
|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u(r,x,t)| &\le \Big|\int_{-\infty}^{x_i} \partial_{x_i}(r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u)dx_i\Big| \\
&\le \int_{-\infty}^{+\infty} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|dx_i := I_i, \quad \text{for } i = 1,\dots,n.
\end{aligned}$$

Similarly, for r- and t- direction, we have

$$\begin{aligned}
&|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u(r,x,t)| \le \int_0^{+\infty} |(r\partial_r)(r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u)|\frac{dr}{r} \\
&\le |N-1-\gamma_1'|\int_0^{+\infty} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r} + \int_0^{+\infty} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r u)|\frac{dr}{r} \\
&:= \mathit{II} + \mathit{III},
\end{aligned}$$

$$|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u(r,x,t)| \le \int_0^{+\infty} |(rt\partial_t)(r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u)|\frac{dt}{rt}$$
$$\le |N-1-\gamma_2'|\int_0^{+\infty} |r^{N-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dt}{rt} + \int_0^{+\infty} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t u)|\frac{dt}{rt}$$
$$:= IV + V.$$

Multiplying the above $N = n+2$ inequalities, one has

$$\begin{aligned}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u(r,x,t)|^N &\le I_1\cdots I_n(II+III)(IV+V)\\ &= I_1\cdots I_n II\,IV + I_1\cdots I_n II\,V + I_1\cdots I_n III\,IV + I_1\cdots I_n III\,V.\end{aligned}$$

Since $\frac{1}{N-1} < 1$, the following inequality holds

$$\begin{aligned}&|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u(r,x,t)|^{\frac{N}{N-1}}\\ &\le (I_1\cdots I_n II\,IV + I_1\cdots I_n II\,V + I_1\cdots I_n III\,IV + I_1\cdots I_n III\,V)^{\frac{1}{N-1}}\\ &\le (I_1\cdots I_n II\,IV)^{\frac{1}{N-1}} + (I_1\cdots I_n II\,V)^{\frac{1}{N-1}} + (I_1\cdots I_n III\,IV)^{\frac{1}{N-1}}\\ &\quad + (I_1\cdots I_n III\,V)^{\frac{1}{N-1}}.\end{aligned}$$

Integrating both sides of the above inequality by $\frac{dr}{r}$, we have

$$\int_{\mathbb{R}_+} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u(r,x,t)|^{\frac{N}{N-1}}\frac{dr}{r} \le \int_{\mathbb{R}_+} (I_1\cdots I_n II\,IV)^{\frac{1}{N-1}}\frac{dr}{r}$$
$$+\int_{\mathbb{R}_+} (I_1\cdots I_n II\,V)^{\frac{1}{N-1}}\frac{dr}{r}+\int_{\mathbb{R}_+} (I_1\cdots I_n III\,IV)^{\frac{1}{N-1}}\frac{dr}{r}+\int_{\mathbb{R}_+} (I_1\cdots I_n III\,V)^{\frac{1}{N-1}}\frac{dr}{r}$$

Now we apply Hölder inequality on the right hand side of the above inequality, then For the first term, one has

$$\int_{\mathbb{R}_+} (I_1\cdots I_n II\,IV)^{\frac{1}{N-1}}\frac{dr}{r} \le |N-1-\gamma_1'|^{\frac{1}{N-1}}|N-1-\gamma_2'|^{\frac{1}{N-1}}(\int_{\mathbb{R}_+} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r})^{\frac{1}{N-1}}$$
$$\prod_{i=1}^n(\int_{\mathbb{R}}\int_{\mathbb{R}_+} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|\frac{dr}{r}dx_i)^{\frac{1}{N-1}}(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |r^{N-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r}\frac{dt}{rt})^{\frac{1}{N-1}}.$$

Similarly, for the other three terms, it follows that

$$\int_{\mathbb{R}_+} (I_1\cdots I_n II\,V)^{\frac{1}{N-1}}\frac{dr}{r} \le |N-1-\gamma_1'|^{\frac{1}{N-1}}(\int_{\mathbb{R}_+} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r})^{\frac{1}{N-1}}$$
$$\prod_{i=1}^n(\int_{\mathbb{R}}\int_{\mathbb{R}_+} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|\frac{dr}{r}dx_i)^{\frac{1}{N-1}}(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+} |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t u)|\frac{dr}{r}\frac{dt}{rt})^{\frac{1}{N-1}},$$

$$\int_{\mathbb{R}_+}(I_1\cdots I_n I\!I\!I I\!V)^{\frac{1}{N-1}}\frac{dr}{r}\leq|N-1-\gamma_2'|^{\frac{1}{N-1}}(\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r u)|\frac{dr}{r})^{\frac{1}{N-1}}$$
$$\prod_{i=1}^{n}(\int_{\mathbb{R}}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|\frac{dr}{r}dx_i)^{\frac{1}{N-1}}(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}|r^{N-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r}\frac{dt}{rt})^{\frac{1}{N-1}},$$

and

$$\int_{\mathbb{R}_+}(I_1\cdots I_n I\!I\!I V)^{\frac{1}{N-1}}\frac{dr}{r}\leq(\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r u)|\frac{dr}{r})^{\frac{1}{N-1}}$$
$$\prod_{i=1}^{n}(\int_{\mathbb{R}}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|\frac{dr}{r}dx_i)^{\frac{1}{N-1}}(\int_{\mathbb{R}_+}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t u)|\frac{dr}{r}\frac{dt}{rt})^{\frac{1}{N-1}}.$$

Repeating the same process with respect to $dx_1,\dots dx_n$ and $\frac{dt}{rt}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|^{\frac{N}{N-1}}\frac{dr}{r}dx\frac{dt}{rt}\\
\leq&|N-1-\gamma_1'|^{\frac{1}{N-1}}|N-1-\gamma_2'|^{\frac{1}{N-1}}(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}\\
&\prod_{i=1}^{n}(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}\\
&(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}\\
+&|N-1-\gamma_1'|^{\frac{1}{N-1}}(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}\\
&\prod_{i=1}^{n}(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}\\
&(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t u)|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}\\
+&|N-1-\gamma_2'|^{\frac{1}{N-1}}(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r u)|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}\\
&\prod_{i=1}^{n}(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}\\
&(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-\gamma_1'}t^{N-1-\gamma_2'}u|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}
\end{aligned}$$

$$+(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r u)|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}$$
$$\prod_{i=1}^{n}(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i} u)|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}}$$
$$(\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}|r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t u)|\frac{dr}{r}dx\frac{dt}{rt})^{\frac{1}{N-1}} := A+B+C+D.$$

Here and in what follows, write $\int \cdots d\sigma$ for $\int_{\mathbb{R}_+}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+}\cdots\frac{dr}{r}dx\frac{dt}{rt}$. Since $\frac{N-1}{N}<1$, the following estimate holds

$$(\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|^{\frac{N}{N-1}}d\sigma)^{\frac{N-1}{N}} \ \le \ A^{\frac{N-1}{N}}+B^{\frac{N-1}{N}}+C^{\frac{N-1}{N}}+D^{\frac{N-1}{N}}.$$

For $a_i\ge 0$, $i=1,..,N$, we have $(\prod_{i=1}^N a_i)^{\frac{1}{N}} \le \frac{1}{N}(\sum_{i=1}^N a_i)$. Applying this inequality on $A^{\frac{N-1}{N}}$, $B^{\frac{N-1}{N}}$, $C^{\frac{N-1}{N}}$, and $D^{\frac{N-1}{N}}$, it then follows that

$$
\begin{aligned}
&(\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|^{\frac{N}{N-1}}d\sigma)^{\frac{N-1}{N}}\\
\leq&\frac{1}{N}|N-1-\gamma_1'|^{\frac{1}{N}}|N-1-\gamma_2'|^{\frac{1}{N}}(\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|d\sigma\\
&+\sum_{i=1}^{n}\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|d\sigma+\int |r^{N-\gamma_1'}t^{N-1-\gamma_2'}u|d\sigma)\\
+&\frac{1}{N}|N-1-\gamma_1'|^{\frac{1}{N}}(\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|d\sigma\\
&+\sum_{i=1}^{n}\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|d\sigma+\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t u)|d\sigma)\\
+&\frac{1}{N}|N-1-\gamma_2'|^{\frac{1}{N}}(\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r u)|d\sigma\\
&+\sum_{i=1}^{n}\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|d\sigma+\int |r^{N-\gamma_1'}t^{N-1-\gamma_2'}u|d\sigma)\\
+&\frac{1}{N}(\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r u)|d\sigma\\
&+\sum_{i=1}^{n}\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|d\sigma+\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t u)|d\sigma)\\
=&(c_3+c_4)\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r u)|d\sigma\\
&+(c_1+c_2+c_3+c_4)\sum_{i=1}^{n}\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}u)|d\sigma\\
&+(c_2+c_4)\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t u)|d\sigma\\
&+(c_1+c_2)\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}u|d\sigma+(c_1+c_3)\int |r^{N-\gamma_1'}t^{N-1-\gamma_2'}u|d\sigma,
\end{aligned}
$$

where $c_1=\frac{1}{N}|N-1-\gamma_1'|^{\frac{1}{N}}|N-1-\gamma_2'|^{\frac{1}{N}}$, $c_2=\frac{1}{N}|N-1-\gamma_1'|^{\frac{1}{N}}$, $c_3=\frac{1}{N}|N-1-\gamma_2'|^{\frac{1}{N}}$, $c_4=\frac{1}{N}$. It implies that the inequality (3.1) holds for $p=1$.

Now we consider the cases of $1<p<N$. Let $\alpha=\frac{(N-1)p}{N-p}>1$, then $|u|^\alpha\in C_0^1(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)$, thus we can use the estimate above to deduce that

$$
\begin{aligned}
&(\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha}|^{\frac{N}{N-1}}d\sigma)^{\frac{N-1}{N}}\\
\le&(c_3+c_4)\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(r\partial_r|u|^{\alpha})|d\sigma\\
&+(c_1+c_2+c_3+c_4)\sum_{i=1}^{n}\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(\partial_{x_i}|u|^{\alpha})|d\sigma\\
&+(c_2+c_4)\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}(rt\partial_t|u|^{\alpha})|d\sigma)\\
&+(c_1+c_2)\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha}|d\sigma+(c_1+c_3)\int |r^{N-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha}|d\sigma.
\end{aligned}
$$

Since $|u|^{\alpha}=(u\cdot\bar{u})^{\frac{\alpha}{2}}$ (here we use the fact that if a function u is real-valued, then $\bar{u}=u$), and $|\partial_{x_j}|u|^{\alpha}|=|\partial_{x_j}(u^{\frac{\alpha}{2}}\cdot\bar{u}^{\frac{\alpha}{2}})|\le\alpha|u|^{\alpha-1}|\partial_{x_j}u|$. Then we have

$$
\begin{aligned}
&(\int |r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha}|^{\frac{N}{N-1}}d\sigma)^{\frac{N-1}{N}}\\
\le&\alpha(c_3+c_4)\int r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha-1}|(r\partial_r u)|d\sigma\\
&+\alpha(c_1+c_2+c_3+c_4)\sum_{i=1}^{n}\int r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha-1}|(\partial_{x_i}u)|d\sigma\\
&+\alpha(c_2+c_4)\int r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha-1}|(rt\partial_t u)|d\sigma)\\
&+(c_1+c_2)\int r^{N-1-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha-1}|u|d\sigma\\
&+(c_1+c_3)\int r^{N-\gamma_1'}t^{N-1-\gamma_2'}|u|^{\alpha-1}|u|d\sigma.
\end{aligned}
$$

Let $\frac{1}{q}=1-\frac{1}{p}$, $\gamma_1^*=\frac{\gamma_1'(N-p)}{(N-1)p}\in\mathbb{R}$ and $\gamma_2^*=\frac{\gamma_2'(N-p)}{(N-1)p}\in\mathbb{R}$, we then have $\frac{\alpha N}{N-1}=p^*$, $\frac{(N-1-\gamma_1')N}{N-1}=\frac{N}{p^*}-\gamma_1^*$, $\frac{(N-1-\gamma_2')N}{N-1}=\frac{N}{p^*}-\gamma_2^*$, and $(\alpha-1)q=p^*$. Moreover, set $\beta_1=(N-1-\gamma_1')\frac{N(p-1)}{(N-1)p}$, $\beta_2=(N-1-\gamma_1')\frac{N-p}{(N-1)p}=\frac{N}{p}-(\gamma_1^*+1)$ and $\varphi_1=(N-1-\gamma_2')\frac{N(p-1)}{(N-1)p}$, $\varphi_2=\frac{N}{p}-(\gamma_2^*+1)$, which imply that $\beta_1+\beta_2=N-1-\gamma_1$, $\varphi_1+\varphi_2=N-1-\gamma_2$. Writing $\gamma_1=\gamma_1^*+1$, $\gamma_2=\gamma_2^*+1$ and from $\frac{N-1}{N}-\frac{1}{q}=\frac{1}{p^*}$, Hölder inequality gives that

$$
\begin{aligned}
\|u\|_{L_{p^*}^{\gamma_1^*,\gamma_2^*}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} &= (\int |r^{\frac{N}{p^*}-\gamma_1^*}t^{\frac{N}{p^*}-\gamma_2^*}u|^{p^*}d\sigma)^{1/p^*} \\
\leq & \alpha(c_3+c_4)(\int |r^{\frac{N}{p}-\gamma_1}t^{\frac{N}{p}-\gamma_2}(r\partial_r u)|^p d\sigma)^{1/p} \\
& + \alpha(c_1+c_2+c_3+c_4)\sum_{i=1}^{n}(\int |r^{\frac{N}{p}-\gamma_1}t^{\frac{N}{p}-\gamma_2}(\partial_{x_i}u)|^p d\sigma)^{1/p} \\
& + \alpha(c_2+c_4)(\int |r^{\frac{N}{p}-\gamma_1}t^{\frac{N}{p}-\gamma_2}(rt\partial_t u)|^p d\sigma)^{1/p} \\
& + (c_1+c_2)(\int |r^{\frac{N}{p}-\gamma_1}t^{\frac{N}{p}-\gamma_1}u|^p d\sigma)^{1/p} \\
& + (c_1+c_3)(\int |r^{\frac{N}{p}-(\gamma_1-1)}t^{\frac{N}{p}-\gamma_2}u|^p d\sigma)^{1/p}.
\end{aligned}
$$

where $c_1 = \frac{1}{N}\left|\frac{(N-1)(N-\gamma_1 p)}{N-p}\right|^{1/N}\left|\frac{(N-1)(N-\gamma_2 p)}{N-p}\right|^{1/N}$, $c_2 = \frac{1}{N}\left|\frac{(N-1)(N-\gamma_1 p)}{N-p}\right|^{1/N}$, $c_3 = \frac{1}{N}\left|\frac{(N-1)(N-\gamma_2 p)}{N-p}\right|^{1/N}$, and $c_4 = \frac{1}{N}$. □

In the case of $\gamma_1 = \gamma_2 = \frac{N}{p}$, we have the constants in (3.1), $c_1 = c_2 = c_3 = 0$. Then, Hölder inequality implies that for $u(r,x,t) \in \mathcal{H}_p^{1,(\gamma_1,\gamma_2)}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+)$

$$
\|u\|_{L_{p^*}^{\gamma_1^*,\gamma_2^*}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \leq c\,\|\nabla_{\mathbb{M}}u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \tag{3.2}
$$

where $\nabla_{\mathbb{M}} = (r\partial_r, \partial_{x_1}, \ldots, \partial_{x_n}, rt\partial_t)$ is the corner type gradient operator on $\mathbb{M} = [0,1) \times X \times [0,1)$, and here the constant $c = \frac{(N-1)p}{(N-p)N}$ is the best constant as we had in standard Sobolev spaces.

Similarly, if $\gamma_2 = \frac{N}{p}$ and $\gamma_1 \in \mathbb{R}$, then $c_1 = c_3 = 0$, which give us

$$
\|u\|_{L_{p^*}^{\gamma_1^*,\gamma_2^*}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)} \leq \bar{c}\,\|u\|_{\mathcal{H}_p^{1,(\gamma_1,\gamma_2)}(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}_+)}
$$

with constant $\bar{c} = \frac{(N-1)p}{N-p}\Big(\frac{1}{N}\left|\frac{(N-1)(N-\gamma_1 p)}{N-p}\right|^{1/N} + \frac{1}{N}\Big)$.

Proposition 3.2 (Poincaré inequality)**.** *For $u(r,x,t) \in \mathcal{H}_{p,0}^{1,(\gamma_1,\gamma_2)}(\mathbb{M})$, $1 \leq p < \infty$, the following estimate holds*

$$
\|u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{M})} \leq d\,\|\nabla_{\mathbb{M}}u\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{M})} \tag{3.3}
$$

where d is the diameter of $\mathbb{M}$.

Proof. Set

$$Q = \{(r,x,t) \in \mathbb{R}^N \mid 0 < r < d, a_i < x_i < a_i{+}d, \text{ for } i = 1,...,n, \text{ and } 0 < t < d\}$$

where $d \in \mathbb{R}_+$ is the diameter of $\mathbb{M}$, i.e. $\mathbb{M} \subset Q$.

Suppose $u(r,x,t) \in C_0^\infty(\mathbb{M}_0)$. For $(r,x,t) \in \mathbb{M} \subset Q$, we have

$$|u(r,x,t)|^p \leq (\int_{a_1}^{x_1} |\partial_{x_1} u(r,s,x_2,...,x_n,t)|ds)^p.$$

Applying Hölder inequality, for $a_1 < x_1' < a_1 + d$, we have

$$|u(r,x,t)|^p \leq d^{p-1}(\int_{a_1}^{x_1+d} |\partial_{x_1} u(r,s,x_2,...,x_n,t)|^p ds).$$

Then, the mean value theorem implies that

$$|u(r,x,t)|^p \leq d^p |\partial_{x_1} u(r,x_1',x_2,...,x_n,t)|^p.$$

Multiplying the both sides with term $r^{N-\gamma_1 p} t^{N-\gamma_2 p}$, and then integrating with respect to $\frac{dr}{r} dx \frac{dt}{rt} := d\sigma$ on Q, we obtain that

$$\int_Q r^{N-\gamma_1 p} t^{N-\gamma_2 p} |u(r,x,t)|^p d\sigma \leq d^p \int_Q r^{N-\gamma_1 p} t^{N-\gamma_2 p} |\partial_{x_1} u(r,x_1',x_2,...,x_n,t)|^p d\sigma.$$

The definition of Q and the assumption $u(r,x,t) \in C_0^\infty(\mathbb{M}_0)$ give that

$$\int_\mathbb{M} r^{N-\gamma_1 p} t^{N-\gamma_2 p} |u(r,x,t)|^p d\sigma \leq d^p \int_\mathbb{M} r^{N-\gamma_1 p} t^{N-\gamma_2 p} |\partial_{x_1} u(r,x_1',x_2,...,x_n,t)|^p d\sigma.$$

Since $C_0^\infty(\mathbb{M}_0)$ is dense in $\mathcal{H}_{p,0}^{1,(\gamma_1,\gamma_2)}(\mathbb{M})$, the estimate above implies that for $u(r,x,t) \in \mathcal{H}_{p,0}^{1,(\gamma_1,\gamma_2)}(\mathbb{M})$,

$$\|u(r,x,t)\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{M})} \leq d\,\|\partial_{x_1} u(r,x,t)\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{M})} \leq d\,\|\nabla_\mathbb{M} u(r,x,t)\|_{L_p^{\gamma_1,\gamma_2}(\mathbb{M})},$$

as required. □

Proposition 3.3. *The following embedding*

$$\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}) \hookrightarrow \mathcal{H}_{l,0}^{0,(\frac{N-1}{l},\frac{N}{l})}(\mathbb{M})$$

is compact, for $1 < l < 2^*$.

Proof. According to Definition 2.3, it is sufficient to show the compactness of the embedding

$$[\omega(r)][\sigma(t)]\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{R}_+\times X\times\mathbb{R}_+) \hookrightarrow [\omega(r)][\sigma(t)]\mathcal{H}_{l,0}^{0,(\frac{N-1}{l},\frac{N}{l})}(\mathbb{R}_+\times X\times\mathbb{R}_+). \tag{3.4}$$

In fact, by virtue of Proposition 3.6 in [5], the following two embeddings are compact for $1 < l < 2^*$

$$[1-\omega(r)][\sigma(t)]\mathcal{H}_{2,0}^{1,\frac{N}{2}}(\Omega_{\varepsilon_1}\times X\times\mathbb{R}_+) \hookrightarrow [1-\omega(r)][\sigma(t)]\mathcal{H}_{l,0}^{0,\frac{N}{l}}(\Omega_{\varepsilon_1}\times X\times\mathbb{R}_+),$$

$$[\omega(r)][1-\sigma(t)]\mathcal{H}_{2,0}^{1,\frac{N-1}{2}}(\mathbb{R}_+\times X\times\Omega_{\varepsilon_2}) \hookrightarrow [\omega(r)][1-\sigma(t)]\mathcal{H}_{l,0}^{0,\frac{N-1}{l}}(\mathbb{R}_+\times X\times\Omega_{\varepsilon_2}),$$

where Ω_{ε_1} and Ω_{ε_2} are given in (2.3).

For the classical Sobolev spaces $W_0^{m,p}(\Omega_{\varepsilon_1}\times X\times\Omega_{\varepsilon_2})$, Rellich-Kondrachov theorem gives the compactness of the following embedding

$$[1-\omega(r)][1-\sigma(t)]W_0^{1,2}(\Omega_{\varepsilon_1}\times X\times\Omega_{\varepsilon_2}) \hookrightarrow [1-\omega(r)][1-\sigma(t)]W_0^{0,l}(\Omega_{\varepsilon_1}\times X\times\Omega_{\varepsilon_2}).$$

It remains to prove the compactness of (3.4). Similar to the transform in (2.7), for $u(r,x,t) \in [\omega(r)][\sigma(t)]\mathcal{H}_{l,0}^{0,(\frac{N-1}{l},\frac{N}{l})}(\mathbb{R}_+\times X\times\mathbb{R}_+)$, the mapping

$$\begin{aligned} S_{l,(\frac{N-1}{l},\frac{N}{l})} : u(r,x,t) \to & e^{-\rho(\frac{N}{l}-\frac{N-1}{l})}e^{-(e^{-\rho}\zeta)(\frac{N}{l}-\frac{N}{l})}u(e^{-\rho},x,e^{-(e^{-\rho}\zeta)}) \\ & := w(\rho,x,\zeta) \end{aligned}$$

gives the isomorphism

$$S_{l,(\frac{N-1}{l},\frac{N}{l})} : [\omega(r)][\sigma(t)]\mathcal{H}_{l,0}^{0,(\frac{N-1}{l},\frac{N}{l})}(\mathbb{R}_+\times X\times\mathbb{R}_+) \to [\tilde{\omega}(\rho)][\tilde{\sigma}(\zeta)]W_0^{0,l}(\mathbb{R}\times X\times\mathbb{R}), \tag{3.5}$$

where $\rho = -\ln r$ for $r \in \operatorname{supp}\omega$, $\tilde{\omega}(\rho) = \omega(e^{-\rho})$, $\tilde{\sigma}(\zeta)$ is the cut-off function in $\zeta = -\frac{\ln t}{r}$ with $r \in \operatorname{supp}\sigma(r)$ and $t \in \operatorname{supp}\sigma(t)$.

Moreover, for $u(r,x,t) \in [\omega(r)][\sigma(t)]\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{R}_+\times X\times\mathbb{R}_+)$ we have

$$\begin{aligned} S_{l,(\frac{N-1}{l},\frac{N}{l})}u(r,x,t) &= e^{-\rho(\frac{N}{l}-\frac{N-1}{l})}e^{-(e^{-\rho}\zeta)(\frac{N}{l}-\frac{N}{l})}u(e^{-\rho},x,e^{-(e^{-\rho}\zeta)}) \\ &= e^{-\rho(\frac{1}{l}-\frac{1}{2})}S_{2,(\frac{N-1}{2},\frac{N}{2})}u(r,x,t) \end{aligned}$$

According to (2.9), $S_{l,(\frac{N-1}{l},\frac{N}{l})}$ induces the following isomorphism

$$S_{l,(\frac{N-1}{l},\frac{N}{l})} : [\omega(r)][\sigma(t)]\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{R}_+\times X\times\mathbb{R}_+) \to [\tilde{\omega}(\rho)][\tilde{\sigma}(\zeta)]W_0^{1,2}(\mathbb{R}\times X\times\mathbb{R}). \tag{3.6}$$

Since we know that, for $\rho \in \operatorname{supp}\tilde{\omega}$ and $\zeta \in \operatorname{supp}\tilde{\sigma}$, $[\tilde{\omega}(\rho)][\tilde{\sigma}(\zeta)]W_0^{1,2}(\mathbb{R}\times X\times\mathbb{R})$ is compactly embedded in $[\tilde{\omega}(\rho)][\tilde{\sigma}(\zeta)]W_0^{0,l}(\mathbb{R}\times X\times\mathbb{R})$ for $1 < l < 2^*$. Thus from the isomorphisms (3.5) and (3.6), the embedding

$$[\omega(r)][\sigma(t)]\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{R}_+\times X\times\mathbb{R}_+) \hookrightarrow [\omega(r)][\sigma(t)]\mathcal{H}_{l,0}^{0,(\frac{N-1}{l},\frac{N}{l})}(\mathbb{R}_+\times X\times\mathbb{R}_+).$$

is compact. Proposition 3.3 is proved. □

4. Proof of Theorem 1.1

Now we recall the $(\mathrm{PS})_c$ condition (Palais-Smale condition, c.f. [1]).

Definition 4.1. *Let E be a Banach space, $I \in C^1(E;\mathbb{R})$ and $c \in \mathbb{R}$, We say that I satisfies the $(PS)_c$ condition, if for any sequence $\{u_k\} \subset E$ with the properties:*

$$I(u_k) \to c \quad \text{and} \quad \| I'(u_k) \|_{E'} \to 0,$$

there exists a subsequence which is convergent, where $I'(\cdot)$ is the Fréchet differentiation of I and E' is the dual space of E. If it holds for any $c \in \mathbb{R}$, we say that I satisfies (PS) condition.

We also need the following well-known result:

Proposition 4.2 (cf. [15])**.** *Suppose E is an infinite dimensional Banach space and the functional $I \in C^1(E;\mathbb{R})$ satisfies (PS) condition, with*

$$I(u) = I(-u) \text{ for all } u \in E, \text{ and } I(0) = 0.$$

Suppose $E = V^+ \oplus V^-$ where V^- is a finite dimensional space and assume the following conditions

(i) *There exist constants $\alpha > 0$, $\rho > 0$, such that for any $u \in V^+$ with $\|u\| = \rho$ we have $I(u) \geq \alpha$.*
(ii) *For any finite dimensional subspace $W \subset E$, there exists a constant R depending on W such that for any $u \in W$ with $\|u\| \geq R$ we have $I(u) \leq 0$.*

Then I admits an unbounded sequence of critical values.

Proof of Theorem 1.1. Define the following energy functional

$$J(u) = \frac{1}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}}u|^2 d\sigma - \int_{\mathbb{M}} rG(z,u)d\sigma, \quad d\sigma := \frac{dr}{r}dx\frac{dt}{rt}.$$

By the assumption that $G(\cdot,u)=\int_0^u g(\cdot,v)dv$, the nonlinear elliptic equation (1.1) is the Euler-Lagrange equation for the energy functional $J(u)$. We say that $u\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ is a weak solution of (1.1) if

$$\begin{aligned}\langle J'(u),v\rangle &= \int_{\mathbb{M}} r(\nabla_{\mathbb{M}}u)(\nabla_{\mathbb{M}}v)d\sigma-\int_{\mathbb{M}} rg(z,u)vd\sigma\\ &= \int_{\mathbb{M}} r[-\Delta_{\mathbb{M}}u-g(z,u)]vd\sigma=0\end{aligned}$$

for any $v\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$, where $\langle\cdot,\cdot\rangle:=\langle\cdot,\cdot\rangle_{L_2^{\frac{N-1}{2},\frac{N}{2}}}$ and $J'(\cdot)$ denotes the Fréchet differentiation. Therefore, the critical points of $J(u)$ in $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ are the weak solutions of Dirichlet problem (1.1).

Now we prove that $J(\cdot)$ satisfies (PS) condition. Let $u_k\in\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$ and $\{u_k\}$ be a (PS) sequence of J, then we have

$$qJ(u_k)-\langle J'(u_k),u_k\rangle\le c_1+o(1)\|u_k\|_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}}$$

where $q>2$, c_1 depends on q and $o(1)\to 0$ as $k\to+\infty$. Furthermore,

$$\begin{aligned}&qJ(u_k)-\langle J'(u_k),u_k\rangle\\ &=\frac{q}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}}u_k|^2d\sigma-q\int_{\mathbb{M}} rG(z,u_k)d\sigma-\Big(\int_{\mathbb{M}} r|\nabla_{\mathbb{M}}u_k|^2d\sigma-\int_{\mathbb{M}} rg(z,u_k)u_kd\sigma\Big)\\ &=\frac{q-2}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}}u_k|^2d\sigma+\int_{\mathbb{M}} rg(z,u_k)u_kd\sigma-\int_{\mathbb{M}} rqG(z,u_k)d\sigma\end{aligned}$$

By Poincaré inequality, we have

$$\int_{\mathbb{M}} r|\nabla_{\mathbb{M}}u_k|^2d\sigma\ge c_2\|u\|^2_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}}$$

where the constant $c_2>0$ depends on the constant d in (3.3). The assumptions (H-2) and (H-3) give us that for $|u_k|\ge R_0$, $\int_{\mathbb{M}} r[g(z,u_k)u_k-qG(z,u_k)]d\sigma>0$ and for $|u_k|<R_0$, $\int_{\mathbb{M}} r[g(z,u_k)u_k-qG(z,u_k)]d\sigma$ is finite. Then there exists a constant $\beta_0>0$ such that

$$\frac{(q-2)c_2}{2}\|u_k\|^2_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}}-\beta_0\le qJ(u_k)-\langle J'(u_k),u_k\rangle\le c_1+o(1)\|u_k\|_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}}$$

which implies that $\{u_k\}$ is bounded in $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Moreover, there exists a weak convergent subsequence of $\{u_k\}$, also denoted by $\{u_k\}$, i.e,

$$u_k\rightharpoonup u\quad\text{in }\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M}).$$

By Proposition 3.3, for $1 < p < 2^*$,

$$u_k \to u \quad \text{in } \mathcal{H}_{p,0}^{0,(\frac{N-1}{p},\frac{N}{p})}(\mathbb{M}).$$

Applying Hölder inequality and corner type Poincaré inequality, we have

$$\begin{aligned}
&\|u_k - u\|^2_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}} = \int_{\mathbb{M}} r|\nabla_{\mathbb{M}}(u_k-u)|^2 d\sigma + \int_{\mathbb{M}} r|u_k-u|^2 d\sigma \\
&\le c_5 \int_{\mathbb{M}} r|\nabla_{\mathbb{M}}(u_k-u)|^2 d\sigma \\
&= c_5\langle J'(u_k-u), u_k-u\rangle + c_5\int_{\mathbb{M}} rg(z,(u_k-u))(u_k-u)d\sigma \\
&\le c_5\langle J'(u_k-u), u_k-u\rangle + c_6\int_{\mathbb{M}} r|u_k-u|d\sigma + c_5\int_{\mathbb{M}} r|u_k-u|^p d\sigma \\
&\le c_5\langle J'(u_k-u), u_k-u\rangle + c_7(\int_{\mathbb{M}} r|u_k-u|^p d\sigma)^{1/p} + c_6\int_{\mathbb{M}} r|u_k-u|^p d\sigma \\
&= c_5\langle J'(u_k-u), u_k-u\rangle + c_7\|u_k-u\|_{\mathcal{H}_{p,0}^{0,(\frac{N-1}{p},\frac{N}{p})}} + c_6\|u_k-u\|^p_{\mathcal{H}_{p,0}^{0,(\frac{N-1}{p},\frac{N}{p})}} \\
&\to 0 \quad \text{as } k\to\infty,
\end{aligned}$$

where $c_5 > 1$, c_6, $c_7 > 0$ depend on d in (3.3) and c_7 also depends on the measure of $\mathbb{M}$, denoted by $|M|$, in the sense of $\frac{dr}{r}dx\frac{dt}{rt}$. That implies that the functional J satisfies (PS) condition.

Now we check the conditions of Proposition 4.2 will be satisfied. Since g is odd, $J(u) = J(-u)$ and $J(0) = 0$. According to Proposition 2.5, the eigenfunctions $\{\varphi_k\}_{k\ge 1}$ constitute the orthogonal basis of $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. For some fixed $k_0 \in \mathbb{N}$, set

$$V^+ = \operatorname{span}\{\varphi_k \mid k \ge k_0\}.$$

Let $u \in V^+$ and $\|u\|_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}} = \rho > 0$, then we have

$$J(u) \ge \frac{1}{2}\int_{\mathbb{M}} r|\nabla_{\mathbb{M}}u|^2 d\sigma - c_0\int_{\mathbb{M}} r|u|d\sigma - c_8\int_{\mathbb{M}} r|u|^p d\sigma,$$

where $c_0 > 0$ in (H-2) and $c_8 > 0$ depends on c_0 and p. By Hölder inequality and $0 < r < 1$, we have

$$\begin{aligned}
&\int_{\mathbb{M}} r|u|^p d\sigma = \int_{\mathbb{M}} (r^{\frac{1}{s}}|u|^{\frac{2}{s}})(r^{1-\frac{1}{s}}|u|^{p-\frac{2}{s}})d\sigma \\
&\le (\int_{\mathbb{M}} r|u|^2 d\sigma)^{\frac{1}{s}}(\int_{\mathbb{M}} r^{(1-\frac{1}{s})q}|u|^{(p-\frac{2}{s})q}d\sigma)^{\frac{1}{q}},
\end{aligned}$$

where $q > 1$ and $\frac{1}{q} + \frac{1}{s} = 1$. Set $(p - \frac{2}{s})q = 2^*$ and $\sigma = \frac{2}{s}$, we have $s = \frac{2^*-2}{2^*-p} > 1$ and $\sigma = \frac{2(2^*-p)}{2^*-2} > 0$. Then, $\frac{1}{q} = 1 - \frac{1}{s} = (p-\sigma)\frac{1}{2^*}$, and it follows

$$\int_{\mathbb{M}} r|u|^p d\sigma \leq (\int_{\mathbb{M}} r|u|^2 d\sigma)^{\frac{1}{2}\sigma}(\int_{\mathbb{M}} r|u|^{2^*} d\sigma)^{\frac{1}{2^*}(p-\sigma)} = \|u\|^{\sigma}_{L_2^{\frac{N-1}{2},\frac{N}{2}}} \|u\|^{p-\sigma}_{L_{2^*}^{\frac{N-1}{2^*},\frac{N}{2^*}}}.$$

According to Proposition 3.2,

$$\|u\|^{\sigma}_{L_2^{\frac{N-1}{2},\frac{N}{2}}} \leq \lambda_{k_0}^{-\frac{\sigma}{2}} \|\nabla_{\mathbb{M}} u\|^{\sigma}_{L_2^{\frac{N-1}{2},\frac{N}{2}}} \leq \lambda_{k_0}^{-\frac{\sigma}{2}} \|u\|^{\sigma}_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}}$$

Then the corner type Sobolev inequality implies that

$$\|u\|^{p-\sigma}_{L_{2^*}^{\frac{N-1}{2^*},\frac{N}{2^*}}} \leq c\|u\|^{p-\sigma}_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}}. \tag{4.1}$$

Furthermore,

$$\begin{aligned} J(u) &\geq c_9\|u\|^2_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}} - c_{10}\lambda_{k_0}^{-\frac{\sigma}{2}}\|u\|^p_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}} - c_{11}\|u\|_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}} \\ &= (c_9 - c_{10}\lambda_{k_0}^{-\frac{\sigma}{2}}\|u\|^{p-2}_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}})\|u\|^2_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}} - c_{11}\|u\|_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}} \end{aligned}$$

where $c_9 > 0$ depends on d in (3.3), $c_{10} > 0$ depends on c_0, p and c in (4.1), and $c_{11} > 0$ depends on c_0 and the measure $|M|$. Let k_0 be large enough such that $c_9 - c_{10}\lambda_{k_0}^{-\frac{\sigma}{2}}\rho^{p-2} > \frac{c_9}{2}$. Then we choose $\rho = \frac{c_{11}+\sqrt{c_{11}^2+2c_9}}{c_9} > 0$ to obtain

$$J(u) \geq \frac{c_9}{2}\rho^2 - c_{11}\rho = 1 := \alpha.$$

Let $V^- := \mathrm{span}\{\varphi_k \mid k < k_0\}$ which is a finite dimensional space such that $V^+ \oplus V^- = \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. By virtue of the assumption (H-3), we have

$$\frac{q}{u} \leq \frac{g(z,u)}{G(z,u)} \quad \text{for } u \geq R_0,$$

$$-\frac{q}{u} \geq \frac{g(z,u)}{G(z,u)} \quad \text{for } u \leq -R_0.$$

Integrating the above two inequalities with respect to u on $[R_0, u]$ and $[u, -R_0]$ respectively, we obtain

$$q \ln \frac{u}{R_0} \leq \ln \frac{G(z,u)}{G(z,R_0)} \quad \text{for } u \geq R_0,$$

$$q \ln \frac{R_0}{-u} \geq \ln \frac{G(z, -R_0)}{G(z,u)} \quad \text{for } u \leq -R_0,$$

which implies that

$$G(z,u) \geq \beta_1(z)|u|^q \quad \text{for } |u| \geq R_0,$$

where $\beta_1(z) > 0$. In fact, $G(z,u)$ is bounded if $|u| < R_0$. Then there exists a constant $\beta_2 > 0$ such that

$$G(z,u) \geq \beta_1(z)|u|^q - \beta_2 \tag{4.2}$$

holds for almost every $(z,u) \in \mathbb{M} \times \mathbb{R}$. For any finite dimensional subspaces $W \subset \mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$, let $u \in W$, and set $\|u\| := \|u\|_{\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}}$ for short. Then we have $u = \|u\|u_0$ with $\|u_0\| = 1$. The estimate (4.2) gives us that

$$\begin{aligned} J(u) &= \frac{\|u\|^2}{2} \int_{\mathbb{M}} r|\nabla_{\mathbb{M}} u_0|^2 d\sigma - \int_{\mathbb{M}} rG(z, \|u\|u_0) d\sigma \\ &\leq \frac{\|u\|^2}{2} - \|u\|^q \int_{\mathbb{M}} r\beta_1(z)|u_0|^q d\sigma + \beta_2|\mathbb{M}|, \end{aligned}$$

where $|\mathbb{M}|$ is the measure of $\mathbb{M}$ in the sense of $d\sigma$. Since $q > 2$, and $\int_{\mathbb{M}} r\beta_1(z)|u_0|^q d\sigma > 0$, then there exists $R > 0$ such that for $\|u\| > R$, $J(u) \leq 0$. Thus the conditions of Proposition 4.2 are satisfied. This implies that the functional J has an unbounded sequence of the critical values. Therefore the problem (1.1) has infinity many weak solutions in the corner type weighted Sobolev space $\mathcal{H}_{2,0}^{1,(\frac{N-1}{2},\frac{N}{2})}(\mathbb{M})$. Theorem 1.1 is proved. $\square$

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