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by

Yan-Ling Wang, Mao-Sheng Li, Shao-Ming Fei, and Zhu-Jun Zheng

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Yan-Ling Wang[1], Mao-Sheng Li[1], Shao-Ming Fei[2,3], Zhu-Jun Zheng[1]
[1] *Department of Mathematics, South China University of Technology, Guangzhou 510640, P.R.China*
[2] *School of Mathematical Sciences, Capital Normal University, Beijing 100048, China*
[3] *Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany*

In this paper, we give a method for the local unitary equivalent problem which is more efficient than that was proposed by Bin Liu *et al* [9].

Keywords: local unitary equivalence, multipartite states

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I. INTRODUCTION

Quantum entanglement plays important roles in many quantum information processing tasks, such as teleportation, quantum error correction and quantum secret sharing [1–3]. In order to understand better the quantum entanglement of quantum states, one needs to investigate equivalence of quantum states under local operations, namely, if a quantum state can be transformed into another one by local operations. Particularly, if one concerns only the local unitary operations, the problem becomes local unitary (LU) equivalence [5].

One of the important methods to solve such LU equivalence problem is to construct the invariants under local unitary transformations. In [4], a complete set of 18 polynomial invariants has been presented for two-qubit mixed states. In [5, 6] Kraus solved the LU equivalence problem for multi-qubit pure states. However, the LU equivalence problem still remains open for arbitrary dimensional states. Even for general bipartite mixed states, there are no efficient approaches to deal with the problem [7, 8].

By using the higher order singular value decomposition (HOSVD) technique, the authors solved the LU equivalence problem for arbitrary dimensional multipartite pure states in [9, 10].

Nevertheless, if the local symmetries of the HOSVD state are blocks with large size, the reduce problem is still difficult to tackle with. Our paper give a refinement of such states. One can find that our method is more efficient than Bin Liu's through the two given examples. In this paper, we first give a necessary condition for the LU equivalence of two quantum states. After considering the simultaneously unitary equivalent, we give a reduce form with local symmetry in smaller size than in [9, 10].

This paper is organized as follows. In section II, we recall some results about the unitary equivalence and some notations about HOSVD that will be used in our work. In section III, we illustrate our method for the tripartite pure states, together with detailed some examples. In section IV, we generalize our method to multipartite pure states and present a reduced form whose symmetry group can be made into smaller blocks than the one in [9].

II. SOME PRELIMINARIES

Definition 1. [14] Let $n \in \mathbb{N}$, $r_1, r_2, ..., r_m \in \mathbb{N}^+$, such that $n = \sum\limits_{i=1}^{m} r_i$ and $\{E_i\}_{i=1}^{s}$ be a partition of $\{1, 2, ..., m\}$. The set $\{H = diag(U_1, U_2, ..., U_m) | U_i \in U(r_i), U_i = U_j, \text{ if } \exists\, 1 \leq r \leq s, i, j \in E_r\}$ is called a direct group of $M_n(\mathbb{C})$ with size $\{r_i\}_{i=1}^{m}$ and restricted by $\{E_i\}_{i=1}^{s}$.

Lemma 1. [12–14] Let H be a direct group of $n \times n$ matrix. Then matrices A and B are unitary equivalent under H if and only if they have the same reduced forms $\widetilde{A}$ and $\widetilde{B}$. Moreover, the invariant group $\{U \in H | U\widetilde{A}U^\dagger = \widetilde{A}\}$ is a direct subgroup of H.

Corollary 1. Given two sets of matrices $\{A_i\}_{i=1}^{m}$ and $\{B_i\}_{i=1}^{m} \in M_n(\mathbb{C})$. The following two statements are equivalent
(I) $\exists U \in U(n)$ such that $B_i = UA_iU^\dagger, i = 1, 2, ..., m$;
(II)$A = diag(A_1, ..., A_m), B = diag(B_1, ..., B_m)$ have the same reduced form under the direct group defined by $\{r_i = n\}_{i=1}^{m}$, $E_1 = \{1, 2, ..., m\}$.

Definition 2. A multipartite pure state $|\psi\rangle$ in $\mathbb{C}^{I_1} \otimes \mathbb{C}^{I_2} \otimes ... \otimes \mathbb{C}^{I_N}$ is called a HOSVD state if $|\psi\rangle_m\langle\psi|$ are diagonal for $m = 1, ..., N$, where $\dagger$ denotes the transpose and conjugation, $|\psi\rangle_m$ is ......

The high order singular value decomposition technique used in [9, 10] can been seen as follows. Let $\lambda_1^{(m)} > \lambda_2^{(m)} > \ldots > \lambda_{t^{(m)}}^{(m)} \geq 0$ be distinct m-mode singular values of $|\psi\rangle$ with multiplicities $\mu_1^{(m)}, \mu_2^{(m)}, \ldots, \mu_{t^{(m)}}^{(m)}$, respectively, where $\sum_{k=1}^{t^{(m)}} \mu_k^{(m)} = I_m$. If Ω is a HOSVD state of $|\psi\rangle$, then

$$\bigotimes_{m=1}^{N}(\bigoplus_{k=1}^{t^{(m)}} U_k^{(m)})\Omega \equiv \bigotimes_{m=1}^{N} S^{(m)}\Omega$$

is also a HOSVD state of $|\psi\rangle$. Here $U_k^{(m)} \in M_{\mu_k^{(m)}}(\mathbb{C})$ are $\mu_k^{(m)} \times \mu_k^{(m)}$ unitary matrices, and constitute the diagonal blocks of $S^{(m)}$ which are conformal to those m-mode singular values of $|\psi\rangle$.

Through out this paper, we say that a state $|\psi\rangle$ is its HOSVD state if $|\psi\rangle_m|\psi\rangle_m^\dagger$ are diagonal to $\underbrace{\lambda_1^{(m)}, ..., \lambda_1^{(m)}}_{\mu_1^{(m)}}, ..., \underbrace{\lambda_{t^{(m)}}^{(m)}, ..., \lambda_{t^{(m)}}^{(m)}}_{\mu_{t^{(m)}}^{(m)}}$, $m = 1, 2, ..., N$, and with local symmetry $\bigotimes_{m=1}^{N} S^{(m)}$.

III. LU EQUIVALENCE FOR TRIPARTITE STATES

We first consider the tripartite case. For a general tripartite pure state $|\psi\rangle$, we have

$$\begin{aligned} |\psi\rangle &= \sum_{k=1}^{t^{(1)}} \sum_{l=1}^{\mu_k^{(1)}} \lambda_k^{(1)} |n_l^{(1),k}\rangle_1 \otimes |v_l^{(1),k}\rangle_{\neg 1} \\ &= \sum_{k=1}^{t^{(2)}} \sum_{l=1}^{\mu_k^{(2)}} \lambda_k^{(2)} |n_l^{(2),k}\rangle_2 \otimes |v_l^{(2),k}\rangle_{\neg 2} \\ &= \sum_{k=1}^{t^{(3)}} \sum_{l=1}^{\mu_k^{(3)}} \lambda_k^{(3)} |n_l^{(3),k}\rangle_3 \otimes |v_l^{(3),k}\rangle_{\neg 3}, \end{aligned}$$

where $n_l^{(i),k} = l + \sum_{s=1}^{k-1} \mu_s^{(i)}$, $|v_l^{(i),k}\rangle_{\neg i}$ is just the normalized vector when we collect the $n_l^{(i),k}$-th term of the i-th system of $|\psi\rangle$. Hence $|\psi\rangle$ can be viewed as the purification of the following mixed bipartite states,

$$\rho^{(1)} = \sum_{k=1}^{t^{(1)}} {\lambda_k^{(1)}}^2 \sum_{l=1}^{\mu_k^{(1)}} |v_l^{(1),k}\rangle_{\neg 1 \neg 1}\langle v_l^{(1),k}|,$$

$$\rho^{(2)} = \sum_{k=1}^{t^{(2)}} {\lambda_k^{(2)}}^2 \sum_{l=1}^{\mu_k^{(2)}} |v_l^{(2),k}\rangle_{\neg 2 \neg 2}\langle v_l^{(2),k}|,$$

$$\rho^{(3)} = \sum_{k=1}^{t^{(3)}} {\lambda_k^{(3)}}^2 \sum_{l=1}^{\mu_k^{(3)}} |v_l^{(3),k}\rangle_{\neg 3 \neg 3}\langle v_l^{(3),k}|.$$

Denote

$$|\psi^{(i),k}\rangle = \lambda_k^{(i)} \sum_{l=1}^{\mu_k^{(i)}} |n_l^{(i),k}\rangle_i \otimes |v_l^{(i),k}\rangle_{\neg i},$$

which can be viewed as the purification of the $\neg i$ bipartite mixed state,

$$\rho_k^{(i)} = {\lambda_k^{(i)}}^2 \sum_{l=1}^{\mu_k^{(i)}} |v_l^{(i),k}\rangle_{\neg i}\langle v_l^{(i),k}|,$$

where super-index (i) stands for the i-th auxiliary system to purify the $\neg i$ bipartite mixed state $\rho_k^{(i)}$, while $\rho_k^{(i)}$ is the k-th part of $\rho^{(i)}$. From the m-mode decomposition of the state $|\psi^{(i),k}\rangle$, we have the matrix $M_{\psi,m}^{i,k}$,

$$M_{\psi,m}^{i,k} \doteq |\psi^{(i),k}\rangle_m \langle\psi^{(i),k}|, \quad 1 \leq m \neq i \leq 3, \quad k = 1,2,...,t^{(i)}.$$

Theorem 1. Let $|\psi\rangle$ and $|\phi\rangle$ be two HOSVD pure states in $\mathbb{C}^{I_1} \otimes \mathbb{C}^{I_2} \otimes \mathbb{C}^{I_3}$. If

$$U_1 \otimes U_2 \otimes U_3 |\psi\rangle = |\phi\rangle, \tag{1}$$

then

$$U_m\, M_{\psi,m}^{i,k}\, U_m^{\dagger} = M_{\phi,m}^{i,k}, \quad 1 \leq m \neq i \leq 3, \quad k = 1,2,...,t^{(i)}.$$

[Proof] For HOSVD states

$$|\psi\rangle = \sum_{k=1}^{t^{(i)}} \sum_{l=1}^{\mu_k^{(i)}} \lambda_k^{(i)} |n_l^{(i),k}\rangle_i, \otimes |v_l^{(i),k}\rangle_{\neg i}, \tag{2}$$

$$|\phi\rangle = \sum_{k=1}^{t^{(i)}} \sum_{l=1}^{\mu_k^{(i)}} \lambda_k^{(i)} |n_l^{(i),k}\rangle_i \otimes |w_l^{(i),k}\rangle_{\neg i}, \tag{3}$$

$U_1 \otimes U_2 \otimes U_3 |\psi\rangle = |\phi\rangle$ implies that $U_i \in S^{(i)}$ are block unitary matrices with size $\{\mu_k^{(i)}\}_{k=1}^{t^{(i)}}$. Suppose

$$U_i = \bigoplus_{k=1}^{t^{(i)}} U_k^{(i)}. \tag{4}$$

From equations (1), (2), (3) and (4), we have

$$U_k^{(i)} \otimes (U_1 \otimes U_2 \otimes U_3)_{\neg i} \sum_{l=1}^{\mu_k^{(i)}} \lambda_k^{(i)} |n_l^{(i),k}\rangle_i \otimes |v_l^{(i),k}\rangle_{\neg i} = \sum_{l=1}^{\mu_k^{(i)}} \lambda_k^{(i)} |n_l^{(i),k}\rangle_i \otimes |w_l^{(i),k}\rangle_{\neg i}.$$

That is

$$U_k^{(i)} \otimes (U_1 \otimes U_2 \otimes U_3)_{\neg i} |\psi_k^{(i)}\rangle = |\phi_k^{(i)}\rangle, \tag{5}$$

where $(U_1 \otimes U_2 \otimes U_3)_{\neg i}$ denotes a matrix with the i-th unitary matrix U_i being removed from the expression $(U_1 \otimes U_2 \otimes U_3)$, for instance, $(U_1 \otimes U_2 \otimes U_3)_{\neg 3} = (U_1 \otimes U_2)$.

Taking into account the m-mode of the purified states, we have $U^{(m)} |\psi_k^{(i)}\rangle_m (U_k^{(i)} \otimes (U_1 \otimes U_2 \otimes U_3)_{\neg i})^t_{\neg m} = |\phi_k^{(i)}\rangle_m$. Hence $U^{(m)} |\psi_k^{(i)}\rangle_m |\psi_k^{(i)}\rangle_m^{\dagger} U^{(m)\dagger} = |\phi_k^{(i)}\rangle_m |\phi_k^{(i)}\rangle_m^{\dagger}$. □

From the theorem 1, for $1 \leq m \leq 3$, there are two groups of matrices $L_m = \sum_{i=1, l \neq m}^{3} t^{(i)}$ in $M_{I_m}(\mathbb{C})$ corresponding to the LU equivalent states $|\psi\rangle$ and $|\phi\rangle$. They are simultaneously unitary equivalent. Denote

$$M_{\psi,1} = diag(M_{\psi,1}^{2,1}, ..., M_{\psi,1}^{2,t^{(2)}}, M_{\psi,1}^{3,1}, ..., M_{\psi,1}^{3,t^{(3)}}),$$
$$M_{\psi,2} = diag(M_{\psi,2}^{1,1}, ..., M_{\psi,2}^{1,t^{(1)}}, M_{\psi,2}^{3,1}, ..., M_{\psi,2}^{3,t^{(3)}}),$$
$$M_{\psi,3} = diag(M_{\psi,3}^{1,1}, ..., M_{\psi,3}^{1,t^{(1)}}, M_{\psi,3}^{2,1}, ..., M_{\psi,3}^{2,t^{(2)}}).$$

Let H_m^M be the direct group of $M_{L_m \times I_m}(\mathbb{C})$ defined by $\{r_k = I_m\}_{k=1}^{L_m}$ and $E_1 = \{1, 2, ..., L_m\}$, and H_m be the set of the I_m-th sequential principal minors of the matrices in H_m^M. Obviously, H_m is a direct group of $M_{I_m}(\mathbb{C})$. Then the matrices in H_m^M are just L_m copies of the matrices in H_m. That is, the matrices in H_m^M are just of the form $diag(\underbrace{H_m, ..., H_m}_{L_m})$.

For a given matrix $M_{\psi,m}$, by the algorithm in [14], there is a matrix $U_{\psi,m}^{\widetilde{\psi},M}$ which transfers $M_{\psi,m}$ into its canonical form $M_{\psi,m}^0$ with invariant subgroup $\widetilde{H}_m^M = diag(\underbrace{\widetilde{H}_m, ..., \widetilde{H}_m}_{L_m})$. Suppose

$$U_{\psi,m}^{\widetilde{\psi},M} = diag(\underbrace{U_{\psi,m}^{\widetilde{\psi},m}, ..., U_{\psi,m}^{\widetilde{\psi},m}}_{L_m}) \in H_m^M$$

with $U_{\psi,m}^{\widetilde{\psi},m}$ in the I_m-th sequential principal minor. By calculating all the three unitary matrices $\{U_{\psi,m}^{\widetilde{\psi},m};\ m = 1, 2, 3\}$, one gets a state $|\widetilde{\psi}\rangle$ defined by

$$|\widetilde{\psi}\rangle = U_{\psi,1}^{\widetilde{\psi},1} \otimes U_{\psi,2}^{\widetilde{\psi},2} \otimes U_{\psi,3}^{\widetilde{\psi},3}\,|\psi\rangle.$$

$|\widetilde{\psi}\rangle$ is called a reduced form of $|\psi\rangle$.

Theorem 2. Let $|\psi\rangle$ and $|\phi\rangle$ be two HOSVD states. $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent if and only if $|\widetilde{\psi}\rangle$ can be transformed to $|\widetilde{\phi}\rangle$ under $\widetilde{H}_1 \otimes \widetilde{H}_2 \otimes \widetilde{H}_3$.

[Proof] If $U_{\psi,1}^{\phi,1} \otimes U_{\psi,2}^{\phi,2} \otimes U_{\psi,3}^{\phi,3}|\psi\rangle = |\phi\rangle$, by the definitions of $|\widetilde{\psi}\rangle$ and $|\widetilde{\phi}\rangle$, we have

$$(U_{\phi,1}^{\widetilde{\phi},1} U_{\psi,1}^{\phi,1} {U_{\psi,1}^{\widetilde{\psi},1}}^\dagger) \otimes (U_{\phi,2}^{\widetilde{\phi},2} U_{\psi,2}^{\phi,2} {U_{\psi,2}^{\widetilde{\psi},2}}^\dagger) \otimes (U_{\phi,3}^{\widetilde{\phi},3} U_{\psi,3}^{\phi,3} {U_{\psi,3}^{\widetilde{\psi},3}}^\dagger)|\widetilde{\psi}\rangle = |\widetilde{\phi}\rangle. \tag{6}$$

With respect to the four states $|\psi\rangle$, $|\phi\rangle$, $|\widetilde{\psi}\rangle$ and $|\widetilde{\phi}\rangle$, we can construct $M_{\psi,m}$, $M_{\phi,m}$, $M_{\widetilde{\psi},m}$, $M_{\widetilde{\phi},m}$, respectively. Then we have the following commutative diagram,

$$\begin{tikzcd}
M_{\psi,m} \arrow[r, "U_{\psi,m}^{\phi,M}"] \arrow[d, "U_{\psi,m}^{\widetilde{\psi},M}"'] & M_{\phi,m} \arrow[d, "U_{\phi,m}^{\widetilde{\phi},M}"] \\
M_{\widetilde{\psi},m} \arrow[r, "U_{\widetilde{\psi},m}^{\widetilde{\phi},M}"'] & M_{\widetilde{\phi},m}
\end{tikzcd}$$

where

$$U_{\widetilde{\psi},m}^{\widetilde{\phi},M} = diag(\underbrace{U_{\phi,m}^{\widetilde{\phi},m} U_{\psi,m}^{\phi,m} {U_{\psi,m}^{\widetilde{\psi},m}}^\dagger, ..., U_{\phi,m}^{\widetilde{\phi},m} U_{\psi,m}^{\phi,m} {U_{\psi,m}^{\widetilde{\psi},m}}^\dagger}_{L_m}).$$

The canonical forms $M_{\psi,m}^0$, $M_{\phi,m}^0$ of $M_{\psi,m}$, $M_{\phi,m}$ are just $M_{\widetilde{\psi},m}, M_{\widetilde{\phi},m}$. Namely $U_{\widetilde{\psi},m}^{\widetilde{\phi},M}$ transforms $M_{\psi,m}^0$ to $M_{\phi,m}^0$.

By lemma 1, we have

$$U_{\widetilde{\psi},m}^{\widetilde{\phi},M} \in \widetilde{H}_m^M.$$

That is,

$$(U_{\phi,m}^{\widetilde{\phi},m} U_{\psi,m}^{\phi,m} {U_{\psi,m}^{\widetilde{\psi},m}}^\dagger) \in \widetilde{H}_m, m = 1, 2, 3.$$

From equation (6), $|\widetilde{\psi}\rangle$ can be transformed to $|\widetilde{\phi}\rangle$ under $\widetilde{H}_1 \otimes \widetilde{H}_2 \otimes \widetilde{H}_3$.

Conversely, if $|\widetilde{\psi}\rangle$ can be transformed to $|\widetilde{\phi}\rangle$ under $\widetilde{H}_1 \otimes \widetilde{H}_2 \otimes \widetilde{H}_3$, it is straightforward to see that $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent. □

Remark: Since $|\psi\rangle$ and $|\phi\rangle$ are HOSVD states, we can choose $H_m = S^{(m)}$. Then the unitary transformations that transform $M_{\psi,m}$ to $M_{\phi,m}$ must lie in $H_m^M = diag(S^{(m)}, ..., S^{(m)})$. Clearly, $\widetilde{H}_m \subseteq H_m$. From this point of view, our method can make the symmetry group into smaller blocks than the HOSVD decomposition technique.

Example 1. Let us consider the state $|\psi\rangle = \sqrt{\frac{1}{6}}|111\rangle + \sqrt{\frac{1}{4}}|123\rangle + \sqrt{\frac{1}{12}}|132\rangle + \sqrt{\frac{1}{8}}|212\rangle + \sqrt{\frac{1}{24}}|221\rangle + \sqrt{\frac{1}{3}}|233\rangle$ in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

By the method in [9], one gets

$$|\psi\rangle_1\langle\psi| = \begin{bmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{bmatrix}, \quad |\psi\rangle_2\langle\psi| = \begin{bmatrix} \frac{7}{24} & & \\ & \frac{7}{24} & \\ & & \frac{5}{12} \end{bmatrix}, \quad |\psi\rangle_3\langle\psi| = \begin{bmatrix} \frac{5}{24} & & \\ & \frac{5}{24} & \\ & & \frac{7}{12} \end{bmatrix}.$$

Hence $|\psi\rangle$ itself is a HOSVD state. From the HOSVD decomposition, $S^{(1)} \otimes S^{(2)} \otimes S^{(3)}$ has the following form

$$U(2) \otimes \begin{bmatrix} U(2) & \\ & e^{i\theta_1} \end{bmatrix} \otimes \begin{bmatrix} U(2) & \\ & e^{i\theta_2} \end{bmatrix}.$$

The LU equivalent problem of $|\psi\rangle$ and another state $|\phi\rangle$ is then reduced to judge if there is a solution in $S^{(1)} \otimes S^{(2)} \otimes S^{(3)}$ which transforms $|\psi\rangle$ to a core state of $|\phi\rangle$.

Nevertheless, by our method, we only need to calculate the three matrices $M_{\psi,1}$, $M_{\psi,2}$ and $M_{\psi,3}$ defined above,

$$M_{\psi,1} = diag\left[\begin{bmatrix} \frac{5}{12} & \\ & \frac{1}{6} \end{bmatrix}, \begin{bmatrix} \frac{1}{12} & \\ & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & \\ & \frac{1}{6} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & \\ & \frac{1}{3} \end{bmatrix}\right],$$

$$M_{\psi,2} = diag\left[\begin{bmatrix} \frac{7}{24} & & \\ & \frac{7}{24} & \\ & & \frac{5}{12} \end{bmatrix}, \begin{bmatrix} \frac{7}{24} & & \\ & \frac{1}{24} & \\ & & \frac{1}{12} \end{bmatrix}, \begin{bmatrix} 0 & & \\ & \frac{1}{4} & \\ & & \frac{1}{3} \end{bmatrix}\right],$$

$$M_{\psi,3} = diag\left[\begin{bmatrix} \frac{5}{24} & & \\ & \frac{5}{24} & \\ & & \frac{7}{12} \end{bmatrix}, \begin{bmatrix} \frac{5}{24} & & \\ & \frac{1}{8} & \\ & & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} 0 & & \\ & \frac{1}{12} & \\ & & \frac{1}{3} \end{bmatrix}\right].$$

Actually all these three matrices are the canonical forms of themselves under their direct groups H_1^M, H_2^M and H_3^M. The direct group $\widetilde{H}_1 \otimes \widetilde{H}_2 \otimes \widetilde{H}_3$ of the canonical form $|\widetilde{\psi}\rangle$ has the following form

$$\begin{bmatrix} e^{i\theta_1} & \\ & e^{i\theta_2} \end{bmatrix} \otimes \begin{bmatrix} e^{i\theta_3} & & \\ & e^{i\theta_4} & \\ & & e^{i\theta_5} \end{bmatrix} \otimes \begin{bmatrix} e^{i\theta_6} & & \\ & e^{i\theta_7} & \\ & & e^{i\theta_8} \end{bmatrix}.$$

Hence the problem to decide whether $|\psi\rangle$ is LU equivalent to another state $|\phi\rangle$ or not can be reduced to judge if there is a solution in $\widetilde{H}_1 \otimes \widetilde{H}_2 \otimes \widetilde{H}_3$ that transforms $|\widetilde{\psi}\rangle$ to $|\widetilde{\phi}\rangle$, which is an easier problem.

Example 2. Let us consider another example, $|\psi\rangle = \sqrt{\frac{2}{15}}|113\rangle + \sqrt{\frac{1}{6}}|121\rangle + \sqrt{\frac{1}{15}}|132\rangle + \sqrt{\frac{1}{5}}|212\rangle + \sqrt{\frac{1}{15}}|223\rangle + \sqrt{\frac{1}{10}}|231\rangle + \sqrt{\frac{1}{15}}|311\rangle + \sqrt{\frac{1}{15}}|323\rangle + \sqrt{\frac{2}{15}}|333\rangle$, a state in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$.

In this case, one has

$$|\psi\rangle_1\langle\psi| = \begin{bmatrix} \frac{11}{30} & & \\ & \frac{11}{30} & \\ & & \frac{4}{15} \end{bmatrix}, \quad |\psi\rangle_2\langle\psi| = \begin{bmatrix} \frac{2}{5} & & \\ & \frac{3}{10} & \\ & & \frac{3}{10} \end{bmatrix}, \quad |\psi\rangle_3\langle\psi| = \begin{bmatrix} \frac{1}{3} & & \\ & \frac{1}{3} & \\ & & \frac{1}{3} \end{bmatrix}.$$

We can see that $|\psi\rangle$ itself is a HOSVD state. By the HOSVD decomposition, $S^{(1)} \otimes S^{(2)} \otimes S^{(3)}$ has the following form

$$\begin{bmatrix} U(2) & \\ & e^{i\theta_1} \end{bmatrix} \otimes \begin{bmatrix} e^{i\theta_2} & \\ & U(2) \end{bmatrix} \otimes U(3).$$

That is, if one uses the approach in [9], the LU equivalent problem of $|\psi\rangle$ and another state $|\phi\rangle$ reduces to whether there is a solution in $S^{(1)} \otimes S^{(2)} \otimes S^{(3)}$ that transforms $|\psi\rangle$ to the core state of $|\phi\rangle$. But this is also a complicated problem.

Now, in terms of our approach, we have the three matrices $M_{\psi,1}$, $M_{\psi,2}$ and $M_{\psi,3}$,

$$M_{\psi,1} = diag\left[\begin{bmatrix} \frac{11}{30} & & \\ & \frac{11}{30} & \\ & & \frac{4}{15} \end{bmatrix}, \begin{bmatrix} \frac{2}{15} & & \\ & \frac{1}{5} & \\ & & \frac{1}{15} \end{bmatrix}, \begin{bmatrix} \frac{7}{30} & & \\ & \frac{1}{6} & \\ & & \frac{1}{5} \end{bmatrix}\right],$$

$$M_{\psi,2} = diag\left[\begin{bmatrix} \frac{2}{5} & & \\ & \frac{3}{10} & \\ & & \frac{3}{10} \end{bmatrix}, \begin{bmatrix} \frac{1}{15} & & \\ & \frac{1}{15} & \\ & & \frac{2}{15} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & & \\ & \frac{7}{30} & \\ & & \frac{1}{6} \end{bmatrix}\right],$$

$$M_{\psi,3} = diag\left[\begin{bmatrix} \frac{4}{15} & & \\ & \frac{4}{15} & \\ & & \frac{1}{5} \end{bmatrix}, \begin{bmatrix} \frac{1}{15} & & \\ & \frac{1}{15} & \\ & & \frac{2}{15} \end{bmatrix}, \begin{bmatrix} \frac{1}{15} & & \\ & \frac{1}{5} & \\ & & \frac{2}{15} \end{bmatrix}, \begin{bmatrix} \frac{4}{15} & & \\ & \frac{2}{15} & \\ & & \frac{1}{5} \end{bmatrix}\right].$$

Clearly, these three matrices are the canonical forms of themselves under their direct groups H_1^M, H_2^M and H_3^M. The direct group $\widetilde{H}_1 \otimes \widetilde{H}_2 \otimes \widetilde{H}_3$ of the canonical form $|\widetilde{\psi}\rangle$ has the following form

$$\begin{bmatrix} e^{i\theta_1} & & \\ & e^{i\theta_2} & \\ & & e^{i\theta_3} \end{bmatrix} \otimes \begin{bmatrix} e^{i\theta_4} & & \\ & e^{i\theta_5} & \\ & & e^{i\theta_6} \end{bmatrix} \otimes \begin{bmatrix} e^{i\theta_7} & & \\ & e^{i\theta_8} & \\ & & e^{i\theta_9} \end{bmatrix}.$$

Hence the problem to decide whether $|\psi\rangle$ is LU equivalent to another state $|\phi\rangle$ or not can be reduced to whether there is a solution in $\widetilde{H}_1 \otimes \widetilde{H}_2 \otimes \widetilde{H}_3$ that transforms $|\widetilde{\psi}\rangle$ to $|\widetilde{\phi}\rangle$.

From the above two examples, it can be seen that our method is more effective than the one in [9].

IV. MULTIPARTITE PURE STATES

Our method can be generalized to the case of multipartite pure states. Given an N-partite pure HOSVD state

$$|\psi\rangle \in \mathbb{C}^{I_1} \otimes \mathbb{C}^{I_2} \otimes \ldots \otimes \mathbb{C}^{I_N},$$

we can write

$$|\psi\rangle = \sum_{k=1}^{t^{(i)}} \sum_{l=1}^{\mu_k^{(i)}} \lambda_k^{(i)} |n_l^{(i),k}\rangle_i \otimes |v_l^{(i),k}\rangle_{\neg i},$$

where $i = 1, 2, ..., N$, $n_l^{(i),k} = l + \sum_{s=1}^{k-1} \mu_s^{(i)}$ and $|v_l^{(i),k}\rangle_{\neg i}$ is just the normal vector when we collect the $n_l^{(i),k}$-th term of the i-th system of $|\psi\rangle$. For each $1 \le i \le N$, $|\psi\rangle$ can be looked as the purification of the mixed N-1 partite state

$$\rho^{(i)} = \sum_{k=1}^{t^{(i)}} {\lambda_k^{(i)}}^2 \sum_{l=1}^{\mu_k^{(i)}} |v_l^{(i),k}\rangle_{\neg i \neg i}\langle v_l^{(i),k}|.$$

Denote

$$\rho_k^{(i)} = {\lambda_k^{(i)}}^2 \sum_{l=1}^{\mu_k^{(i)}} |v_l^{(i),k}\rangle_{\neg i \neg i}\langle v_l^{(i),k}|.$$

Purifying $\rho_k^{(i)}$, we get an N-partite pure state

$$|\psi^{(i),k}\rangle = \sum_{l=1}^{\mu_k^{(i)}} |n_l^{(i),k}\rangle_i \otimes |v_l^{(i),k}\rangle_{\neg i}.$$

From the m-mode of $|\psi^{(i),k}\rangle$, we have the following matrices

$$M_{\psi,m}^{i,k} = |\psi^{(i),k}\rangle_m \langle \psi^{(i),k}|, \quad 1 \le m \ne i \le N, \quad k = 1, ..., t^{(i)}.$$

These matrices give rise to

$$M_{\psi,m} = diag(M_{\psi,m}^{1,1}, ..., M_{\psi,m}^{1,t^{(1)}}, ..., \overbrace{M_{\psi,m}^{m,1}, ..., M_{\psi,m}^{m,t^{(m)}}}, ..., M_{\psi,m}^{N,1}, ..., M_{\psi,m}^{N,t^{(N)}}),$$

where the $\overbrace{...}$ means that the matrices under the cap are absent from expression.

Let H_m^M be the direct group of $M_{L_m \times I_m}(\mathbb{C})$ defined by $\{r_k = I_m\}_{k=1}^{L_m}$ and $E_1 = \{1, 2, ..., L_m\}$. Given a matrix $M_{\psi,m}$, by the algorithm in [14], we have a matrix $U_{\psi,m}^{\widetilde{\psi},M}$ transforming $M_{\psi,m}$ to its canonical form $M_{\psi,m}^0$ with invariant subgroup $\widetilde{H}_m^M = diag(\underbrace{\widetilde{H}_m, ..., \widetilde{H}_m}_{L_m})$. Suppose

$$U_{\psi,m}^{\widetilde{\psi},M} = diag(\underbrace{U_{\psi,m}^{\widetilde{\psi},m}, ..., U_{\psi,m}^{\widetilde{\psi},m}}_{L_m}) \in H_m^M$$

with $U_{\psi,m}^{\widetilde{\psi},m}$ in the I_m-th sequential principal minor.

A state

$$|\widetilde{\psi}\rangle \doteq U_{\psi,1}^{\widetilde{\psi},1} \otimes U_{\psi,2}^{\widetilde{\psi},2} \otimes ... \otimes U_{\psi,N}^{\widetilde{\psi},N} |\psi\rangle.$$

is called a reduced form of $|\psi\rangle$, where all the unitary matrices $\{U_{\psi,m}^{\widetilde{\psi},m}, 1 \leq m \leq N\}$ can be calculated explicitly.

Theorem 3. Let $|\psi\rangle$ and $|\phi\rangle$ be two HOSVD states. Then $|\psi\rangle$ and $|\phi\rangle$ are LU equivalent if and only if $|\widetilde{\psi}\rangle$ can be transformed to $|\widetilde{\phi}\rangle$ under $\widetilde{H}_1 \otimes \widetilde{H}_2 \otimes ... \otimes \widetilde{H}_N$.

The proof is of theorem 3 is similar to that of theorem 2.

V. CONCLUSION

We have considered the problem of LU equivalence for tripartite and multipartite pure states. After analyzing the necessary conditions of the LU equivalence, we have obtained the conclusion of the simultaneously unitary equivalence of two order sets of matrices. A reduced form for each quantum state has been derived by virtue of the algorithm for dealing with the unitary equivalence under direct groups. Accordingly, the LU equivalence problem can be reduced into a simpler one. Our method is more efficient than that in [9]. Hence the algorithm for unitary equivalence under direct group is a more efficient way to deal with the LU equivalent problem.

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