Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig

Quantum Separability Criteria for Arbitrary Dimensional Multipartite States

Ming Li[1,2], Jing Wang[1,3], Shao-Ming Fei[2,3] and Xianqing Li-Jost[2]

[1] College of the Science, China University of Petroleum, 266580 Qingdao

[2] Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig

[3] School of Mathematical Sciences, Capital Normal University, 100048 Beijing

Abstract

We present new separability criteria for both bipartite and multipartite quantum states. These criteria include the criteria based on the correlation matrix and its generalized form as special cases. We show by detailed examples that our criteria are more powerful than the positive partial transposition criterion, the realignment criterion and the criteria based on the correlation matrices.

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I. INTRODUCTION

Quantum entanglement, as the remarkable nonlocal feature of quantum mechanics, is recognized as a valuable resource in the rapidly expanding field of quantum information science, with various applications such as quantum computation [1, 2], quantum teleportation [3], dense coding [4], quantum cryptographic schemes [5], quantum radar [6], entanglement swapping [7] and remote state preparation (RSP) [8–11]. Quantum states without entanglement are called separable states, which constitute a convex subset of all the quantum states. Distinguishing quantum entangled states from the separable ones is a basic and longer standing problem in the theory of quantum entanglement. It has attracted great interest in the last twenty years.

For mixed states we still have no general criterion. A strong criterion, named PPT (partial positive transposition), to recognize mixed entangled quantum state was proposed by Peres in 1996 in [12]. It says that for any bipartite separable quantum states the density matrix must be semi-positive under partial transposition. Afterwards, by using the method of positive maps the family Horodecki [13] showed that the Peres' criterion is also sufficient

for 2×2 and 2×3 bipartite systems. For high-dimensional states, the PPT criterion is only necessary. Horodecki [14] has constructed some classes of families of inseparable states with positive partial transposes for 3×3 and 2×4 systems. States of this kind are said to be bound entangled (BE). Another powerful operational criterion for separability is the realignment criterion [15, 16]. It demonstrates a remarkable ability in detecting the entanglement of many bound entangled states and even genuinely tripartite entanglement [17]. Considerable efforts have been made in proposing stronger variants and multipartite generalizations for this criterion [18, 19]. It was shown that PPT criterion and realignment criterion are equivalent to the permutations of the density matrix's indices [17].

Recently, some more elegant results for the separability problem have been derived. In [20–22], a separability criteria based on the local uncertainty relations (LUR) was obtained. The authors show that for any separable state $\rho \in \mathcal{H}_A \otimes \mathcal{H}_B$,

$$1 - \sum_k \langle G_k^A \otimes G_k^B \rangle - \frac{1}{2} \langle G_k^A \otimes I - I \otimes G_k^B \rangle^2 \geq 0,$$

where G_k^A or G_k^B are arbitary local orthogonal and normalized operators (LOOs) in $\mathcal{H}_A \otimes \mathcal{H}_B$. This criterion is strictly stronger than the realignment criterion. Thus more bound entangled quantum states can be recognized by the LUR criterion. The criterion is optimized in [23] by choosing the optimal LOOs. The covariance matrix of a quantum state is also used to study separability in [24]. It has been pointed out in [25] that the LUR criterion, including the optimized one, can be derived from the covariance matrix criterion. In [26] the author has given a criterion based on the correlation matrix of a state. The correlation matrix (CM) criterion is then shown to be independent of PPT and realignment criterion in [27], i.e. there exist quantum states that can be recognized by the correlation criterion while the PPT, realignment criterion and the covariance matrix criterion fail. In [28], by defining matricizations of the correlation tensors, the authors introduced a general framework for detecting genuine multipartite entanglement and non-fully separability in multipartite quantum systems.

In this paper, we present a generalized form of the correlation matrix criterion for bipartite quantum systems [26, 27] and for multipartite quantum systems [29]. Our new criterion includes the criterion based on the correlation matrix as a special case and is more powerful than the later for detecting entanglement, as shown by detailed examples. Thus our criterion will be more efficient than the Positive partial transposition criterion, the realignment criterion and the covariance matrix criterion for some quantum states.

II. SEPARABILITY CRITERION FOR BIPARTITE QUANTUM STATES

Let $H_A^{d_1}$ and $H_B^{d_2}$ be two vector spaces with dimensions d_1 and d_2 respectively. By using the generators of $SU(d)$, λ_i, $i = 1, 2, ..., d^2 - 1$, any quantum state $\rho \in H_A^{d_1} \otimes H_B^{d_2}$ can be writing as:

$$\rho = \frac{1}{d_1 d_2} I \otimes I + \sum_{k=1}^{d_1^2-1} r_k \lambda_k \otimes I + \sum_{l=1}^{d_2^2-1} s_l I \otimes \lambda_l + \sum_{k=1}^{d_1^2-1} \sum_{l=1}^{d_2^2-1} t_{kl} \lambda_k \otimes \lambda_l, \tag{1}$$

where $r_k = \frac{1}{2d_2} Tr(\rho \lambda_k \otimes I)$, $s_l = \frac{1}{2d_1} Tr(\rho I \otimes \lambda_l)$ and $t_{kl} = \frac{1}{4} Tr(\rho \lambda_k \otimes \lambda_l)$. We denote T the matrix with entries t_{kl} and define

$$\tilde{T} = \begin{pmatrix} \frac{1}{d_1 d_2} & s_1 & s_2 & \cdots & s_{d_2^2-1} \\ r_1 & t_{11} & t_{12} & \cdots & t_{1(d_2^2-1)} \\ r_2 & t_{21} & t_{22} & \cdots & t_{2(d_2^2-1)} \\ \cdots & & & & \\ r_{d_1^2-1} & t_{(d_1^2-1)1} & t_{(d_1^2-1)2} & \cdots & t_{(d_1^2-1)(d_2^2-1)} \end{pmatrix}. \tag{2}$$

Theorem 1: If $\rho \in H_A^{d_1} \otimes H_B^{d_2}$ is separable, then for any $d_1^2 \otimes d_2^2$ matrix M and $(d_1^2 - 1) \otimes (d_2^2 - 1)$ matrix N with real entries m_{ij} and n_{ij} respectively,

$$|\sum_{kl} m_{kl} \widetilde{T}_{kl}| \le \frac{\sqrt{(d_1^2 - d_1 + 2)(d_2^2 - d_2 + 2)}}{2d_1 d_2} \sigma_{max}(M) \tag{3}$$

$$|\sum_{kl} n_{kl} t_{kl}| \le \sqrt{\frac{(d_1 - 1)(d_2 - 1)}{4d_1 d_2}} \sigma_{max}(N), \tag{4}$$

where $\sigma_{max}(M)$ and $\sigma_{max}(N)$ are the maximal singular values of M and N respectively.

Proof: A separable quantum state ρ can be expressed as:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i|. \tag{5}$$

By writing the pure states $|\psi_i\rangle$ and $|\phi_i\rangle$ in their Bloch forms, we have that

$$\begin{aligned} \rho &= \sum_i p_i |\psi_i\rangle\langle\psi_i| \otimes |\phi_i\rangle\langle\phi_i| \\ &= \sum_i p_i (\frac{1}{d_1} I + \sum_k x_{ik} \lambda_k) \otimes (\frac{1}{d_2} I + \sum_l y_{il} \lambda_l) \\ &= \frac{1}{d_1 d_2} I \otimes I + \frac{1}{d_2} \sum_i p_i \sum_k x_{ik} \lambda_k \otimes I + \frac{1}{d_1} \sum_i p_i \sum_l y_{il} I \otimes \lambda_k \\ &\quad + \sum_i p_i \sum_{kl} x_{ik} y_{il} \lambda_k \otimes \lambda_l. \end{aligned} \tag{6}$$

Comparing (1) with (6), we have

$$r_k = \frac{1}{d_2} \sum_i p_i x_{ik}, \quad s_l = \frac{1}{d_1} \sum_i p_i y_{il}, \quad t_{kl} = \sum_i p_i \sum_{kl} x_{ik} y_{il}. \tag{7}$$

Define $\vec{\tilde{x}}_i = (\frac{1}{d_1}, x_{i1}, \cdots, x_{i(d_1^2-1)})^t$ and $\vec{\tilde{y}}_i = (\frac{1}{d_2}, y_{i1}, \cdots, y_{i(d_2^2-1)})^t$, where t stands for the transposition. Since $|\psi_i\rangle \in H_A^{d_1}$ and $|\phi_i\rangle \in H_B^{d_2}$ are all pure states, one has

$$Tr(|\psi_i\rangle\langle\psi_i|)^2 = Tr(\frac{1}{d_1}I + \sum_k x_{ik}\lambda_k)^2 = \frac{1}{d_1} + 2\sum_k x_{ik}^2 = 1, \tag{8}$$

i.e. $||\vec{x}_i|| = \sqrt{\sum_k x_{ik}^2} = \sqrt{\frac{d_1-1}{2d_1}}$. Hence $||\vec{\tilde{x}}_i|| = \sqrt{\frac{d_1^2-d_1+2}{2d_1^2}}$. Similarly we have $||\vec{\tilde{y}}_i|| = \sqrt{\frac{d_2^2-d_2+2}{2d_2^2}}$. Therefore for any real matrices M and N, one obtains that

$$|\sum_{kl} m_{kl}\widetilde{T}_{kl}| = |\sum_{ikl} p_i m_{kl}\tilde{x}_{ik}\tilde{y}_{il}| \le \sum_i p_i|\langle\vec{\tilde{x}}_i, M\vec{\tilde{y}}_i\rangle| \le \frac{\sqrt{(d_1^2-d_1+2)(d_2^2-d_2+2)}}{2d_1d_2}\sigma_{max}(M);$$

$$|\sum_{kl} n_{kl}t_{kl}| = |\sum_{ikl} p_i n_{kl}x_{ik}y_{il}| \le \sum_i p_i|\langle\vec{x}_i, N\vec{y}_i\rangle| \le \sqrt{\frac{(d_1-1)(d_2-1)}{4d_1d_2}}\sigma_{max}(N).$$

∎

The correlation matrix criterion in [26] illustrates that if quantum state ρ is separable, then the Key-Fan norm $||T||_{KF} \le \sqrt{\frac{(d_1-1)(d_2-1)}{4d_1d_2}}$. In the following we show the power of Theorem 1 in detecting entanglement by two corollaries.

Corollary 1: The criterion based on the correlation matrix is included in Theorem 1.

Proof: Let $T = U\Sigma V^\dagger$ be the singular value decomposition of T. Since T is a real matrix, one can always choose U and V to be orthogonal matrices. Without loss of generality, we assume that $d_1 \le d_2$. Set $N = (V\Delta U^\dagger)^t$, where Δ is a block matrix of the form $\begin{pmatrix} I & 0 \end{pmatrix}^t$, I is the $(d_1^2-1) \times (d_1^2-1)$ identity matrix, 0 stands for a $(d_2^2-d_1^2) \times (d_2^2-d_1^2)$ zero matrix. The singular values of N must be either 1 or 0. One obtains

$$||T||_{KF} = |Tr(U\Sigma V^\dagger V\Delta U^\dagger)| = |Tr(TN^t)| = |\sum_{kl} n_{kl}t_{kl}|$$

$$\le \sqrt{\frac{(d_1-1)(d_2-1)}{4d_1d_2}}\sigma_{max}(N) = \sqrt{\frac{(d_1-1)(d_2-1)}{4d_1d_2}}.$$

This means that one can get the correlation matrix criterion from Theorem 1. ∎

Corollary 2: If a bipartite quantum state $\rho \in H_A^{d_1} \otimes H_B^{d_2}$ is separable, then the following inequality must hold:

$$||\tilde{T}||_{KF} \le \frac{\sqrt{(d_1^2-d_1+2)(d_2^2-d_2+2)}}{2d_1d_2}, \tag{9}$$

where $||\Omega||_{KF} = Tr\sqrt{\Omega\Omega^\dagger}$ stands for the trace norm of Ω.

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Proof: Assume $d_1 \leq d_2$. Let $\tilde{T} = X\Sigma Y^\dagger$ be the singular value decomposition of $\tilde{T}$, with X and Y the corresponding orthogonal matrices. Set $M = (Y\Gamma X^\dagger)^t$, where $\Gamma = \begin{pmatrix} I & 0 \end{pmatrix}^t$, I and 0 are the $d_1^2 \times d_1^2$ identity matrix and the $(d_2^2 - d_1^2) \times (d_2^2 - d_1^2)$ zero matrix, respectively. The singular values of M are either 1 or 0. Then we obtain that

$$
\begin{aligned}
||\tilde{T}||_{KF} &= |Tr(X\Sigma Y^\dagger Y\Gamma X^\dagger)| = |Tr(\tilde{T}M^t)| = |\sum_{kl} m_{kl}\tilde{T}_{kl}| \\
&\leq \frac{\sqrt{(d_1^2 - d_1 + 2)(d_2^2 - d_2 + 2)}}{2d_1 d_2}\sigma_{max}(M) = \frac{\sqrt{(d_1^2 - d_1 + 2)(d_2^2 - d_2 + 2)}}{2d_1 d_2},
\end{aligned}
$$

which ends the proof of the corollary. ∎

Corollary 1 shows that Theorem 1 is not weaker than the correlation matrix criterion in detecting entanglement for quantum states in $H_A^{d_1} \otimes H_B^{d_2}$. In fact, by the following example we can show that Theorem 1 is strictly stronger than the correlation matrix criterion, the realignment criterion and the PPT criterion.

Example: A 3×3 PPT entangled state is given in [30]:

$$
\rho = \frac{1}{4}(I_9 - \sum_{i=0}^{4} |\psi_i\rangle\langle\psi_i|), \tag{10}
$$

where $|\psi_0\rangle = |0\rangle(|0\rangle - |1\rangle)/\sqrt{2}$, $|\psi_1\rangle = (|0\rangle - |1\rangle)|2\rangle/\sqrt{2}$, $|\psi_2\rangle = |2\rangle(|1\rangle - |2\rangle)/\sqrt{2}$, $|\psi_3\rangle = (|1\rangle - |2\rangle)|0\rangle/\sqrt{2}$ and $|\psi_4\rangle = (|0\rangle + |1\rangle + |2\rangle)(|0\rangle + |1\rangle + |2\rangle)/3$. The state is shown to violate the correlation matrix criterion. Let us mix ρ with white noise:

$$
\sigma(x) = x\rho + \frac{1-x}{9}I_9. \tag{11}
$$

The correlation matrix criterion detects the entanglement for $0.9493 < x \leq 1$. If we choose the matrix M in theorem 1 to be

$$
\begin{pmatrix}
0.8134 & 0.1905 & -0.11 & 0.18 & -0.4067 & 0.1798 & 0 & 0 & 0 \\
0.1905 & 0.3849 & -0.243 & -0.806 & 0.2608 & -0.0989 & 0 & 0 & 0 \\
-0.11 & -0.243 & 0.1043 & -0.3511 & -0.1506 & 0.8736 & 0 & 0 & 0 \\
0.1798 & -0.0989 & 0.8736 & -0.3258 & -0.1634 & -0.2898 & 0 & 0 & 0 \\
-0.4067 & 0.2608 & -0.1506 & -0.1634 & -0.867 & -0.1634 & 0 & 0 & 0 \\
0.1798 & -0.806 & -0.3511 & -0.2898 & -0.1634 & -0.3258 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.964 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.964 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.964
\end{pmatrix},
$$

which has the maximal singular value 1.036. From (3) the state $\sigma(x)$ is entangled for $0.94 < x \leq 1$. Furthermore, by corollary 2 one can show that $\sigma(x)$ is entangled for $0.89254 < x \leq 1$. Here one finds that our criterion is much better than the correlation matrix criterion.

III. SEPARABILITY CRITERION FOR MULTIPARTITE QUANTUM STATES

In this section we consider the separability problem for N-partite quantum systems $H_1 \otimes H_2 \otimes \cdots \otimes H_N$ with $dim \, H_i = d_i$, $i = 1, 2, \cdots, N$.

Let $\lambda_{\alpha_k}^{\{\mu_k\}} = I_{d_1} \otimes I_{d_2} \otimes \cdots \otimes \lambda_{\alpha_k} \otimes I_{d_{\mu_k+1}} \otimes \cdots \otimes I_{d_N}$ with λ_{α_k}, the generators of $SU(d_i)$, appearing at the μ_kth position and

$$\mathcal{T}_{\alpha_1 \alpha_2 \cdots \alpha_M}^{\{\mu_1 \mu_2 \cdots \mu_M\}} = \frac{\prod_{i=1}^{M} d_{\mu_i}}{2^M \prod_{i=1}^{N} d_i} \text{Tr}[\rho \lambda_{\alpha_1}^{\{\mu_1\}} \lambda_{\alpha_2}^{\{\mu_2\}} \cdots \lambda_{\alpha_M}^{\{\mu_M\}}],$$

which can be viewed as the entries of the tensors $\mathcal{T}^{\{\mu_1 \mu_2 \cdots \mu_M\}}$.

For $\alpha_M = \cdots = \alpha_N = 0$ with $1 \leq M \leq N$, we define that $\tilde{\mathcal{T}}_{\alpha_1 \alpha_2 \cdots \alpha_N} = \mathcal{T}_{\alpha_1 \cdots \alpha_M}^{\mu_1 \cdots \mu_M}$, and for $\alpha_1 = \cdots = \alpha_N = 0$, define that $\tilde{\mathcal{T}}_{\alpha_1 \cdots \alpha_N} = \frac{1}{\prod_{k=1}^{N} d_k}$. Hence we have a tensor $\tilde{\mathcal{T}}$ with elements $\{\tilde{\mathcal{T}}_{\alpha_1 \cdots \alpha_N}, \alpha_k = 0, 1, \cdots, d_k^2 - 1\}$.

If we set $\lambda_0^{\{k\}} = I_{d_k}$ for any $1 \leq k \leq N$, then any multipartite state $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ can be generally expressed by the tensor $\tilde{\mathcal{T}}$ as [29],

$$\rho = \sum_{\alpha_1 \alpha_2 \cdots \alpha_N} \tilde{\mathcal{T}}_{\alpha_1 \alpha_2 \cdots \alpha_N} \lambda_{\alpha_1}^{\{1\}} \lambda_{\alpha_2}^{\{2\}} \cdots \lambda_{\alpha_N}^{\{N\}}, \tag{12}$$

where the summation is taken for all $\alpha_k = 0, 1, \cdots, d_k^2 - 1$.

To obtain the criterion for N-partite quantum systems, we adopt the definition of the nth matrix unfolding $\mathcal{T}^n$ of a tensor $\mathcal{T}$, which is a matrix with i_n to be the row index and the rest subscripts of $\mathcal{T}$ to be column indices(detailed description can be found in Refs. [29, 31]). The Ky Fan norm of the tensor $\mathcal{T}$ over N matrix unfoldings is defined as

$$||\mathcal{T}||_{KF} = \max\{||\mathcal{T}_n||_{KF}\}, \quad n = 1, 2, \cdots, N. \tag{13}$$

Theorem 2: If a quantum state $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ is fully separable, then for any tensors M and W with real entries $m_{i_1 i_2 \cdots i_N}$, $i_k = 1, 2, \cdots, d_k^2 - 1$, and $w_{j_1 j_2 \cdots j_N}$, $j_l = 1, 2, \cdots, d_k^2$, we have:

$$|\sum_{i_1 i_2 \cdots i_N} m_{i_1 i_2 \cdots i_N} \mathcal{T}_{i_1 i_2 \cdots i_N}| \leq \Pi_{k=1}^{N} \sqrt{\frac{d_k - 1}{2 d_k}} \sigma_{max}(M), \tag{14}$$

$$|\sum_{j_1 j_2 \cdots j_N} w_{j_1 j_2 \cdots j_N} \tilde{\mathcal{T}}_{i_1 i_2 \cdots i_N}| \leq \Pi_{k=1}^{N} \sqrt{\frac{d_k^2 - d_k + 2}{2 d_k^2}} \sigma_{max}(W), \tag{15}$$

where $\sigma_{max}(M)$ and $\sigma_{max}(W)$ stand for the maximal eigenvalue of the matrix unfolding M_n and W_n. The maximum is taken over all kinds of mode n matricization.

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Proof: Assume that $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ is fully separable, one can always find the following decomposition:

$$\rho = \sum_i p_i |\psi_i^1\rangle\langle\psi_i^1| \otimes |\psi_i^2\rangle\langle\psi_i^2| \otimes \cdots \otimes |\psi_i^N\rangle\langle\psi_i^N|, \tag{16}$$

where $|\psi_i^m\rangle\langle\psi_i^m|$ are density matrices of pure states in H_m. Using the Bloch representation of density matrix, we have that

$$|\psi_i^m\rangle\langle\psi_i^m| = \frac{1}{d_m}I + \sum_{\alpha_m} x_{i\alpha_m}^m \lambda_{\alpha_m}, \tag{17}$$

where $x_{i\alpha_m}^m = Tr(|\psi_i^m\rangle\langle\psi_i^m|\lambda_{\alpha_m})/2$. By (8) one has that $||\vec{x}_i^m|| = \sqrt{\frac{d_m-1}{2d_m}}$. Denote $\tilde{\vec{x}}_i^m = (\frac{1}{d_m}, x_{i1}^m, \cdots, x_{i(d_1^2-1)}^m)^t$. We obtain that $||\tilde{\vec{x}}_i^m|| = \sqrt{\frac{d_m^2-d_m+2}{2d_m^2}}$. Substituting (17) into (16) one has that:

$$\begin{aligned}
\rho &= \frac{1}{\Pi_{k=1}^N d_k} \otimes_{k=1}^N I_k + \sum_{\mu_1\alpha_1} \frac{d_{\mu_1}}{\Pi_{k=1}^N} \sum_i p_i x_{i\alpha_1}^{\mu_1} \lambda_{\alpha_1}^{\mu_1} + \sum_{\mu_1\mu_2\alpha_1\alpha_2} \frac{d_{\mu_1}d_{\mu_2}}{\Pi_{k=1}^N} \sum_i p_i x_{i\alpha_1}^{\mu_1} x_{i\alpha_2}^{\mu_2} \lambda_{\alpha_1}^{\mu_1} \lambda_{\alpha_2}^{\mu_2} \\
&\quad + \cdots + \sum_{\mu_1\cdots\mu_M, \alpha_1\cdots\alpha_M} \frac{\Pi_{k=1}^M d_{\mu_k}}{\Pi_{k=1}^N} \sum_i p_i x_{i\alpha_1}^{\mu_1} \cdots x_{i\alpha_M}^{\mu_M} \lambda_{\alpha_1}^{\mu_1} \cdots \lambda_{\alpha_M}^{\mu_M} \\
&\quad + \sum_{\alpha_1\cdots\alpha_N} \sum_i p_i x_{i\alpha_1}^1 \cdots x_{i\alpha_N}^N \lambda_{\alpha_1}^1 \cdots \lambda_{\alpha_N}^N. \tag{18}
\end{aligned}$$

Comparing (12) and (18), one gets

$$\mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_M}^{\{\mu_1\mu_2\cdots\mu_M\}} = \frac{\Pi_{k=1}^M d_{\mu_k}}{\Pi_{k=1}^N} \sum_i p_i x_{i\alpha_1}^{\mu_1} \cdots x_{i\alpha_M}^{\mu_M}. \tag{19}$$

According to the definitions of $\vec{x}_i^m, \tilde{\vec{x}}_i^m$ and $\mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_N}, \tilde{\mathcal{T}}_{\alpha_1\alpha_2\cdots\alpha_N}$, we have that

$$\mathcal{T}_{\alpha_1\alpha_2\cdots\alpha_N} = \sum_i p_i x_{i\alpha_1}^1 \cdots x_{i\alpha_N}^N = \sum_i p_i \vec{x}_i^1 \circ \vec{x}_i^2 \circ \cdots \circ \vec{x}_i^N \tag{20}$$

$$\tilde{\mathcal{T}}_{\alpha_1\alpha_2\cdots\alpha_N} = \sum_i p_i \tilde{x}_{i\alpha_1}^1 \cdots \tilde{x}_{i\alpha_N}^N = \sum_i p_i \tilde{\vec{x}}_i^1 \circ \tilde{\vec{x}}_i^2 \circ \cdots \circ \tilde{\vec{x}}_i^N, \tag{21}$$

where $\circ$ stands for the out product.

Let M_n be mode n matricization of M. Then for any tensor M we have that

$$\sum_{i_1 i_2 \cdots i_N} m_{i_1 i_2 \cdots i_N} \mathcal{T}_{i_1 i_2 \cdots i_N} = \sum_i p_i \langle \vec{x}_i^n, M_n(\vec{x}_i^1 \circ \cdots \circ \vec{x}_i^{\hat{n}} \circ \cdots \circ \vec{x}_i^N)^t \rangle \le \Pi_{k=1}^N \sqrt{\frac{d_k-1}{2d_k}} \sigma_{max}(M).$$

Inequality (15) can be derived similarly. ■

In [29], the authors have derived a generalized form of the correlation matrix criterion which says that if a quantum state $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ is fully separable, then

$$||\mathcal{T}||_{KF} = ||\mathcal{T}_n||_{KF} \le \Pi_{k=1}^N \sqrt{\frac{d_k-1}{2d_k}}. \tag{22}$$

Here we show that one can obtain the generalized correlation matrix criterion from Theorem 2.

Corollary 3: Inequality (22) is included in theorem 2. Moreover, if quantum state $\rho \in H_1 \otimes H_2 \otimes \cdots \otimes H_N$ is fully separable, then the following inequality holds:

$$||\tilde{\mathcal{T}}||_{KF} = ||\tilde{\mathcal{T}}_n||_{KF} \leq \Pi_{k=1}^N \sqrt{\frac{d_k^2 - d_k + 2}{2d_k^2}}. \tag{23}$$

Proof: Assume that the nth unfold $\mathcal{T}_n$ is just the one to attain the $||\mathcal{T}||_{KF}$. One immediately derives a singular value decomposition of $\mathcal{T}_n$, $\mathcal{T}_n = V_n \Sigma_n U_n^\dagger$ for some orthogonal matrices V_n and U_n. Let M be the tensor with the nth matrix unfolding $M_n = V_n \Pi_n U_n^\dagger$, where $\Pi_n = \begin{pmatrix} I & 0 \end{pmatrix}$, I is the $(d_n^2 - 1) \times (d_n^2 - 1)$ identity matrix and 0 is the zero matrix with order such that Π_n is a $(d_n^2 - 1) \times \frac{\Pi_{k=1}^N (d_k^2 - 1)}{(d_n^2 - 1)}$ matrix. Since both V_n and U_n are orthogonal matrices, the maximal singular value must be 1. From Theorem 2 we have

$$|\sum_{i_1 i_2 \cdots i_N} m_{i_1 i_2 \cdots i_N} \mathcal{T}_{i_1 i_2 \cdots i_N}| = Tr(M_n T_n^\dagger) = Tr(V_n \Pi_n U_n^\dagger U_n \Sigma_n V_n^\dagger)$$

$$= Tr(\Sigma_n) = ||\mathcal{T}||_{KF} \leq \Pi_{k=1}^N \sqrt{\frac{d_k - 1}{2d_k}},$$

which leads to the inequality (22). Inequality (23) can be proved similarly. ∎

Corollary 3 can detect some PPT entangled quantum states in multipartite quantum systems, such as the three-qutrit bound entangled states $\rho_c \otimes |\psi\rangle\langle\psi|$ condidered by L. Clarisse and P. Wocjan [32], where

$$\rho_c = \frac{1}{12} \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the chess-board state and $|\psi\rangle$ is an uncorrelated ancilla. If we mix $\rho_c \otimes |\psi\rangle\langle\psi|$ with white noise and define $\sigma = p\rho_c \otimes |\psi\rangle\langle\psi| + \frac{1-p}{27}I$, the entanglement is detected for $0.83265 < p \leq 1$ by corollary 3.

IV. CONCLUSIONS AND REMARKS

It is a basic and fundamental question to distinguish separable quantum states from entangled ones. Although the quantum separability problem has been shown to be NP-hard, it is possible to derive some necessary criteria of separability. We have derived separability criteria of quantum states for both bipartite and multipartite quantum ones. The criteria are shown to be more efficient in detecting quantum entanglement of some quantum states than the (generalized) criterion based on the correlation matrix, the PPT criterion, the realignment criterion, and the covariance matrix criterion. Similar to the case of previous separability criteria, our criteria can also be used to derive lower bounds for concurrence.

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